LOGARITHM

ASIC MATHEMATICS :

:mainder Theorem :

Let $p(x)$ be any polynomial of degree geater than or equal to one and 'a' be any real numl
If $p(x)$ is divided by $(x-a)$, then the remainder is equal to $p(a)$.

ıctor Theorem :

Let $p(x)$ be a polynomial of degree greater than or equal to 1 and 'a' be a real number such t
$p(a)=0$, then $(x-a)$ is a factor of $p(x)$. Conversely, if $(x-a)$ is a factor of $p(x)$, then $p(a)=0$

te : Let $p(x)$ be any polynomial of degree greater than or equal to one. If leading coefficient of $p(x)$ i
then $p(x)$ is called monic. (Leading coefficient means coefficient of highest power.)

SOME IMPORTANT IDENTITIES :

(1) $(a+b)^2 = a^2 + 2ab + b^2 = (a-b)^2 + 4ab$

(2) $(a-b)^2 = a^2 - 2ab + b^2 = (a+b)^2 - 4ab$

(3) $a^2 - b^2 = (a+b)(a-b)$

(4) $(a+b)^3 = a^3 + b^3 + 3ab(a+b)$

(5) $(a-b)^3 = a^3 - b^3 - 3ab(a-b)$

(6) $a^3 + b^3 = (a+b)^3 - 3ab(a+b) = (a+b)(a^2 + b^2 - ab)$

(7) $a^3 - b^3 = (a-b)^3 + 3ab(a-b) = (a-b)(a^2 + b^2 + ab)$

(8) $(a+b+c)^2 = a^2 + b^2 + c^2 + 2(ab + bc + ca) = a^2 + b^2 + c^2 + 2abc\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)$.

(9) $a^2 + b^2 + c^2 - ab - bc - ca = \frac{1}{2}\left[(a-b)^2 + (b-c)^2 + (c-a)^2\right]$

(10) $a^3 + b^3 + c^3 - 3abc = (a+b+c)(a^2 + b^2 + c^2 - ab - bc - ca)$

$$= \frac{1}{2}(a+b+c)\left[(a-b)^2 + (b-c)^2 + (c-a)^2\right]$$

If $(a+b+c)=0$, then $a^3 + b^3 + c^2 = 3abc$.

(11) $a^4 - b^4 = (a^2 + b^2)(a^2 - b^2) = (a^2 + b^2)(a-b)(a+b)$

(12) If $a, b \geq 0$ then $(a-b) = \left(\sqrt{a} + \sqrt{b}\right)\left(\sqrt{a} - \sqrt{b}\right)$

(13) $a^4 + a^2 + 1 = (a^4 + 2a^2 + 1) - a^2 = (a^2 + 1)^2 - a^2 = (a^2 + a + 1)(a^2 - a + 1)$

Definition of Indices :

The product of m factors each equal to a is represented by a^m. So, $a^m = a \cdot a \cdot a$ .... a (m times). Here a is called the base and m is the index (or power or exponent).

Law of Indices :

(1) $a^{m+n} = a^m \cdot a^n$, where m and n are rational numbers.

(2) $a^{-m} = \dfrac{1}{a^m}$, provided $a \neq 0$.

(3) $a^0 = 1$, provided $a \neq 0$.

(4) $a^{m-n} = \dfrac{a^m}{a^n}$, where m and n are rational numbers, $a \neq 0$.

(5) $(a^m)^n = a^{mn}$.

(6) $a^{\frac{p}{q}} = \sqrt[q]{a^p}$

(7) $(ab)^n = a^n b^n$.

Intervals :

Intervals are basically subsets of R (the set of all real numbers) and are commonly used in solving inequaltities. If $a, b \in R$ such that $a < b$, then we can defined four types of intervals as follows :

Name	**Representation**	**Discription.**
Open interval	(a, b)	$\{x : a < x < b\}$ i.e., end points are not included.
Close interval	$[a, b]$	$\{x : a \leq x \leq b\}$ i.e., end points are also included. This is possible only when both a and b are finite.
Open-closed interval	$(a, b]$	$\{x : a < x \leq b\}$ i.e., a is excluded and b is included.
Closed-open interval	$[a, b)$	$\{x : a \leq x < b\}$ i.e., a is included and b is excluded.

Note :

(1) The infinite intervals are defined as follows :

(i) $(a, \infty) = \{x : x > a\}$

(ii) $[a, \infty) = \{x \mid x \geq a\}$

(iii) $(-\infty, b) = \{x : x < b\}$

(iv) $(-\infty, b] = \{x : x \leq b\}$

(v) $(-\infty, \infty) = \{x : x \in R\}$

(2) $x \in \{1, 2\}$ denotes some particular values of x, i.e., $x = 1, 2$.

(3) If their is no value of x, then we say $x \in \phi$ (i.e., null set or void set or empty set).

Proportion :

When two ratios are equal, then the four quantities compositing then are said to be proportional.

If $\frac{a}{b}=\frac{c}{d}$, then it is written as $a:b=c:d$ or $a:b::c:d$.

Note :

(1) a and d are known as extremes while b and c are known as means.

(2) Product of extremes = product of means.

(3) If $\frac{a}{b}=\frac{c}{d} \Rightarrow \frac{b}{a}=\frac{d}{c}$ (Invertando)

(4) If $\frac{a}{b}=\frac{c}{d} \Rightarrow \frac{a}{c}=\frac{b}{d}$ (Alternando)

(5) If $\frac{a}{b}=\frac{c}{d} \Rightarrow \frac{a}{b}+1=\frac{c}{d}+1 \Rightarrow \frac{a+b}{b}=\frac{c+d}{d}$ (Componendo)

(6) If $\frac{a}{b}=\frac{c}{d} \Rightarrow \frac{a}{b}-1=\frac{c}{d}-1 \Rightarrow \frac{a-b}{b}=\frac{c-d}{d}$ (Dividendo).

(7) If $\frac{a}{b}=\frac{c}{d} \Rightarrow \frac{a+b}{a-b}=\frac{c+d}{c-d}$ (Componendo and dividendo)

(8) If $\frac{a}{b}=\frac{b}{c}$ then $b^2=ac$. Here b is called mean proportional of a and c.

Histrorical Development of Number System :

I. Natural Number's

Number's used for counting are called as Natural number's.

{1, 2, 3, 4, ...}

II. Whole number's

Including zero (0) | cypher | शून्य | duck |love| knot along with natural numbers called as whole numbers.

w = {0,1, 2, 3 ....}

i.e. $N \subset W$

0 is neither positive nor negative

III Integer's

Integer's given by

I = {...........–2, –1, 0, 1, 2, 3.......}

i.e. $N \subset W \subset I$

Type of Integer's

(a) None negative integers	{ 0, 1, 2, 3, ...}
(b) Negative integers (I^-)	{...........–3, –2, –1}
(c) Non positive integers	{...........–3, –2 –1, 0}
(d) Positve integers (I^+)	{1, 2, 3 ....}

IV. Rational Number's

Number's which are of the form p/q where p, q, $\in$ I & $q \neq 0$ called as rational number's.

Rational numbers are also represented by recurring & terminating or repeating decimal's

e.g. $1.\overline{3} = 1.333$ ....

$$x = 1.3333 \ldots$$
$$10x = 13.33\ldots$$
$$9x = 12$$
$$x = \frac{4}{3}$$

Every rational is either a terminating or a recurring decimal

V Irrational number's

The number's which cannot be expressed in the form p/q (p,q $\in$ I) are called as irrational numbers.
The decimal representation of these number is non-terminating and non repeating.

$$\sqrt{2} = 1.414 \ldots\ldots\ldots\ldots$$

π is an irrational number

VI Real Number's

Set of real number's is union of the set of rational number's and the set of irrational numbers.

$$\text{Real} \rightarrow \text{Rational} + \text{Irrational}$$
$$N \subset W \subset I \subset Q \subset R \subset Z$$

VII. Prime Number's

Number's which are devisible by 1 or itself

e.g. {2, 3, 5, 7, 11, 13 ....}

VIII Composite Number's

Number's which are multiples of prime are called composite number's

{4, 6, 8, 9 ....}

IX Coprime or relatively prime number's

The number's having heighest common factor 1 are called relatively prime.

e.g. (2, 9), (16, 25 ......)

X Twin primes :

The prime number's which having the diffrence of 2

e.g. (5, 3), (7, 5), (13,11) ....

1 is niether a prime nor a composite number.

When studying logarithms it is important to note that all the propertise of logarithms are consequences of the corresponding properties of power, which means that sudent should have a good working knowledge of powers are a foundation for tacking logarithms

RITHM :

Definition : Every positive real number N can be expressed in exponential form as

$N = a^x$ ....(1) ***e.g.*** $49 = 7^2$

where 'a' is also a positive real different than unity and is called the base and 'x' is called the expor

We can write the relation (1) in logarithmic form as

$\log_a N = x$ ....(2)

Hence the two relations

$$\left.\begin{array}{r} a^x = N \\ \text{and} \quad \log_a N = x \end{array}\right]$$

re identical where $N > 0, a > 0, a \neq 1$

Hence logarithm of a number to some base is the exponent by which the base must be raised in orc
get that number. Logarithm of zero does not exist and logarithm of (–) ve reals are not defined i
ystem of real numbers.

e a is raised what power to get N

Illustration :

Find value of

(i) $\log_{81}27$ *(ii)* $\log_{10}100$ *(iii)* $\log_{1/3} 9\sqrt{3}$

Sol.(i) *Let* $\log_{81}27 = x$

$\Rightarrow 27 = 81^x$

$\Rightarrow 3^3 = 3^{4x}$ *gives* $x = 3/4$

(ii) *Let* $\log_{10}100 = x$

$\Rightarrow 100 = 10^x$

$\Rightarrow 10^2 = 10^x$ *gives* $x = 2$

(iii) *Let* $\log_{1/3} 9\sqrt{3} = x$

$$\Rightarrow 9\sqrt{3} = \left(\frac{1}{3}\right)^x$$

$\Rightarrow 3^{5/2} = 3^{-x}$ *gives* $x = -5/2$

Note that :

(a) Unity has been excluded from the base of the logarithm as in this case

$\log_1 N$ will not be possible and if $N = 1$

then $\log_1 1$ will have infinitely many solutions and will not be unique which is necessary in the functional notation.

(b) $a^{\log_a N} = N$ is an identify for all $N > 0$ and $a > 0, a \neq 1$ e.g. $2^{\log_2 5} = 5$

(c) The number N in (2) is called the antilog of 'x' to the base 'a'. Hence

If $\log_2 512$ is 9 then $\text{antilog}_2 9$ is equal to $2^9 = 512$

(d) Using the basic definition of log we have 3 important deductions :

(i) $\log_N N = 1$ i.e. logarithm of a number to the same base is 1.

(ii) $\log_{\frac{1}{N}} N = -1$ i.e. logarithm of a number to its reciprocal is – 1.

(iii) $\log_a 1 = 0$ i.e. logarithm of unity to any base is zero.

(basic constraints on number and base must be observed.)

(iv) $a^{\log_a n} = n$ is an identify for all N > 0 and a> 0 ; $a \neq 1$ e.g. $2^{\log_2 5} = 5$

(e) Whenever the number and base are on the same side of unity then logarithm of that number to the base is (+ve), however if the number and base are located an diffrent side of unity then logarithm of that number to the base is (–ve)

e.g. (i) $\log_{10} 100 = 2$

(ii) $\log_{1/10} 100 = -2$

(f) For a non negative number 'a' & $n \geq 2, n \in N$ $\sqrt[n]{a} = a^{1/n}$

Illustration :

(i) $\log_{\sin 30^\circ} \cos 60^\circ = 1$ *(ii)* $\log_{3/4} 1.\overline{3} = -1$ *(iii)* $\log_{2-\sqrt{3}} 2+\sqrt{3} = -1$

(iv) $\log_5 \sqrt{5\sqrt{5\sqrt{5.......\infty}}} = 1$

Sol. *Let* $\sqrt{5\sqrt{5\sqrt{5.....\infty}}} = x$

$\Rightarrow \sqrt{5x} = x \Rightarrow x^2 = 5x \Rightarrow x = 5 \Rightarrow \log_5 5 = 1$

(v) *(log tan 1°) (log tan 2°) (log tan 3°) ....(log tan 89°) = 0*

Sol. *Since tan 45° = 1 thus log tan 45° = 0*

(vi) $7^{\log_7 x} + 2x + 9 = 0$

Sol. $3x + 9 = 0 \Rightarrow (x = -3)$ *as it makes initial problem undefined*

$x = \phi$

(vii) $2^{\log_2 (x-3)} + 2(x-3) - 12 = 0$

Sol. $x - 3 + 2x - 6 - 12 = 0$

$3x = 21 \Rightarrow x = 7$

(viii) $\log_2 (x-3) = 4$

Sol. $x - 3 = 2^4$

$x = 19$

Practice Problem

Q.1 Find the logarithms of the following numbers to the base 2:

(i) $\sqrt[3]{8}$ (ii) $2\sqrt{2}$ (iii) $\frac{1}{\sqrt[5]{2}}$ (iv) $\frac{1}{\sqrt[7]{8}}$

Q.2 Find the logarithms of the following numbers to the base $\frac{1}{3}$

(i) 81 (ii) $\sqrt[3]{3}$ (iii) $\frac{1}{\sqrt[7]{3}}$ (iv) $9\sqrt{3}$ (v) $\frac{1}{9\sqrt[4]{3}}$

Q.3 Find all number a for which each of the following equalities hold true?

(i) $\log_2 a = 2$ (ii) $\log_{10}(a(a+3)) = 1$

(iii) $\log_{1/3}(a^2-1) = -1$ (iv) $\log_2(a^2-5) = 2$

Q.4 Find all values of x for which the following equalities hold true?

(i) $\log_2 x^2 = 1$ (ii) $\log_3 x = \log_3(2-x)$ (iii) $\log_4 x^2 = \log_4 x$

(iv) $\log_{1/2}(2x+1) = \log_{1/2}(x+1)$ (v) $\log_{1/3}(x^2+8) = -2$

Q.5 If $2\left(\sqrt{3+\sqrt{5-\sqrt{13+\sqrt{48}}}}\right) = \sqrt{a} + \sqrt{b}$ where a and b are natural number find $(a+b)$.

Answer key

Q.1 (i) 1, (ii) 3/2, (iii) – 1/5, (iv) – 3/7 Q.2 (i) – 4, (ii) – 1/3, (iii) 1/7, (iv) – 5/2, (v) 9/4

Q.3 (i) 4, (ii) –5, 2, (iii) –2, 2, (iv) –3, 3 Q.4 (i) $\sqrt{2}, -\sqrt{2}$, (ii) 1, (iii) 1, (iv) 0, (v) 1, – 1 Q.5 8

Answer key

Q.1 (i) 1, (ii) 3/2, (iii) – 1/5, (iv) – 3/7 Q.2 (i) – 4, (ii) – 1/3, (iii) 1/7, (iv) – 5/2, (v) 9/4

Q.3 (i) 4, (ii) –5, 2, (iii) –2, 2, (iv) –3, 3 Q.4 (i) $\sqrt{2}, -\sqrt{2}$, (ii) 1, (iii) 1, (iv) 0, (v) 1, – 1 Q.5 8

PRINCIPAL PROPERTISE OF LOGARITHM :

If m, n are arbitrary positive real numbers where

$a > 0$; $a \neq 1$

(1) $\log_a m + \log_a n = \log_a mn$ $(m > 0, n > 0)$

Proof: Let $x_1 = \log_a m$; $m = a^{x}$

$x_2 = \log_a n$; $n = a^{x_2}$

Now $mn = a^{x}$; a^{x_2}

$mn = a^{x_1+x_2}$

$x_1 + x_2 = \log_a mn$

$\log_a m + \log_a n = \log_a mn$

(2) $\log_a \frac{m}{n} = \log_a m - \log_a n$

$\frac{m}{n} = a^{x_1-x_2}$

$x_1 - x_2 = \log_a \frac{m}{n}$

$\log_a m - \log_a n = \log_a \frac{m}{n}$

(3) $\log_a m^x = x \log_a m$

$\log_a m = p$; $m = a^p$

$m^x = a^{px}$

taking log both the side with base a

$\log_a m^x = \log_a a^{px} = px = x \log_a m$

(4) $\log_{a^x} m = \dfrac{1}{x} \log_a m$

Ex :(i) Find the solution of $\mathbf{\log_2 x^2 = 4}$ & $\mathbf{2\log_2 x = 4}$ and verify solutions.

Sol. $\log_2 x^2 = 4$ $\qquad$ $2\log_2 x = 4$

$\Rightarrow$ $x^2 = 16$ $\qquad$ $\Rightarrow$ $\log_2 x = 2$

$\Rightarrow$ $x = \pm 4$ (two solution) $\qquad$ $\Rightarrow$ $x = 4$ only possible soln.

Ex: (ii) $\log_2 x^2 + 2\log_2 x = 8$

Sol. $4\log_2 x = 8$ $\quad (x > 0)$

$\log_2 x = 2$

$x = 4$

BASE CHANGING THEOREM :

Can be stated as "quotient of the logarithm of two numbers is independent of their common base."

Symbolically, $\dfrac{\log_c a}{\log_c b} = \log_b a$

proof Let $\log_c a = x$; $\log_c b = y$ & $\log_b a = z$

$a = c^x$; $b = c^y$, $a = b^z$

$c^x = b^z$

$a = c^x$; $b = c^y$, $a = b^z$

$c^x = b^z$

$c^x = c^{yz} \Rightarrow x = yz$

i.e $z = \dfrac{x}{y}$

$$\boxed{\log_b a = \dfrac{\log_c a}{\log_c b}}$$

e.g. Find value of $\log_{64} 16$

$\log_{64} 16 = \dfrac{\log_4 16}{\log_4 64} = \dfrac{2}{3}$

Case-I : $\log_b a = \dfrac{1}{\log_a b}$

We have proved that $\dfrac{\log_c a}{\log_c b} = \log_b a$

put $c = a$

Similarly $\dfrac{\log_a a}{\log_a b} = \log_b a$

or $\boxed{\dfrac{1}{\log_a b} = \log_b a}$

e.g. prove $\dfrac{1}{\log_a abc} + \dfrac{1}{\log_b abc} + \dfrac{1}{\log_c abc} = 1$

Sol. $\log_{abc} a + \log_{abc} b + \log_{abc} C = \log_{abc} abc = 1$

Case-II : $(\log_b a).(\log_c b).(\log_d c) = \log_d a$

Proof $\dfrac{\log a}{\log b} \times \dfrac{\log b}{\log c}.\dfrac{\log c}{\log d} = \dfrac{\log a}{\log b} = \log_d a$

e.g. $(\log_3 5).(\log_{25} 27) = \dfrac{3}{2}$

Case-III : Very imp form

$a^{\log_b c} = c^{\log_b a}$

Proof $a^{\log_b c} = a^{(\log_b c)\ (\log_a c)\ (\log_c a)} = a^{\log_a c\ (\log_b c.\ \log_c a)} = c^{\log_b a}$

$\Rightarrow \boxed{a^{\log_b c} = c^{\log_b a}}$

Illustration :

$$2^{-\log_{\frac{1}{2}} 7} = 7$$

Sol. $2^{\log_2 7} = 7$

Illustration :

$$8^{\frac{-1}{\log_3 2}} = \frac{1}{27} = 8^{\frac{-1}{\log_3 2}}$$

Sol. $8^{-\log_2 3} = 8^{\log_2 \frac{1}{3}} = \dfrac{1}{27}$

Illustration :

$(\log_2 3)(\log_3 4)(\log_4 5)....\log_n(n+1) = 10$. *Find* $n = ?$

Sol. $\log_2(n+1) = 10$

$n + 1 = 1024\ ;\ n = 1023$

Illustration :

$$\log 2 + 16 \log\frac{16}{15} + 12 \log\frac{25}{24} + 7 \log\frac{81}{80}.$$

Sol. $\log 2 + 16 \log 16 - 16 \log 15 + 12 \log 25 - 12 \log 24 + 7 \log 81 - 7 \log 80$

$= \log 2 + 64 \log 2 - 16 \log 5 - 16 \log 3 + 24 \log 5 - 12 \times 3 \log 2 - 12 \log 3$

$+ 28 \log 3 - 7 \log 5 - 28 \log 2$

$= \log 2 + \log 5 = \log 10 = 1$

Illustration :

$$\frac{1}{\log_3 2} + \frac{2}{\log_9 4} + \frac{3}{\log_{27} 8} = 0$$

Sol. $\log_2 3 + 2 \log_4 9 - 3\log_8 27$

$= 3\log_2 3 - 3\log_2 3 = 0$

Illustration :

Let $a > 1$ be a real number then solve

$$a^{2\log_2 x} = 5 + 4x^{\log_2 a}$$

Sol. $(x^{\log_2 a})^2 - 4(x^{\log_2 a}) - 5 = 0 \quad ; \qquad t^2 - 4t - 5 = 0$

$t = 5 \quad x^{\log_a 2} = 5 \qquad (\log_2 a).\log_5 x = 1$

Take log w.r.t. base 5.

$\log_5 x = \log_a 2$

$x = (5)^{\log_a 2}$

Illustration :

Prove that $2^{\sqrt{\log_2 3}} = 3^{\sqrt{\log_3 2}}$

Sol. $2^{\sqrt{\log_2 3}} = 2^{(\log_2 3).\frac{1}{\sqrt{\log_2 3}}} = \left(2^{\log_2 3}\right)^{\frac{1}{\sqrt{\log_2 3}}} = 3^{\frac{1}{\sqrt{\log_2 3}}} = 3^{\sqrt{\log_3 2}}$

Illustration :

If $a > 0; c > 0$; $b = \sqrt{ac}$; $ac \neq 1$

Prove that $\quad \dfrac{\log_a N}{\log_c N} = \dfrac{\log_a N - \log_b N}{\log_b N - \log_c N}$

Sol. L.H.S. $= \dfrac{\log_a N}{\log_c N} = \dfrac{\log_N c}{\log_N a} = \log_a c$

R.H.S. $= \dfrac{\log_a N - \log_b N}{\log_b N - \log_c N}$

$= \dfrac{(\log_N b - \log_N a)}{\log_N c - \log_N b} \times \dfrac{\log_N c}{\log_N b} \times \dfrac{\log_N b}{\log_N a}$

$= \dfrac{\log_N b/a}{\log_N c/b} \times \log_a c$

$= \log_{c/b} b/a \times \log_a c$

$= \left(\log_{c/a}^{c/a}\right)\log_a c$

$= \log_a c \qquad$ *Ans.*

$$\left(\begin{array}{l} b = \sqrt{ac} \\ b^2 = ac \\ \dfrac{b}{a} = \dfrac{c}{b} \end{array}\right)$$

LOGARITHMIC EQUATIONS :

Illustration :

Prove that $x^2 + 7^{\log_7 x} - 2 = 0$

Sol. $\Rightarrow x^2 + x - 2 = 0 \qquad \left[a^{\log_a x} = x(>0)\right]$

$= \ (x + 2)\ (x - 1) = 0$

$= \ x = -2$ or 1

$=$ By definition, $x > 0$

$\therefore \quad x = 1$

Illustration :

Find the value of x : $(x+1)^{\log_{10}(x+1)} = 100(x+1)$

Sol. *By definition* $(x+1) > 0$

$\therefore$ *Taking log with base 10 both side*

$= \log (x+1)^{\log_{10}(x+1)} = \log 100(x+1)$

$= \log_{10}(x+1).\ \log_{10}(x+1) = \log_{10} 100 + \log_{10}(x+1)$

$= (\log_{10}(x+1))^2 - \log (x+1) = 2$

$=$ let $\log_{10}(x+1) = y$

$= y^2 - y - 2 = 0$

$= (y-2)(y+1) = 0$

$= y = 2$ or -1

$= \log_{10}(x+1) = 2$ *or* $\log_{10}(x+1) = -1$

$\therefore\ x + 1 = 100$ *or* $x + 1 = 10^{-1} = 0.1$

$x = 99$ *or* $x = -0.9$

Illustration :

Find the value of x : $3^{\log_3^2 x} + x^{\log_3 x} = 162$

Sol. $\left(3^{\log_3 x}\right)^{\log_3 x} + x^{\log_3 x} = 162$

$\Rightarrow\ x^{\log_3 x} + x^{\log_3 x} = 162$ $\quad \left[a^{\log_a x} = x\right]$

$\Rightarrow\ x^{\log_3 x} = 81$

Taking log both side with base 3

$\Rightarrow\ x^{\log_3 x} = 81$

Taking log both side with base 3

$\Rightarrow\ (\log_3 x)^2 = \log_3 81 = 4 \quad \Rightarrow \quad \log_3 x = \pm 2$

$\Rightarrow\ x = 9$ or $\dfrac{1}{9}$

Illustration :

Find the value of x : $\log_5(5^{1/x} + 125) = \log_5(6) + 1 + \dfrac{1}{2x}$.

Sol. $= \log_5(5^{1/x} + 125) = \log_5 6 + \left(1 + \dfrac{1}{2x}\right)$

$= \log_5\left[\dfrac{5^{1/x} + 125}{6}\right] = \left(1 + \dfrac{1}{2x}\right)$

$= 5^{1/x} + 125 = (5^1 \cdot 5^{1/2x}) \cdot 6$

$= 5^{1/x} + 125 = 5 \cdot 6 \cdot 5^{1/2x}$

$=$ let $5^{\frac{1}{2x}} = y$

$= y^2 - 30y + 125 = 0$

$= (y-25)(y-5) = 0$

$= y = 5$ *or* 25

$\therefore\ 5^{\frac{1}{2x}} = 5^1$ *or* 5^2

$\therefore\ x = \dfrac{1}{2}$ *or* $\dfrac{1}{4}$

Note : [If given problem is $log_5\left(\sqrt[x]{5}+125\right)=\log_5(6)+1+\frac{1}{2x}$ then the equation will have no solution since for $\sqrt[x]{5}$, $x \in N$ and $N \geq 2$]

Illustration :

Prove that : $log_2 7$ is irrational.

Sol. *Let $log_2 7$ is Rational.*

So $log_2 7 = a/b$ where (a & b are integers & $b \neq 0$)

$$= \frac{log_{10} 7}{log_{10} 2} = \frac{a}{b}$$

$$= b\ log_{10} 7 = a\ log_{10} 2$$

$$= 7^b = 2^a \qquad \text{(7 \& 2 are co-prime)}$$

So there is no such type of integers for a & b so there is a contradiction.

Common and natural logarithm :

$\log_{10}N$ is referred as a common logarithm and $\log_e N$ is called as natural logarithm or logarithm of N to the base Napierian and is popularly written as ln N. Note that e is an irrational quantity lying between 2.7 to 2.8 which you will study later. **Note that** $e^{ln\,x} = x$

Characteristic and Mantissa :

We observe that $\log_{10}10 = 1$ and $\log_{10}100 = 2$.

Hence logarithm of a number lying between 10 to 100 = 1 + a positive quantity

$\log_{10}(0.1) = -1$ and $\log_{10}(0.01) = -2$

hence log (a number between 0.01 to 0.1) = – 2 + a positive quantity

Hence the common logarithm of a number consists of two parts, integral and fractional, of which the integral part may be zero or an integer (+ve or –ve) and the fractional part, a decimal, less than one and always positive.

The integral part is called the <u>characteristic</u> and the decimal part is called the <u>mantissa</u>.

e.g. $\log_{10}33.8 = 1.5289 \Rightarrow 33.8 = 10^{1.5289} = 10.10^{0.5289}$

$\log_{10}0.338 = -1 + 0.5289 = \bar{1}.5289$

It should be noted that, if the characteristic of the logarithm of N is

$$\left.\begin{array}{lll} 1 & \Rightarrow & \text{that N has two significant digits before decimal.} \\ 2 & \Rightarrow & \text{that N has three significant digits before decimal.} \end{array}\right\} \textit{very Important}$$

(Hence number of significant digit in N = p + 1 if p is the non negative characteristic of log N.)

if characteristic

– 1 ⇒ N has no zeros after decimal before a significant digit starts

– 2 ⇒ N has 1 zero after decimal before a significant digit starts and so on.

Using log 2 = 0.3010 and log 3 = 0.4771, and log 7 = 0.8451

Illustration :

Find the number of digits $(2.5)^{200}$

Sol. Let $N = (2.5)^{200}$

Taking log both side with base 10

$$\log_{10}N = 200\log_{10}(2.5) = 200\log_{10}\left(\frac{5}{2}\right) = 200\,[\log_{10}5 - \log_{10}2]$$

$$= 200\,[1 - 2\log_{10}2] = 200\,[1 - 2 \times 0.3010] = 200\,[0.3990] = 79.80$$

Characteristic = 79

Number of digits = 79 + 1 = 80

Illustration :

Find the number of digits 6^{50}

Sol. Let $N = 6^{50}$

Taking log both side with base 10

$$\log N = 50\,[\log_{10}6]$$

$$= 50\,[\log_{10}2 + \log_{10}3] = 50\,[0.3010 + 0.4771] = 50\,[0.7781] = 38.9050$$

Characteristic = 38

Number of digits = 38 + 1 = 39

Number of digits = 38 + 1 = 39

Illustration :

Find the number of digits 5^{25}

Sol. Let $N = 5^{25}$

Taking log both side with base 10

$$\log N = 25\log_{10}5 = 25\,[1 - \log_{10}2] = 25\,[1 - 0.3010] = 17.4750$$

Characteristic = 17

Number of digits = 17 + 1 = 18

Illustration :

Find the number of zeros after decimal before a significant figure start in $\left(\frac{9}{8}\right)^{-100}$

Sol. Let $N = \left(\frac{9}{8}\right)^{-100}$

Taking log both side with base 10

$$\log_{10}\left(\frac{9}{8}\right)^{-100} = -100\,[\log_{10}9/8] = -100\,[\log_{10}9 - \log_{10}8] = -100\,[2\log_{10}3 - 3\log_{10}2]$$

$$= -100\,[2 \times 0.4771 - 3 \times 0.3010] = -100\,[0.0512] = -5.12$$

Characteristic = – 6

Number of zero after decimal before a significant figure start = 5

ustration :

Find the number of zeros after decimal before a significant figure start in 3^{-50}.

l. *Let $N = 3^{-50}$*

Taking log both side with base 10

$$\log_{10}3^{-50} = -50\log_{10}3 = -50\,(0.4771) = -23.8550$$

Characteristic = – 24

Number of zeros after decimal befores a significant figure start = 23

ustration :

Find the number of zeros after decimal before a significant figure start in $(0.35)^{12}$

l. *Let $N = (0.35)^{12}$*

Taking log both side with base 10

$$\log_{10}(0.35)^{12} = 12\log_{10}(0.35) = 12\log_{10}\frac{7}{20} = 12\,[\log_{10}7 - \log_{10}20]$$

$$= 12\,[\log_{10}7 - \log_{10}2 - 1] = 12\,[0.8451 - 0.3010 - 1] = -5.4708$$

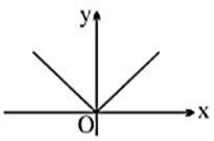
Characteristic = –6

Number of zeros after decimal befores a significant figure start = 5]

bsolute value function :

Absolute value function :

(a) $y = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$



(b) $\sqrt{x^2} = |x|$

(c) $\log x^{2n} = 2n\log|x|$, where $n \in I$

General Note : Equations of the form

$[a(x)]^{b(x)} = [a(x)]^{c(x)}$ (Variable exponent on a variable base)

with the set of permissible values defined by the condition $a(x) > 0$, can be reduced to the equivalent equation

$$b(x)\log_d[a(x)] = c(x)\log_d[a(x)]$$

by taking logarithms of its both sides. The last equation is equivalent to two equations.

$\log_d[a(x)] = 0,\ b(x) = c(x).$

e.g. $|x-2|^{10x^2-1} = |x-2|^{3x}$

Sol. Taking log both side w.r.t. base 10

$$\log_{10}|x-2|^{10x^2-1} = \log|x-2|^{3x}$$

$\Rightarrow (10x^2 - 1)\log|x-2| = 3x\log|x-2|$

$\Rightarrow \log|x-2|\,(10x^2 - 3x - 1) = 0$

$\Rightarrow \log|x-2| = 0$ or $10x^2 - 3x - 1 = 0$

$\Rightarrow |x-2| = 1$ or $(2x-1)(5x+1) = 0$

$\Rightarrow x - 2 = \pm 1$

$x = 3\,;\,1$ $\qquad$ $x = 1/2, -1/5$

Illustration :

Solve for x : $|3x-2|+x=11$

Sol. $|3x-2| = 11 - x$ $\qquad [11 - x \geq 0]$

<u>Case</u> 1 : $3x - 2 \geq 0$

$\Rightarrow \quad 3x - 2 = 11 - x$

$\Rightarrow \quad 4x = 13$

$\Rightarrow \quad x = 13/4$

<u>Case</u> 2 : $3x - 2 < 0$

$\Rightarrow \quad -(3x-2) = 11 - x$

$\Rightarrow \quad -3x + 2 = 11 - x$

$\Rightarrow \quad 2x = -9$

$\Rightarrow \quad x = -9/2$

Illustration :

Solve for x : $|x| - |x-2| = 2$

Sol. *Case 1:* $x \geq 2 \quad \Rightarrow (x) - (x-2) = 2 \quad \Rightarrow \quad 2 = 2$

for all $x \geq 2$

Case 2 : $0 \leq x < 2$

$\Rightarrow x + x - 2 = 2 \quad \Rightarrow 2x = 4 \quad \Rightarrow x = 2$

No solution $\qquad (0 \leq x < 2)$

Case 3 : $x < 0$

$\Rightarrow -x + x - 2 = 2 \quad \Rightarrow \quad -2 = 2$

Not possible

$\therefore \quad x \in [2, \infty]$

$\Rightarrow -x + x - 2 = 2 \quad \Rightarrow \quad -2 = 2$

Not possible

$\therefore \quad x \in [2, \infty]$

Illustration :

Solve for x : $|x-3|^{3x^2-10x+3} = 1$

Sol. *Taking log both side with 10*

$\Rightarrow \quad \log |x-3|^{3x^2-10x+3} = \log 1$

$\Rightarrow \quad (3x^2 - 10x + 3) \log_{10}|x-3| = 0 \qquad [x \neq 3]$

$\Rightarrow \quad (3x-1)(x-3)\log_{10}|x-3| = 0$

$\Rightarrow \quad x = \frac{1}{3}, \; |x-3| = 1$

again when $x > 3$

$\Rightarrow \quad x - 3 = 1$

$\Rightarrow \quad x = 4$

when $x < 3$

$-(x-3) = 1$

$\Rightarrow \quad x = 2$

$\therefore \quad x = \frac{1}{3}, 2, 4$

Illustration :

Solve for x : $\log_4(x^2 - 1) - \log_4(x - 1)^2 = \log_4\sqrt{(4-x)^2}$

Sol. $\log_4 (x^2 - 1) - \log_4 (x - 1)^2 = \log_4\sqrt{(4-x)^2}$ $\quad [x \neq [-1, 1] \cup \{4\}]$

$\log_4 (x - 1)(x + 1) - \log_4 (x + 1)^2 = \log_4 |4 - x|$

$\log_4 \left[\dfrac{(x-1)(x+1)}{(x+1)^2}\right] = \log_4 |4 - x|$

$\Rightarrow \quad \dfrac{x+1}{x-1} = |4 - x|$

$\Rightarrow \quad (x + 1) = (x - 1)|x - 4| \quad (\because |4 - x| = |x - 4|)$

Case (i) : $x \geq 4$

$\Rightarrow \quad (x + 1) = (x - 1)(x - 4)$

$\Rightarrow \quad x + 1 = x^2 - 5x + 4$

$\Rightarrow \quad x^2 - 6x + 3 = 0$

$\Rightarrow \quad x = \dfrac{6 \pm \sqrt{24}}{2}$

$x = 3 \pm \sqrt{6}$

$x \neq 3 - \sqrt{6}$ (not in domain)

$\Rightarrow \quad x = 3 + \sqrt{6}$

Case (ii) : $x < 4$

$\Rightarrow (x + 1) = (x - 1)(4 - x)$

$\Rightarrow x + 1 = 4x - x^2 - 4 + x$

$\Rightarrow x^2 - 4x + 5 = 0$

Discriminant < 0

$\because x^2 - 4x + 5 \neq 0$

= no solution

from case (i) and case (ii)

$= \quad x = 3 + \sqrt{6}$

$x = 3 \pm \sqrt{6}$ $\quad \because x^2 - 4x + 5 \neq 0$

$x \neq 3 - \sqrt{6}$ (not in domain) $\quad$ = no solution

$\Rightarrow \quad x = 3 + \sqrt{6}$

from case (i) and case (ii)

$= \quad x = 3 + \sqrt{6}$

Illustration :

Solve for x : $2\log_3(x - 2) + \log_3(x - 4)^2 = 0$

Sol. $2\log_3 (x - 2) + \log_3 (x - 4)^2 = 0$ $\quad [x - 2 > 0, x \neq 4]$

$\Rightarrow \quad 2\log_3 (x - 2) + 2\log_3 |x - 4| = 0$

$\Rightarrow \quad 2(\log_3 (x - 2)|x - 4|) = 0$

$\Rightarrow \quad (x - 2)|x - 4| = 1$

Case (i) : $x \geq 4$

$\Rightarrow \quad (x - 2)(x - 4) = 1 \quad \Rightarrow \quad x^2 - 6x + 7 = 0$

$\Rightarrow \quad x = \dfrac{6 \pm \sqrt{6}}{2} \quad \Rightarrow \quad x = 3 \pm \sqrt{2}$

$\Rightarrow \quad x = 3 - \sqrt{2}$ (Not in domain)

$\Rightarrow \quad x = 3 + \sqrt{2}$

Case (ii): $x < 4$

$\Rightarrow \quad (x - 2)(4 - x) = 1 \quad \Rightarrow \quad 4x - x^2 - 8 + 2x = 1$

$\Rightarrow \quad x^2 - 6x + 9 = 0 \quad \Rightarrow \quad (x - 3)^2 = 0$

$x = 3$

from case (i) and case (ii)

$x \{3, 3+\sqrt{2}\}$

Solved Examples

Q.1 Find the value of x satisfying $\log_{10}(2^x + x - 41) = x(1 - \log_{10}5)$.

Sol. We have,

$$\log_{10}(2^x + x - 41) = x(1 - \log_{10}5)$$

$$\Rightarrow \quad \log_{10}(2^x + x - 41) = x\log_{10}2 = \log_{10}(2^x)$$

$$\Rightarrow \quad 2^x + x - 41 = 2^x \Rightarrow x = 41. \text{ Ans.}$$

Q.2 If the product of the roots of the equation, $x^{\left(\frac{3}{4}\right)(\log_2 x)^2 + \log_2 x - \left(\frac{5}{4}\right)} = \sqrt{2}$ is $\frac{1}{\sqrt[b]{a}}$ (where $a, b \in N$) then the value of $(a + b)$.

Sol. Take log on both the sides with base 2

$$\left(\frac{3}{4}(\log_2 x)^2 + \log_2 x - \frac{5}{4}\right)\log_2 x = \frac{1}{2}$$

$\log_2 x = y$

$$3y^3 + 4y^2 - 5y - 2 = 0 \Rightarrow \quad 3y^2(y-1) + 7y(y-1) + 2(y-1) = 0$$

$$\Rightarrow \quad (y-1)(3y^2 + 7y + 2) = 0 \quad \Rightarrow \quad (y-1)(3y+1)(y+2) = 0$$

$$\Rightarrow \quad y = 1 \text{ or } y = -2 \text{ or } y = \frac{-1}{3}$$

$$\therefore \quad x = 2;\ \frac{1}{4};\ \frac{1}{2^{1/3}} \Rightarrow x_1x_2x_3 = \frac{1}{\sqrt[3]{16}} \Rightarrow a + b = 19$$

Q.3 For $0 < a \neq 1$, find the number of ordered pair (x, y) satisfying the equation $\log_{a^2}|x + y| = \frac{1}{2}$ and

$$\therefore \quad x = 2;\ \frac{1}{4};\ \frac{1}{2^{1/3}} \Rightarrow x_1x_2x_3 = \frac{1}{\sqrt[3]{16}} \Rightarrow a + b = 19$$

Q.3 For $0 < a \neq 1$, find the number of ordered pair (x, y) satisfying the equation $\log_{a^2}|x + y| = \frac{1}{2}$ and

$\log_a y - \log_a|x| = \log_{a^2}4$.

Sol. We have $\log_{a^2}|x + y| = \frac{1}{2} \quad \Rightarrow \quad |x + y| = a \quad \Rightarrow \quad x + y = \pm a \qquad .....(1)$

Also, $\log_a\left(\frac{y}{|x|}\right) = \log_{a^2}4 \quad \Rightarrow \quad y = 2|x| \qquad .....(2)$

If $x > 0$, then $x = \frac{a}{3}$, $y = \frac{2a}{3}$

If $x < 0$, then $y = 2a$, $x = -a$

$\therefore$ possible ordered pairs $= \left(\frac{a}{3}, \frac{2a}{3}\right)$ and $(-a, 2a)$

Q.4 The system of equations

$$\log_{10}(2000xy) - \log_{10}x \cdot \log_{10}y = 4$$

$$\log_{10}(2yz) - \log_{10}y \cdot \log_{10}z = 1$$

and $\log_{10}(zx) - \log_{10}z \cdot \log_{10}x = 0$

has two solutions (x_1, y_1, z_1) and (x_2, y_2, z_2). Find $(y_1 + y_2)$.

Sol. From (1),

$$3 + \log_{10}(2xy) - \log_{10}x \cdot \log_{10}y = 4 \qquad ....(i)$$

or $\log_{10}(xy) - \log_{10}x \cdot \log_{10}y = 1 - \log_{10}(2)$

From (2)

$$\log_{10}(yz) - \log_{10}y \cdot \log_{10}z = 1 - \log_{10}(2) \qquad ....(ii)$$

From (i) and (ii), we get

$$\log x + \log y - \log x \cdot \log y = \log y + \log z - \log y \cdot \log z$$

$\Rightarrow \quad \log x\,(1 - \log y) = \log z\,(1 - \log y) \;\Rightarrow\; (\log x - \log z)(1 - \log y) = 0$

$\therefore \quad$ Either, $\log x = \log z$

or $\quad \log_{10}y = 1 \;\Rightarrow\; y = 10$

but $y = 10$ does not satisfy equation (1), hence rejected.

$\therefore \quad \log x = \log z$

From (3), we get

$$(\log_{10}x)^2 = 2(\log_{10}x) \;\Rightarrow\; \log_{10}x\,[\log_{10}x - 2] = 0$$

$\therefore \quad x = 1$ or $x = 100$

if $\; x = z = 1$ then $y = 5 \;\Rightarrow\; (x_1, y_1, z_1) \equiv (1, 5, 1)$

$x = z = 100$ then $y = 20 \;\Rightarrow\; (x_2, y_2, z_2) \equiv (100, 20, 100)$

Hence $(y_1 + y_2) = 25$ Ans.

Q.5 A circle has a radius of $\log_{10}(a^2)$ and a circumference of $\log_{10}(b^4)$. The value of $\log_a b$ is equal to

(A) $\frac{1}{4\pi}$ (B) $\frac{1}{\pi}$ (C) π (D) 2π

Sol. $C = 4\log_{10}b = 2\pi r$

$\therefore \quad 4\log_{10}b = 2\pi \cdot 2\log_{10}a \qquad (\text{as } r = 2\log_{10}a)$

$$\frac{\log_{10} b}{\log_{10} a} = \pi$$

$\therefore \quad \log_a b = \pi \qquad$ Ans. (C)

$\therefore \quad 4\log_{10}b = 2\pi \cdot 2\log_{10}a \qquad (\text{as } r = 2\log_{10}a)$

$$\frac{\log_{10} b}{\log_{10} a} = \pi$$

$\therefore \quad \log_a b = \pi \qquad$ Ans. (C)

Q.6 If $\log_{10}\sin x + \log_{10}\cos x = -1$ and $\log_{10}(\sin x + \cos x) = \frac{(\log_{10} n) - 1}{2}$ then the value of 'n' is

(A) 24 (B) 36 (C) 20 (D) 12

Sol. Given $\log_{10}\left(\frac{\sin 2x}{2}\right) = -1$

$$\Rightarrow \quad \frac{\sin 2x}{2} = \frac{1}{10} \;\Rightarrow\; \sin 2x = \frac{1}{5} \qquad ....(1)$$

Also $\quad \log_{10}(\sin x + \cos x) = \dfrac{\log_{10}\left(\frac{n}{10}\right)}{2}$

$$\Rightarrow \quad \log_{10}(\sin x + \cos x)^2 = \log_{10}\left(\frac{n}{10}\right)$$

$$\Rightarrow 1 + \sin 2x = \frac{n}{10} \;\Rightarrow\; 1 + \frac{1}{5} = \frac{n}{10}$$

$$\Rightarrow \frac{6}{5} = \frac{n}{10} \Rightarrow n = 12 \qquad \text{Ans. (D)}$$

Q.7 The ratio $\dfrac{2^{\log_{2^{1/4}} a} - 3^{\log_{27}(a^2+1)^3} - 2a}{7^{4\log_{49} a} - a - 1}$ simplifies to

(A) $a^2 - a - 1$ (B) $a^2 + a - 1$ (C) $a^2 - a + 1$ (D) $a^2 + a + 1$

Hint: $\dfrac{2^{\log_{2^{1/4}} a} - 3^{\log_{27}(a^2+1)^3} - 2a}{7^{4\log_{49} a} - a - 1} = \dfrac{2^{4\log_2 a} - 3^{3\log_{3^3}(a^2+1)}}{7^{4\log_{7^2} a} - a - 1}$

$= \dfrac{a^4 - a^2 - 2a - 1}{a^2 - a - 1}$ $[a^4 - (a-1)^2$ Ans. (D)

Q.8 The number of values of x satisfying the equation $2^{\log_5 16 \cdot \log_4 x + \log_{x\sqrt{2}} 5} + 5^x + x^{\log_3 4 + 5} + x^5 = 0$ is :

(A) 0 (B) 1 (C) 2 (D) 3

Sol. $2^{\log_5 16 \cdot \log_4 x + \log_{x\sqrt{2}} 5} + 5^x + x^{\log_3 4 + 5} + x^5 = 0$

$2^{2\log_5 4 \cdot \log_4 x + x\log_2 5} + 5^x + x^{\log_5 4} \cdot x^5 + x^5 = 0$

$2^{2\log_5 x}\, 2^{x\log_2 5} + 5^x + x^{2\log_5 2} \cdot x^5 + x^5 = 0$

$\left(2^{\log_5 x}\right)^2 \cdot 5^x + 5^x + \left(2^{\log_5 x}\right)^2 \cdot x^5 + x^5 = 0$

$5^x\left[\left(2^{\log_5 x}\right)^2 + 1\right] + x^5\left[\left(2^{\log_5 x}\right)^2 + 1\right] = 0$

$\left(5^x + x^5\right)\left[\left(2^{\log_5 x}\right)^2 + 1\right] = 0$

$5^x\left[\left(2^{\log_5 x}\right)^2 + 1\right] + x^5\left[\left(2^{\log_5 x}\right)^2 + 1\right] = 0$

$\left(5^x + x^5\right)\left[\left(2^{\log_5 x}\right)^2 + 1\right] = 0$

$5^x + x^5 = 0$ $\qquad$ $\left(2^{\log_5 x}\right)^2 + 1 = 0$

This possible only when x will be –ve $\qquad$ No solution

while according to question $x \geq 2$

$\therefore$ number of values of x = zero $\qquad$ Ans. (A)

Q.9 The number $N = 6\log_{10}2 + \log_{10}31$, lies between two successive integers whose sum is equal to

(A) 5 (B) 7 (C) 9 (D) 10

Hint: $N = \log_{10}64 + \log_{10}31 = \log_{10}1984$

$\therefore \quad 3 < N < 4 \quad \Rightarrow \quad 7$ Ans. $\qquad$ Ans. (B)

Q.10 If $2^a = 7^b$ then number of ordered pairs (a, b) of real numbers is

(A) zero (B) one (C) two (D) more than 2

Sol. $2^a = 7^b \Rightarrow a = b = 0$ if a and b are integers

In case a and b are not integers then

$2^{\log_2 7} = 7^b \Rightarrow a = \log_2 7$ and $b = 1$

or $2^{\log_2 49} = 7^b \Rightarrow a = \log_2 49$ and $b = 2$

or $2^a = 7^{\log_7 2}$

$\Rightarrow a = 1$ and $b = \log_7 2$

$\therefore$ Infinite solutions. $\quad$ Ans.(D)

Q.11 The number $2^{2\log_2\left(3^{3\log_3 4}\right)}$ simplifies as :

(A) 12 (B) 16 (C) 24 (D) 72

Sol. $2^{2\log_2\left(3^{3\log_3 4}\right)} = 2^{2\log_2\left(3^{\log_3 4^3}\right)}$

$= 2^{2\log_2\left(4^3\right)} = 2^{\log_2\left(4^3\right)^2} = 2^{\log_2\left((2)^2\right)^6} = 2^{12}$. Ans. (A)

Q.12 If $\log_2\log_3\log_4\log_5 A = x$, then the value of A is

(A) 120^x (B) 2^{60x} (C) $2^{3^{4^{5^x}}}$ (D) $5^{4^{3^{2^x}}}$

Sol. $\log_3\log_4\log_5 A = 2^x$

$\log_4\log_5 A = 3^{2^x}$

$\log_5 A = 4^{3^{2^x}}$

$A = 5^{4^{3^{2^x}}}$.

Q.13 Suppose $\log_a 2 = m$, $\log_a 3 = r$, $\log_a 5 = s$ and $\log_a 11 = t$. The value of $\log_a 990$, is

(A) 2mrst (B) m + 2r + s + t (C) m + r + s + t (D) m + 2r +5 + t

Sol. $a^m = 2$; $a^r = 3$; $a^s = 5$ and $a^t = 11$

now, $\log_a 990 = \log_a 11 + 2\log_a 3 + \log_a 2 + \log_a 5 = t + 2r + m + s$. Ans. (D)

Q.14 Suppose $\log_{32} p = q$ and $\log_2 r = q$ for positive numbers p, q and r. Which of the following must be true?

I. $5\log_2 r = \log_2 p$ II. $p = \dfrac{q}{5}$ III. $5\log_{32} r = \log_2 p$

(A) Only (I) must be true (B) Only (II) must be true

(C) Only (III) must be true (D) none of the statements must be true

Q.15 Solve following log equation $\log_4(2\log_3(1+\log_2(1+3\log_3 x))) = \dfrac{1}{2}$

Sol. $\Rightarrow \dfrac{1}{2}\log_2(2\log_3(1+\log_2(1+3\log_3 x))) = \dfrac{1}{2}$

$\Rightarrow 2\log_3(1+\log_2(1+3\log_3 x)) = 2^1 = 2$ $\Rightarrow (1+\log_2(1+3\log_3 4) = 3^1$

$\Rightarrow \log_2(1+3\log_3 x) = 3-1 = 2$ $\Rightarrow 1+3\log_3 x = 2^2$

$\Rightarrow 3\log_3 x = 3$ $\Rightarrow \log_3 x = 1$

$\Rightarrow x = 3^1 = 3$ $\therefore$ Final solution $x \in \{3\}$

Q.16 Solve following log equation $3^{\log_3 \log\sqrt{x}} - \log x + \log^2 x - 3 = 0$

Sol. $\log\sqrt{x} - \log x + \log^2 x - 3 = 0$

$\Rightarrow \quad \frac{1}{2}\log_{10} x - \log_{10} x + (\log_{10} x)^2 - 3 = 0$

$\Rightarrow \quad \log_{10}x - 2\log_{10} x + 2(\log_{10}x)^2 - 6 = 0 \Rightarrow \quad 2(\log_{10}x)^2 - \log_{10}x - 6 = 0$

$\Rightarrow \quad 2(\log_{10}x)^2 - 4\log_{10}x + 3\log_{10}x - 6 = 0 \Rightarrow \quad (2\log_{10}x + 3)(\log_{10}x - 2) = 0$

$\Rightarrow \quad \log_{10}x = 2 \quad$ or $\quad \log_{10}x = -\frac{3}{2}$

$\Rightarrow \quad x = 10^2 = 100 \quad$ or $\quad x = 10^{-3/2}$

Q.17 Solve following log equation $(x-2)^{\log^2(x-2)+\log(x-2)^5-12} = 10^{2\log(x-2)}$

Sol. $\log\,(x-2)^{\log^2(x-2)+\log(x-2)(5-12)} = \log 10^{2\log(x-2)}$ (Let $\log(x-2) = t$)

$\Rightarrow \quad \log(x-2)\,(\log^2(x-2) + \log(x-2)^5 - 12)) = \log_{10}10^{2\log(x-2)}$

$\Rightarrow \quad t\,[t^2 + 5t - 12)] = 2t \quad \Rightarrow \quad t^2 + 5t^2 - 14t = 0 \quad \Rightarrow \quad t(t^2 + 5t - 14) = 0$

$\Rightarrow \quad t(t+7)(t-2) = 0 \quad \Rightarrow \quad t = 0, -7, 2 \quad \Rightarrow \quad \log(x-2) = 0, -7, 2$

$\Rightarrow \quad x = 3, 2 + 10^{-7}, 102$

Q.18 Solve following log equation $x^{\frac{\log x+5}{3}} = 10^{5+\log x}$

Q.18 Solve following log equation $x^{\frac{\log x+5}{3}} = 10^{5+\log x}$

Sol. $\log x \,.\, \frac{\log x + 5}{3} = \log_{10} 10^{5+\log x} \quad \Rightarrow \quad \log x \,.\, \frac{\log x + 5}{3} = 5 + \log x$

$\Rightarrow \quad \log^2 x + 5\log_{10}x = 15 + 3\log_{10} x \quad \Rightarrow \quad \log^2 x + 2\log_{10}x - 15 = 0$

$\Rightarrow \quad \log_{10}x(\log_{10}x + 5) - 3(\log_{10} x + 5) = 0 \quad \Rightarrow \quad (\log_{10}x - 3)(\log_{10} x + 5) = 0$

$\Rightarrow \quad \log_{10}x = 3 \qquad \log_{10}x = -5$

$x = 10^3 \qquad x = 10^{-5}$

Q.19 Solve following log equation $x^{\frac{\log x+7}{4}} = 10^{(\log x)+1}$

Sol. taking log both the side with base 10

$\Rightarrow \quad \frac{\log x + 7}{4}.\log x = \log x + 1 \quad \Rightarrow \quad \log_{10}^2 x + 3\log_{10}x - 4 = 0$

$\Rightarrow \quad \log_{10}^2 x + 4\log_{10}x - \log_{10}x - 4 = 0 \quad \Rightarrow \quad (\log_{10}x + 4)(\log_{10}x - 1) = 0$

$\Rightarrow \quad x = 10^{-4}$ and $x = 10^1$

Q.20 Solve following log equation $\log^2 x - 3\log x = \log(x^2) - 4$

Sol. $\log^2 x - 3\log x - 2\log x + 4 = 0 \quad \Rightarrow \quad \log_{10}{}^2 x - 5\log_{10}x + 4 = 0$

$\Rightarrow \quad \log_{10}{}^2 x - \log_{10}x - 4\log_{10}x + 4 = 0 \quad \Rightarrow \quad (\log_{10}x - 4)(\log_{10}x - 1) = 0$

$\Rightarrow \quad \log_{10}x = 4 \qquad \log_{10} x = 1$

$\Rightarrow \quad x = 10^4 \qquad x = 10$

$\Rightarrow \quad x = 10$ or 10^4

Q.21 Solve following log equation $\log_{1/3}x - 3\sqrt{\log_{1/3} x} + 2 = 0$

Sol. $-\log_3 x - 3\sqrt{-\log_3 x} + 2 = 0$

$\Rightarrow (2 - \log_2 x)^2 = \left(3\sqrt{-\log_3 x}\right)^2$

$\Rightarrow 4 + \log_2{}^2 x - 4\log_3 x = -9\log_3 x$

$\Rightarrow (\log_3 x + 4)(\log_3 x + 1) = 0$

$\Rightarrow \log_3 x = -4 \qquad \log_3 x = -1$

$\Rightarrow x = 3^{-4} \qquad x = 3^{-1}$

$\Rightarrow x = \frac{1}{81}, \frac{1}{3}$

Q.22 Solve the value of x : $2(\log_x \sqrt{5})^2 - 3\log_x \sqrt{5} + 1 = 0$

Sol. $\Rightarrow \frac{2}{(\log_{\sqrt{5}} x)^2} - \frac{3}{\log_{\sqrt{5}} x} + 1 = 0$

$\Rightarrow \log^2_{\sqrt{5}} x - 2\log_{\sqrt{5}} x - \log_{\sqrt{5}} x + 2 = 0$

$\Rightarrow \log_{\sqrt{5}} x(\log_{\sqrt{5}} x - 2) - (\log_{\sqrt{5}} x - 2) = 0$

$\Rightarrow (\log_{\sqrt{5}} x - 1)(\log_{\sqrt{5}} x - 2) = 0$

$\Rightarrow \log_{\sqrt{5}} x = 1$ or $\log_{\sqrt{5}} x = 2$

$x = \sqrt{5}$ or $x = \left(\sqrt{5}\right)^2 = 5$

$\Rightarrow \log_{\sqrt{5}} x = 1$ or $\log_{\sqrt{5}} x = 2$

$x = \sqrt{5}$ or $x = \left(\sqrt{5}\right)^2 = 5$

23 Solve the value of x : $(\log 100\,x)^2 + (\log 10\,x)^2 = 14 + \log\frac{1}{x}$

1. $= (\log 100 + \log x)^2 + (\log 10 + \log x)^2 = 14 - \log x$

$= (2 + \log x)^2 + (1 + \log x)^2 = 14 - \log x$

$=$ put $\log x = t$

$= (2 + t)^2 + (1 + t)^2 = 14 - t$

$= 2t^2 + 7t - 9 = 0$

$= 2t^2 + 9t - 2t - 9 = 0$

$= t(2t + 9) - 1(2t + 9) = 0$

$\Rightarrow t = 1 \qquad$ or $\qquad t = -\frac{9}{2}$

$= \log_{10} x = 1 \qquad$ or $\qquad \log_{10} x = -\frac{9}{2}$

$x = 10 \qquad$ or $\qquad x = +10^{-\frac{9}{2}} = \sqrt{10^{-9}}$

COMPOUND ANGLE (Trigonometry Phase-I)

What is trigonometry?

The word trigonometry is derived from three greek words

Tri	Gon	Metron
↓	↓	↓
three	sides	Measure

In the ancient sense trigonometry defines relations between elements of a triangle. In a triangle there are six basic elements, three sides and three angles. Any three line segments will form a triangle iff they satisfy three triangular inequalities i.e. the sum of any two lines segment is greater than third side. In Euclidean geometry the sum of three angles of a triangle is 180º. These requirements impose limitations on the manner in which the relations between the elements are defined.

Basic definition of six trigonometric functions :

three triangular inequalities i.e. the sum of any two lines segment is greater than third side. In Euclidean geometry the sum of three angles of a triangle is 180º. These requirements impose limitations on the manner in which the relations between the elements are defined.

Basic definition of six trigonometric functions :

$\sin\theta$	$\cos\theta$	$\tan\theta$	$\sec\theta$	$\text{cosec}\,\theta$	$\cot\theta$
$\frac{P}{H}$	$\frac{B}{H}$	$\frac{P}{B}$	$\frac{H}{B}$	$\frac{H}{P}$	$\frac{B}{P}$

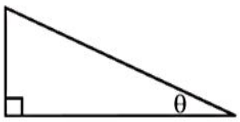


Illustration :

Which of the following reduces to unity for $0 < A < 90^o$?

(i) $\cos A \ \text{cosec} A \sqrt{\sec^2 A - 1}$ (ii) $(1 + \tan^2 A)(1 - \sin^2 A)$

(iii) $\dfrac{1}{1+\sin^2 A} + \dfrac{1}{1+\text{cosec}^2 A}$ (iv) $\dfrac{\cot^2 A \cos^2 A}{\cot^2 A - \cos^2 A}$

Sol. (i) $\cos A \dfrac{1}{\sin A} \tan A = \dfrac{\cos A}{\sin A} \times \dfrac{\sin A}{\cos A} = 1$

(ii) $(\sec^2 A)(\cos^2 A) = 1$

(iii) $\dfrac{1}{1+\sin^2 A} + \dfrac{\sin^2 A}{1+\sin^2 A} = 1$

(iv) $\dfrac{\dfrac{\cos^2 A}{\sin^2 A} \times \cos^2 A}{\cos^2 A\left(\dfrac{1}{\sin^2 A} - 1\right)} = \dfrac{\dfrac{\cos^4 A}{\sin^2 A}}{\dfrac{\cos^4 A}{\sin^2 A}} = 1$

Illustration :

Prove that $(\sec\theta + \text{cosec}\,\theta)(\sin\theta + \cos\theta) = \sec\theta \cdot \text{cosec}\,\theta + 2$

Sol $\left(\dfrac{1}{\ } + \dfrac{1}{\ }\right)(\sin\theta + \cos\theta)$

Prove that $(\sec\theta + \text{cosec}\,\theta)(\sin\theta + \cos\theta) = \sec\theta \cdot \text{cosec}\,\theta + 2$

Sol. $\left(\dfrac{1}{\cos\theta} + \dfrac{1}{\sin\theta}\right)(\sin\theta + \cos\theta)$

$\Rightarrow \dfrac{\sin\theta}{\cos\theta} + \dfrac{\cos\theta}{\sin\theta} + 2$

$\Rightarrow \dfrac{1}{\sin\theta\cos\theta} + 2 = \sec\theta\,\text{cosec}\,\theta + 2$

Illustration :

If $(1 + \sin A)(1 + \sin B)(1 + \sin C) = (1 - \sin A)(1 - \sin B)(1 - \sin C)$.

Prove that each side equal to $\cos A \cos B \cos C$.

Sol. *Multiply both side by $(1 - \sin A)(1 - \sin B)(1 - \sin C)$*

$= (1 - \sin^2 A)(1 - \sin^2 B)(1 - \sin^2 C) = (1 - \sin A)^2 (1 - \sin B)^2 (1 - \sin C)^2$

$= \cos^2 A \cos^2 B \cos^2 C = (1 - \sin A)^2 (1 - \sin B)^2 (1 - \sin C)^2$

$= \cos A \cos B \cos C = (1 - \sin A)(1 - \sin B)(1 - \sin C)$

Illustration :

Prove that $\dfrac{\tan A + \sec A - 1}{\tan A - \sec A + 1} = \dfrac{1 + \sin A}{\cos A}$

Sol. $= \dfrac{(\tan A + \sec A) - (\sec^2 A - \tan^2 A)}{(1 - \sec A + \tan A)} = \dfrac{(\tan A + \sec A)(1 - \sec A + \tan A)}{(1 - \sec A + \tan A)} = \dfrac{1 + \sin A}{\cos A}$

Illustration :

Prove that $\left(\dfrac{1}{sec^2 a - cos^2 a} + \dfrac{1}{cosec^2 - sec^2 a}\right) sin^2 a\, cos^2 a = \dfrac{1 - cos^2 a\, sin^2 a}{2 + cos^2 a\, sin^2 a}$

Sol. $\left(\dfrac{cos^2 a}{1 - cos^4 a} + \dfrac{sin^2 a}{1 - sin^4 a}\right) sin^2 a\, cos^2 a$

$$= \left(\frac{cos^4 a \cdot sin^2 a}{1 - cos^4 a} + \frac{sin^4 a \cdot cos^4 a}{1 - sin^4 a}\right) = \frac{cos^4 a \cdot sin^2 a}{(1 + cos^2 a)(1 - cos^2 a)} + \frac{sin^4 a \cdot cos^2 a}{(1 - sin^2 a)(1 + sin^2 a)}$$

$$= \frac{cos^4 a + sin^4 a + cos^2 a\, sin^2 a\,(sin^2 a + cos^2 a)}{1 + sin^2 a + cos^2 a + sin^2 a\, cos^2 a}$$

$$= \frac{\left(cos^2 a + sin^2 a\right)^2 - 2 sin^2 a\, cos^2 a + cos^2 a\, sin^2 a\,(sin^2 a + cos^2 a)}{2 + sin^2 a\, cos^2 a} = \frac{1 - sin^2 a\, cos^2 a}{2 + sin^2 a\, cos^2 a}$$

Practice Problem

Q.1 Prove that $\dfrac{\cot\theta + \text{cosec}\,\theta - 1}{\cot\theta - \text{cosec}\,\theta} = (\cot\theta + \text{cosec}\theta)(\cot\theta + \text{cosec}\theta - 1)$

Q.2 If $\sin\theta + \sin^2\theta = 1$. Prove that $\cos^{12}\theta + 3\cos^{10}\theta + 3\cos^8\theta + \cos^6\theta - 1 = 0$

Q.3 Prove that $(1 + \cot A - \text{cosec}\, A)(1 + \tan A + \sec A) = 2$

Q.3 Prove that $(1 + \cot A - \text{cosec}\, A)(1 + \tan A + \sec A) = 2$

Q.4 Prove that $(1 + \cot A + \tan A)(\sin A - \cos A) = \dfrac{\sec A}{\text{cosec}^2 A} - \dfrac{\text{cosec} A}{\sec^2 A}$

MEASUREMENT OF ANGLE AND SIGN CONVENTION :

Angle :

The measure of angle is the amount of rotation from the direction of one ray of the angle to the other. The initial and final position of the revolving ray are respectively called the initial side and terminal side.

Systems of Angle Measurement

English system (Degree)	French system (Grades)	Circular system (Radians)

English System :

One right angle = 90º (degree)

1º = 60' (minutes)

1' = 60" (seconds)

Circular system :

If length of arc of a circle equal's to radius then angle impose by that arc on centre of circle is called one radian.

Otherwise $\ell = r \cdot \theta$

Note : Important Relation :-

(i) Radian and Degree's

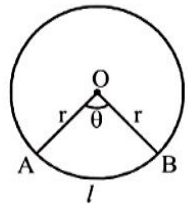
$\pi = 180^o$

(ii) Length of an arc of a circle

$\ell = r\,\theta$

(iii) Area of sector of a circle

$$A = \frac{1}{2}r^2\theta = \frac{1}{2}r\ell$$



l = length of arc,
r = radius of circle
θ = angle in radian

REDUCTION FORMULAE :

(I) $(90 + \theta)$ Relation

Δ OPB and Δ OP'B' are congruent by ASA property one $\angle\theta$, side r, $\angle(90° - \theta)$

$\therefore$ In Δ OP'B', P'B' = x as side opposite to $90° - \theta$ is x in Δ OPB

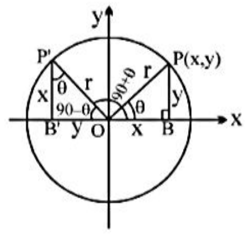
In Δ OP'B', OB' = y as side opposite to θ in Δ OPB is y.

In Δ OP'B'

$\sin(90° + \theta) = \dfrac{x}{r} = \cos\theta$; $\cos(90° + \theta) = \dfrac{-y}{r} = -\sin\theta$;

$\tan(90° + \theta) = -\cot\theta$; $\cot(90° + \theta) = -\tan\theta$;

$\sec(90° + \theta) = -\mathrm{cosec}\,\theta$; $\mathrm{cosec}(90° + \theta) = \sec\theta$



In all $(90° + \theta)$ relations

In all $(90° + \theta)$ relations

sin changes to cos
cos changes to sin
cosec changes to sec
tan changes to cot
cot changes to tan
and sec changes to cosec
with appropriate sign

$\therefore$ $\sin(120°) = \sin(90° + 30°) = \cos 30° = \dfrac{\sqrt{3}}{2}$

$\tan(135°) = \tan(90° + 45°) = -\cot 45° = -1$

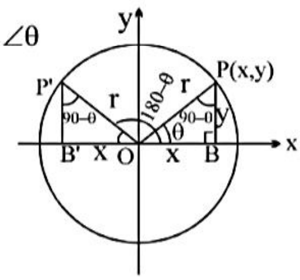
$\cos(150°) = \cos(90° + 60°) = -\sin 60° = -\dfrac{\sqrt{3}}{2}$

(II) Reduction $(180° - \theta)$

Δ's OPB and OP'B' are congruent by A S A. $\angle(90° - \theta)$, side r, $\angle\theta$

$\therefore$ side opposite to $90° - \theta$ = x same as in ΔOPB

and side opposite to θ = y same as in ΔOPB



$\sin(180° - \theta) = \dfrac{y}{r} = \sin\theta$; $\cos(180° - \theta) = \dfrac{-x}{r} = -\cos\theta$;

$\tan(180° - \theta) = -\tan\theta$; $\cot(180° - \theta) = -\cot\theta$;

$\mathrm{cosec}(180° - \theta) = \mathrm{cosec}\theta$; $\sec(180° - \theta) = -\sec\theta$;

Sines of supplementary angles are equal supplementary angles are those whose sum is 180°.

Sum of the cosines, tangents, cotangents, secants of supplementary angles is zero.

since $\cos(180° - \theta) = -\cos\theta$

$\therefore \quad \cos(180° - \theta) + \cos\theta = 0$

same for tan, cot and sec

$$\therefore \quad \sin(120°) = \sin(180° - 60°) = \sin 60° = \frac{\sqrt{3}}{2}$$

$$\cos(150°) = \cos(180° - 30°) = -\cos 30° = -\frac{\sqrt{3}}{2}$$

For (180° – θ)

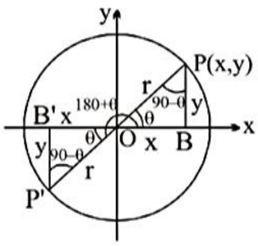
sin	remains sin
cos	remains cos
tan	remains tan
cot	remains cot
sec	remains sec
cosec	remains cosec with appropriate signs.

(III) Reduction (180° + θ)

ΔOPB and ΔOP'B' are congruent by A S A in which $\angle(90° - \theta)$, side r, $\angle\theta$.

(III) Reduction (180° + θ)

ΔOPB and ΔOP'B' are congruent by A S A in which $\angle(90° - \theta)$, side r, $\angle\theta$.



$$\therefore \quad \sin(180 + \theta) = \frac{-y}{r} = -\sin\theta;$$

$$\cos(180 + \theta) = \frac{-x}{r} = -\cos\theta;$$

$\tan(180 + \theta) = \tan\theta; \qquad \cot(180 + \theta) = \cot\theta;$

$\text{cosec}(180 + \theta) = -\text{cosec}\theta; \qquad \sec(180 + \theta) = -\sec\theta;$

$\sin(210°) = \sin(180° + 30°) = -\sin 30° = -1/2$

$\cos(240°) = \cos(180° + 60°) = -\cos 60° = -1/2$

$\tan(225°) = \tan(180° + 45°) = \tan 45° = 1$

In (180 + θ) relations

sin	remains sin
cos	remains cos
tan	remains tan
cot	remains cot
sec	remains sec
cosec	remains cosec with appropriate signs.

$\sin(270°) = \sin(180° + 90°) = -\sin 90° = -1$

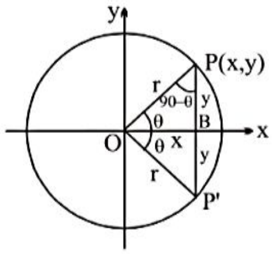
$\cos(270°) = \tan(180° + 90°) = -\cos 90° = 0$

(IV) Reduction (360 – θ) or (2π – θ)

Any angle of the form $2\pi - \theta$ can be written as $-\theta$ because if we say $2\pi - \theta$ then it means we are moving clockwise from origin and by convention all angles measured clockwise are – ve.

$$\therefore \quad \sin(2\pi - \theta) = \sin(-\theta)$$
$$\cos(2\pi - \theta) = \cos(-\theta)$$
$$\tan(2\pi - \theta) = \tan(-\theta)$$

Again ΔOPB and ΔOP'B are congruent by ASA



$$\sin(-\theta) = \frac{-y}{r} = -\sin\theta; \qquad \cos(-\theta) = \frac{x}{r} = \cos\theta;$$
$$\tan(-\theta) = -\tan\theta; \qquad \cot(-\theta) = -\cot\theta;$$
$$\operatorname{cosec}(-\theta) = -\operatorname{cosec}\theta; \qquad \sec(-\theta) = -\sec\theta;$$

$$\cos(315°) = \cos(360° - 45°) = \cos(-45°) = \cos(45°) = \frac{1}{\sqrt{2}}$$

$$\tan(330°) = \tan(360° - 30°) = \tan(-30°) = -\tan 30° = -\frac{1}{\sqrt{3}}$$

In (2π – θ) relations

sin remains sin
cos remains cos
tan remains tan
cot remains cot
sec remains sec
cosec remains cosec with appropriate signs.

sec remains sec
cosec remains cosec with appropriate signs.

To remember the signs we use

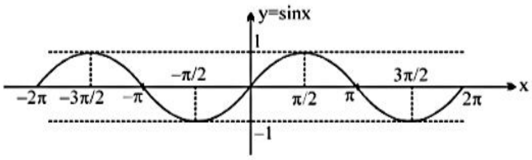
sin +ve Students	All All +ve
Take tan +ve	Coffee cos +ve

$$\tan(-120°) = -\tan 120° = -\tan(180° - 60°) = \tan 60° = \sqrt{3}$$
$$\cos(180°) = \cos(90° + 90°) = \cos(180° - 0) = \cos(180 + 0°)$$
$$= -1 \qquad\qquad = -1 \qquad\qquad = -1$$

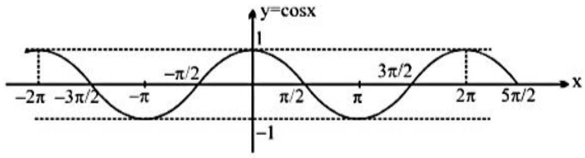
Radians	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	2π
Degree	0	30°	45°	60°	90°	120°	135°	150°	180°	210°	225°	240°	270°	300°	315°	330°	360°
sin	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{\sqrt{2}}$	$-\frac{\sqrt{3}}{2}$	–1	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{\sqrt{2}}$	$-\frac{1}{2}$	0
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{\sqrt{2}}$	$-\frac{\sqrt{3}}{2}$	–1	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{\sqrt{2}}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
tan	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	ND	$-\sqrt{3}$	–1	$-\frac{1}{\sqrt{3}}$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	ND	$-\sqrt{3}$	–1	$-\frac{1}{\sqrt{3}}$	0
cot	ND	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0	$-\frac{1}{\sqrt{3}}$	–1	$-\sqrt{3}$	ND	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0	$-\frac{1}{\sqrt{3}}$	–1	$-\sqrt{3}$	ND

GRAPHS OF 6 TRIGONOMETRIC FUNCTIONS :

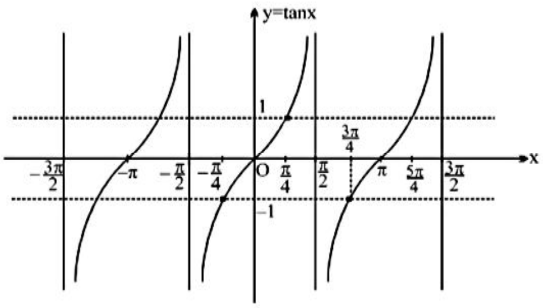
(1) $y = \sin x$, where $y \in [-1, 1]$, $x \in R$



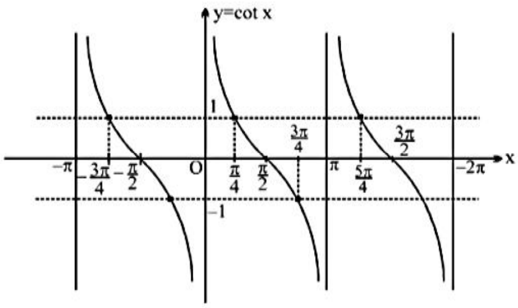
(2) $y = \cos x$, where $y \in [-1, 1]$, $x \in R$



(3) $y = \tan x$, $x \in R$, $y \in (-\infty, \infty)$,
$x \neq (2n-1)\dfrac{\pi}{2}$ for $n \in I$



(4) $y = \cot x$, $x \in R$, $y \in (-\infty, \infty)$,
$x \neq n\pi$ for $n \in I$

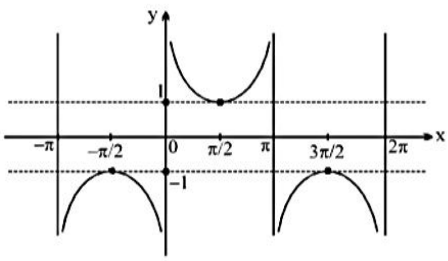


(5) $y = \text{cosec } x$, $y \in (-\infty, -1] \cup [1, \infty)$,
$x \in R - n\pi$, $x \neq n\pi$ for $n \in I$

(6) $y = \sec x$, $x \neq (2n+1)\dfrac{\pi}{2}$ for $n \in I$

(5) $y = \text{cosec } x$, $y \in (-\infty, -1] \cup [1, \infty)$,
$x \in R - n\pi$, $x \neq n\pi$ for $n \in I$

(6) $y = \sec x$, $x \neq (2n+1)\dfrac{\pi}{2}$ for $n \in I$



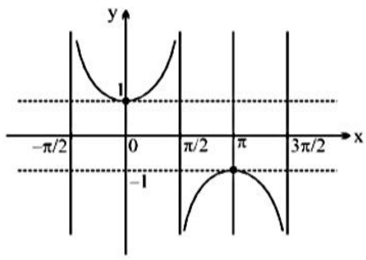


Illustration :

Consider a square of side 4cm. Now if a man runs at a distance of 1cm from the sides of the square. How much distance will he travel.

Sol. *Lienar distance = 16*

curved distance $= r\,\theta = 1.\dfrac{\pi}{2}$

Total curve part $4.\dfrac{\pi}{2} = 2\pi$

Total distance $16 + 2\pi$

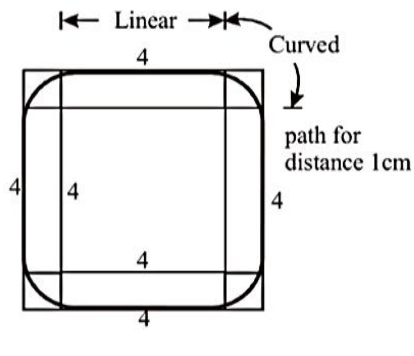
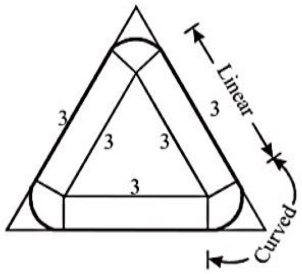


Illustration :

Consider an equilateral triangle with side 3cm. Now if a man runs around the triangle in such a ways that he is always at a distance of 1cm from the sides of the triangle then how much distance will he travel.



Sol. Linear distance = 9

$$\text{curved distance } \ell = 1 \times \frac{2\pi}{3}$$

$$3\ell = 2\pi$$

$$\text{Total distance} = 9 + 2\pi$$

Illustration :

Prove that

$\cos 10° + \cos 20° + \cos 30° + ... + \cos 80° + \cos 100° + \cos 150° + \cos 160° + \cos 170° = 0.$

Sol. $\cos 10° + \cos 20° + \cos 30° + ... + \cos 80° + \cos(180° - 80°) + \cos(180° - 30°)$
$+ \cos(180° - 20°) + \cos(180° - 10°) = 0$

$\cos 10° + \cos 20° + \cos 30° + ... + \cos 80° - \cos 80° - \cos 30° - \cos 20° - \cos 10° = 0$

Illustration :

$$\text{Prove that } \tan\frac{\pi}{11} + \tan\frac{2\pi}{11} + \tan\frac{4\pi}{11} + \tan\frac{7\pi}{11} + \tan\frac{9\pi}{11} + \tan\frac{10\pi}{11} = 0$$

Sol. *Sum of tangents of supplementary angles is zero.*

Illustration :

11 11 11 11 11 11

Sol. *Sum of tangents of supplementary angles is zero.*

Illustration :

Prove that $\sin 420° \cos 390° + \cos(-300°) \sin(-330°) = 1.$

Sol. $= \sin(360° + 60°) \cos(360° + 30°) + \cos(-360° + 60°) \sin(-360° + 30°)$

$= \sin 60° \cos 30° + \cos 60° \sin 30°$

$= \sin 90° = 1$

Illustration :

Prove that $\sin 240° \sin 510° - \sin 330° \cos 390° = 0$

Sol. $\sin(270° - 30°) \sin(540° - 30°) \cos(360° + 30°)$

$= -\cos 30° \sin 30° + \sin 30° \cos 30°$

$= 0$

Illustration :

What sign has $(\sin A + \cos A)$ *for the following vlaues of A?*

(a) $140°$ (b) $-1125°$

Sol. $\sin A + \cos A$

(a) $\sin 140° + \cos 140°$

$= \sin 40° - \cos 40°$

$= -ve$ (For $A < 45°$; $\cos 40° > \sin 40°$)

(b) $\sin(-1125°) + \cos(-1125°)$

$= \sin(-1080° - 45°) + \cos(-1080° - 45°)$

$= \sin(-45°) + \cos(-45°)$

$= -\sin 45° + \cos 45° = 0$

Illustration :

Prove that $\cos A + \sin(270° + A) - \sin(270° - A) + \cos(180° + A) = 0$

Sol. $\cos A + \sin(270° + A) - \sin(270° - A) + \cos(180° + A)$

$= \cos A - \cos A - (-\cos A) + (-\cos A)$

$= 0$

Practice Problem

Q.1 Prove that $\tan 225° \cot 405° + \tan 765° \cot 675° = 0$

Q.2 What sign has $\sin A - \cos A$ for the following vlaues of A?
(a) 215° (b) –457°

Q.3 Prove that $\sec(270° - A)\sec(90° - A) - \tan(270° - A)\tan(90° + A) + 1 = 0$.

Q.4 Prove that $\sin^2 5° + \sin^2 10° + \sin^2 15° + ... + \sin^2 85° + \sin^2 90° = 9\frac{1}{2}$.

Q.5 Consider an triangle with sides 3, 6, 8. Now if a man run's around the triangle is such a way that he is always at a distance of 1cm from the sides of triangle then how much distance will he travel.

Answer key

Q.2 (a) Positive, (b) Negative Q.5 $17 + 2\pi$

Q.2 (a) Positive, (b) Negative Q.5 $17 + 2\pi$

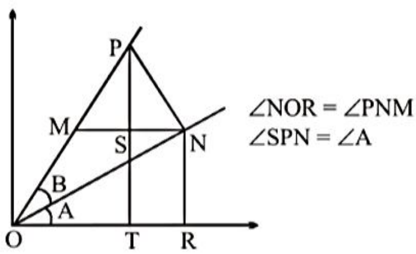
TRIGONOMETRY OF COMPOUND ANGLES :

Trigonometric ratios i.e., 'sin', 'cos', 'tan', cot, 'sec' and cosec are not distributed over addition and subtraction of 2 angles.

i.e., $\sin(A + B) \neq \sin A + \sin B$

Note that $A = 60°$, $B = 30°$

$\sin(A + B) = \sin A \cdot \cos B + \cos A \cdot \sin B$



$$\sin(A + B) = \frac{PT}{OP} = \frac{PS + ST}{OP}$$

$$= \frac{PS}{OP} + \frac{ST}{OP}$$

$$= \frac{PS}{PN} \cdot \frac{PN}{OP} + \frac{NR}{ON} \cdot \frac{ON}{OP}$$

$$= \frac{PS}{PN} \cdot \frac{PN}{OP} + \frac{NR}{ON} \cdot \frac{ON}{OP}$$

$$= \cos A \cdot \sin B + \sin A \cdot \cos B$$

Hence we got $\boxed{\sin(A + B) = \sin A \cdot \cos B + \cos A \sin B}$ ....(i)

Now replace B by –B

$\boxed{\sin(A - B) = \sin A \cdot \cos B - \cos A \cdot \sin B}$ ... (ii)

In (i) replace A by $\frac{\pi}{2}+A$

$$\sin\left(\frac{\pi}{2}+A+B\right)=\sin\left(\frac{\pi}{2}+A\right)\cdot\cos B+\cos\left(\frac{\pi}{2}+A\right)\cdot\sin B$$

$$\boxed{\cos(A+B)=\cos A\cdot\cos B-\sin A\cdot\sin B} \quad \text{...(iii)}$$

In (iii) relation replace B by –B

$$\boxed{\cos(A-B)=\cos A\cdot\cos B+\sin A\cdot\sin B} \quad \text{...(iv)}$$

To deduce the value of tan (A + B) and cot (A + B) :

(1) $\boxed{\tan(A+B)=\dfrac{\tan A+\tan B}{1-\tan A\cdot\tan B}}$

(2) $\boxed{\tan(A-B)=\dfrac{\tan A-\tan B}{1+\tan A\tan B}}$

Note : $\boxed{\tan\left(\dfrac{\pi}{4}\pm A\right)=\dfrac{1\pm\tan A}{1\mp\tan A}}$

(3) $\boxed{\cot(A+B)=\dfrac{\cot A\cot B-1}{\cot B+\cot A}}$ or $\boxed{\cot A\cot B-1=\cot(A+B)(\cot B+\cot A)}$

(4) $\boxed{\cot(A-B)=\dfrac{\cot A\cot B+1}{\cot B-\cot A}}$

Important identities :

Important identities :

(a) $\boxed{\sin(A+B)\cdot\sin(A-B)=\sin^2A-\sin^2B}$

(b) $\boxed{\cos(A+B)\cdot\cos(A-B)=\cos^2A-\sin^2B}$

(c) $\boxed{\tan(A+B+C)=\dfrac{\tan A+\tan B+\tan C-\tan A\tan B\tan C}{1-\tan A\tan B-\tan B\tan C-\tan C\tan A}}$

Illustration :

Prove that $\dfrac{\cos 10^o+\sin 10^o}{\cos 10^o-\sin 10^o}=\tan 55^o$.

Sol. $\dfrac{\cos 10^o+\sin 10^o}{\cos 10^o-\sin 10^o}=\dfrac{1+\tan 10^o}{1-\tan 10^o}=\dfrac{\tan 45^o+\tan 10^o}{1-\tan 45^o\tan 10^o}=\tan(45^o+10^o)=\tan 55^o$

Illustration :

If $3\tan\theta\tan\varphi=1$, *prove that* $2\cos(\theta+\varphi)=\cos(\theta-\varphi)$

Sol. *Given* $3\tan\theta\tan\varphi=1$ *or* $\cot\theta\cot\varphi=3$

or $\dfrac{\sin\theta\cos\varphi}{\sin\theta\sin\varphi}=\dfrac{3}{1}$

By componendo and dividendo,

we have, $\dfrac{\cos\theta\cos\varphi+\sin\theta\sin\varphi}{\cos\theta\cos\varphi-\sin\theta\sin\varphi}\times\dfrac{3+1}{3-1} \Rightarrow \dfrac{\cos(\theta-\varphi)}{\cos(\theta+\varphi)}=2$

$\Rightarrow \quad 2\cos(\theta+\varphi)=\cos(\theta-\varphi)$

Illustration :

Show that $\cos^2\theta + \cos^2(\alpha + \theta) - 2\cos\alpha\cos\theta\cos(\alpha + \theta)$ *is independent of* θ.

Sol. $\cos^2\theta + \cos^2(\alpha + \theta) - 2\cos\alpha\cos\theta\cos(\alpha + \theta) = \cos^2\theta + \cos(\alpha + \theta)[\cos(\alpha + \theta) - 2\cos\alpha\cos\theta]$

$= \cos^2\theta + \cos(\alpha + \theta)\,[\cos\alpha\cos\theta - \sin\alpha\sin\theta - 2\cos^2\alpha\cos\theta]$

$= \cos^2\theta - \cos(\alpha + \theta)\,[\cos\alpha\cos\theta + \sin\alpha\sin\theta]$

$= \cos^2\theta - \cos(\alpha + \theta)\cos(\alpha - \theta)$

$= \cos^2\theta - [\cos^2\alpha - \sin^2\theta] = \cos^2\theta + \sin^2\theta - \cos^2\alpha$

$= 1 - \cos^2\alpha$, *which is independent of* θ.

Illustration :

To prove that $\cot 16^o \cdot \cot 44^o + \cot 44^o \cdot \cot 76^o - \cot 76^o \cdot \cot 16^o = 3$

or

$(\cot 16^o)(\cot 44^o) - 1 + (\cot 44^o \cdot \cot 76^o - 1) + (\cot 44^o \cdot \cot 76^o - 1) - (\cot 76^o \cdot \cot 16^o + 1) = 0$

Sol. $\cot A \cot B \mp 1 = \dfrac{\cos(A \pm B)}{\sin A \sin B}$

and $3 + 1 + 1 + 1$

$44° + 16° = 60°,\ 44° + 76° = 120°$

$76° - 16° = 60°$

$\therefore$ L.H.S.

$$= \frac{\cos 60° \sin 76° + \cos 120° \sin 16° - \cos 60° \sin 44°}{\sin 16° \sin 44° \sin 76°}$$

$$= \frac{1}{2}\left[\frac{\sin 76° + \sin 16° - \sin 44°}{\sin 16^o \sin 44^o \sin 76^o}\right] = \frac{1}{2}\frac{(2\sin 30° \cos 46° - \sin 44°)}{\sin 16^o \sin 44^o \sin 76^o}$$

$2[\ \sin 16^o \sin 44^o \sin 76^o\ ]\quad 2\quad \sin 16^o \sin 44^o \sin 76^o$

$$= \frac{1}{2\sin 16^o \sin 44^o \sin 76^o}(\cos 46° - \cos 46°), \text{ by comp. rule} = 0.$$

Practice Problem

Q.1 If $x - y = \dfrac{\pi}{4}$ and $\cot x + \cot y = 2$; Find smallest +ve angles x and y.

Q.2 If $\sin\alpha\sin\beta - \cos\alpha\cos\beta + 1 = 0$, prove that $1 + \cot\alpha\tan\beta = 0$

Q.3 If $\sin\alpha = \dfrac{15}{17}$, $\cos\beta = \dfrac{-5}{13}$. Find $\sin(\alpha - \beta)$

Q.4 If $\dfrac{\tan(\alpha + \beta - \gamma)}{\tan(\alpha - \beta + \gamma)} = \dfrac{\tan\gamma}{\tan\beta}$ then prove that $\sin(\beta - \gamma) = 0$ or $\sin 2\alpha + \sin 2\beta + \sin 2\gamma = 0$

Q.5 If α and β are the solutions of the equation $a\tan\theta + b\sec\theta = c$, then show that

$$\tan(\alpha + \beta) = \frac{2ac}{(a^2 - c^2)}.$$

Answer key

Q.1 $x = 75^o$, $y = 30^o$

TRANSFORMATION FORMULAE:

Transform the product into sum or difference :

We know that

$\sin(A+B) = \sin A \cdot \cos B + \cos A \cdot \sin B$
$\sin(A-B) = \sin A \cdot \cos B - \cos A \cdot \sin B$
$\cos(A+B) = \cos A \cdot \cos B - \sin A \cdot \sin B$
$\cos(A-B) = \cos A \cdot \cos B + \sin A \cdot \sin B$

(1) $\boxed{\mathbf{2\sin A\cos B = \sin(A+B) + \sin(A-B)}}$

(2) $\boxed{\mathbf{2\cos A\sin B = \sin(A+B) - \sin(A-B)}}$

(3) $\boxed{\mathbf{2\cos A\cos B = \cos(A+B) + \cos(A-B)}}$

(4) $\boxed{\mathbf{2\sin A\sin B = \cos(A-B) - \cos(A+B)}}$

Transforming sum or differences into products :

Putting $A + B = C$ & $A - B = D$

$\Rightarrow \quad A = \frac{C+D}{2}; \qquad B = \frac{C-D}{2}$

in 1, 2, 3, 4

(5) $\boxed{\mathbf{\sin C + \sin D = 2\sin\frac{(C+D)}{2}\cos\frac{(C-D)}{2}}}$

(5) $\boxed{\mathbf{\sin C + \sin D = 2\sin\frac{(C+D)}{2}\cos\frac{(C-D)}{2}}}$

(6) $\boxed{\mathbf{\sin C - \sin D = 2\cos\frac{(C+D)}{2}\ \sin\frac{(C-D)}{2}}}$

(7) $\boxed{\mathbf{\cos C + \cos D = 2\cos\frac{(C+D)}{2}\ \cos\frac{(C-D)}{2}}}$

(8) $\boxed{\mathbf{\cos C - \cos D = -2\sin\frac{(C+D)}{2}\sin\frac{(C-D)}{2}}}$ **(Imp.)**

VALUES OF TRIGONOMETRIC RATIO OF STANDARD ANGLES :

Finding values of 15º & 75º

(1) $\sin 15^\circ = \sin\frac{\pi}{12} = \sin(45^\circ - 30^\circ) = \sin 45^\circ \cdot \cos 30^\circ - \cos 45^\circ \cdot \sin 30^\circ$

$$\frac{\sqrt{3}-1}{2\sqrt{2}} = \frac{\sqrt{6}-\sqrt{2}}{4} = \cos 75^\circ = \mathbf{\cos\frac{5\pi}{12}}$$

(2) $\sin 75^\circ = \sin\frac{5\pi}{12} = \sin(45^\circ + 30^\circ) = \sin 45^\circ \cdot \cos 30^\circ + \cos 45^\circ \cdot \sin 30^\circ$

$$= \frac{\sqrt{3}+1}{2\sqrt{2}} = \frac{\sqrt{6}+\sqrt{2}}{4} = \cos 15^\circ = \mathbf{\cos\frac{\pi}{12}}$$

(3) $\tan 15^\circ = \dfrac{\sqrt{3}-1}{\sqrt{3}+1} = 2-\sqrt{3} = \textbf{cot } \mathbf{75^\circ}$

(4) $\tan 75^\circ = \dfrac{\sqrt{3}+1}{\sqrt{3}-1} = 2+\sqrt{3} = \textbf{cot } \mathbf{15^\circ}$

NOTE :- $\sin A \ \sin(60^\circ - A)\ \sin(60^\circ + A) = \dfrac{1}{4}\sin 3A$

$\cos A\ \cos(60^\circ - A)\ \cos(60^\circ + A) = \dfrac{1}{4}\cos 3A$

$\tan A\ \tan(60^\circ - A)\ \tan(60^\circ + A) = \tan 3A$

Illustration :

Prove that $\cos(36^\circ - A)\cos(36^\circ + A) + \cos(54^\circ + A)\cos(54^\circ - A) = \cos 2A$

Sol. $\cos(36^\circ - A)\cos(36^\circ + A) + \cos[90^\circ - (54^\circ + A)]\sin[90^\circ - (54^\circ - A)] = \cos 2A$

$= \cos(36^\circ - A)\cos(36^\circ + A) + \sin(36^\circ - A)\sin(36^\circ + A)$

$= \cos[(36^\circ + A) - (36^\circ - A)] = \cos 2A$

Illustration :

Prove that $\dfrac{\cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10}{\cos 5\theta + 5\cos 3\theta + 10\cos\theta} = 2\cos\theta$

Sol. *In the* N^r *write 6 as 1 + 5 and 15 as 10 + 5*

$\cos 5\theta + 5\cos 3\theta + 10\cos\theta$

Sol. *In the* N^r *write 6 as 1 + 5 and 15 as 10 + 5*

$N^r = \cos 6\theta + \cos 4\theta + 5\cos 4\theta + 5\cos 2\theta + 10\cos 2\theta + 10$

$= 2\cos 5\theta\cos\theta + 5.2\cos\cos 3\theta\cos\theta + 10.2\cos 2\theta$

$= 2\cos\theta[\cos 5\theta + 5\cos 3\theta + 10\cos\theta] = 2\cos\theta[D^r]$

$\therefore \quad \dfrac{N'}{D'} = 2\cos\theta$

Illustration :

If $\dfrac{\tan(\theta+x)}{a} = \dfrac{\tan(\theta+y)}{b} = \dfrac{\tan(\theta+z)}{c}$

then show that $\dfrac{a+b}{a-b}\sin^2(x-y) + \dfrac{b+c}{b-c}\sin^2(y-z) + \dfrac{c+a}{c-a}\sin^2(z-x) = 0$

Sol. $\dfrac{a}{b} = \dfrac{\tan(\theta+x)}{\tan(\theta+y)}$. *Applying componendo and dividendo*

$\therefore \quad \dfrac{a+b}{a-b} = \dfrac{\sin(2\theta+x+y)}{\sin(x-y)}$

$\therefore \quad \dfrac{a+b}{a-b}\sin^2(x-y) = \sin(2\theta+x+y)\sin(x-y) = \dfrac{1}{2}[\cos(2\theta+2y) - \cos(2\theta+2x)]$

$\therefore \quad \sum\dfrac{a+b}{a-b}\sin^2(x-y) = 0$ *as the term in R.H.S will be cancelled.*

Illustration :

Prove that $\sin 20^o \sin 40^o \sin 80^o = \dfrac{\sqrt{3}}{8}$

Sol. $\dfrac{1}{2}(2\sin 20^o \sin 40^o)\sin 80^o = \dfrac{1}{2}(\cos 20^o - \cos 60^o)\sin 80^o$

$= \dfrac{1}{2}(\cos 20^o \sin 80^o - \dfrac{1}{2}\sin 80^o) = \dfrac{1}{4}(2\cos 20^o \sin 80^o - \sin 80^o)$

$= \dfrac{1}{4}(\sin 100^o + \sin 60^o - \sin 80^o) = \dfrac{1}{4}(2\cos 90^o \sin 10^o + \sin 60^o) = \dfrac{1}{4}\sin 60^o = \dfrac{\sqrt{3}}{8}$

Alternate method :-

$\sin 20^o \sin 40^o \sin 80^o = \sin 20^o \sin(60^o - 20^o)\sin(60^o + 20^o) = \dfrac{1}{4}\sin(3 \times 20^o) = \dfrac{\sqrt{3}}{8}$

Practice Problem

Q.1 Find the value of $\cos^2 73^o + \cos^2 47^o + \cos 73^o \cdot \cos 47^o$

Q.2 If α in first quadrant $= \dfrac{\pi}{19}$. Find value of $\dfrac{\sin 23\alpha - \sin 3\alpha}{\sin 16\alpha + \sin 4\alpha}$.

Q.3 If $x\sin\theta = y\sin\left(\theta + \dfrac{2\pi}{3}\right) = z\sin\left(\theta + \dfrac{4\pi}{3}\right)$ then. Prove that $\sum xy = 0$.

Q.4 If α, β are two values of θ satisfying equation $\dfrac{\cos\theta}{a} + \dfrac{\sin\theta}{b} = \dfrac{1}{c}$ then. Prove that $\cot\left(\dfrac{\alpha+\beta}{2}\right) = \dfrac{b}{a}$.

Q.5 Prove that $\dfrac{\sin A + 2\sin 3A + \sin 5A}{\sin 3A + 2\sin 5A + \sin 7A} = \dfrac{\sin 3A}{\sin 5B}$.

Answer key

Q.1 3/4 Q.2 –1

Trigonometric Ratio's of Multiple and submultiple angles :

Multiple angles are 2θ and 3θ and sub multiple angles are $\dfrac{\theta}{2}, \dfrac{\theta}{4}, \dfrac{\theta}{8}$.

1. $\mathbf{\sin 2A = \sin(A + A) = 2\sin A \cos A}$

2. $\mathbf{\cos 2A}$

$$\boxed{\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A}$$

$$\boxed{\cos A = \cos^2\left(\frac{A}{2}\right) - \sin^2\left(\frac{A}{2}\right)}$$

Formulaes in terms of $\tan^2 A$

3. $\tan 2A = \tan (A + A) = \dfrac{2\tan A}{1 - \tan^2 A}$

4. $\sin 2A = \dfrac{(2\tan A)}{1 + \tan^2 A}$

5. $\cos 2A = \dfrac{1 - \tan^2 A}{1 + \tan^2 A}$

Sine, cosines and tangent of 3A

6. $\sin 3A = 3\sin A - 4\sin^3 A$
7. $\cos 3A = 4\cos^3 A - 3\cos A$

8. $\tan 3A = \dfrac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}$

9. $\tan (A_1 + A_2 + \ldots\ldots A_n) = \dfrac{S_1 - S_3 + S_5 - S_1 + \ldots\ldots}{1 - S_2 + S_4 - S_6 + \ldots\ldots}$,

where
$S_1 = \tan A_1 + \tan A_2 + \ldots\ldots + \tan A_n$ = Sum of the tangents of the separate angles,
$S_2 = \tan A_1 \tan A_2 + \tan A_1 + \tan A_3 + \ldots\ldots$ = Sum of the tangents taken two at a time,
$S_3 = \tan A_1 \tan A_2 \tan A_3 + \tan A_2 \tan A_3 \tan A_4 + \ldots\ldots$ = Sum of the tangents taken three at a

$S_1 = \tan A_1 + \tan A_2 + \ldots\ldots + \tan A_n$ = Sum of the tangents of the separate angles,
$S_2 = \tan A_1 \tan A_2 + \tan A_1 + \tan A_3 + \ldots\ldots$ = Sum of the tangents taken two at a time,
$S_3 = \tan A_1 \tan A_2 \tan A_3 + \tan A_2 \tan A_3 \tan A_4 + \ldots\ldots$ = Sum of the tangents taken three at a
time, and so on.

Continued product of cosine Series :

$$\cos A \cos 2A \cos 4A \cos 8A \ldots \cos 2^{n-1} A = \frac{1}{2^n \sin A} \sin (2^n A)$$

Proof : Multiplying above and below by $2^n \sin A$

$\therefore$ LHS

$$= \frac{2^{n-1}}{2^n \sin A} [2 \sin A \cos A \cos 2A \cos 4A \ldots \cos 2^{n-1} A]$$

$$= \frac{2^{n-2}}{2^n \sin A} [2 \sin 2A \cos 2A \cos 4A \ldots \cos 2^{n-1} A]$$

$$= \frac{2^{n-3}}{2^n \sin A} [2 \sin 4A \cos 4A \ldots \cos 2^{n-1} A]$$

$$= \frac{1}{2^n \sin A} [2 \sin 2^{n-1} A \cos 2^{n-1} A]$$

$$= \frac{1}{2^n \sin A} \sin (2 \cdot 2^{n-1} A) = \frac{\sin(2^n A)}{2^n \sin A}$$

Illustration :

Prove that $\cos\frac{\pi}{15}\cos\frac{2\pi}{15}\cos\frac{3\pi}{15}\cos\frac{4\pi}{15}\cos\frac{5\pi}{15}\cos\frac{6\pi}{15}\cos\frac{7\pi}{15} = \frac{1}{128}$

Sol. $\cos\frac{5\pi}{15} = \cos\frac{\pi}{3} = \frac{1}{2}$, $\cos\frac{7\pi}{15} = -\cos\frac{8\pi}{15}$

$$L.H.S = \left[\cos\frac{\pi}{15}\cos\frac{2\pi}{15}\cos\frac{4\pi}{15}\left(-\cos\frac{8\pi}{15}\right)\right]\left(\cos\frac{3\pi}{15}\cos\frac{6\pi}{15}\right)\cdot\frac{1}{2}$$

$$= -\frac{1}{2}\cdot\frac{\sin 2^4\left(\frac{\pi}{15}\right)}{2^4\sin\frac{\pi}{15}}\cdot\frac{\sin\left(2^2\frac{3\pi}{15}\right)}{2^2\sin\frac{3\pi}{15}} = \frac{1}{128}, \text{ as } \sin(\pi+\theta) = -\sin\theta$$

Illustration :

Prove that $\sin\frac{\pi}{14}\sin\frac{3\pi}{14}\sin\frac{5\pi}{14}\sin\frac{7\pi}{14} = \frac{1}{8}$

Sol. $\sin\frac{\pi}{14} = \sin\left(\frac{\pi}{2}-\frac{6\pi}{14}\right) = \cos\frac{6\pi}{14} = \cos\left(\pi-\frac{8\pi}{14}\right) = -\cos\frac{8\pi}{14}$

$\sin\frac{3\pi}{14} = \sin\left(\frac{\pi}{2}-\frac{4\pi}{14}\right) = \cos\frac{4\pi}{14}$

$\sin\frac{3\pi}{14} = \sin\left(\frac{\pi}{2}-\frac{4\pi}{14}\right) = \cos\frac{4\pi}{14}$

$\sin\frac{5\pi}{14} = \sin\left(\frac{\pi}{2}-\frac{2\pi}{14}\right) = \cos\frac{2\pi}{14}$

$\therefore$ L.H.S. $= -\cos\frac{2\pi}{14}\cos\frac{4\pi}{14}\cos\frac{8\pi}{14} = -\frac{1}{2^3\sin A}\sin(2^3 A)$, $A = \frac{2\pi}{14}$

$$= -\frac{1}{8\sin\frac{\pi}{7}}\sin\frac{8\pi}{7} = -\frac{1}{8\sin\frac{\pi}{7}}\sin\left(\pi+\frac{\pi}{7}\right) = -\frac{1}{8}(1) = \frac{1}{8}$$

$\because \sin(\pi+\theta) = -\sin\theta$

Illustration :

Prove that $\frac{\cos 2\theta}{1+\sin 2\theta} = \tan\left(\frac{\pi}{4}-\theta\right)$

Sol. L.H.S. $= \frac{\cos 2\theta}{1+\sin 2\theta}$

$$= \frac{\sin\left(\frac{\pi}{2}-2\theta\right)}{1+\cos\left(\frac{\pi}{2}-2\theta\right)} \qquad \left[\because \cos A = \sin\left(\frac{\pi}{2}-A\right), \sin A = \cos\left(\frac{\pi}{2}-A\right)\right]$$

$$= \frac{2\sin\left(\frac{\pi}{4}-\theta\right)\cos\left(\frac{\pi}{4}-\theta\right)}{2\cos^2\left(\frac{\pi}{4}-\theta\right)} \qquad \left[\begin{array}{l}\because \sin A = 2\sin\frac{A}{2}\cos\frac{A}{2} \\ 1+\cos A = 2\cos^2\frac{A}{2}\end{array}\right]$$

$$= \tan\left(\frac{\pi}{4}-\theta\right) = R.H.S.$$

Illustration :

Prove that $\dfrac{\sin 8\theta - 1}{\sec 4\theta - 1} = \dfrac{\tan 8\theta}{\tan 2\theta}$

Sol. $L.H.S. = \dfrac{\sin 8\theta - 1}{\sec 4\theta - 1} = \dfrac{\frac{1}{\cos 8\theta}-1}{\frac{1}{\cos 4\theta}-1} = \dfrac{1-\cos 8\theta}{\cos 8\theta}\cdot\dfrac{\cos 4\theta}{1-\cos 4\theta}$

$$= \frac{2^2\sin 4\theta}{\cos 8\theta}\times\frac{\cos 4\theta}{2\sin^2 2\theta} = \frac{(2\sin 4\theta\cos 4\theta)}{\cos 8\theta}\times\frac{\sin 4\theta}{2\sin^2 2\theta} \qquad \left[\begin{array}{l}\because 1-\cos 8\theta = 2\sin^2\frac{8\theta}{2} = 2\sin^2 4\theta \\ \text{and } 1-\cos 4\theta = 2\sin^2\frac{4\theta}{2} = 2\sin^2 2\theta\end{array}\right]$$

$$= \left(\frac{2\sin 4\theta\cos 4\theta}{\cos 8\theta}\right)\times\left(\frac{2\sin 2\theta\cos 2\theta}{2\sin^2 2\theta}\right)$$

$$= \left(\frac{\sin 2(4\theta)}{\cos 8\theta}\right)\times\left(\frac{\cos 2\theta}{2\sin^2 2\theta}\right) = \left(\frac{\sin 8\theta)}{}\right)\times\left(\frac{\cos 2\theta}{}\right)$$

$$= \left(\frac{\sin 2(4\theta)}{\cos 8\theta}\right)\times\left(\frac{\cos 2\theta}{\sin 2\theta}\right) = \left(\frac{\sin 8\theta)}{\cos 8\theta}\right)\times\left(\frac{\cos 2\theta}{\sin 2\theta}\right)$$

$$= \tan 8\theta\cot 2\theta = \frac{\cos 8\theta}{\sin 2\theta} = R.H.S.$$

Illustration :

Prove that $\cos^3\theta + \cos^3\theta\,(120° + \theta) + \cos^3(240° + \theta) = \dfrac{3}{4}\cos 3\theta$

Sol. *We know that* $\cos 3A = 4\cos 3A - 3\cos A$

$\therefore \quad \cos^3 A = \dfrac{1}{4}(\cos 3A + 3\cos A)$

Also, $\cos(2n\pi + \theta) = \cos\theta$

Applying the above we have

$$= \frac{1}{4}\,[(3\cos\theta + \cos 3\theta) + 3\cos(120° + \theta) + \cos(360° + 3\theta) + 3\cos(240° + \theta) + \cos(720° + 3\theta)]$$

$$= \frac{1}{4}[3\cos 3\theta] + \frac{3}{4}\,[\cos\theta + \cos(120° + \theta) + \cos(240° + \theta)]$$

$$= \frac{3}{4}\cos 3\theta + \frac{3}{4}[\cos\theta + 2\cos(180° + \theta)\cos 60°]$$

$$= \frac{3}{4}\cos 3\theta + \frac{3}{4}[\cos\theta + 2\cdot\frac{1}{2}(-\cos\theta)] = \frac{3}{4}\cos 3\theta$$

Illustration :

If A, B, C are the angles of triangle ABC and tan A, tan B, tan C are the roots of the equation $x^4 - 3x^3 + 3x^2 + 2x + 5 = 0$, then find the fourth root of the equation.

Sol. *Let the fourth root of the equation be tan D*

Now $A + B + C = \pi$

Consider $\tan(A + B + C + D) = \dfrac{\sum \tan A - \sum \tan A \tan B \tan C}{1 - \sum \tan A \tan B + \prod \tan A} = \dfrac{3-(-2)}{1-3+5} = \dfrac{5}{3}$

$\tan(\pi + D) = \tan D = \dfrac{5}{3}$ ***Ans.***

Practice Problem

Q.1 Prove that $\dfrac{1+\sin\theta-\cos\theta}{1+\sin\theta+\cos\theta} = \tan\dfrac{\theta}{2}$

Q.2 For a positive integer n, let $f_n(\theta) = \left(\tan\dfrac{\theta}{2}\right)(1+\sec\theta)(1+\sec^1\theta)(1+\sec^2\theta)\ldots\ldots(1+\sec 2^n\theta)$ then

Q.2 For a positive integer n, let $f_n(\theta) = \left(\tan\dfrac{\theta}{2}\right)(1+\sec\theta)(1+\sec^1\theta)(1+\sec^2\theta)\ldots\ldots(1+\sec 2^n\theta)$ then

(A) $f_2\left(\dfrac{\pi}{16}\right)=1$ (B) $f_3\left(\dfrac{\pi}{32}\right)=1$ (C) $f_4\left(\dfrac{\pi}{64}\right)=1$ (D) $f_5\left(\dfrac{\pi}{128}\right)=1$

Q.3 Show that $\sqrt{2+\sqrt{2+\sqrt{2+2\cos 8\theta}}} = 2\cos\theta$.

Q.4 Prove that $\tan^2\dfrac{\pi}{16} + \tan^2\dfrac{2\pi}{16} + \tan^2\dfrac{3\pi}{16} + \ldots\ldots + \tan^2\dfrac{7\pi}{16} = 35$

Q.5 If $\sin x + \sin y = 3(\cos y - \cos x)$, prove that $\sin 3x + \sin 3y = 0$

Q.6 If $x + 270\left[\cos\dfrac{2\pi}{7} + \cos\dfrac{4\pi}{7} + \cos\dfrac{6\pi}{7}\right] = 3746$, then find the value of x.

Q.7 Show that $\sin\dfrac{\pi}{14}\sin\dfrac{3\pi}{14}\sin\dfrac{5\pi}{14}\sin\dfrac{7\pi}{14}\sin\dfrac{9\pi}{14}\sin\dfrac{11\pi}{14}\sin\dfrac{13\pi}{14} = \dfrac{1}{64}$.

Answer key

Q.2 A, B, C, D Q.6 $x = 3881$

To find the trigonometrical funcitons of an angle of 18° :

Let θ stand for 18°, so that 2θ is 36° and 3θ is 54°.

Hence $2\theta = 90° - 3\theta$.

and therfore

$$\sin 2\theta = \sin(90° - 3\theta) = \cos 3\theta$$

$$\therefore \quad 2 \sin 6 \cos \theta = 4 \cos^3\theta - 3 \cos \theta$$

Hence, either $\cos \theta = 0$, which gives $\theta = 90°$, or

$$2 \sin \theta = 4 \cos^2 \theta - 3 = 1 - 4 \sin^2\theta$$

$$\therefore \quad 4 \sin^2 \theta + 2 \sin \theta = 1$$

By solving this quadratic equation, we have

(In our case sin θ is necessarily a positive quantity. Hence we take the upper sign, and have)

$$\sin \theta = \frac{\sqrt{5}-1}{4} = \sin 18°$$

Hence $\mathbf{\cos 18°} = \sqrt{1-\sin^2 18°} = \sqrt{1-\frac{6-2\sqrt{5}}{16}} = \sqrt{\frac{10+2\sqrt{5}}{16}}$

The remaining trigonometrical ratio of 18° may be now found.

Since 72° is the completement of 18°, the vlaue of the ratios for 72° may be obtained.

To find the trigonometrical functions of an angle of 36°

Since $\cos 2\theta = 1 - 2 \sin^2\theta$,

Since 72° is the completement of 18°, the vlaue of the ratios for 72° may be obtained.

To find the trigonometrical functions of an angle of 36°

Since $\cos 2\theta = 1 - 2 \sin^2\theta$,

$$\therefore \cos 36° = 1 - 2 \sin^2 18° = \frac{\sqrt{5}+1}{4}$$

The remaining trigonometrical functions of 36° may now be found.

Also, since 54° may be found.

Values of standard angles :

Angle → / ↓T. Ratio	$\left(\frac{\pi}{12}\right)$	$\left(\frac{\pi}{10}\right)$	$\left(\frac{\pi}{8}\right)$	$\left(\frac{\pi}{5}\right)$	$\left(\frac{3\pi}{8}\right)$	$\left(\frac{5\pi}{12}\right)$
	15°	18°	22.5°	36°	67.5°	75°
sin	$\frac{\sqrt{3}-1}{2\sqrt{2}}$	$\frac{\sqrt{5}-1}{4}$	$\frac{\sqrt{2-\sqrt{2}}}{2}$	$\frac{\sqrt{10-2\sqrt{5}}}{4}$	$\frac{\sqrt{2+\sqrt{2}}}{2}$	$\frac{\sqrt{3}+1}{2\sqrt{2}}$
cos	$\frac{\sqrt{3}+1}{2\sqrt{2}}$	$\frac{\sqrt{10+2\sqrt{5}}}{4}$	$\frac{\sqrt{2+\sqrt{2}}}{2}$	$\frac{\sqrt{5}+1}{4}$	$\frac{\sqrt{2-\sqrt{2}}}{2}$	$\frac{\sqrt{3}-1}{2\sqrt{2}}$
tan	$2-\sqrt{3}$	$\frac{1}{\sqrt{(5+2\sqrt{5})}}$	$\sqrt{2}-1$	$\sqrt{5-2\sqrt{5}}$	$\sqrt{2}+1$	$2+\sqrt{3}$
cot	$2+\sqrt{3}$	$\sqrt{(5+2\sqrt{5})}$	$\sqrt{2}+1$	$\sqrt{\left(1+\frac{2}{\sqrt{2}}\right)}$	$\sqrt{2}-1$	$2-\sqrt{3}$

Illustration :

Prove that $\sin\frac{\pi}{5}\sin\frac{2\pi}{5}\sin\frac{3\pi}{5}\sin\frac{4\pi}{5} = \frac{5}{16}$

Sol. $L.H.S. = \sin 36^\circ \sin 72^\circ \sin 108^\circ \sin 144^\circ$

$= \sin 36^\circ \sin 72^\circ \sin(180^\circ - 72^\circ)\sin(108^\circ - 36^\circ)$

$= \sin 36^\circ \sin 72^\circ \sin 72^\circ \sin 36^\circ = \sin^2 36^\circ \sin^2 72^\circ$

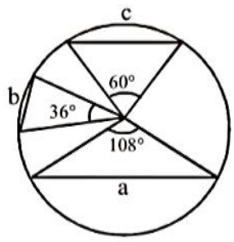
$$= \frac{5-\sqrt{5}}{8} \times \frac{5+\sqrt{5}}{8} = \frac{25-5}{64} = \frac{20}{64} = \frac{5}{16}.$$

Illustration :

In any circle prove that the chord which subtends 108° at the centre is equal to the sum of the two chords which subtend angles of 36° and 60°.

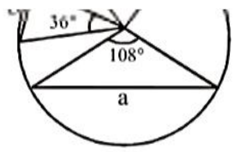
Sol. *If r be the radius of the circle and a, b, c and chords which subtend at the centre angles = 108°, 36° and 60°, then*

$$a = 2r\sin\frac{108^\circ}{2} = r\sin 54^\circ$$



$$b = 2r\sin\frac{36^\circ}{2} = r\sin 18^\circ$$

$$c = 2r\sin\frac{60^\circ}{\ } = r\sin 30^\circ$$



$$b = 2r\sin\frac{36^\circ}{2} = r\sin 18^\circ$$

$$c = 2r\sin\frac{60^\circ}{2} = r\sin 30^\circ$$

$$= 2r\sin 54^\circ = 2r\sin 36^\circ$$

$$= 2r\frac{\sqrt{5}+1}{4}$$

or $$2r(\sin 18^\circ) + (2r\sin 30^\circ) = 2r\left(\frac{\sqrt{5}-1}{4} + \frac{1}{2}\right) = 2r\frac{\sqrt{5}+1}{4}$$

Hence the chords which subtends 108° at the centre is equal to the sum of the two chords which subtends angle of 36° and 60°.

Practice Problem

Q.1 If $\sec^2 60^\circ + k(\sin 54^\circ - \cos 72^\circ) = 10$, then find k.

Q.2 If $\frac{\sqrt{3}-1}{\sin x} + \frac{\sqrt{3}+1}{\cos x} = 4\sqrt{2}$. Then find x.

Answer key

Q.1 k = 12 Q.2 x = 15°, 55°

APPLICATION OF TRIGONOMETRY IN MAXIMISING AND MINIMISING i.e. (Optimisation) :

1. Maximising and minimising by using the property of boundness of trigonometric functions.
   (a) Sine and cosine have bounded values between – 1 and 1.
   (b) Tangent and cotangent are unbounded functions.
   (c) Cosec and sec have values greater than 1 and less than – 1.
   (d) $0 \le \sin^2 x \le 1$, $0 \le \cos^2 x \le 1$, $\tan^2 x \ge 0$, $\sec^2 x \ge 1$.

Note : If maximum value of a function is **'b'**, and minimum value **'a'** then range is [a, b].

TYPE - I :

Illustration :

Find minimum and maximum values of $y = 2 + \cos x$.

Sol. $y = 2 + \cos x$

$\because -1 \le \cos x \le 1$

$\therefore y = 2 \pm 1$

$y_{max} = 3$ and $y_{min} = 1$

Illustration :

Find minimum and maximum values of $y = 1 + \cos x + \cos^2 x + \cos^3 x + \cos^4 x$.

Sol. for $x = 0°$ $\cos x = 1$

for $x = \pi$ $\cos x = -1$

Sol. for $x = 0°$ $\qquad \cos x = 1$

for $x = \pi$ $\qquad \cos x = -1$

for $x = \frac{\pi}{2}$ $\qquad \cos x = 0$

for $\cos x = 1,$ $\qquad y_{max} = 5$

for $\cos x = -1$ or 0 $\qquad y_{min} = 1$

Special Cases:

When argument of sine and cosine are same

General form $y = a \sin x + b \cos x + C$. Find min. and max. value of y.

[Ans. $y_{max.} = \sqrt{a^2 + b^2} + C$; $y_{min} = -\sqrt{a^2 + b^2} + C$, where $\theta = \cos^{-1} \frac{b}{\sqrt{a^2 + b^2}}$]

Illustration :

Find minimum and maximum value of $7 \cos \theta + 24 \sin \theta$.

Sol. $y = 7 \cos \theta + 24 \sin \theta$.

$$-\sqrt{a^2 + b^2} \le a\cos\theta + b\sin\theta \le \sqrt{a^2 + b^2}$$

$$\therefore -\sqrt{7^2 + 24^2} \le 7\cos\theta + 24\sin\theta \le \sqrt{7^2 + 24^2}$$

$$-25 \le 7\cos\theta + 24\sin\theta \le 25.$$

Illustration :

$$y = sin^2\left(\frac{15\pi}{8} - 4x\right) - sin^2\left(\frac{17\pi}{8} - 4x\right).\ \textit{Find range of y.}$$

Sol. $\Rightarrow\ sin\left(\frac{15\pi}{8} - 4x + \frac{17\pi}{8} - 4x\right) sin\left(\frac{15\pi}{8} - \frac{17\pi}{8}\right)$

$$\Rightarrow\ sin\,(4\pi - 8x)\ sin\left(-\frac{\pi}{4}\right)\ \Rightarrow - sin\,\frac{\pi}{4}\cdot sin\,8x\ \Rightarrow\ -\frac{1}{\sqrt{2}}\cdot sin\,8x$$

$$\Rightarrow\ \boxed{-\frac{1}{\sqrt{2}} \le y \le \frac{1}{\sqrt{2}}}$$

Ans. $y \in \left[-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right]$

Illustration :

$$y = log_2\left(\frac{3\sin x - 4\cos x + 15}{10}\right).\ \textit{Find range (minimum and maximum) value of y.}$$

Sol. $\therefore\quad -5 \le 3\ sin\ x - 4\ cos\ x \le 5$

$$1 \le \frac{3\sin x - 4\cos x + 15}{10} \le 2$$

$$log_2{}^1 \le log_2\left(\frac{3\sin x - 4\cos x + 15}{10}\right) \le log_2 2$$

10

$$log_2{}^1 \le log_2\left(\frac{3\sin x - 4\cos x + 15}{10}\right) \le log_2 2$$

$$0 \le log_2\left(-\frac{3\sin x - 4\cos x + 15}{10}\right) \le 1$$

Ans. $y_{min} = log_2 1 = 0;\ y_{max} = 1;\ y \in [0, 1]$

TYPE-II :

Argument of sine and cosine are different or a quadratic in sine / cos is given then we make a perfect square in sine / cosine and interpret.

Illustration :

$y = cos\ 2x + 3\ sin\ x$. *Find range of y.*

Sol. $\Rightarrow 1 - 2sin^2x + 3sinx$

$$\Rightarrow 1 - 2\left[sin^2 x - \frac{3}{2}sin\,x + \frac{9}{16} - \frac{9}{16}\right] \Rightarrow\quad 1 - 2\left[sin\,x - \frac{3}{4}\right]^2 + \frac{9}{8}$$

$$y = \frac{17}{8} - 2\left[sin\,x - \frac{3}{4}\right]^2$$

$$y_{max} = \frac{17}{8}\quad at\ sin\ x = \frac{3}{4}$$

$$y_{min} = -4\quad at\ sin\ x = -1$$

[Ans. $y \in \left[-4, \frac{17}{8}\right];\ y_{max} = \frac{17}{8}\ ;\ y_{min} = -4$]

TYPE-III:

Making use of reciprocal relationship between tan and cot, sin/cosec and cos/sec.

Illustration :

If $y = a^2\tan^2x + b^2\cot^2x$ $(a, b \geq 0)$. *Find minimum value of y.*

Sol. $y = a^2\tan^2x + \dfrac{b^2}{\tan^2 x} - 2ab + 2ab \Rightarrow \left(a\tan x - \dfrac{b}{\tan x}\right)^2 + 2ab \geq 2ab$

$y_{min.} = 2ab$

Miscellaneous problems

Illustration :

If $x^2 + y^2 = 4$ *and* $a^2 + b^2 = 8$. *Find minimum and maximum value of* $(ax + by)$

Sol. *Let* $x = r_1\cos\theta_1$, $y = r_1\sin\theta$

and $a = r_2\cos\phi$; $b = r_2\sin\phi$

$\therefore$ $r_1 = 2$, $r_2 = 2\sqrt{2}$

Then $(ax + by) = r_1r_2\cos(\theta - \phi)$

$-r_1r_2 \leq (ax + by) \leq r_1r_2$

$-4\sqrt{2} \leq (ax + by) \leq 4\sqrt{2}$ *Ans.* $y_{max} = 4\sqrt{2}$ *and* $y_{min} = -4\sqrt{2}$

Practice Problem

Practice Problem

Q.1 $y = 3\cos\left(x + \dfrac{\pi}{3}\right) + 5\cos x + 3$. Find minimum and maximum value of y.

Q.2 $y = \cos^2x - 4\cos x + 13$. Find minimum and maximum value of y.

Q.3 $y = 8\sec^2x + 18\cos^2x$, find y_{min}.

Q.4 $y = \dfrac{\tan 3x}{\tan x}$, find range of y.

Q.5 The maximum possible value of $(xv - yu)^2$ over the surface given by the equations $x^2 + y^2 = 4$ and $u^2 + v^2 = 9$ is

(A) 13 (B) 26 (C) 36 (D) 40

Answer key

Q.1 $y_{max} = 10$ and $y_{min} = -4$ Q.2 $y_{max} = 18$; $y_{min} = 10$ Q.3 $y_{min} = 24$

Q.4 $\left(-\infty, \dfrac{1}{3}\right) \cup [3, \infty)$ Q.5 C

CONDITIONAL IDENTITIES AND INEQUALITIES :

(i) If A, B, C are the angles of a triangle, then

$$A + B + C = 180°$$

$\therefore$ $B + C = 180° - A$, $C + A = 180° - B$, $A + B = 180° - C$,

Hence $\sin(B + C) = \sin(180° - A) = \sin A$

Similarly $\sin(C + A) = \sin B$ and $\sin(A + B) = \sin C$.

Thus in a triangle, sine of any angle is equal to the sine of the sum of the remaining angles.

(ii) Again $\cos(B + C) = \cos(180° - A) = -\cos A$

In the same manner

$$\cos(C + A) = -\cos B \text{ and } \cos(A + B) = -\cos C.$$

Thus in a triangle, the cosine of any one angle is equal to minus times the cosine of the remaining two angles.

(iii) If $\frac{A}{2} + \frac{B}{2} + \frac{C}{2} = \frac{\pi}{2}$ then $\frac{A}{2} + \frac{B}{2} = \frac{\pi}{2} - \frac{C}{2}$

$$\sin\left(\frac{A}{2} + \frac{B}{2}\right) = \sin\left(\frac{\pi}{2} - \frac{C}{2}\right) = \cos\frac{C}{2}$$

Note : Some standard identities in triangle ($A + B + C = \pi$) :-

(1) $\sin 2A + \sin 2B + \sin 2C = 4 \sin A \sin B \sin C$

(2) $\cos 2A + \cos 2B + \cos 2C = -1 - 4 \cos A \cos B \cos C$

(3) $\sin A + \sin B + \sin C = 4 \cos\frac{A}{2} \cos\frac{B}{2} \cos\frac{C}{2}$

(2) $\cos 2A + \cos 2B + \cos 2C = -1 - 4 \cos A \cos B \cos C$

(3) $\sin A + \sin B + \sin C = 4 \cos\frac{A}{2} \cos\frac{B}{2} \cos\frac{C}{2}$

(4) $\cos A + \cos B + \cos C = 1 + 4 \sin\frac{A}{2} \sin\frac{B}{2} \sin\frac{C}{2}$

(5) $\tan A + \tan B + \tan C = \tan A \tan B \tan C$

(6) $\tan\frac{A}{2} \tan\frac{B}{2} + \tan\frac{C}{2} \tan\frac{B}{2} + \tan\frac{C}{2} \tan\frac{A}{2} = 1$

Illustration :

Prove that $\sin 2A + \sin 2B + \sin 2C = 4 \sin A \sin B \sin C$

Sol. $\sin 2A + \sin 2B + \sin 2C = 2\sin(A + B)\cos(A - B) + 2 \sin C \cos C$

Since $A + B + C = 180°$

We have $A + B = 180° - C$,

and therfore $\sin(A + B) = \sin C$,

and $\cos(A + B) = -\cos C$

Hence the expression

$$= 2\sin C \cos(A - B) + 2 \sin C \cos C$$
$$= 2\sin C [\cos(A - B) + \cos C]$$
$$= 2\sin C [\cos(A - B) - \cos(A + B)]$$
$$= 2\sin C, 2 \sin A \sin B$$
$$= 4\sin A \sin B \sin C.$$

Note: Remember this identity: $\sin 2A + \sin 2B + \sin 2C = 4 \sin A \sin B \sin C$

Illustration :

Prove that $\cos A + \cos B + \cos C = 1 + 4 \sin\frac{A}{2} \sin\frac{B}{2} \sin\frac{C}{2}$

Sol. $(\cos A + \cos B) + \cos C$

$$= 2\cos\frac{A+B}{2}\cos\frac{A-B}{2} + \cos C - 1$$

$$= 2\cos\left(\frac{\pi}{2}-\frac{C}{2}\right)\cos\frac{A-B}{2} + \cos C - 1$$

$$= 2\sin\frac{C}{2}\cos\frac{A-B}{2} + 1 - 2\sin^2\frac{C}{2} - 1$$

$$= 2\sin\frac{C}{2}\cos\frac{A-B}{2} - 2\sin^2\frac{C}{2}$$

$$= 2\sin\frac{C}{2}\left[\cos\frac{A-B}{2} - \sin\left(\frac{\pi}{2}-\frac{A+B}{2}\right)\right]$$

$$= 2\sin\frac{C}{2}\ 2\sin\frac{A}{2}\sin\frac{B}{2} = 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$$

Illustration :

If A, B, C are the angles of a triangle, show that $\tan A + \tan B + \tan C = \tan A \tan B \tan C$

Sol. *We have*

Illustration :

If A, B, C are the angles of a triangle, show that $\tan A + \tan B + \tan C = \tan A \tan B \tan C$

Sol. *We have*

$$\tan(A + B + C) = \frac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - (\tan B \tan C + \tan C \tan A + \tan A \tan B)}$$

But $\tan(A + B + C) = \tan 180^\circ = 0$

Hence $0 = \tan A + \tan B + \tan C - \tan A \tan B \tan C$

i.e, $\tan A + \tan B + \tan C = \tan A \tan B \tan C$

This may also be proved independently. For

$$\tan(A + B) = \tan(180^\circ - C) = -\tan C.$$

$$\therefore \quad \frac{\tan A + \tan B}{1 - \tan A \tan B} = -\tan C$$

$$\tan A + \tan B = -\tan C + \tan A \tan B \tan C$$

$$\tan A + \tan B + \tan C = \tan A \tan B \tan C$$

Illustration :

In a ΔABC *show that* $\cos A \cos B \cos C \le \frac{1}{8}$

Sol. *Let* $y = \cos A \cos B \cos C$

$$= \frac{1}{2}\cos A\left[\cos\frac{(B+C)}{2} + \cos\frac{(B-C)}{2}\right]$$

$$= \frac{1}{2}\cos A\,[-\cos A + \cos(B - C)]$$

$= \frac{1}{2} \cos A\, [\cos (B - C) - \cos A]$

$\leq \frac{\cos A}{2}\ (1 - \cos A)$

$\therefore \cos (B - C)_{max} = 1)$

$\leq -\frac{1}{2}\ (\cos^2 A - \cos A)$

$\leq -\frac{1}{2}\left[\left(\cos A - \frac{1}{2}\right)^2 - \frac{1}{4}\right]$

$\leq \frac{1}{8} - \frac{1}{2}\left(\cos A - \frac{1}{2}\right)^2$

$y \leq \frac{1}{8} - \frac{1}{2}\left(\cos A - \frac{1}{2}\right)^2$

max vlaues of y occurs when $\cos A = \frac{1}{2}$

$A = 60°$.

max vlaues of y occurs when $\cos A = \frac{1}{2}$

$A = 60°$.

$y_{max} = \frac{1}{8} \qquad \boxed{y \leq \frac{1}{8}}$

Practice Problem

Q.1 In Δ ABC, prove that $\cos 2A + \cos 2B + \cos 2C = -1 - 4 \cos A \cos B \cos C$

Q.2 In Δ ABC, prove that $\cot \frac{A}{2} + \cot \frac{B}{2} + \cot \frac{C}{2} = \cot \frac{A}{2} \cot \frac{B}{2} \cot \frac{C}{2}$

Q.3 In Δ ABC, prove that $\cot B \cot C + \cot C \cot A + \cot A \cot B = 1$.

Q.4 If $\alpha + \beta = \gamma$, prove that $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1 + 2 \cos \alpha \cdot \cos \beta \cdot \cos \gamma$.

Q.5 In ΔABC, prove that $\tan \frac{A}{2} \tan \frac{B}{2} + \tan \frac{B}{2} \cdot \tan \frac{C}{2} + \tan \frac{C}{2} \cdot \tan \frac{A}{2} = 1$.

Q.6 In a ΔABC show that $1 < \cos A + \cos B + \cos C \leq \frac{3}{2}$.

Q.7 For all real θ prove that $\cos (\sin \theta) > \sin (\cos \theta)$

SUMMATION OF TRIGONOMETRIC SERIES :

Type-I

Sum of the sin and cosine series when the angles are in A.P.

(1) $$\sin\alpha + \sin(\alpha+\beta) + \sin(\alpha+2\beta) + \ldots + \sin(\alpha + \overline{n-1}\,\beta) = \frac{\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}} \times \sin\left[\alpha + (n-1)\frac{\beta}{2}\right]$$

Proof :

Let $S = \sin\alpha + \sin(\alpha+\beta) + \sin(\alpha+2\beta) + \ldots + \sin(\alpha + \overline{n-1}\,\beta)$

Here angle are in A.P. and common difference of angles = β

$\therefore$ multiplying both sides by $2\sin\frac{\beta}{2}$, we get

$$2S\sin\frac{\beta}{2} = 2\sin\alpha\sin\frac{\beta}{2} + 2\sin(\alpha+\beta)\sin\frac{\beta}{2} + \ldots\ldots + 2\sin(\alpha+\overline{n-1}\,\beta)\sin\frac{\beta}{2} \qquad \ldots(i)$$

Now, $$2\sin\alpha\sin\frac{\beta}{2} = \cos\left(\alpha - \frac{\beta}{2}\right) - \cos\left(\alpha + \frac{\beta}{2}\right)$$

$$2\sin(\alpha+\beta)\sin\frac{\beta}{2} = \cos\left(\alpha + \frac{\beta}{2}\right) - \cos\left(\alpha + \frac{3\beta}{2}\right)$$

… … … … … … … … … …

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$$2\sin(\alpha+\overline{n-1}\,\beta)\sin\frac{\beta}{2} = \cos\left[\alpha + (2n-3)\frac{\beta}{2}\right] - \cos\left[a(2n-1)\frac{\beta}{2}\right]$$

---

adding, we get R.H.S. of equation (i) $= \cos\left(\alpha - \frac{\beta}{2}\right) - \cos\left[\alpha + (2n-1)\frac{\beta}{2}\right]$

hence, $$S = \frac{\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}}\sin\left[\alpha + (n-1)\frac{\beta}{2}\right]$$

(2) $$\cos\alpha + \cos(\alpha+\beta) + \cos(\alpha+2\beta) + \ldots\ldots + \cos(\alpha + \overline{n-1}\,\beta) = \frac{\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}} \times \cos\left[\alpha + (n-1)\frac{\beta}{2}\right]$$

Illustration :

Find the sum of series $\cos\frac{\pi}{11} + \cos\frac{3\pi}{11} + \cos\frac{5\pi}{11} + \cos\frac{7\pi}{11} + \cos\frac{9\pi}{11}$.

Sol. *Here the angles are in A.P. of 5 terms*

$n = 5,\ \alpha = \frac{\pi}{11},\ \beta = \frac{2\pi}{11}\ \therefore\ \frac{\beta}{2} = \frac{\pi}{11}$

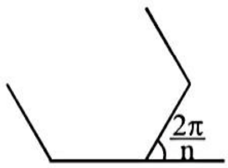
$$S = \frac{\sin \cdot \frac{5\pi}{11}}{\sin \frac{\pi}{11}} \cdot \cos \frac{1}{2}\left[\frac{\pi}{11} + \frac{9\pi}{11}\right]$$

$$S = \frac{2\sin \cdot \frac{5\pi}{11} \cos \cdot \frac{5\pi}{11}}{2\sin \frac{\pi}{11}} = \frac{\sin \cdot \frac{10\pi}{11}}{2\sin \frac{\pi}{11}} = \frac{\sin\left(\pi - \frac{\pi}{11}\right)}{2\sin \frac{\pi}{11}} = \frac{1}{2}$$

Type - II

For n sided regular polygon

Sum of all exterior angles = 2π



(i) The value of one exterior angle = $\frac{2\pi}{n}$

(ii) The value of one interior angle = $\left(\pi - \frac{2\pi}{n}\right) = \pi\frac{(n-2)}{n}$

(iii) Sum of interior angles = $\pi\,(n-2)$

Illustration :

In a regular polygon of n-sides with $A_1, A_2, \ldots\ldots. A_n$ vertices prove that

Illustration :

In a regular polygon of n-sides with $A_1, A_2, \ldots\ldots. A_n$ vertices prove that

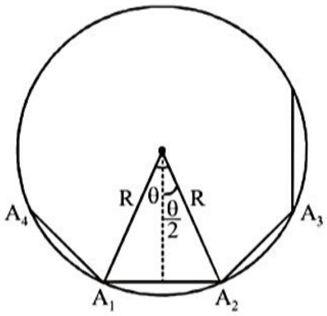
$$(A_1A_2)^2 + (A_1A_3)^2 + (A_1A_4)^2 + \ldots\ldots\ldots + (A_1A_n)^2 = 2nR^2$$

Where R is the radius of circumcircle circumscribing it.

Sol. $(A_1A_2)^2 + (A_1A_3)^2 + \ldots.\ (A_1A_n)^2 = 2nR^2$

$$\theta = \frac{2\pi}{n}$$

By trigonometry



$$(A_1A_2)^2 = 4R^2 \cdot \sin^2\frac{\pi}{n}$$

$$(A_1A_3)^2 = 4R^2 \cdot \sin^2\frac{2\pi}{n}$$

$$(A_1A_4)^2 = 4R^2 \cdot \sin^2\frac{3\pi}{n}$$

...

$$(A_1A_n)^2 = 4R^2 \cdot \sin^2\frac{(n-1)\pi}{n}$$

$$\therefore\ (A_1A_2)^2 + (A_1A_3)^2 + \ldots. + (A_1A_n)^2$$

$$4R^2\left[\sin^2\frac{\pi}{n} + \sin^2 x\frac{2\pi}{n} + \sin^2\frac{3\pi}{n} + \ldots..\sin^2\frac{(n-1)\pi}{n}\right]$$

$$\frac{1}{2}4R^2\left[1 - \cos\frac{2\pi}{n} + 1 - \cos\frac{4\pi}{n} + 1 - \cos\frac{6\pi}{n} + \ldots..1 - \cos\frac{2(n-1)\pi}{n}\right]$$

$$2R^2(n-1) - 2R^2\left[\frac{\sin\frac{(n-1)\pi}{n}}{\sin\left(\frac{\pi}{n}\right)}\cdot\cos\left(\frac{\frac{2\pi}{n}+\frac{2(n-1)\pi}{n}}{2}\right)\right]$$

$\Rightarrow \quad 2R^2(n-1) - 2R^2(-1) \qquad \Rightarrow \quad 2R^2n - 2R^2 + 2R^2$

$\Rightarrow \quad 2nR^2$

Type-III of summation of sine/cosine series

Splitting the sum series as difference of 2 terms.

Illustration :

$S = cosec\, x + cosec\, 2x + cosec\, 4x + \ldots\ldots + cosec\, 2^n x = \cot\frac{x}{2} - \cot 2^n x.$

Sol. Let $cosec\, x$

$$= \frac{\sin\left(\frac{x}{2}\right)}{\sin\frac{x}{2}\cdot\sin x} = \frac{\sin\left(x-\frac{x}{2}\right)}{\sin\left(\frac{x}{2}\right)\cdot\sin x} = \frac{\sin\cdot\cos\frac{x}{2} - \cos x\cdot\sin\frac{x}{2}}{\sin\frac{x}{2}\cdot\sin x} = \cot\frac{x}{2} - \cot x$$

$$S = \left(\cot\frac{x}{2} - \cot x\right) + (\cot x - \cot 2x) + (\cot 2x - \cot 2^2 x) + \ldots (\cot 2^{n-1} x - \cot 2^n x)$$

$$\Rightarrow S = \left(\cot\frac{x}{2} - \cot 2^n\cdot x\right)$$

$$S = \left(\cot\frac{x}{2} - \cot x\right) + (\cot x - \cot 2x) + (\cot 2x - \cot 2^2 x) + \ldots (\cot 2^{n-1} x - \cot 2^n x)$$

$$\Rightarrow S = \left(\cot\frac{x}{2} - \cot 2^n\cdot x\right)$$

Illustration :

Given $\theta \in (0, 2\pi)$ and if $\sum_{r=1}^{r=3}\sec\left(\frac{r\pi}{6}+\theta\right)\sec\left(\frac{(r-1)\pi}{6}+\theta\right) = 4$, then which of the following alternative(s) is/are correct?

(A) $\theta = \frac{3\pi}{4}$ (B) $\theta = \frac{\pi}{8}$ (C) $\theta = \frac{7\pi}{4}$ (D) $\theta = \frac{3\pi}{2}$

Sol. $$\frac{1}{\sin\frac{\pi}{6}}\left[\frac{1}{\cos\left(\frac{r\pi}{6}+\theta\right)}\cdot\frac{\sin\frac{\pi}{6}}{\cos\left(\frac{(r-1)\pi}{6}+\theta\right)}\right] = \frac{1}{\sin\frac{\pi}{6}}\left[\frac{\sin\left(\left(\frac{\pi}{6}+\theta\right)-\left(\frac{(r-1)\pi}{6}+\theta\right)\right)}{\cos\left(\frac{r\pi}{6}+\theta\right)\cos\left(\frac{(r-1)\pi}{6}+\theta\right)}\right]$$

$$= \frac{1}{\sin\frac{\pi}{6}}\left[\frac{\sin A\cdot\cos B - \cos A\cdot\sin B}{\cos A\cdot\cos B}\right] \qquad \text{Let} \quad \left(\frac{\pi}{6}+\theta\right) = A,\ \left(\frac{(r-1)\pi}{6}+\theta\right) = B$$

$$= \frac{1}{\sin\frac{\pi}{6}}[\tan A - \tan B] = \frac{1}{\sin\frac{\pi}{6}}\left[\tan\left(\frac{r\pi}{6}+\theta\right) - \tan\left(\frac{(r-1)\pi}{6}+\theta\right)\right]$$

$$\sum_{r=1}^{3} S = -2\,[\cot\theta + \tan\theta] \quad \Rightarrow \quad \frac{-4}{\sin 2\theta} = +4$$

$$\text{or} \quad \sin 2\theta = -1 \qquad \therefore\ \theta = \frac{3\pi}{4} \text{ or } \frac{7\pi}{4}$$

Practice Problem

Q.1 Prove that $\displaystyle\sum_{r=1}^{n-1} \sin^2 \frac{r\pi}{n} = \frac{n}{2}$.

Q.2 Prove that $\tan\frac{x}{2}\sec x + \tan\frac{x}{2^2}\sec\frac{x}{2} + \tan\frac{x}{2^3}\sec\frac{x}{2^2} + \ldots\ldots$ up to n terms $= \tan x - \tan\frac{x}{2^n}$

Q.3 Find the sum to n terms of the series $\dfrac{\sin x}{\cos x + \cos 2x} + \dfrac{\sin 2x}{\cos x + \cos 4x} + \dfrac{\sin 3x}{\cos x + \cos 6x} + \ldots\ldots.$

Answer key

Q.3 $\dfrac{1}{4}\operatorname{cosec}\dfrac{x}{2}\left[\sec(2n+1)\dfrac{x}{2} - \sec\dfrac{x}{2}\right]$

ELIMINATION :

ELIMINATION :

Between any two equations involving one unknown quantity we can, in theory, always eliminate that quantity. In practice, a considerable amount of artifice and ingenuity is often required in seemingly simple cases. So, between any three equations involving two unknown quantities, we can theoretically eliminate both of the unknown quantities.

Eliminate θ from the equations

Illustration :

Eliminate θ from the equations $a\cos\theta + b\sin\theta = c$ and $b\cos\theta - a\sin\theta = d$.

Sol. $a\cos\theta + b\sin\theta = c$ ... (i) ; $b\cos\theta - a\sin\theta = d$. ... (ii)

Square (i) and (ii) and adding,

$a^2(\cos^2\theta + \sin^2\theta) + b^2(\sin^2\theta + \cos^2\theta) = c^2 + d^2$

or $a^2 + b^2 = c^2 + d^2$.

Illustration :

Eliminate θ from the equations

$a\sin\alpha - b\cos\alpha = 2b\sin\theta$, and $a\sin 2\alpha - b\cos 2\theta = a$,

Sol. $a\sin\alpha - b\cos\alpha = 2b\sin\theta$ ... (i)

$a\sin 2\alpha - b\cos 2\theta = a$ ... (ii)

From (i) $\sin\theta = \dfrac{a\sin\alpha - b\cos\alpha}{2b}$

From (ii) $a \sin 2\alpha - b\,(1 - 2\sin^2\theta) = a$

or $\quad a\sin 2\alpha - b\left\{1 - 2\left(\frac{a\sin\alpha - b\cos\alpha}{2b}\right)^2\right\} = a$

or $\quad a^2\sin^2\alpha + b^2\cos^2\alpha + 4ab\sin\alpha\cos\alpha - 2ab\sin\alpha\cos\alpha = 2ab + 2b^2$

or $\quad a\sin\alpha + b\cos\alpha = \sqrt{\{2b(a+b)\}}$

Illustration :

Eliminate θ from the equations

$$x\sin\theta - y\cos\theta = \sqrt{(x^2+y^2)},\ \text{and}\ \frac{\sin^2\theta}{a^2} + \frac{\cos^2\theta}{b^2} = \frac{1}{x^2+y^2}.$$

Sol. $x\sin\theta - y\sin\theta = \sqrt{x^2+y^2}$ ... (1)

and $\quad \frac{\sin^2\theta}{a^2} + \frac{\cos^2\theta}{b^2} = \frac{1}{x^2+y^2}$ ... (2)

Squaring (1) we have

$x^2\sin^2\theta + y^2\cos^2\theta - 2xy\sin\theta\cos\theta = x^2 + y^2$

or $\quad x\cos\theta + y\sin\theta = 0$

or $\quad \tan\theta = \frac{-x}{y}$; $\sin^2\theta = \left\{\frac{x}{\sqrt{(x^2+y^2)}}\right\}^2 = \frac{x^2}{x^2+y^2}$

and $\quad \cos^2\theta = \frac{y^2}{x^2+y^2}$

and $\quad \cos^2\theta = \frac{y^2}{x^2+y^2}$

Substituting value in (2) we have

or $\quad \frac{x^2}{a^2(x^2+y^2)} + \frac{y^2}{b^2(x^2+y^2)} = \frac{1}{x^2+y^2} \quad \Rightarrow \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Practice Problem

Q.1 Eliminate θ from the following

$\sin\theta - \cos\theta = p$, and $\operatorname{cosec}\theta - \sin\theta = q$.

Q.2 Eliminate θ from the following

$a\cos 2\theta = b\sin\theta$, and $c\sin 2\theta = d\cos\theta$.

Q.3 Eliminate x, y from the following

$\cos x + \cos y = a$

$\cos 2x + \cos 2y = b$

$\cos 3x + \cos 3y = c$.

Answer key

Q.1 $(p^2+1)^2 + 2q\,(p^2+1) = 4(p+q)^2$ Q.2 $a(2c^2 - d^2) = bcd$

Q.3 $2a^3 + c = 3a(b+1)$

Solved Examples

Q.1 Compute the square of the value of the expression $\frac{4+\sec 20°}{\text{cosec } 20°}$.

Sol. $\frac{4+\sec 20°}{\text{cosec} 20°}$

$$=\left(\frac{4\cos 20°+1}{\cos 20°}\right)\sin 20° = \frac{4\sin 20° \cos 20° + \sin 20°}{\cos 20°}$$

$$= \frac{2\sin 40° + \sin 20°}{\cos 20°} = \frac{\sin 40° + 2\sin 30° \cos 10°}{\cos 20°}$$

$$= \frac{\sin 40° + \cos 10°}{\cos 20°} = \frac{\sin 40° + \sin 80°}{\cos 20°} = \frac{2\sin 60° \cos 20°}{\cos 20°} = \sqrt{3}$$

$$= \left(\sqrt{3}\right)^2 = 3 \textbf{ Ans.}$$

Q.2 If $\alpha+\beta+\gamma=\pi$ and $\prod \tan\left(\frac{\alpha+\beta-\gamma}{4}\right) = 1$, then find $\sum \sin\alpha + \prod \sin\alpha$.

Sol. $\Pi \tan\left(\frac{\alpha+\beta-\gamma}{4}\right)=1$

$(\pi - 2\gamma)$

Sol. $\Pi \tan\left(\frac{\alpha+\beta-\gamma}{4}\right)=1$

$$\Rightarrow \Pi \tan\left(\frac{\pi-2\gamma}{4}\right)=1 \qquad \text{(since } \alpha+\beta+\gamma=\pi\text{)}$$

$$\Rightarrow \Pi \tan\left(\frac{\pi}{4}-\frac{\gamma}{2}\right)=1 \qquad \Rightarrow \qquad \Pi\left(\frac{\cos\frac{\gamma}{2}-\sin\frac{\gamma}{2}}{\cos\frac{\gamma}{2}+\sin\frac{\gamma}{2}}\right)=1$$

$$\Rightarrow \Pi\left(\cos\frac{\gamma}{2}-\sin\frac{\gamma}{2}\right)=\Pi\left(\cos\frac{\gamma}{2}+\sin\frac{\gamma}{2}\right)$$

After squaring both sides

$$\Rightarrow \Pi(1-\sin\gamma)=\Pi(1+\sin\gamma)$$

$$\Rightarrow 1-\sum\sin\gamma+\sum\sin\gamma\sin\alpha-\Pi\sin\alpha = 1+\sum\sin\gamma+\sum\sin\gamma\sin\alpha+\Pi\sin\alpha$$

$$\Rightarrow \sum\sin\gamma+\Pi\sin\gamma=0 \textbf{ Ans.}$$

Q.3 Let $f(x)=\frac{1+\cos 2x+8\sin^2 x}{\sin 2x}$, $x \in (0, \pi/2)$. Find the minimum value of $f(x)$.

Sol. $f(x)=\frac{1+\cos 2x+8\sin^2 x}{\sin 2x}$

$$f(x)=\frac{2\cos^2 x+8\sin^2 x}{2\sin x\cos x}$$

Divide by $2\cos^2 x$, we get

$$f(x) = \frac{1+4\tan^2 x}{\tan x} \quad \Rightarrow \quad f(x) = \frac{1}{\tan x} + 4\tan x$$

Apply A.M. $\geq$ G.M.

$$\therefore \quad \frac{\frac{1}{\tan x} + 4\tan x}{2} \geq \sqrt{4} \qquad \therefore \quad \frac{1}{\tan x} + 4\tan x \geq 4$$

Q.4 α and β are the positive acute angles and satisfying simultaneously the equations

$5\sin 2\beta = 3\sin 2\alpha$ and $\tan\beta = 3\tan\alpha$,

find the value of $\tan\alpha + \tan\beta$.

Sol. $\frac{\sin 2\alpha}{\sin 2\beta} = \frac{5}{3}$

$$\Rightarrow \frac{(2\tan\alpha)}{(1+\tan^2\alpha)}\frac{(1+\tan^2\beta)}{(2\tan\beta)} = \frac{5}{3} \qquad \Rightarrow \qquad \frac{(2\tan\alpha)}{(1+\tan^2\alpha)}\frac{(1+9\tan^2\alpha)}{(6\tan\alpha)} = \frac{5}{3} \quad (\tan\beta = 3\tan\alpha)$$

$\Rightarrow 1 + 9\tan^2\alpha = 5 + 5\tan^2\alpha \qquad \Rightarrow \qquad 4\tan^2\alpha = 4$

$\Rightarrow \boxed{\tan\alpha = +1}$

$\Rightarrow \tan\beta = 3$

$\therefore \tan\alpha + \tan\beta = 1 + 3 = 4$ **Ans.**

Q.5 If $\frac{\cos 42° + \sin 42°}{\cos 42° - \sin 42°} = (\tan\alpha + \sec\alpha)$ where $0 < \alpha < 90°$, then find the value of α in degree.

Sol. RHS $= \tan\alpha + \sec\alpha$

$$= \frac{1+\sin\alpha}{\cos\alpha} = \frac{\left(\cos\frac{\alpha}{2} + \sin\frac{\alpha}{2}\right)^2}{\left(\cos^2\frac{\alpha}{2} - \sin^2\frac{\alpha}{2}\right)} = \frac{\cos\frac{\alpha}{2} + \sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2} - \sin\frac{\alpha}{2}}$$

$\therefore \frac{\alpha}{2} = 42° \Rightarrow \alpha = 84°$ **Ans.**

Q.6 Let $x = \frac{\sum_{n=1}^{44}\cos n°}{\sum_{n=1}^{44}\sin n°}$ find the greatest integer that does not exceed $100x$.

Sol. $x = \frac{\sum_{n=1}^{44}\cos n°}{\sum_{n=1}^{44}\sin n°} = \frac{\cos 1° + \cos 2° + ...\cos 44°}{\sin 1° + \sin 2° + ...\sin 44°}$

$$\frac{\dfrac{\sin\left(\frac{44}{2}\right)^{\circ}}{\sin\left(\frac{1}{2}\right)^{\circ}}\cos\left(\dfrac{1^{\circ}+44^{\circ}}{2}\right)}{\dfrac{\sin\left(\frac{44}{2}\right)^{\circ}}{\sin\left(\frac{1}{2}\right)^{\circ}}\sin\left(\dfrac{1^{\circ}+44^{\circ}}{2}\right)} \Rightarrow \cot\left(\frac{45^{\circ}}{2}\right) \Rightarrow \cot\left(22\frac{1}{2}^{\circ}\right)$$

$\Rightarrow x = \left(\sqrt{2}+1\right) = 2.414$

$\Rightarrow [100\,x] = 241$ **Ans.**

Q.7 For $A \in \left(0, \frac{\pi}{4}\right)$, if $\left(1+\frac{\cos 3A}{\cos A}\right)+\left(1+\frac{\cos 6A}{\cos 2A}\right)+\left(1+\frac{\cos 9A}{\cos 3A}\right)+\left(1+\frac{\cos 12A}{\cos 4A}\right) = 0$, then find the value of $(\operatorname{cosec} A - \sec 2A)$.

Sol. $\left(1+\frac{\cos 3A}{\cos A}\right)+\left(1+\frac{\cos 6A}{\cos 2A}\right)+\left(1+\frac{\cos 9A}{\cos 3A}\right)+\left(1+\frac{\cos 12A}{\cos 4A}\right)=0$

$\Rightarrow \quad \left(\frac{\cos A+\cos 3A}{\cos A}\right)+\left(\frac{\cos 2A+\cos 6A}{\cos 2A}\right)+\left(\frac{\cos 3+\cos 9A}{\cos 3A}\right)+\left(\frac{\cos 4A+\cos 12A}{\cos 4A}\right)=0$

Sol. $\left(1+\frac{\cos 3A}{\cos A}\right)+\left(1+\frac{\cos 6A}{\cos 2A}\right)+\left(1+\frac{\cos 9A}{\cos 3A}\right)+\left(1+\frac{\cos 12A}{\cos 4A}\right)=0$

$\Rightarrow \quad \left(\frac{\cos A+\cos 3A}{\cos A}\right)+\left(\frac{\cos 2A+\cos 6A}{\cos 2A}\right)+\left(\frac{\cos 3+\cos 9A}{\cos 3A}\right)+\left(\frac{\cos 4A+\cos 12A}{\cos 4A}\right)=0$

$\Rightarrow \quad \left(\frac{2\cos 2A\cdot\cos A}{\cos A}\right)+\left(\frac{2\cos 4A\cdot\cos 2A}{\cos 2A}\right)+\left(\frac{\cos 6A\cdot\cos 3A}{\cos 3A}\right)+\left(\frac{\cos 8A\cdot\cos 4A}{\cos 4A}\right)=0$

$\Rightarrow \quad 2[\cos 2A+\cos 4A+\cos 6A+\cos 8A = 0$

$\Rightarrow \quad 2\left[\frac{\sin 4A}{\sin A}\cos(5A)\right]=0$

$\Rightarrow \quad \frac{4\sin 2A\cos 2A\cos 5A}{\sin A}=0$

$\cos 5A = 0$ or $\cos 2A = 0$ or $\cos A = 0$

$\therefore \quad 5A = \frac{\pi}{2}$ or $2A = \frac{\pi}{2}$ or $A = \frac{\pi}{2}$

$A = \frac{\pi}{10}$ or $A = \frac{\pi}{4}$

$A \in \left(0, \frac{\pi}{4}\right) \Rightarrow A = \frac{\pi}{10}$

$\therefore \quad (\operatorname{cosec} A - \sec 2A)$

$=\left(\frac{1}{\sin 18^{\circ}}-\frac{1}{\cos 36^{\circ}}\right)=\left(\frac{4}{\sqrt{5}-1}-\frac{4}{\sqrt{5}+1}\right)=2$ **Ans.**

Q.8 Prove that $\dfrac{2\cos 2^n\theta+1}{2\cos\theta+1} = (2\cos\theta-1)(2\cos 2\theta-1)(2\cos 2^2\theta-1).......(2\cos 2^{n-1}\theta-1)$

Sol. We have to prove $\dfrac{2\cos 2^n\theta+1}{2\cos\theta+1} = (2\cos\theta-1)(2\cos 2\theta-1)(2\cos 2^2\theta-1).......(2\cos 2^{n-1}\theta-1)$

or $2\cos 2^n\theta+1 = [(2\cos\theta+1)(2\cos\theta-1)](2\cos 2\theta-1)(2\cos 2^2\theta-1)\ ....\ (2\cos 2^{n-1}\theta-1)$

Now $[(2\cos\theta+1)(2\cos\theta-1)](2\cos 2\theta-1)(2\cos 2^2\theta-1)\ ....\ (2\cos 2^{n-1}\theta-1)$

$= (4\cos^2\theta-1)(2\cos 2\theta-1)(2\cos 2^2\theta-1)\ ....\ (2\cos 2^{n-1}\theta-1)$

$= (2\cos 2\theta+1)(2\cos 2\theta-1)(2\cos 2^2\theta-1)....\ (2\cos 2^{n-1}\theta-1)$ [using $\cos 2\theta = 2\cos^2\theta-1$]

$= (4\cos^2\theta-1)(2\cos 2^2\theta-1)\ ....\ (2\cos 2^{n-1}\theta-1)$

$= (2\cos^2\theta+1)(2\cos 2^2\theta-1)\ ....\ (2\cos 2^{n-1}\theta-1)$

$= (4\cos^2 2^2\theta-1)(2\cos 2^3\theta-1)\ ....\ (2\cos 2^{n-1}\theta-1)$

$\vdots$

$= (2\cos 2^{n-1}\theta+1)(2\cos 2^{n-1}\theta-1)$

$= 4\cos 2\ 2^{n-1}\theta-1$

$= 2\cos 2^n\theta+1$ **Hence proved.**

Q.9 Find the sum

$$\tan\theta+\frac{1}{2}\tan\frac{\theta}{2}+\frac{1}{2^2}\cdot\tan\frac{\theta}{2^2}+\frac{1}{2^3}\cdot\tan\frac{\theta}{2^3}+\ldots\ldots\infty$$

Sol. Let t_n denote the n^{th} term of the given series.

Now, first term $= \tan\theta$

$\sin\theta \quad \sin\theta$

Sol. Let t_n denote the n^{th} term of the given series.

Now, first term $= \tan\theta$

$$= \frac{\sin\theta}{\cos\theta}\cdot\frac{\sin\theta}{\sin\theta}$$

$$= \frac{\sin^2\theta}{\sin\theta\cos\theta}$$

$$= \frac{\cos^2\theta-(\cos^2\theta-\sin^2\theta)}{\sin\theta\cos\theta}$$

$= \cot\theta-2\cot 2\theta$

i.e., $t_1 = \cot\theta-2\cot 2\theta$

similarly, $t_2 = \dfrac{1}{2}\cot\dfrac{\theta}{2}-\cot\theta$

$$t_3 = \frac{1}{2^2}\cdot\cot\frac{\theta}{2^2}-\frac{1}{2}\cot\frac{\theta}{2}$$

....

....

$$t_n = \frac{1}{2^{n-1}}\cot\frac{\theta}{2^{n-1}}-\frac{1}{2^{n-2}}\cdot\cot\frac{\theta}{2^{n-2}}$$

---

adding we get, $S_n = \dfrac{1}{2^{n-1}}\cdot\cot\dfrac{\theta}{2^{n-1}}-2\cot 2\theta$

Required sum $S = \underset{n\to\infty}{\text{Lt}}\, S_n$

$$= \underset{n\to\infty}{\text{Lt}}\left(\frac{1}{2^{n-1}}\cot\frac{\theta}{2^{n-1}} - 2\cot 2\theta\right)$$

$$= \underset{n\to\infty}{\text{Lt}}\left(\frac{1}{2^{n-1}}\cdot\frac{1}{\tan\frac{\theta}{2^{n-1}}} - 2\cot 2\theta\right)$$

$$= \underset{n\to\infty}{\text{Lt}}\left[\frac{1}{\theta}\cdot\frac{\theta}{2^{n-1}}\cdot\frac{1}{\tan\frac{\theta}{2^{n-1}}} - 2\cot 2\theta\right]$$

$$= \frac{1}{\theta} - 2\cot 2\theta \textbf{ Ans.}$$

Q.10 If $A + B + C = \pi$, then prove : $\dfrac{\sin 2A + \sin 2B + \sin 2C}{\sin A + \sin B + \sin C} = 8\sin\dfrac{A}{2}\sin\dfrac{B}{2}\sin\dfrac{C}{2}$.

Sol. L.H.S. $= \dfrac{\sin 2A + \sin 2B + \sin 2C}{\sin A + \sin B + \sin C}$

Sol. L.H.S. $= \dfrac{\sin 2A + \sin 2B + \sin 2C}{\sin A + \sin B + \sin C}$

$$= \frac{2\sin(A+B)\cos(A-B) + 2\sin C\cos C}{2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right) + 2\sin\frac{C}{2}\cos\frac{C}{2}}$$

$$= \frac{2\sin C[\cos(A-B) - \cos(A+B)]}{2\cos\frac{C}{2}\left[\cos\left(\frac{A-B}{2}\right) + \cos\left(\frac{A+B}{2}\right)\right]}$$

$$= \frac{2\sin C\cdot 2\sin A\sin B}{2\cos\frac{C}{2}\ 2\cos\frac{A}{2}\cos\frac{B}{2}} = \frac{\sin A\sin B\sin C}{\cos\frac{A}{2}\ \cos\frac{B}{2}\ \cos\frac{C}{2}}$$

$$= 8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} = \text{R.HS. } \textbf{Hence proved.}$$

Q.11 If $A + B + C = 180^\circ$, prove $\tan A + \tan B + \tan C \le 3\sqrt{3}$, where A, B, C are acute angles.

Sol. $\tan(A + B) = \tan(180^\circ - C)$

or, $\dfrac{\tan A + \tan B}{1 - \tan A\tan B} = -\tan C$

or, $\tan A + \tan B + \tan C = \tan A\tan B\tan C$

$\dfrac{\tan A + \tan B + \tan C}{3} \ge \sqrt[3]{\tan A\tan B\tan C}$ [since A.M. $\ge$ G.M.]

or, $\quad \dfrac{\tan A \tan B \tan C}{3} \geq \sqrt[3]{\tan A \tan B \tan C}$

or, $\quad \tan^2 A \tan^2 B \tan^2 C \geq 27 \qquad$ [cubing both sides]

or $\quad \tan A \tan B \tan C \geq 3\sqrt{3} \;\Rightarrow\; \tan A + \tan B + \tan C \geq 3\sqrt{3}$. **Hence proved.**

Q.12 Let A, B, C be three angles such that $A = \dfrac{\pi}{4}$ and $\tan B \tan C = p$. Find all possible values of p such that A, B, C are the angles of a triangle.

Sol. $A + B + C = \pi$

$$\Rightarrow \quad B + C = \frac{3\pi}{4} \quad \Rightarrow \quad 0 < B, C < \frac{3\pi}{4}$$

Also $\tan B \tan C = p$

$$\Rightarrow \quad \frac{\sin B \sin C}{\cos B \cos C} = \frac{p}{1} \quad \Rightarrow \quad \frac{\cos B \cos C - \sin B \sin C}{\cos B \cos C + \sin B \sin C} = \frac{1-p}{1+p}$$

$$\Rightarrow \quad \frac{\cos(B+C)}{\cos(B-C)} = \frac{1-p}{1+p} \quad \Rightarrow \quad \frac{1+p}{\sqrt{2}(p-1)} = \cos(B-C) \qquad \ldots(i)$$

Since B or C can vary from 0 to $\dfrac{3\pi}{4}$,

$$0 \leq B - C < \frac{3\pi}{4} \quad \Rightarrow \quad -\frac{1}{\sqrt{2}} < \cos(B-C) \leq 1.$$

$$\Rightarrow \quad \frac{\cos(B+C)}{\cos(B-C)} = \frac{1-p}{1+p} \quad \Rightarrow \quad \frac{1+p}{\sqrt{2}(p-1)} = \cos(B-C) \qquad \ldots(i)$$

Since B or C can vary from 0 to $\dfrac{3\pi}{4}$,

$$0 \leq B - C < \frac{3\pi}{4} \quad \Rightarrow \quad -\frac{1}{\sqrt{2}} < \cos(B-C) \leq 1.$$

Equation (i) will now lead to

$$-\frac{1}{\sqrt{2}} < \frac{p+1}{\sqrt{2}(p-1)} \leq 1 \quad \Rightarrow \quad 0 < 1 + \frac{p+1}{p-1}$$

$$\Rightarrow \quad \frac{2p}{(p-1)} > 0 \quad \Rightarrow \quad p < 0 \text{ or } p > 1 \qquad \ldots(ii)$$

Also $\quad \dfrac{p+1-\sqrt{2}(p-1)}{\sqrt{2}(p-1)} \leq 0 \quad \Rightarrow \quad \dfrac{\left(p-\left(\sqrt{2+1}\right)^2\right)}{(p-1)} \geq 0$

$$\Rightarrow \quad p < 1 \;\text{ or }\; p \leq \left(\sqrt{2+1}\right)^2 \qquad \ldots(iii)$$

Combining (ii) and (iii), we get $p < 0$ or $p \leq \left(\sqrt{2+1}\right)^2$

PROGRESSION & SERIES

1. INTRODUCTION :

A succession of numbers $t_1, t_2, \ldots t_n$ formed according to the some definite rule is called sequence. "A sequence is a function of natural numbers with codomain as the set of Real numbers or complex numbers"

Domain of sequence $= N$
if Range of sequence $\subseteq R \Rightarrow$ Real sequence
if Range of sequence $\subseteq C \Rightarrow$ Complex sequence

Sequence is called finite or infinite depending upon its having number of terms as finite or infinite respectively.
For example: 2, 3, 5, 7, 11, ... is a sequence of prime numbers. It is an infinite sequence.
A progression is a sequence having its terms in a definite pattern e.g.: 1, 4, 9, 16, ... is a progression as each successive term is obtained by squaring the next natural number.
However a sequence may not always have an explicit formula of n^{th} term.
Series is constructed by adding or subtracting the terms of a sequence e.g., 2 + 4 + 6 + 8 + ... is a series. The term at n^{th} place is denoted by T_n and is called general term of a sequence or

Series is constructed by adding or subtracting the terms of a sequence e.g., 2 + 4 + 6 + 8 + ... is a series. The term at n^{th} place is denoted by T_n and is called general term of a sequence or progression or series.

2. ARITHMETIC PROGRESSION (A.P.) :

2.1 Definition and Standard Appearance of A.P. :

It is sequence in which the difference between any term and its just preceding term remains constant throughout. This constant is called the "common difference" of the A.P. and is denoted by 'd' generally.
Standard appearance of an A.P. is

$$a, (a + d), (a + 2d) \ldots\ldots\ldots (a + \overline{n-1}\,d)$$

where 'a' denotes the first term of the AP

2.2 General term/n^{th} term/Last term of A.P. :

It is given by $T_n = a + (n - 1)d$
where a = first term, d = common difference and n = position of the term which we require.

Note : If $d > 0 \Rightarrow$ increasing A.P.
If $d < 0 \Rightarrow$ decreasing A.P.
If $d = 0 \Rightarrow$ all the terms remain same

Illustration :

If 5^{th} and 6^{th} terms of an A.P. are respectively 6 and 5. Find the 11^{th} term of the A.P.

Sol. $T_5 = 6,\ T_6 = 5$

$$a + 4d = 6$$
$$a + 5d = 5$$
$$d = -1$$
$$a = 10$$
$$T_{11} = a + 10d$$
$$= 10 - 10 = 0.$$

[Ans. 0]

Illustration :

In an A.P. if $a_2 + a_5 - a_3 = 10$ and $a_2 + a_9 = 17$ then find the 1^{st} term and the common difference.

Sol. *In an A.P.*

Let a_1 = first term & d = common difference

$$a_2 + a_5 - a_3 = 10$$
$$a_1 + d + a_1 + 4d - (a_1 + 2d_1) = 10$$
$$a_1 + 3d_1 = 10 \quad \ldots(i)$$
$$a_2 + a_9 = 17$$
$$a_1 + d + a_1 + 8d_1 = 17$$

$$a_1 + 3d_1 = 10 \quad \ldots(i)$$
$$a_2 + a_9 = 17$$
$$a_1 + d + a_1 + 8d_1 = 17$$

$$2a_1 + 9d_1 = 17 \quad \ldots(ii)$$

On solving (i) & (ii) we get

$$a_1 = 13, \qquad d = -1$$

[Ans. a_1 = 13 and d = – 1]

Illustration :

If p^{th}, q^{th} and r^{th} term of an A.P. are respectively a, b, and c then find the value of

$$a(q - r) + b(r - p) + c(p - q)$$

Sol. $T_p = a,\ T_q = b,\ T_r = c$

Let first term be α and common difference be d.

$$\alpha + (p - 1)d = a, \qquad \alpha + (q - 1)d = b, \qquad \alpha + (r - 1)d = c$$
$$(p - q)d = a - b$$
$$(q - r)d = b - c$$
$$(r - p)d = c - a$$
$$a(q - r) + b(r - p) + c(p - q)$$
$$a\left(\frac{b-c}{d}\right) + b\left(\frac{c-a}{d}\right) + c\left(\frac{a-b}{d}\right)$$
$$= \frac{1}{d}(ab - ac + bc - ba + ca - cb)$$
$$= 0.$$

[Ans. 0]

2.3 Sum of n terms of an A.P. :

$$S_n = a \quad + (a + d) \quad + (a + 2d) + \ldots\ldots\ldots\ldots + (a + \overline{n-1}d)$$
$$S_n = (a + \overline{n-1}d) + (a + \overline{n-2}d) + (a + d) + \ldots\ldots\ldots\ldots\ldots + a$$

$$2S_n = n[2a + (n-1)d]$$
$$S_n = \frac{n}{2}[2a + \overline{n-1}d]$$

or

$$S_n = \frac{n}{2}[a + a + (n-1)d] \Rightarrow S_n = \frac{n}{2}(a + l) \quad \text{where last term, } l = a + \overline{n-1}d$$

Illustration :

The first term of an A.P. is 5, the last is 45, and the sum 400. Find the number of terms and the common difference.

Sol. $a = 5, \ \ell = 45$

Let common difference = d

$$a + (n-1)d = 45$$
$$S_n = 400$$

$$\frac{n}{2}[a + \ell] = 400$$

$$\frac{n}{2}[5 + 45] = 400$$

$$n = 16$$
$$a + (n-1)d = 45$$
$$5 + 15d = 45$$
$$d = \frac{8}{3}.$$

[Ans. $n = 16, d = \frac{8}{3}$*]*

Illustration :

Solve the equation $(x + 1) + (x + 4) + (x + 7) + \ldots\ldots + (x + 28) = 155$.

Sol. *We have,*

$$(x + 1) + (x + 4) + \ldots\ldots + (x + 28) = 155$$

Let n be the number of terms in the A.P. on L.H.S. Then,

$$(x + 1) + (x + 4) + \ldots\ldots + (x + 28) = 155$$

$$\Rightarrow \quad \frac{10}{2}[(x + 1) + (x + 28)] = 155$$

$$\Rightarrow \quad x = 1.$$

[Ans. 1]

Illustration :

How many terms of the sequence, $20 + 19\frac{1}{3} + 18\frac{2}{3} + ...$ *must be taken so that there sum is 300. Explain the reason of double answer*

Sol. $20 + 19\frac{1}{3} + 18\frac{2}{3} + ...$

$$a = 20, d = \frac{-2}{3}$$

$$S_n = \frac{n}{2}[2a + (n-1)d]$$

$$300 = \frac{n}{2}\left[2 \times 20 + (n-1)\left(\frac{-2}{3}\right)\right]$$

$$300 = 20\,n - \frac{n(n-1)}{3}$$

$$n^2 - 61n + 900 = 0$$

$$n = 25, 36$$

we got two values of n because when n = 25 all terms will be positive, but when n = 36, terms will become negative. *[Ans. 36 or 25]*

we got two values of n because when n = 25 all terms will be positive, but when n = 36, terms will become negative. *[Ans. 36 or 25]*

Illustration :

The sum of n terms of two A.P.'s are in the ratio of $7n + 1 : 4n + 27$*, find the ratio of their* 11^{th} *terms*

Sol. *Let the two series be* $a_1, a_1 + d_1, a + 2d_1 ......$ & $a_2\, a_2 + d_2, a_2 + 2d_2, ...$

$$S_1 = \frac{n}{2}[2a_1 + (n-1)d_1]$$

$$S_2 = \frac{n}{2}[2a_2 + (n-1)d_2]$$

$$\frac{S_1}{S_2} = \frac{7n+1}{4n+27}$$

$$\frac{2a_1 + (n-1)d_1}{2a_2 + (n-1)d_2} = \frac{7n+1}{4n+27}$$

$$\frac{a_1 + \left(\frac{n-1}{2}\right)d_1}{a_2 + \left(\frac{n-1}{2}\right)d_2} = \frac{7n+1}{4n+27}$$

$T_{11} = a + 10d$

So to find ratio of 11^{th} term we should

Put $\frac{(n-1)}{2} = 10$

$n = 21$

$$\frac{a_1 + 10d_1}{a_2 + 10d_2} = \frac{7(21)+1}{4(21)+27} = \frac{4}{3}.$$ *[Ans. 4/3]*

Illustration :

In an A.P. t_n denotes n^{th} term and s_n denotes sum of first n terms.

If $t_7 = \frac{1}{9}$ and $t_9 = \frac{1}{7}$ then find the value of S_{63}.

Sol. *Let a and d are first term and common difference of the A.P.*

$t_7 = \frac{1}{9}$ *and* $t_9 = \frac{1}{7}$

$a + 6d = \frac{1}{9}$ ....(i)

$a + 8d = \frac{1}{7}$ ....(ii)

$a + 8d = \frac{1}{7}$ ....(ii)

Solving (i) and (ii), we get

$a = \frac{1}{63}, d = \frac{1}{63}$

$$S_{63} = \frac{63}{2}\left(\frac{2}{63} + 62 \times \frac{1}{63}\right) = 32.$$ *[Ans. 32]*

Practice Problem

Q.1 In an A.P. if p^{th} term is q and q^{th} term is p then find its r^{th} term.

Q.2 Find the first negative term of the sequence $20, 19\frac{1}{4}, 18\frac{1}{2}, 17\frac{3}{4}, \ldots\ldots$

Q.3 In an A.P. if $S_p = q$ and $S_q = p$ then find S_{p+q}. (S_n denotes the sum of first n term of the A.P.)

Answer key

Q.1 $p + q - r$ Q.2 28^{th} Q.3 $-(p+q)$

2.4 Highlights of an A.P. :

(i) If each term of an A.P. is increased, decreased, multiplied or divided by the same non zero number, then the resulting sequence is also an AP.

(ii) Three numbers in AP can be taken as $a-d$, a, $a+d$; four numbers in AP can be taken as $a-3d$, $a-d$, $a+d$, $a+3d$; five numbers in AP are $a-2d$, $a-d$, a, $a+d$, $a+2d$ & six terms in AP are $a-5d$, $a-3d$, $a-d$, $a+d$, $a+3d$, $a+5d$ etc.

(iii) The common difference can be zero, positive or negative.

(iv) The sum of the two terms of an AP equidistant from the beginning & end is constant and equal to the sum of first & last terms.

(v) For any series, $T_n = S_n - S_{n-1}$. In a series if S_n is a quadratic function of n or T_n is a linear function of n, then the series is an A.P.

(vi) If a, b, c are in A.P. $\Rightarrow$ $2b = a + c$.

Illustration :

Illustration :

The sum of first 3 terms of an A.P. is 27 and the sum of their squares is 293. Find absolute value of S_{15}

Sol. *Let the terms be*

$a-d,\ a,\ a+d$

$3a = 27$

$a = 9$

$(a-d)^2 + a^2 + (a+d)^2 = 293$

$3a^2 + 2d^2 = 293$

$2d^2 = 50$

$d = \pm 5$

when $d = 5$ then A.P. is 4, 9, 14, ...

$\therefore \quad S_{15} = \frac{15}{2}[2 \times 4 + 14 \times 5] = 585$

when $d = -5$ then A.P. is 14, 9, 4,..........

$\therefore \quad S_{15} = \frac{15}{2}[2 \times 14 + 14 \times (-5)] = -315$

$\therefore \quad |S_{15}| = 315.$ *[Ans. 315 or 585]*

Illustration :

Find the nature and 30^{th} term of the sequence whose sum to n terms is $5n^2 + 2n$

Sol. $S_n = 5n^2 + 2n$

$S_{n-1} = 5(n-1)^2 + 2(n-1)$

$T_n = S_n - S_{n-1} = 5(2n-1) + 2$

$= 10n - 3$

Series $= 7, 17, 27, 37 \ldots.$

Series is A.P.

$T_{30} = 10 \times 30 - 3 = 297.$ [*Ans. A.P.* $7, 17, 27, 37 \ldots\ldots\ldots, T_{30} = 297$]

Illustration :

If a^2, b^2, c^2 *are in A.P. then prove that*

(a) $\frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b}$ *are in A.P.* (b) $\frac{a}{b+c}, \frac{b}{c+a}, \frac{c}{a+b}$ *are in A.P.*

Sol. (a) *Let* $\frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b}$ *are in A.P.*

Sol. (a) *Let* $\frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b}$ *are in A.P.*

$$\frac{2}{c+a} = \frac{1}{a+b} + \frac{1}{b+c}$$

$2(a+b)(b+c) = (c+a)(2b+a+c)$

$2b^2 = a^2 + c^2$

hence, a^2, b^2, c^2 *are in A.P.*

so if a^2, b^2, c^2 *are in A.P., then*

$\frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b}$ *are in A.P.*

(b) *If* a^2, b^2, c^2 *are in A.P. then* $\frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b}$ *are in A.P.* (*Proved earlier*)

Multiply each term with $a + b + c$

$$\frac{a+b+c}{b+c}, \frac{a+b+c}{c+a}, \frac{a+b+c}{a+b}$$

Subtract 1 from each term

$\frac{a}{b+c}, \frac{b}{c+a}, \frac{c}{a+b}$ *are in A.P.*

Illustration :

Find the condition that the roots of the equation $x^3 - px^2 + qx - r = 0$ may be in A.P. and hence solve the equation $x^3 - 12x^2 + 39x - 28 = 0$.

Sol. $x^3 - px^2 + qx - r = 0$

Let the three roots be $a - d, a, a + d$

$$3a = p \quad \Rightarrow \quad a = \frac{p}{3}$$

$$(a - d)\, a\, (a + d) = r \quad \Rightarrow \quad a\,(a^2 - d^2) = r$$

$$a^2 - d^2 = \frac{\mathrm{r}}{\mathrm{a}} = \frac{3r}{p}$$

$$a\,(a - d) + a\,(a + d) + (a - d)\,(a + d) = q$$

$$3a^2 - d^2 = q \quad \Rightarrow \quad 2a^2 + \frac{3r}{p} = q$$

$$2\left(\frac{p}{3}\right)^2 + \frac{3r}{p} = q$$

$$x^3 - 12x^2 + 39x - 28 = 0$$

Let the root be $a - d, a, a + d$

$$3a = 12 \quad \Rightarrow \quad a = 4$$

$$4(16 - d^2) = 28$$

$$16 - d^2 = 7$$

$$d^2 = 9$$

$$d = \pm 3$$

hence roots are 1, 4, 7. *[Ans. roots are 1, 4, 7]*

Illustration :

(a) If $\log_3 2$, $\log_3 (2^x - 5)$ & $\log_3\left(2^x - \frac{7}{2}\right)$ are in AP, determine x.

(b) Solve the equation $\frac{x-1}{x} + \frac{x-2}{x} + \ldots\ldots + \frac{1}{x} = 3$

Sol. (a) $2 \log_3 (2^x - 5) = \log_3 2 + \log_3\left(2^x - \frac{7}{2}\right)$

$$\log_3 (2^x - 5)^2 = \log_3 (2.2x - 7)$$

$$(2^x - 5)^2 = 2.2^x - 7$$

Put $2^x = t$

$$(t - 5)^2 = 2t - 7$$

$$t^2 - 10t + 25 = 2t - 7$$

$$t^2 - 12t + 32 = 0$$

$$t = 4, 8$$

$$2^x = 4, 8$$

$$x = 2, 3$$

but at $x = 2$, $2^x - 5$ is negative hence $x = 3$ is only solution. *[Ans: x = 3]*

(b) $= \dfrac{(x-1)+(x-2)+(x-3)+....+3+2+1}{x} = \dfrac{1+2+3+....+(x-1)}{x}$

$= \dfrac{\frac{(x-1)(x)}{2}}{x} = \dfrac{x-1}{2} = 3$

$\Rightarrow x = 7$ [Ans. $x = 7$]

Illustration :

If the sum of the roots of the equation $ax^2 + bx + c = 0$ is equal to the sum of the squares of their reciprocals, then show that bc^2, ca^2, ab^2 are in AP.

Sol. $ax^2 + bx + c = 0$ Let α, β be the roots

$$\alpha + \beta = \frac{1}{\alpha^2} + \frac{1}{\beta^2}$$

$$\alpha^2 \beta^2 (\alpha + \beta) = (\alpha + \beta)^2 - 2\alpha\beta$$

$$\frac{c^2}{a^2} \cdot \left(\frac{-b}{a}\right) = \frac{b^2}{a^2} - \frac{2c}{a}$$

$$2a^2 c = bc^2 + ab^2$$

hence bc^2, a^2c, ab^2 are in A.P.

Illustration :

Illustration :

Given $a_1, a_2, a_3.....a_n$ in A.P and

if $\dfrac{1}{a_1 a_n} + \dfrac{1}{a_2 a_{n-1}} + \dfrac{1}{a_3 a_{n-1}} + ... + \dfrac{1}{a_n a_1} = \dfrac{\lambda}{a_1 + a_n}\left[\dfrac{1}{a_1} + \dfrac{1}{a_2} + ... + \dfrac{1}{a_n}\right]$ then find the value of λ.

Sol. $S = \dfrac{1}{a_1 a_n} + \dfrac{1}{a_2 a_{n-1}} + \dfrac{1}{a_3 a_{n-1}} + ... + \dfrac{1}{a_n a_1}$

$$S = \frac{1}{(a_1 + a_n)}\left[\frac{a_1 + a_n}{a_1 a_n} + \frac{a_1 + a_n}{a_2 a_{n-1}} + \frac{a_1 + a_n}{a_3 a_{n-2}} + ... + \frac{a_1 + a_n}{a_n a_1}\right]$$

but $a_1 + a_n = a_2 + a_{n-1} = a_3 + a_{n-2} = ...$

Hence

$$S = \frac{1}{(a_1 + a_n)}\left[\frac{a_1 + a_n}{a_1 a_n} + \frac{a_2 + a_{n-1}}{a_2 a_{n-1}} + \frac{a_3 + a_{n-2}}{a_3 a_{n-2}} + ... + \frac{a_1 + a_n}{a_n a_1}\right]$$

$$= \frac{1}{(a_1 + a_n)}\left[\frac{1}{a_n} + \frac{1}{a_1} + \frac{1}{a_{n-1}} + \frac{1}{a_2} + ... + \frac{1}{a_n} + \frac{1}{a_1}\right]$$

$$= \frac{2}{a_1 + a_n}\left[\frac{1}{a_1} + \frac{1}{a_2} + ... + \frac{1}{a_n}\right]$$

Hence $\lambda = 2$. [Ans. 2]

2.5 Arithmetic Mean (A.M.) :

2.5.1 Definition :

When three quantities are in A.P. then the middle one is called the *Arithmetic Mean* of the other two.
e.g. a, b, c are in A.P. then 'b' is the arithmetic mean between 'a' and 'c' and $a + c = 2b$.
It is to be noted that between two given quantities it is always possible to insert any number of terms such that the whole series thus formed shall be in A.P. and the terms thus inserted are called the *arithmetic means*.

2.5.2 To insert 'n' AM's between a and b :

Let $A_1, A_2, A_3 \ldots\ldots A_n$ are the n means between a and b.
Hence $a, A_1A_2, \ldots\ldots A_nb$ is an A.P. and b is the $(n + 2)^{th}$ terms.

Hence $b = a + (n + 1)d \quad \Rightarrow \quad d = \dfrac{b-a}{n+1}$

Now
$$A_1 = a + d$$
$$A_2 = a + 2d$$
$$\vdots$$
$$A_n = a + nd$$

$$\sum_{i=1}^{n} A_i = na + (1 + 2 + 3 + \ldots. + n)d$$

$$= na + \frac{n(n+1)}{2}d = na + \frac{n(n+1)}{2}\cdot\frac{b-a}{n+1}$$

$$= \frac{n}{2}[2a + b - a] = n\left(\frac{a+b}{2}\right) = na$$

Hence the sum of n AM's inserted between a and b is equal to n times a single AM between them.

Illustration :

Insert 20 AM's between 4 and 67.

Sol. $a = 4,\ b = 67,\ n = 20\ \ d = \dfrac{b-a}{n+1} = \dfrac{67-4}{21} = 3$

$$A_1 = a + d = 4 + 3 = 7$$
$$A_2 = a + 2d = 4 + 6 = 13$$
$$A_3 = 19$$
$$A_4 = 25$$
$$A_{20} = a + 20d = 4 + 20 \times 3 = 64.$$

[Ans. 7, 13, 19, ..., 64]

Illustration :

If p arithmetic means are inserted between 5 and 41 so that the ratio $\frac{A_3}{A_{p-1}} = \frac{2}{5}$ *then find the value of p.*

Sol. *a = 5, b = 41, number of airthmetic means = p*

$$d = \frac{b-a}{n+1} = \frac{36}{p+1}$$

$$A_3 = a + 3d = 5 + 3\left(\frac{36}{p+1}\right) = \frac{5p+5+108}{p+1}$$

$$A_{p-1} = 41 - d = 41 - 2\left(\frac{36}{p+1}\right) = \frac{41p-31}{p+1} \quad = \frac{5p+113}{41p-31} = \frac{2}{5}$$

$25p + 565 = 82p - 62 \Rightarrow 57p = 627 \quad \Rightarrow \quad p = 11.$ *[Ans. p = 11]*

Illustration :

If eleven A.M.'s are inserted between 28 and 10, then find the number of integral A.M.'s.

Sol. *Assume* $A_1, A_2, A_3 \ldots\ldots, A_{11}$ *be the eleven A.M.'s between 28 and 10, so 28,* $A_1, A_2, \ldots\ldots A_{11}$*, 10 are in A.P. Let d be the common difference of the A.P. The number of terms is 13. Now,*

$$10 = T_{13} = T_1 + 12d = 28 + 12d$$

Sol. *Assume* $A_1, A_2, A_3 \ldots\ldots, A_{11}$ *be the eleven A.M.'s between 28 and 10, so 28,* $A_1, A_2, \ldots\ldots A_{11}$*, 10 are in A.P. Let d be the common difference of the A.P. The number of terms is 13. Now,*

$$10 = T_{13} = T_1 + 12d = 28 + 12d$$

$$\Rightarrow \quad d = \frac{10-28}{12} = -\frac{18}{12} = -\frac{3}{2}$$

Hence, the number of integral A.M.'s is 5. *[Ans. 5]*

Practice Problem

Q.1 If the first 3 terms of an increasing A.P. are the roots of the cubic

$$4x^3 - 24x^2 + 23x + 18 = 0, \text{ then find } S_{19}.$$

Q.2 Divide 32 into four parts which are in A.P. such that the ratio of the product of extremes to the product of mean is 7: 15.

Q.3 If a, b, c are in A.P. then prove that

(a) $b + c$; $c + a$; $a + b$ are in A.P.

(b) $(b + c)^2 - a^2$; $(c + a)^2 - b^2$; $(a + b)^2 - c^2$ are in A.P.

Q.4 If n arithmetic means are inserted between 2 and 38, then the sum of the resulting series is obtained as 200. Then find the value of n.

Answer key

Q.1	418	Q.2	2, 6, 10, 14	Q.4	10

3. GEOMETRIC PROGRESSION (G.P.) :

3.1 Definition and Standard Appearance of G.P. :

In a sequence if each term (except the first non zero term) bears the same constant ratio with its immediately preceding term then the series is called a G.P. and the constant ratio is called the common ratio.
Standard appearance of a G.P. is

$$a, ar, ar^2, ar^3, \ldots\ldots\ldots., ar^{n-1}\ .$$

3.2 General term/nth term/Last term of G.P. :

It is given by $T_n = a \cdot r^{n-1}$
where a = first term, r = common ratio and n = position of the term which we required.

Illustration :

In a G.P. if $t_3 = 2$ and $t_6 = -\frac{1}{4}$ find t_{10}.

Sol. $t_3 = 2,\ t_6 = -\frac{1}{4}$

$$ar^2 = 2\ ,\ ar^5 = \frac{1}{4}$$

$$1 \qquad\qquad 1$$

$$\frac{1}{r^3} = -8 \qquad \Rightarrow \qquad \frac{1}{r} = -2$$

$$r = \frac{-1}{2} \qquad\qquad a = 8$$

$$t_{10} = ar^9 = 8\left(\frac{-1}{2}\right)^9 = \frac{-1}{64}$$ [Ans. $-\frac{1}{64}$]

Illustration :

Find the four successive terms of a G.P. of which the 2nd term is smaller than the first by 35 and the 3rd term is larger than the 4th by 560.

Sol. *Let the four terms be*

$$a_1,\ ar_1,\ ar^2,\ ar^3$$
$$a - ar = 35$$
$$ar^2 - ar^3 = 560$$
$$\frac{a(1-r)}{ar^2(1-r)} = \frac{35}{560}$$
$$r^2 = 16$$
$$r = \pm 4$$

Since 2nd term is less than 1st so $r = -4$

$$a = 7$$

terms are 7, –28, 112, –448 [Ans. 7, – 28, 112, – 448]

Illustration :

If p^{th}, q^{th} and r^{th} terms of a G.P. are x, y and z respectively then find the value of $x^{q-r} \cdot y^{r-p} \cdot z^{p-q}$

Sol. *Let a and k be the first term and common ratio of the G.P. respectively.*

$$x = a\,k^{p-1}$$
$$y = a\,k^{q-1}$$
$$z = a\,k^{r-1}$$
$$x^{q-r} \cdot y^{r-p} \cdot z^{p-q}$$
$$= \left(\frac{x}{z}\right)^q \cdot \left(\frac{y}{x}\right)^r \cdot \left(\frac{z}{y}\right)^p$$
$$= (k^{p-r})^q\ (k^{q-p})r(k^{r-q})^p$$
$$= k^{pq-qr+qr-rp+rp-pq}$$
$$= k^0 = 1.$$

[Ans. 1]

3.3 Sum of n terms of a G.P. :

$$S = a + ar + ar^2 + \text{..........} + ar^{n-1}$$
$$Sr = \quad + ar\ + ar^2 + \text{.....................} + ar^n$$
subtract $\quad - \quad - \quad\quad\quad -$

$$S(1-r) = a - ar^n = a\,(1-r^n)$$

$$S(1-r) = a - ar^n = a\,(1-r^n)$$

$$S = \frac{a(1-r^n)}{1-r}, \text{ where } r \neq 1, (\text{if } r = 1 \text{ then } S = na)$$

3.4 Sum of infinite terms of a G.P. :

If $|r| < 1$ and $n \to \infty$ then $r^n \to 0$ and in this case geometric series will be summable upto infinity and its sum is given by

$$S_\infty = \frac{a}{1-r}$$

Illustration :

Determine the number of terms in a G.P. if $a_1 = 3$, $a_n = 96$ and $S_n = 189$.

Sol. $a_1 = 3$

$$a_1\,r^{n-1} = 96$$
$$r^{n-1} = 32$$

$$\frac{a(1-r^n)}{1-r} = 189 \quad \Rightarrow \quad \frac{3(1-32r)}{1-r} = 189$$

$$93r = 186$$
$$r = 2$$
$$2^{n-1} = 32$$
$$n = 6.$$

[Ans. 6]

Illustration :

Find the least value of n ($n \in N$) for which sum of the series $1 + 3 + 3^2 + 3^3 + \ldots\ldots$ upto n terms exceeds 9000.

Sol. $S_n = 1 + 3 + 3^2 + \ldots\ldots + n$ *term*

$$S_n = \frac{1\cdot\left(3^{n-1}\right)}{3-1} > 9000$$

$$3^n - 1 > 18000$$

$$3^n > 18001$$

$$n \geq 9$$

least value of $n = 9$. *[Ans. 9]*

Illustration :

Find the sum of n terms of a sequence $\left(x+\frac{1}{x}\right)^2, \left(x^2+\frac{1}{x^2}\right)^2, \left(x^3+\frac{1}{x^3}\right)^2, \ldots\ldots$

Sol. $\left(x+\frac{1}{x}\right)^2 \left(x^2+\frac{1}{x^2}\right)^2 \left(x^3+\frac{1}{x^3}\right)^2 \ldots\ldots \left(x^n+\frac{1}{x^n}\right)^2$

Sol. $\left(x+\frac{1}{x}\right)^2, \left(x^2+\frac{1}{x^2}\right)^2, \left(x^3+\frac{1}{x^3}\right)^2 \ldots\ldots \left(x^n+\frac{1}{x^n}\right)^2$

$$x^2 + \frac{1}{x^2} + 2,\ x^4 + \frac{1}{x^4} + 2,\ x^6 + \frac{1}{x^6} + 2 + \ldots\ldots + x^{2n} + \frac{1}{x^{2n}} + 2$$

$$S = (x^2 + x^4 + \ldots\ldots + x^{2n}) + \left(\frac{1}{x^2}+\frac{1}{x^4}+\ldots\ldots+\frac{1}{x^{2n}}\right) + 2n$$

$$= x^2\left(\frac{x^{2n}-1}{x^2-1}\right) + \frac{1}{x^2}\frac{\left(1-\frac{1}{x}2n\right)}{\left(1-\frac{1}{x^2}\right)} + 2n$$

$$= x^2\frac{(x^{2n}-1)}{x^2-1} + \left(\frac{x^{2n}-1}{x^2-1}\right)\frac{1}{x^{2n}} + 2n$$

$$= \left(\frac{x^{2n}-1}{x^2-1}\right)\left(x^2+\frac{1}{x^{2n}}\right) + 2n.$$

[Ans. $\left(\frac{x^{2n}-1}{x^2-1}\right)\left(x^2+\frac{1}{x^{2n}}\right) + 2n$*]*

Illustration :

The sum of an infinite number of terms of a G.P. is 15 and the sum of their squares is 45. Find the series.

Sol. $a + ar + ar^2 + ar^2 + \ldots\ldots = 15$

$$\frac{a}{1-r} = 15 \qquad \ldots(i)$$

$a^2 + (ar)^2 + (ar^2)^2 + \ldots\ldots = 45$

$$\frac{a^2}{1-r^2} = 45 \qquad \ldots(ii)$$

$$\frac{a^2}{(1-r^2)} = 15 \qquad \ldots(iii)$$

$$\frac{(1-r)^2}{1-r^2} = \frac{45}{225} \qquad \Rightarrow \qquad \frac{1-r}{1+r} = \frac{1}{5}$$

$5 - 5r = 1 + r$

$r = 2/3$

$$a = 15\left(1-\frac{2}{3}\right) = 5$$

$$5, \frac{10}{3}, \frac{20}{9}, \frac{40}{27} \ldots\ldots$$

$\left[Ans.\ 5, \frac{10}{3}, \frac{20}{9}, \ldots\ldots\ldots\right]$

Illustration :

If $x = a + a/r + a/r^2 + \ldots\ldots\infty$, $y = b - b/r + b/r^2 - \ldots\ldots\ \infty$ and $z = c + c/r^2 + c/r^4 + \ldots\ldots\ \infty$, then prove that $xy/z = ab/c$.

Sol. *We have,*

$$x = \frac{a}{1-\frac{1}{r}} = \frac{ar}{r-1}$$

$$y = \frac{b}{1-\left(-\frac{1}{r}\right)} = \frac{br}{1+r}$$

$$z = \frac{c}{1-\frac{1}{r^2}} = \frac{cr^2}{r^2-1}$$

$$\therefore \qquad xy = \left(\frac{ar}{r-1}\right)\left(\frac{br}{r+1}\right) = \frac{abr^2}{r^2-1}$$

$$\therefore \qquad \frac{xy}{z} = \left[\frac{\frac{abr^2}{r^2-1}}{\frac{cr^2}{r^2-1}}\right] = \frac{ab}{c}.$$

Practice Problem

Q.1 Which term of the G.P. $2, 1, \frac{1}{2}, \frac{1}{4}, ...$ is $\frac{1}{28}$?

Q.2 The first term of a G.P. is 1. The sum of the third and fifth terms is 90. Find the common ratio the G.P.

Q.3 Fifth term of a G.P. is 2. Find the product of its first nine terms.

Q.4 Prove that $6^{\frac{1}{2}} \times 6^{\frac{1}{4}} \times 6^{\frac{1}{8}} .....\infty = 6$.

Q.5 Find the sum of the following series :

(a) $\left(\sqrt{2}+1\right)+1+\left(\sqrt{2}-1\right)+......\infty$ (b) $\frac{1}{2}+\frac{1}{3^2}+\frac{1}{2^3}+\frac{1}{3^4}+\frac{1}{2^5}+\frac{1}{3^6}+.....\infty$

Q.6 Sum of infinite number of terms in G.P. is 20 and sum of their squares is 100.
Then find the common ratio of G.P.

Answer key

Q.1 9^{th} Q.2 ± 3 Q.3 512 Q.5 (a) $\sqrt{2}-1$ (b) $\frac{19}{24}$ Q.6 $\frac{3}{5}$

3.5 Highlights of G.P. :

(i) If each term of a GP be multiplied or divided by the same non-zero quantity, the resulting sequence is also a GP.

(ii) If each term of a G.P. raised the same power then resulting sequence is also a G.P.

(iii) Any 3 consecutive terms of a GP can be taken as $a/r, a, ar$; any 4 consecutive terms of a GP can be taken as $a/r^3, a/r, ar, ar^3$ & so on.

(iv) In a finite G.P. the product of the terms equidistant from the begining and the end are equal.
$a_1a_n = a_2a_{n-1} = a_3a_{n-2} = ...$

(v) If a, b, c are in GP $\Rightarrow$ $b^2 = ac$.

Illustration :

The sum of first 3 consecutive terms of a G.P. is 19 and their product is 216. Find S_{20}, also compute s_∞ if it exist.

Sol. *Let the 3 consecutive terms be*

$$\frac{a}{r}, a, ar$$

$$\frac{a}{r} + a + ar = 19$$

$$\frac{a}{r} \cdot a \cdot ar = 216$$
$$a^3 = 216$$
$$a = 6$$

$$\frac{6}{r} + 6 + 6r = 19$$
$$6(1 + r + r^2) = 19r$$
$$6r^2 - 13r + 6 = 0$$
$$(3r - 2)(2r - 3) = 0$$
$$r = \frac{2}{3}, \frac{3}{2}$$

when $r = \frac{3}{2}$, $S_{20} = 8\left[\left(\frac{3}{2}\right)^{20} - 1\right]$ and S_∞ does not exist.

when $r = \frac{2}{3}$, $S_{20} = 27\left(1 - \left(\frac{2}{3}\right)^{20}\right)$ and $S_\infty = \dfrac{9}{1 - \frac{2}{3}} = 2$

[Ans. $S_{20} = 8\left(\left(\frac{3}{2}\right)^{20} - 1\right)$, S_∞ does not exist ; $S_{20} = 27\left(1 - \left(\frac{2}{3}\right)^{20}\right)$, $S_\infty = 27$]

Illustration :

If $a_n = n^{th}$ term of G.P. and

$a_1 + a_2 + a_3 = 13$ and $a_1^2 + a_2^2 + a_3^2 = 91$ then find a_{50}.

Sol. Given

$a_1 + a_2 + a_3 = 13$, Let $a_1 = \frac{a}{r}$, $a_2 = a$, $a_3 = ar$

$$\frac{a}{r} + a + ar = 13 \quad \Rightarrow \quad a\left(\frac{1}{r} + 1 + r\right) = 13$$

$$r + \frac{1}{r} = \frac{13}{a} - 1$$

$$r + \frac{1}{r} = \frac{13 - a}{a}$$

$$r^2 + \frac{1}{r^2} + 2 = \left(\frac{13 - a}{a}\right)^2$$

Now given

$$a_1^2 + a_2^2 + a_3^2 = 91$$

$$a^2\left(\frac{1}{r^2} + 1 + r^2\right) = 91$$

$$a^2\left(\frac{(13-a)^2}{a^2}-1\right) = 91$$

$$(13-a)^2 - a^2 = 91 \quad \Rightarrow \quad 13^2 - 26a = 91$$
$$26a = 169 - 91 \quad \Rightarrow \quad 26a = 78$$
$$a = 3$$
$$3r^2 + 3 = 10r \quad \Rightarrow \quad 3r^2 - 19r + 3 = r$$
$$3r^2 - 9r - r + 3 = 0$$
$$r = 3, \frac{1}{3}$$

When $a = 3$ *and* $r = 3$ *then* $a_{50} = 3 \cdot 3^{49} = 3^{50}$.

and when $a = 3$ *and* $r = \frac{1}{3}$ *then* $a_{50} = 3 \cdot \left(\frac{1}{3}\right)^{49} = \left(\frac{1}{3}\right)^{48}$. [Ans. 3^{50} or $\left(\frac{1}{3}\right)^{48}$]

Illustration :

If a, b, c, d are in G.P., then prove that $(a^3 + b^3)^{-1}$, $(b^3 + c^3)^{-1}$, $(c^3 + d^3)^{-1}$ *are also in G.P.*

Sol. *Let* $b = ar$, $c = ar^2$ *and* $d = ar^3$. *Then*

$$\frac{1}{a^3+b^3} = \frac{1}{a^3(1+r^3)}$$

$$\frac{1}{b^3+c^3} = \frac{1}{a^3r^3(1+r^3)}$$

$$\frac{1}{c^3+d^3} = \frac{1}{a^3r^6(1+r^3)}$$

Clearly, $(a^3 + b^3)^{-1}$, $(b^3 + c^3)^{-1}$ *and* $(c^3 + d^3)^{-1}$ *are in G.P. with common ratio* $1/r^3$.

3.6 Sequences convertible to G.P. :

Illustration :

Use infinite series to compute the rational number corresponding to $0.4\overline{23}$

Sol.

$$x = 0.4\overline{23}$$
$$= 0.4 + 0.023 + 0.00023 + \ldots\ldots$$
$$= 0.4 + \frac{23}{10^3} + \frac{23}{10^5} + \frac{23}{10^7} + \ldots\ldots$$
$$= \frac{4}{10} + \frac{23}{10^3}\left(1 + \frac{1}{10^2} + \frac{1}{10^4} + \ldots\ldots\right)$$
$$= \frac{4}{10} + \frac{23}{10^3}\left(\frac{1}{1-1/100}\right)$$
$$x = \frac{4}{10} + \frac{23}{990} = \frac{419}{990}$$

Illustration :

(a) *If* $9 + 99 + 999 + \ldots\ldots + \text{upto } 49 \text{ terms} = 10\dfrac{(10^{\lambda}-1)}{\mu} - 49$, *where* $\lambda, \mu \in N$ *then find the value of* $\lambda + \mu$

(b) $0.9 + 0.99 + 0.999 + \ldots\ldots \text{ up to } 51 \text{ terms} = 51 - \dfrac{1}{p}\left(1 - \dfrac{1}{10^{q}}\right)$ *where* $p, q \in N$ *then find the value of* $p + q$.

Sol. (a) $S = 9 + 99 + 999 + \ldots\ldots + \text{upto } 49 \text{ terms}$

$S = 10 - 1 + 10^2 - 1 + 10^3 - 1 + \ldots\ldots + 10^{49} - 1$

$= (10 + 10^2 + 10^3 + \ldots\ldots 10^{49}) - 49$

$$S = 10 \cdot \left(\frac{10^{49}-1}{9}\right) - 49$$

$\lambda + \mu = 49 + 9 = 58$

(b) $S = 0.9 + 0.99 + 0.999 + \ldots\ldots\text{up to } 51 \text{ terms}$

$$= \frac{9}{10} + \frac{99}{100} + \frac{999}{1000} + \ldots\ldots \text{up to } 51 \text{ terms}$$

$$= 1 - \frac{1}{10} + 1 - \frac{1}{10^2} + 1 - \frac{1}{10^3} + \ldots\ldots + 1 - \frac{1}{10^{51}}$$

$$= 1 - \frac{1}{10} + 1 - \frac{1}{10^2} + 1 - \frac{1}{10^3} + \ldots\ldots + 1 - \frac{1}{10^{51}}$$

$$= 51 - \left(\frac{1}{10} + \frac{1}{10^2} + \frac{1}{10^3} + \ldots\ldots + \frac{1}{10^{51}}\right)$$

$$= 51 - \frac{\frac{1}{10}\left(1 - \frac{1}{10^{51}}\right)}{1 - \frac{1}{10}} = 51 - \frac{1}{9}\left(1 - \frac{1}{10^{51}}\right).$$

$\therefore \quad p + q = 60$ [Ans. (a) 58 (b) 60]

Illustration :

Find the sum $S = (x + y) + (x^2 + xy + y^2) + (x^3 + x^2y + xy^2 + y^3) + \ldots\ldots n$ *terms.*

Sol. *It is easy to observe that*

$$\frac{x^2 - y^2}{x - y} = x + y, \quad \frac{x^3 - y^3}{x - y} = x^2 + xy + y^2, \quad \frac{x^n - y^n}{x - y} = x^{n-1} + x^{n-2}y + \ldots\ldots + xy^{n-2} + y^{n-1}$$

$$S = \frac{1}{x - y}[(x^2 - y^2) + (x^3 - y^3) + \ldots\ldots n \text{ terms}]$$

$$= \frac{1}{x - y}\left[\frac{x^2(1 - x^n)}{1 - x} - \frac{y^2(1 - y^n)}{1 - y}\right].$$

[Ans. $\dfrac{1}{x - y}\left[\dfrac{x^2(1 - x^n)}{1 - x} - \dfrac{y^2(1 - y^n)}{1 - y}\right]$]

Illustration :

Find the sum of series $\frac{3}{19} + \frac{33}{19^2} + \frac{333}{19^3} + \frac{3333}{19^4} + \ldots\ldots\ldots \infty$

Sol. $S = \frac{3}{19} + \frac{33}{19^2} + \frac{333}{19^3} + \frac{3333}{19^4} + \ldots\ldots\ldots \infty$

or $S = \frac{3}{9}\left[\frac{9}{19} + \frac{99}{19^2} + \frac{999}{19^3} + \ldots\ldots \infty\right]$

$\Rightarrow \quad \frac{3}{9}\left[\frac{10-1}{19} + \frac{10^2-1}{19^2} + \frac{10^3-1}{19^3} + \ldots\ldots \infty\right]$

$\Rightarrow \quad \frac{3}{9}\left[\left(\left(\frac{10}{19}\right) + \left(\frac{10}{19}\right)^2 + \left(\frac{10}{19}\right)^3 + \ldots\ldots \infty\right) - \left(\frac{1}{19} + \frac{1}{19^2} + \ldots\ldots \infty\right)\right]$

$S = \frac{3}{9}\left[\frac{10/19}{1-10/19} - \left(\frac{1/19}{1-1/19}\right)\right]$

$S = \frac{3}{9}\left[\frac{10/19}{9/19} - \frac{1}{18}\right]$

$= \frac{3}{9}\left[\frac{19}{18}\right] \Rightarrow \frac{19}{54}.$ [Ans. $\frac{19}{54}$]

$= \frac{3}{9}\left[\frac{19}{18}\right] \Rightarrow \frac{19}{54}.$ [Ans. $\frac{19}{54}$]

Practice Problem

Q.1 If the continued product of three numbers in a G.P. is 216 and the sum of their products in pairs is 156, find the numbers.

Q.2 Three number are in G.P. whose sum is 70. If the extremes be each multiplied by 4 and the means by 5, they will be in A.P. Find the numbers.

Q.3 Using infinite G.P. express the number $2.3\overline{57}$ in rational form.

Q.4 Find the sum of the following series.
(i) $5 + 55 + 555 + \ldots\ldots$ to n terms
(ii) $0.3 + 0.33 + 0.333 + \ldots\ldots$ to n terms

Answer key

Q.1 2, 6, 18 or 18, 6, 2 Q.2 10, 20, 40 Q.3 $\frac{389}{165}$

Q.4 (i) $\frac{5}{9}\left[10\left(\frac{10^n-1}{9}\right)-n\right]$; (ii) $\frac{1}{3}\left[n-\frac{1}{9}\left(1-\frac{1}{10^n}\right)\right]$

3.7 Geometric Mean (G.M.) :

3.7.1 Definition :

If a, b, c are three positive numbers in G.P. then b is called the geometrical mean between a and c and $b^2 = ac$. If a and b are two positive real numbers and G is the G.M. between them, then

$$G^2 = ab$$

3.7.2 To insert 'n' GM's between a and b :

Let a and b are two positive numbers are $G_1, G_2, ...G_n$ are 'n' GM's then

$a\ G_1\ G_2\ ....G_n\ b$ is a G.P. with 'b' as its $(n+2)^{th}$ term.

Hence $\quad b = ar^{n+1}$

$$\therefore \quad r = \left(\frac{b}{a}\right)^{\frac{1}{n+1}}$$

Now $\quad G_1 = ar,\ G_2 = ar^2, ...G_n = ar^n$

hence $$\prod_{i=1}^{n} G_i = a^n \quad r^{1+2+....+n} = a^n \cdot r^{\frac{n(n+1)}{2}} = a^n\left[\left(\frac{b}{a}\right)^{\frac{1}{n+1}}\right]^{\frac{n(n+1)}{2}}$$

$$= a^n \cdot \frac{b^{n/2}}{a^{n/2}} = a^{n/2} \cdot b^{n/2} = \left(\sqrt{ab}\right)^n = G^n$$

where G is the angle GM between a and b.

Hence product of n GM's inserted between of a and b is equal to the n^{th} power of a single GM between them.

Illustration :

Insert 4 GM's between 5 and 160.

Sol. Four GM between 5 & 160

$$r = \left(\frac{b}{a}\right)^{\frac{1}{n+1}} = \left(\frac{160}{5}\right)^{\frac{1}{5}}$$

$$= (32)^{1/5} = 2$$

$G_1 = ar = 10$

$G_2 = 20$

$G_3 = 40$

$G_4 = 80.$ $\qquad$ [Ans. 10, 20, 40, 80]

Illustration :

If AM between two positive numbers a and b is 15 and GM between a and b is 9. Find the numbers.

Sol. $\frac{a+b}{2}=15$

$a + b = 30$

$\sqrt{ab}=9$

$\sqrt{a(30-a)}=9$

$a(30 - a) = 81$

$a^2 = 30a + 81 = 0$

$a = 3, 27$

Hence two nos. are 3, 27. *[Ans. 3, 27]*

Illustration :

If sum of two numbers a and b (a > b) is 3 times their GM and given that $a : b = \left(p+\sqrt{q}\right) : \left(p-\sqrt{q}\right)$ *where p and q are prime numbers. Find p + q*

Sol. $a + b = 3\sqrt{ab}$

$\frac{a+b}{\quad} = \frac{3}{\quad}$

Sol. $a + b = 3\sqrt{ab}$

$\frac{a+b}{\sqrt{ab}}=\frac{3}{1}$

$\frac{a+b}{2\sqrt{ab}}=\frac{3}{2}$

Applying componendo and dividendo

$\frac{a+b+2\sqrt{ab}}{a+b-2\sqrt{ab}}=\frac{3+2}{3-2}$

$\left(\frac{\sqrt{a}+\sqrt{b}}{\sqrt{a}-\sqrt{b}}\right)^2=\frac{5}{1}$

$\frac{\sqrt{a}+\sqrt{b}}{\sqrt{a}-\sqrt{b}}=\frac{\sqrt{5}}{1}$

Again, applying componendo - dividendo.

$\frac{\sqrt{a}}{\sqrt{b}}=\frac{\sqrt{5}+1}{\sqrt{5}-1}$

$\frac{a}{b}=\frac{6+2\sqrt{5}}{6-2\sqrt{5}} = \frac{3+\sqrt{5}}{3-\sqrt{5}}$

$\therefore \quad a : b = 3+\sqrt{5} : 3-\sqrt{5}$

$\Rightarrow \quad p + q = 3 + 5 = 8.$ *[Ans. 8]*

Illustration :

If a, b, c are in G.P. and x, y are the AM's between a, b and b, c respectively then prove that

$$\frac{1}{x} + \frac{1}{y} = \frac{2}{b} \quad \text{and} \quad \frac{a}{x} + \frac{c}{y} = 2$$

Sol. *a, b, c are in G.P.* $\Rightarrow b^2 = ac$

Now

$$x = \frac{a+b}{2}, \qquad y = \frac{b+e}{2}$$

$$\frac{1}{x} + \frac{1}{y} = \frac{2}{a+b} + \frac{2}{b+c}$$

$$= 2\left[\frac{b+c+a+b}{(a+b)(b+c)}\right] = 2\left[\frac{2b+a+c}{ab+ac+b^2+bc}\right]$$

$$= 2\left[\frac{2b+a+c}{2b^2+ab+bc}\right] \quad (ac = b^2)$$

$$= \frac{2\,(2b+a+c)}{b\,(2b+a+c)} = \frac{2}{b}$$

$$\frac{a}{x} + \frac{c}{y} = \frac{2a}{a+b} + \frac{2c}{b+c}$$

$$\frac{a}{x} + \frac{c}{y} = \frac{2a}{a+b} + \frac{2c}{b+c}$$

$$= 2\left[\frac{ab+ac+ac+bc}{2b^2+ab+bc}\right] \qquad (\text{since } ac = b^2)$$

$$= 2\left[\frac{ab+bc+2b^2}{2b^2+ab+bc}\right]$$

$$= 2$$

3.8 Relation between A.M. and G.M. :

A.M. ≥ G.M

Let A and G be the AM and GM between two positive numbers a and b then $A \geq G$

(Sign of equality holds when $a = b$)

$$\text{AM between a \& b, } A = \frac{a+b}{2}$$

$$\text{GM between a \& b, } G = \sqrt{ab}$$

$$A - G = \frac{a+b}{2} - \sqrt{ab} = \frac{1}{2}\left(\sqrt{a} - \sqrt{b}\right)^2$$

$\Rightarrow$ $A - G > 0 \Rightarrow A \geq G$

(If $a = b$ then $A - G = 0$ $\Rightarrow A = G$)

Illustration :

If $x > 0,\ y > 0, z > 0$ *then prove that* $(x + y)(y + z)(z + x) \geq 8xyz$

Sol. $(x + y)\ (y + z)\ (z + x)$

$$\frac{x+y}{2} \geq \sqrt{xy} \qquad (A.M. \geq G.M.)$$

$$\frac{y+z}{2} \geq \sqrt{yz}$$

$$\frac{z+x}{2} \geq \sqrt{zx}$$

$$\frac{(x+y)(y+z)(z+x)}{8} \geq xyz$$

$$(x + y)\ (y + z)\ (z + x) \geq 8xyz$$

Illustration :

Prove that a ΔABC *is equilateral if and only if*

$$\tan A + \tan B + \tan C = 3\sqrt{3}$$

Sol. $\dfrac{\tan A + \tan B + \tan C}{3} \geq (\tan A \tan B \tan C)^{1/3}$

since $A + B + C = \pi$

$\tan A + \tan B + \tan C = \tan A \tan B \tan C$

$\dfrac{\tan A + \tan B + \tan C}{3}\ ^3\ (\tan A + \tan B + \tan C)^{1/3}$

$(\tan A + \tan B + \tan C)^3 \geq 27\ (\tan A + \tan B + \tan C)$

$(\tan A + \tan B + \tan C)^2 \geq 27$

$\tan A + \tan B + \tan C \geq 3\sqrt{3}$

Illustration :

If $a + b + c = 3$ *and a, b, c are positive then prove that* $a^2b^3c^2 \leq \dfrac{3^{10} \cdot 2^4}{7^7}$

Sol. $a + b + c = 3$

We can write it as

$$\frac{a}{2}+\frac{a}{2}+\frac{b}{3}+\frac{b}{3}+\frac{b}{3}+\frac{c}{2}+\frac{c}{2}=3$$

Now A.M. ≥ G.M.

$$\frac{\frac{a}{2}+\frac{a}{2}+\frac{b}{3}+\frac{b}{3}+\frac{c}{2}+\frac{c}{2}}{7} \geq \left(\frac{a^2}{4}\frac{b^3}{27}\frac{c^2}{4}\right)^{1/7}$$

$$\frac{3}{7} \geq \left(\frac{a^2b^3c^2}{2^4 \times 3^3}\right)^{1/7}$$

$$a^2\,b^3\,c^2 \leq \frac{3^{10}\cdot 2^4}{7^7}$$

Illustration :

If a, b, c are positive real number then prove that $\frac{a^3}{4b}+\frac{b}{8c^2}+\frac{1+c}{2a} \geq \frac{5}{4}$

Sol. $\frac{a^3}{4b}, \frac{b}{8c^2}, \frac{1}{2a}, \frac{c}{4a}, \frac{c}{4a}$

Applying A.M. ≥ G.M.

$$\frac{\frac{a^3}{4b}+\frac{b}{8c^2}+\frac{1}{2a}+\frac{c}{4a}+\frac{c}{4a}}{5} \geq \left(\frac{a^3}{4b}\cdot\frac{b}{8c^2}\cdot\frac{1}{2a}\cdot\left(\frac{c}{4a}\right)^2\right)^{1/5}$$

$$\frac{a^3}{4b}+\frac{b}{8c^2}+\frac{1}{2a}+\frac{c}{2a} \geq \frac{5}{4}$$

Practice Problem

Q.1 Find the product of three geometric means between 4 and 1/4.

Q.2 Find two numbers whose arithmetic mean is 34 and the geometric mean is 16.

Q.3 If x, y, z are positive numbers then prove that $(x+y+z)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right) \geq 9$

Q.4 If x & y are positive number such that $\log_2 x + \log_2 y \geq 6$ then find the least value of $x + y$

Q.5 If $a+b+c+d = s$ where a, b, c, d are distinct positive numbers then show that $(s-a)(s-b)(s-c)(s-d) > 81\,abcd$

Answer key

Q.1	1	Q.2	64 and 4	Q.4	16

4. ARITHMETIC GEOMETRIC PROGRESSION (A.G.P.) :

4.1 Standard appearance of an A.G.P. :

$$S = a + (a + d)r + (a + 2d)r^2 + (a + 3d)r^3 + \ldots\ldots\ldots\ldots$$

Here each term is the product of corresponding terms in an arithmetic and geometric series.

4.2 n^{th} term of A.G.P. :

$T_n = [a + (n - 1) d]\, r^{n-1}$

Where a = first term, d = common difference, r = common ratio and n = position of the term which we require.

4.3 Sum of n terms and infinite terms of an A.G.P. :

Let

$$S = a + (a + d)r + (a + 2d)r^2 + (a + 3d)r^3 + \ldots\ldots\ldots\ldots + \left(a + \overline{n-1}\,d\right)r^{n-1}$$

$$Sr = \quad + ar \quad + (a + d)r^2 + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots + \left(a + \overline{n-2}\,d\right)r^{n-1} + \left(a + \overline{n-1}\,d\right)r^n$$

$$S(1 - r) = a + dr + dr^2 + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots + dr^{n-1} - [a + (n - 1) d]\, r^n$$

$$= a + dr\left(\frac{1 - r^{n-1}}{1 - r}\right) - [a + (n - 1) d]\, r^n$$

$$S = \frac{a}{1-r} + dr\left(\frac{1 - r^{n-1}}{(1-r)^2}\right) - \frac{[a + (n-1)d]\, r^n}{1-r}$$

If $0 < |r| < 1$ and $n \to \infty$ then

$r^n, r^{n-1} \to 0$

$$\therefore \qquad s_\infty = \frac{a}{1-r} + \frac{dr}{(1-r)^2}$$

Students are suggested not to learn the formula, process should be keep in mind. See the illustrations –

Illustration :

Find the sum to n terms and also S_∞.

$$\frac{3}{5} + \frac{5}{15} + \frac{7}{45} + \frac{9}{135} + \ldots\ldots$$

Sol. $S = \dfrac{3}{5} + \dfrac{5}{15} + \dfrac{7}{45} + \dfrac{9}{135} + \ldots\ldots \dfrac{(2n+1)}{5 \cdot 3^{n-1}}$

$$S = \frac{3}{5}+\frac{5}{5\cdot 3}+\frac{7}{5\cdot 3^2}+\frac{9}{5\cdot 3^3}+\ldots\ldots\frac{(2n+1)}{5\cdot 3^{n-1}} \qquad \ldots\ldots.(1)$$

$$\frac{1}{3}S = \frac{3}{5\cdot 3}+\frac{5}{5\cdot 3^2}+\frac{7}{5\cdot 3^3}+\ldots\ldots+\frac{2n-1}{5\cdot 3^{n-1}}+\frac{(2n+1)}{5\cdot 3^n} \qquad \ldots\ldots.(2)$$

$$\frac{2}{3}S = \frac{3}{5}+\frac{2}{5\cdot 3}+\frac{2}{5\cdot 3^2}+\frac{2}{5\cdot 3^3}+\ldots\ldots+\frac{2}{5\cdot 3^{n-1}}-\frac{(2n+1)}{5\cdot 3^n}$$

$$= \frac{3}{5}+\frac{2}{5\cdot 3}\left(1+\frac{1}{3}+\frac{1}{3^2}+\ldots\ldots\frac{1}{3^{n-2}}\right)-\frac{2n+1}{5\cdot 3^n}$$

$$= \frac{3}{5}+\frac{2}{15}\left(\frac{1-\frac{1}{3^{n-1}}}{1-\frac{1}{3}}\right)-\frac{(2n+1)}{5\cdot 3^n} = \frac{3}{5}+\frac{2}{15}\cdot\frac{3}{2}\left(1-\frac{1}{3^{n-1}}\right)-\frac{(2n+1)}{5\cdot 3^n}$$

$$= \frac{3}{5}+\frac{1}{5}-\frac{1}{5\cdot 3^{n-1}}-\frac{2n+1}{5\cdot 3^n} = \frac{4}{5}-\frac{3}{5\cdot 3^n}-\frac{2n+1}{5\cdot 3^n} = \frac{4}{5}-\frac{2n+4}{5\cdot 3^n} = \frac{2}{5}\left(2-\frac{n+2}{3^n}\right)$$

$$= \frac{3}{5}+\frac{1}{5}-\frac{1}{5\cdot 3^{n-1}}-\frac{2n+1}{5\cdot 3^n} = \frac{4}{5}-\frac{3}{5\cdot 3^n}-\frac{2n+1}{5\cdot 3^n} = \frac{4}{5}-\frac{2n+4}{5\cdot 3^n} = \frac{2}{5}\left(2-\frac{n+2}{3^n}\right)$$

$$\therefore\ S = \frac{3}{5}\left(2-\frac{n+2}{3^n}\right)$$

and $S_\infty = \dfrac{6}{5}$. $\qquad$ *[Ans.* $S = \dfrac{3}{5}\left(2-\dfrac{n+2}{3^n}\right)$ *and* $S_\infty = \dfrac{6}{5}$*]*

Illustration :

If the sum to infinity of the series $3 + (3 + d)\frac{1}{4} + (3 + 2d)\frac{1}{4^2} + \ldots\ldots \infty$ *is* $\frac{44}{9}$, *then find d.*

Sol. $S = 3 + (3 + d)\frac{1}{4} + (3 + 2d)\frac{1}{4^2} + \ldots\ldots \infty$

$$\Rightarrow \quad \frac{1}{4}S = (3)\frac{1}{4}+(3+d)\frac{1}{4^2}+\ldots\ldots\infty$$

Subtracting (2) from (1), we have

$$\frac{3}{4}S = 3+(d)\frac{1}{4}+(d)\frac{1}{4^2}+\ldots\ldots\infty$$

$$= 3 + \frac{\frac{d}{4}}{1 - \frac{1}{4}}$$

$$= 3 + \frac{d}{3}$$

$\Rightarrow \quad S = 4 + \dfrac{4d}{9}$

Given,

$$4 + \frac{4d}{9} = \frac{44}{9}$$

$\Rightarrow \quad \dfrac{4d}{9} = \dfrac{8}{9}$

$\Rightarrow \quad d = 2.$ *[Ans. 2]*

Illustration :

If $|x| < 1$ *then compute the sum*

(a) $1 + 2x + 3x^2 + 4x^3 + \ldots\ldots\ldots\ \infty$

If $|x| < 1$ *then compute the sum*

(a) $1 + 2x + 3x^2 + 4x^3 + \ldots\ldots\ldots\ \infty$

(b) $1 + 3x + 6x^2 + 10x^3 + \ldots\ldots\ldots\ \infty$

Sol. (a) $S = 1 + 2x + 3x^2 + 4x^3 + \ldots\ldots\ \infty$

$xS = x + 2x^2 + 3x^2 + \ldots\ldots\ \infty$

$S(1 - x) = 1 + x + x^2 + x^3 + \ldots\ldots$

$$S(1 - x) = \frac{1}{1 - x}$$

$$S = \frac{1}{(1 - x)^2}$$

(b) $S = 1 + 3x + 6x^2 + 10x^3 + \ldots\ldots\ldots\ \infty$

$xS = \quad x + 3x^2 + 6x^3 + \ldots\ldots\ldots\ \infty$

$(1 - x)\,S = 1 + 2x + 3x^2 + 4x^3 + \ldots\ldots\ldots\ \infty$

or by equation (1)

$$(1 - x)\,S = \frac{1}{(1 - x)^2}$$

$$S = \frac{1}{(1 - x)^3}.$$

[Ans. (a) $\dfrac{1}{(1-x)^2}$ *(b)* $\dfrac{1}{(1-x)^3}$*]*

5. MISCELLANEOUS SEQUENCES

5.1 Type-1 :

Sequences dealing with $\sum n$; $\sum n^2$; $\sum n^3$

(1) $\sum n = \dfrac{n(n+1)}{2}$

(2) $^{*}\sum n^2 = \dfrac{n(n+1)(2n+1)}{6}$

Proof:

Sum of the squares of the first n natural numbers

$S = 1^2 + 2^2 + 3^2 + \ldots\ldots + n^2$.

We have, $n^3 - (n-1)^3 = 3n^2 - 3n + 1$; and by changing n to n – 1, we get

$(n-1)^3 - (n-2)^3 = 3(n-1)^2 - 3(n-1) + 1$

$(n-2)^3 - (n-3)^3 = 3(n-2)^2 - 3(n-2) + 1$

....

....

$3^3 - 2^3 = 3 \times 3^2 - 3 \times 3 + 1$

$2^3 - 1^2 = 3 \times 2^2 - 3 \times 2 + 1$

$1^2 - 0^2 = 3 \times 1^2 - 3 \times 1 + 1$

Hence, by adding

$n^3 = 3(1^2 + 2^2 + 3^2 + \ldots\ldots + n^2) - 3\,(1 + 2 + 3 + \ldots\ldots + n) + n$

[illegible]

$$= 3S - \frac{3n(n+1)}{2} + n$$

$$\Rightarrow \qquad 3S = n^2 - n + \frac{3n(n+1)}{2}$$

$$= n\,(n+1)\left(n - 1 + \frac{3}{2}\right)$$

$$\Rightarrow \qquad S = \frac{n(n+1)(2n+1)}{6}$$

(3) $^{**}\sum n^3 = \left[\dfrac{n(n+1)}{2}\right]^2 = \left(\sum n\right)^2$

For proof :

* Consider the identity $k^4 - (k-1)^4 = 4k^3 - 6k^2 + 4k - 1$

Note : (i) $\displaystyle\sum_{r=1}^{n}(a_r \pm b_r) = \sum_{r=1}^{n} a_r \pm \sum_{r=1}^{n} b_r$;

(ii) $\displaystyle\sum_{r=1}^{n} k\,a_r = k\sum_{r=1}^{n} a_r$

Important: (iii) $\displaystyle\sum_{r=1}^{n} k = k\sum_{r=1}^{n} = kn$

Illustration :

Compute the value of $\sum_{i=1}^{n}\sum_{j=1}^{i}\sum_{k=1}^{j} 1$.

Sol. $\sum_{i=1}^{n}\sum_{j=1}^{i}\sum_{k=1}^{j} 1 = \sum_{i=1}^{n}\sum_{j=1}^{i}(j) = \sum_{i=1}^{n}\frac{i(i+1)}{2} = \frac{1}{2}\left[\sum n^2 + \sum n\right]$

$$= \frac{1}{2}\left[\frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2}\right] = \frac{n(n+1)(2n+4)}{12} = \frac{n(n+1)(n+2)}{6}.$$

[Ans. $\frac{n(n+1)(n+2)}{6}$]

Illustration :

Find the value(s) of the positive integer n *for which the quadratic equation,* $\sum_{k=1}^{n}(x+k-1)(x+k) = 10n$ *has solutions* α *and* $\alpha + 1$ *for some* α.

n

Sol. $\sum_{k=1}^{n}(x+k-1)(x+k) = 10n$ ...(1)

$\Rightarrow \quad \sum\left[x^2 + (2k-1)x + (k-1)k\right] = 10n$

$\Rightarrow \quad x^2 \cdot n + n^2x + \frac{(n-1)n(n+1)}{3} - 10n = 0$ ...(2)

$\Rightarrow$ *If roots are* α *and* $\alpha + 1$ *(p) then difference = 1*   *By equation (2)*

$\therefore \quad (\alpha+\beta)^2 - 4\alpha\beta = 1$ $\alpha + \beta = n$

or $\quad n^2 - 4\left(\frac{(n-1)(n+1)}{3} - 10\right) = 1$ $\alpha\beta = \frac{(n-1)(n+1)}{3} - 10$

or $\quad n = 11.$ [Ans. 11]

Illustration :

Compute the sum $(31)^2 + (32)^2 + (33)^2 + ... + (50)^2$

Sol. $S = (31^2) + (32^2) + (33^2) + ... + (n+30)^2 + ... (50)^2$

$S = \sum_{n=1}^{20}(n+30)^2 \quad \Rightarrow \quad n^2 + 60n + 900 \quad \Rightarrow \quad \frac{n(n+1)(2n+1)}{6} + \frac{60n(n+1)}{2} + 900n$

$\Rightarrow \quad \frac{20\times 21\times 41}{6} + 30 \times 20 \times 21 + 900 \times 20$

$\Rightarrow \quad 33470.$ [Ans. 33470]

Illustration :

Find the sum to n terms of the series $1 \times 2 \times 3 + 2 \times 3 \times 4 + 3 \times 4 \times 5 + \ldots\ldots$.

Sol. $T_n = n(n+1)(n+2)$

Let S_n denote the sum to n terms of the given series. Then,

$$S_n = \sum_{k=1}^{n} T_k = \sum_{k=1}^{n} k(k+1)(k+2)$$

$$= \sum_{k=1}^{n}(k^3 + 3k^2 + 2k)$$

$$= \left(\sum_{k=1}^{n} k^3\right) + 3\left(\sum_{k=1}^{n} k^2\right) + 2\left(\sum_{k=1}^{n} k\right)$$

$$= \left(\frac{n(n+1)}{2}\right)^2 + \frac{3n(n+1)(2n+1)}{6} + \frac{2n(n+1)}{2}$$

$$= \frac{n(n+1)}{2}\left\{\frac{n(n+1)}{2} + (2n+1) + 2\right\}$$

$$= \frac{n(n+1)}{2}\{n^2 + n + 4n + 2 + 4\} \quad = \frac{n(n+1)}{2}(n^2 + 5n + 6)$$

$$= \frac{n(n+1)}{2}\{n^2 + n + 4n + 2 + 4\} \quad = \frac{n(n+1)}{2}(n^2 + 5n + 6)$$

$$= \frac{n(n+1)(n+2)(n+3)}{4}$$

[Ans. $\frac{n(n+1)(n+2)(n+3)}{4}$]

Illustration :

Find the sum of the series : $\frac{1^2}{1} + \frac{1^2+2^2}{1+2} + \frac{1^2+2^2+3^2}{1+2+3} + \ldots\ldots$ upto 31 terms.

Sol. $T(r) = \frac{1^2 + 2^2 + \ldots\ldots + r^2}{1 + 2 + \ldots\ldots + r}$

$$= \frac{r(r+1)(2r+1)2}{6r(r+1)}$$

$$= \frac{1}{3}(2r+1)$$

$$\Rightarrow \quad \sum_{r=1}^{n} T(r) = \left(\frac{2}{3}\sum_{r=1}^{n} r\right) + \frac{n}{3} \quad = \frac{1}{3}n(n+1) + \frac{n}{3}$$

$$S_n = \frac{n(n+2)}{3}$$

$$\therefore \quad S_{31} = \frac{31 \times 33}{3} = 341.$$

[Ans. 341]

Practice Problem

Q.1 Find the sum of $1+\frac{4}{5}+\frac{7}{5^2}+\frac{10}{5^3}+\ldots\ldots$ to n terms and to ∞.

Q.2 Find the sum of series –
$1.1 + 3.01 + 5.001 + 7.0001 + \ldots\ldots +$ to n terms

Q.3 Find the sum of the series
$1.n + 2.(n-1) + 3.(n-2) + \ldots\ldots\ n.1$

Q.4 Find the sum of n terms of the series
$1^2 + 3^2 + 5^2 + 7^2 + \ldots\ldots$

Answer key

Q.1 $S_n=\frac{35}{16}-\frac{(12n+7)}{16\cdot 5^{n-1}}$; $S_\infty=\frac{35}{16}$

Q.2 $n^2+\left(\frac{1}{9}\right)\left(1-\frac{1}{10}n\right)$

Q.3 $\frac{n(n+1)(n+2)}{6}$

Q.4 $\frac{n(4n^2-1)}{3}$

5.2 TYPE-2 (Using method of difference) :

5.2 TYPE-2 (Using method of difference) :

If $T_1, T_2, T_3, \ldots\ldots$ are the terms of a sequence then the terms
$T_2-T_1, T_3-T_2, T_4-T_3 \ldots\ldots\ldots$
some times are in A.P. and some times in G.P. For such series we first compute their n^{th} term and then compute the sum to n terms, using sigma notation.

Illustration :

Find the sum of series .
(i) $6 + 13 + 22 + 33 + \ldots\ldots\ldots\ldots$ *n terms*
(ii) $5 + 7 + 13 + 31 + 85 + \ldots\ldots\ldots$ *up to n terms.*

Sol. *(i) Let* $S = 6 + 13 + 22 + 33 + \ldots\ldots + T_n$
$S = \quad 6 + 13 + 22 + \ldots\ldots + \quad + T_n$

$O = 6 + 7 + 9 + 11 + \ldots\ldots \quad - T_n$

or $T_n = 6 + (7 + 9 + 11 + \ldots\ldots)$

$$=6+\left[\frac{n-1}{2}[2\times 7+(n-2)2]\right] \quad =6+\left[\frac{(n-1)}{2}(14+24-4)\right]$$

$= 6 + (n-1)(n+5) \quad = 6 + n^2 + 4n - 5$

$T_n = n^2 + 4n + 1$

$$\therefore \quad S_n=\sum_{n=1}^{n}T_n \quad \Rightarrow \quad \frac{n(n+1)(2n+1)}{6}+\frac{4(n)(n+1)}{2}+n$$

$$\Rightarrow \quad \frac{n(n+1)}{2}\left[\frac{2n+1}{3}+4\right]+n \;\Rightarrow\; \frac{n(n+1)}{2}\left[\frac{2n+13}{3}\right]+n \;\Rightarrow\; \frac{n(n+1)(2n+13)}{6}+n$$

(ii) $S = 5 + 7 + 13 + 31 + 85 \ldots\ldots + T_n$

$S = \quad 5 + 7 + 13 + 31 \ldots\ldots - T_n$

$O = 5 + 2 + 6 + 18 + 54 \ldots\ldots - T_n$

$Tn = 5 + 2 + 6 + 18 + 54 + \ldots\ldots\ldots\ldots$

$$T_n = 5 + \frac{2[3^{n-1}-1]}{3-1} \quad \Rightarrow \quad T_n = 5 + (3^{n-1} - 1)$$

$$S_n = \sum_{n=1}^{n} T_n \;\Rightarrow S_n + \frac{3^n-1}{2} - n$$

$S_n = 4n + \frac{1}{2}(3^n - 1)$. [Ans. (i) $\frac{n(n+1)(6n+13)}{6}+n$ (ii) $4n + \frac{1}{2}(3^n + 1)$]

5.3 TYPE -3 (Splitting the n^{th} term as a difference of two) :

Here is a series in which each term is composed of the reciprocal of the product of r factors in A.P., the first factor of the several terms being in the same A.P.

Illustration :

Illustration :

Find the sum of n terms of the series and also find S_∞.

$$\frac{1}{1\cdot2\cdot3\cdot4} + \frac{1}{2\cdot3\cdot4\cdot5} + \frac{1}{3\cdot4\cdot5\cdot6} + \ldots\ldots$$

Sol. $\frac{1}{1\cdot2\cdot3\cdot4} + \frac{1}{2\cdot3\cdot4\cdot5} + \ldots\ldots + \frac{1}{n(n+1)(n+2)(n+3)}$.

$$S = \frac{1}{3}\left[\frac{4-1}{1\cdot2\cdot3\cdot4}+\frac{5-2}{2\cdot3\cdot4\cdot5}+\ldots\ldots+\frac{(n+3)-n}{n(n+1)(n+2)(n+3)}\right]$$

$$T_1 = \frac{1}{3}\left(\frac{1}{1\cdot2\cdot3}-\frac{1}{2\cdot3\cdot4}\right)$$

$$T_2 = \left(\frac{1}{2\cdot3\cdot4}-\frac{1}{3\cdot4\cdot5}\right)$$

....

....

$$T_n = \frac{1}{3}\left(\frac{1}{n(n+1)(n+2)}-\frac{1}{(n+1)(n+2)(n+3)}\right)$$

$$S = T_1 + T_2 + \ldots\ldots + T_n = \frac{1}{3}\left[\frac{1}{1\cdot2\cdot3}-\frac{1}{(n+1)(n+2)(n+3)}\right]$$

[Ans. $S_n = \frac{1}{18} - \frac{1}{3(n+1)(n+2)(n+3)}$; $S_\infty = \frac{1}{18}$]

Illustration :

Find the sum of n terms of the series $\frac{3}{1\cdot 2\cdot 4} + \frac{4}{2\cdot 3\cdot 5} + \frac{5}{3\cdot 4\cdot 6} + ...$ *and also find sum of infinite terms* (S_∞)

Sol. $\frac{3}{1\cdot 2\cdot 4} + \frac{4}{2\cdot 3\cdot 5} + \frac{5}{3\cdot 4\cdot 6} + ... \frac{(n+2)}{n(n+1)(n+3)}$

$$= \frac{3^2}{1\cdot 2\cdot 3\cdot 4} + \frac{4^2}{2\cdot 3\cdot 4\cdot 5} + ... + \frac{(n+2)^2}{n(n+1)(n+2)(n+3)}$$

$$T_n = \frac{n^2+4n+4}{n(n+1)(n+2)(n+3)} = \frac{n^3+4n+3}{n(n+1)(n+2)(n+3)} + \frac{1}{n(n+1)(n+2)(n+3)}$$

$$T_n = \frac{1}{n(n+2)} + \frac{1}{n(n+1)(n+2)(n+3)}$$

$$S_1 = \sum_{n=1}^{n} \frac{1}{n(n+2)},\ S_2 = \sum_{n=1}^{n} \frac{1}{n(n+1)(n+2)(n+3)}$$

$$S_1 = \frac{1}{1\cdot 3} + \frac{1}{2\cdot 4} + \frac{1}{3\cdot 5} + ... \frac{1}{n(n+2)}$$

$$S_1 = \frac{1}{2}\left[\frac{3-1}{1\cdot 3} + \frac{4-2}{2\cdot 4} + \frac{5-3}{3\cdot 5} + ... + \frac{(n+2)-n}{n(n+2)}\right] = \frac{1}{2}\left[\frac{1}{1} - \frac{1}{3} + \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{5} + ... + \frac{1}{n} - \frac{1}{n+2}\right]$$

$$S_1 = \frac{}{2}\left[\frac{}{1\cdot 3} + \frac{}{2\cdot 4} + \frac{}{3\cdot 5} + ... + \frac{}{n(n+2)}\right] = \frac{}{2}\left[\frac{}{1} - \frac{}{3} + \frac{}{2} - \frac{}{4} + \frac{}{3} - \frac{}{5} + ... + \frac{}{n} - \frac{}{n+2}\right]$$

$$= \frac{1}{2}\left[1 + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2}\right]$$

$$S_2 = \frac{1}{3}\left[\frac{1}{1\cdot 2\cdot 3} - \frac{1}{(n+1)(n+2)(n+3)}\right]$$

$S = S_1 + S_2$.

Illustration :

Find sum of n terms (S_n) *for* $\frac{1}{2\cdot 4} + \frac{1\cdot 3}{2\cdot 4\cdot 6} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 8} + ...$

Sol. $S_n = \frac{1}{2\cdot 4} + \frac{1\cdot 3}{2\cdot 4\cdot 6} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 8} + ... + \frac{1\cdot 3\cdot 5......(2n-1)}{2\cdot 4\cdot 6\cdot 8......(2n+2)}$

$$T_n = \frac{1\cdot 3\cdot 5......(2n-1)}{2\cdot 4\cdot 6\cdot 8......(2n+2)}$$

$$= \frac{1\cdot 3\cdot 5...(2n-1)\cdot[(2n+2)-(2n+1)]}{2\cdot 4\cdot 6\cdot 8......(2n+2)}$$

$$T_n = \frac{1\cdot 3\cdot 5...(2n-1)}{2\cdot 4\cdot 6\cdot 8......2n} - \frac{1\cdot 3\cdot 5...(2n+1)}{2\cdot 4\cdot 6\cdot 8......(2n+2)}$$

$$T_1 = \frac{1}{2} - \frac{1\cdot 3}{2\cdot 4}$$

$$T_2 = \frac{1\cdot 3}{2\cdot 4} - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}$$

$$T_n = \frac{1\cdot 3\cdot 5 \ldots\ldots (2n-1)}{2\cdot 4\cdot 6 \ldots\ldots 2n} - \frac{1\cdot 3\cdot 5 \ldots\ldots (2n+1)}{2\cdot 4\cdot 6\cdot 8 \ldots\ldots (2n+2)}$$

$$S_n = \frac{1}{2} - \frac{1\cdot 3\cdot 5 \ldots\ldots (2n-1)}{2\cdot 4\cdot 6 \ldots\ldots (2n+2)}.$$

$$\left[Ans. S_n = \frac{1}{2} - \frac{1\cdot 3\cdot 5 \ldots\ldots (2n-1)}{2\cdot 4\cdot 6 \ldots\ldots (2n+2)}\right]$$

for infinte terms – $$S_\infty = \frac{1}{2} - \lim_{n\to\infty}\left(\frac{1}{2}\right)\left(\frac{3}{4}\right)\left(\frac{5}{6}\right)\ldots\ldots\left(\frac{2n-1}{2n}\right)\left(\frac{1}{2n+2}\right)$$

$$\therefore \quad S_\infty = \frac{1}{2}$$

5.4 TYPE-4:

Here is a series in which each terms is composed of r factor in A.P., the first factor of the several terms being in the same A.P.

Illustration :

$1\cdot 2\cdot 3\cdot 4 + 2\cdot 3\cdot 4\cdot 5 + 3\cdot 4\cdot 5\cdot 6 + \ldots\ldots$ *up to n terms*

Sol. $T_n = n(n+1)(n+2)(n+3)$

$$T_n = \frac{1}{5} n(n+1)(n+2)(n+3)[(n+4)-(n-1)]$$

$$T_n = \frac{n(n+1)(n+2)(n+3)(n+4)}{5} - \frac{(n-1)(n)(n+1)(n+2)(n+3)}{5}$$

$$T_1 = \frac{1\cdot 2\cdot 3\cdot 4\cdot 5}{5} - 0$$

$$T_2 = \frac{2\cdot 3\cdot 4\cdot 5\cdot 6}{5} - \frac{1\cdot 2\cdot 3\cdot 4\cdot 5}{5}$$

$$T_3 = \frac{3\cdot 4\cdot 5\cdot 6\cdot 7}{5} - \frac{2\cdot 3\cdot 4\cdot 5\cdot 6}{5}$$

... .

... .

$$T_n = \frac{n(n+1)(n+2)(n+3)(n+4)}{5} - \frac{(n-1)(n)(n+1)(n+2)(n+3)}{5}$$

$$S_n = T_1 + T_2 + T_3 + \ldots\ldots + T_n = \frac{n(n+1)(n+2)(n+3)(n+4)}{5}$$

$$\left[Ans.\ S_n = \frac{1}{5}[n(n+1)(n+2)(n+3)(n+4)]\ \right]$$

Practice Problem

Q.1 If $I(r) = r(r^2 - 1)$, then find $\sum_{r=2}^{\infty} \frac{1}{I(r)}$.

Q.2 If $S = \frac{1}{1\cdot3\cdot5} + \frac{1}{3\cdot5\cdot7} + \frac{1}{5\cdot7\cdot9} + ...$ to infinity, then find the vlaues of [36S], where [·] represents the greatest integer function.

Q.3 If $\sum_{r=1}^{n} t_r = \frac{n}{8}(n+1)(n+2)(n+3)$, then $\sum_{r=1}^{n} \frac{1}{t_r}$.

Q.4 Find the sum of n terms of the series
$3 + 8 + 22 + 72 + 266 + 1036 + \ldots\ldots$

Q.5 Show that the sum of the n terms of the series
$$\frac{3}{1^2} + \frac{5}{1^2+2^2} + \frac{7}{1^2+2^2+3^2} + \frac{9}{1^2+2^2+3^2+4^2} + \ldots\ldots = \frac{6n}{n+1}$$

Answer key

Q.1 $\frac{1}{4}$ Q.2 3 Q.3 $\frac{1}{2} - \frac{1}{(n+1)(n+2)}$ Q.4 $\frac{1}{3}(4^n - 1) + n(n+1)$

Q.1 $\frac{1}{4}$ Q.2 3 Q.3 $\frac{1}{2} - \frac{1}{(n+1)(n+2)}$ Q.4 $\frac{1}{3}(4^n - 1) + n(n+1)$

6. HARMONIC PROGRESSION (H.P.) :

6.1 Definition and Standard Appearance of H.P. :

A sequence is said to be in H.P. if the reciprocals of its terms are in A.P.

e.g. if $a_1, a_2, a_3, ...$ are in H.P. then $\frac{1}{a_1}, \frac{1}{a_2}, \frac{1}{a_3}$ .... are in A.P.

A standard H.P. is $\frac{1}{a} + \frac{1}{a+d} + \frac{1}{a+2d} + ... + \frac{1}{a+\overline{n-1}d}$

For every HP there is a corresponding A.P.
Terms of harmonic series are the outcomes of an A.P.

6.2 General term/n^{th} term/last term of H.P. :

$$T_n = \frac{1}{a+(n-1)d}$$

where a and d are respectively the first term and the common difference of the corresponding A.P. and n = position of the term which we required.

Note:

(i) There is no general formula for finding the sum to n terms of H.P.

(ii) If a, b, c are in H.P. $\Rightarrow$ $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in **A.P**

$\therefore \quad \frac{2}{b} = \frac{1}{a} + \frac{1}{c} \quad \Rightarrow \quad b = \frac{2ac}{a+c} \quad \Rightarrow \quad$ a, b, c are in HP

also $\frac{1}{b} - \frac{1}{a} = \frac{1}{c} - \frac{1}{b}$ i.e. $\frac{a-b}{ab} = \frac{b-c}{bc}$ i.e. $\frac{a}{c} = \frac{a-b}{b-c}$

Illustration :

If the 3^{rd}, 6^{th} and last term of a H.P. are $\frac{1}{3}, \frac{1}{5}, \frac{3}{203}$, find the number of terms.

Sol. $T_3 = \frac{1}{3}, \quad T_6 = \frac{1}{5}, \quad T_n = \frac{3}{203}$

then 3^{rd}, 6^{th} and n^{th} termof A.P. series are 3, 6, $\frac{203}{3}$.

$a + 2d = 3 \quad \Rightarrow \quad a = 5d = 5$

$d = \frac{2}{}\ \ a = \frac{5}{}$

$a + 2d = 3 \quad \Rightarrow \quad a = 5d = 5$

$d = \frac{2}{3}, a = \frac{5}{3}$

$a + (n-1)d = \frac{203}{3} \quad \Rightarrow \quad \square\frac{5}{3} + (n-1)\frac{2}{3} = \frac{203}{3}$

$(n-1)^2 = 198$

$n = 100.$ *[Ans. 100]*

Illustration :

If m^{th} term of an H.P. is n, and n^{th} term is equal to m then prove that $(m+n)^{th}$ term is $\frac{mn}{m+n}$

Sol. m^{th} *term of A.P.* $= \frac{1}{n}$

n^{th} *term of A.P.* $= \frac{1}{m} \quad \Rightarrow \quad a + (m-1)d = \frac{1}{n}$

$a + (n-1)d = \frac{1}{m} \quad \Rightarrow \quad d = \frac{1}{mn}, \ a = \frac{1}{mn}$

$T_{m+n} = a + (m+n-1)d = \frac{1}{mn} + (m-m-1)\frac{1}{mn}$

$T_{m+n} = \frac{mn}{m+n}.$

Illustration :

If a, b, c are in HP, find the value of $\dfrac{b+a}{b-a}+\dfrac{b+c}{b-c}$.

Sol. *a, b, c are in HP, then*

$\dfrac{1}{a}, \dfrac{1}{b}, \dfrac{1}{c}$ *are in A.P.*

$$S = \frac{b+a}{b-a}+\frac{b+c}{b-c} = \frac{\frac{1}{a}+\frac{1}{b}}{\frac{1}{a}-\frac{1}{b}} + \frac{\frac{1}{c}+\frac{1}{b}}{\frac{1}{c}-\frac{1}{b}}$$

Let $\dfrac{1}{a}-\dfrac{1}{b} = \dfrac{1}{b}-\dfrac{1}{c} = d$

$$S = \frac{\left(\frac{1}{a}+\frac{1}{b}\right)-\left(\frac{1}{c}+\frac{1}{b}\right)}{d} = \frac{\left(\frac{1}{a}-\frac{1}{c}\right)}{d} = \frac{2d}{d} = 2$$

[Ans. 2]

6.3 Harmonic Mean (H.M.) :

6.3 Harmonic Mean (H.M.) :

If a, b, c are in H.P. then middle term is called the harmonic mean between them. Hence if H is the harmonic mean (H.M.) between a and b then a, H, b are in H.P. and $H = \dfrac{2ab}{a+b}$.

(Recall that $AM = \dfrac{a+b}{2}$ and $GM = \sqrt{ab}$ if $a > 0, b > 0$)

6.3.1 To insert n HM between a and b :

Let $H_1, H_2........H_n$ are n HM's between a and b
hence $a, H_1, H_2, ...H_n$ b are in H.P.

$\dfrac{1}{a}, \dfrac{1}{H_1}, \dfrac{1}{H_2} ...... \dfrac{1}{H_n}, \dfrac{1}{b}$ are in A.P.

$\dfrac{1}{b} = \dfrac{1}{a} + (n+1)d$; $-\dfrac{1}{a} = (n+1)d$; $d = \dfrac{a-b}{ab(n+1)}$

$$\frac{1}{H_1} = \frac{1}{a} + d$$

$$\frac{1}{H_2} = \frac{1}{a} + 2d$$

$$\frac{1}{H_3} = \frac{1}{a} + 3d$$

$$\vdots$$

$$\frac{1}{H_n} = \frac{1}{a} + nd$$

$$\sum_{i=1}^{n} \frac{1}{H_i} = \frac{n}{a} + \frac{d(n)(n+1)}{2} = \frac{n}{a}s + \frac{n(n+1)}{2} \cdot \frac{(a-b)}{ab(n+1)}$$

$$= n\left[\frac{1}{a} + \frac{a-b}{2ab}\right] = \frac{n}{2ab}[2b + a - b] = \frac{n(a+b)}{2ab} = n \cdot \frac{1}{H}$$

Hence sum of the reciprocals of all the n HM's between a and b is equal to *n* times the reciprocal of single HM between a and b.

Note: For 3 numbers a, b, c MH is defined as reciprocals of a, b and c i.e. of reciprocals $\frac{1}{3}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$.

$$\text{H.M.} = \frac{3}{\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)}.$$

\a b c/

If $A_1\ A_2$ ....A_n are n quantities then $\text{HM} = \dfrac{1}{\frac{1}{A_1}+\frac{1}{A_2}+.......+\frac{1}{A_n}}$.

Illustration :

If a^2, b^2, c^2 are in A.P. show that $b + c, c + a, a + b$ are in H.P.

Sol. *a^2, b^2, c^2 are in A.P.*

Let $b + c,\ \ c + a,\ \ a + b$ *are in H.P.*

then $\dfrac{1}{b+c}, \dfrac{1}{c+a}, \dfrac{1}{a+b}$ *are in A.P.*

$$\frac{2}{c+a} = \frac{1}{b+c} + \frac{1}{a+b}$$

$$2(a + b)(b + c) = (2b + a + c)(a + c)$$

$$2b^2 = a^2 + c^2$$

hence a^2, b^2, c^2 *are in A.P.*

So *if a^2, b^2, c^2 are in A.P. then* $b + c,\ \ c + a,\ \ a + b$ *are in H.P.*

Illustration :

If $\frac{a-x}{px} = \frac{a-y}{qy} = \frac{a-z}{rz}$ *and p, q, r are in A.P. then prove that x, y, z are in H.P.*

Sol. *Let* $\frac{a-x}{px} = \frac{a-y}{qy} = \frac{a-z}{rz} = k$

$$P = \frac{a-x}{kx},\ q = \frac{a-y}{kx},\ r = \frac{a-z}{kz}$$

$$2\left(\frac{a-y}{ky}\right) = \frac{a-x}{kx} + \frac{a-z}{kz}$$

$$2\left(\frac{a}{y} - 1\right) = \frac{a}{x} - 1 + \frac{a}{z} - 1$$

$$\frac{2a}{y} = \frac{a}{x} + \frac{a}{z}$$

$$\frac{2}{\ } = \frac{1}{\ } + \frac{1}{\ }$$

$$\frac{2}{y} = \frac{1}{x} + \frac{1}{z}$$

Hence x, y, z are in H.P.

Illustration :

If the roots of the equation $x^3 - 11x^2 + 36x - 36 = 0$ *are in H.P. find the middle root.*

Sol. $x^3 - 11x^2 + 36x - 36 = 0$

If roots are in H.P. then roots of new equation

$$\frac{1}{x^3} - \frac{11}{x^2} + \frac{36}{x} - 36 = 0 \text{ are in A.P.}$$

$$-36x^3 + 36x^2 - 11x + 1 = 0$$

$$36x^3 - 36x^2 + 11x - 1 = 0$$

Let the roots be α, β, γ.

$$\alpha + \beta + \gamma = 1$$

$$3\beta = 1 \qquad (2\beta = \alpha + \gamma)$$

$$\beta = \frac{1}{3}$$

So middle root is 3. *[Ans. 6, 3, 2]*

6.4 Relation between A.M, G.M. and H.M :

If a and b are two positive numbers then $A \geq G \geq H$ and A, G, H are in G.P. i.e. $G^2=AH$

Proof: We have $A = \dfrac{a+b}{2}$, $G = \sqrt{ab}$ and $H = \dfrac{2ab}{a+b}$

G
a A b
H

now $AH = ab = G^2$ $\Rightarrow$ A G H are in G.P.

also $\dfrac{A}{G} = \dfrac{G}{H}$; $\therefore$ $A \geq G \Rightarrow G \geq H$

Hence $A \geq G \geq H$ Infact $AM \geq GM \geq HM$

Illustration :

If 9 arithmetic and harmonic means be inserted between 2 and 3, prove that $A + 6/H = 5$ where A is any of the A.M.'s and H the corresponding H.M.

Sol. *Let A_i, H_i $(i = 1, 2, \ldots\ldots 9)$ denote the 9 A.M.'s and 9 H.M.'s between 2 and 3. If d denote the common difference of A.P. then*

$$3 = T_{11} = 2 + 10\,d \qquad \text{or} \qquad d = 1/10.$$

L et A denote the its mean, then

$$A = T_{i+1} = 2 + di = 2 + i/10$$

Again Let 2, H_1, H_2, ……, H_9, 3 be in H.P.

$$1 \quad 1 \quad 1 \qquad 1 \quad 1$$

or $$\frac{1}{2}, \frac{1}{H_1}, \frac{1}{H_2}, \ldots\ldots, \frac{1}{H_9}, \frac{1}{3}$$

If d_1 is the common difference of this A.P., then

$$\frac{1}{3} = T_{11} = \frac{1}{2} + 10d_1 \qquad \text{or} \qquad d_1 = -\frac{1}{60}$$

If H is the ith H.m., then

$$\frac{1}{H} = \frac{1}{2} + d_1 i = \frac{1}{2} - \frac{i}{60}$$

Now $$A + \frac{6}{H} = \left(2 + \frac{i}{10}\right) + 6\left(\frac{1}{2} - \frac{i}{60}\right) = 5 + \frac{i}{10} - \frac{i}{10} = 5.$$

Illustration :

If a, b & c are in A.P. & a^2, b^2 & c^2 are in H.P., then prove that either $a = b = c$ or a, b & $-\dfrac{c}{2}$ are in G.P.

Sol. $2b = a + c$

$$\frac{1}{b^2} - \frac{1}{a^2} = \frac{1}{c^2} - \frac{1}{b^2}$$

$$\Rightarrow \qquad \frac{(a-b)(a+b)}{b^2a^2} = \frac{(b-c)(b+c)}{b^2c^2}$$

$\Rightarrow \quad ac^2 + bc^2 = a^2 b + a^2 c$

$ac(c-a) + b(c-a)(c+a) = 0$

$(c-a)(ab + bc + ca) = 0$

for, $\quad c = a, \;\; a = b = c$

for $\quad ab + bc + ca = 0$

$(a + b) + ca = 0$

$2b^2 + ca = 0$

$$b^2 = -\frac{ac}{2}$$

$a, b, -\dfrac{c}{2}$ *are in G.P.*

Illustration :

If a, b, c are positive real number representing the sides of a triangle, prove that

$$ab + bc + ca < a^2 + b^2 + c^2 < 2(ab + bc + ca)$$

$$\text{or} \qquad 1 < \frac{a^2+b^2+c^2}{ab+bc+ca} < 2$$

Hence prove that

$$3(ab + bc + ca) < (a + b + c)^2 < 4(ab + bc + ca)$$

$$\text{or} \qquad 3 < \frac{(a+b+c)}{ab+bc+ca} < 4.$$

$$3(ab + bc + ca) < (a + b + c)^2 < 4(ab + bc + ca)$$

$$\text{or} \qquad 3 < \frac{(a+b+c)}{ab+bc+ca} < 4.$$

Sol. $\dfrac{a^2+b^2}{2} > ab,\; \dfrac{b^2+c^2}{2} > bc \quad$ *and* $\quad \dfrac{c^2+a^2}{2} > ca \qquad [\because \text{ A.M.} > \text{G.M.}]$

$\Rightarrow \quad a^2 + b^2 > 2ab,\; b^2 + c^2 > 2bc$ *and* $\;c^2 + a^2 > 2ca$

$\Rightarrow \quad a^2 + b^2 + b^2 + c^2 + c^2 + a^2 > 2(ab + bc + ca)$

$\Rightarrow \quad a^2 + b^2 + c^2 > ab + bc + ca$

$\Rightarrow \quad ab + bc + ca < a^2 + b^2 + c^2 \qquad \ldots(i)$

In a triangle ABC with sides BC = a, CA = b, AB = c, we have $b^2 + c^2 - a^2 = 2bc \cos A$

$\Rightarrow \quad b^2 + c^2 - a^2 < 2bc \qquad [\because \cos A < 1]$

Similarly, we have $c^2 + a^2 - b^2 < 2ca$*and* $\quad a^2 + b^2 - c^2 < 2ab$

Adding these three, we get

$$a^2 + b^2 + c^2 < 2(ab + bc + ca) \qquad \ldots(ii)$$

From (i) and (ii), we get

$$ab + bc + ca < a^2 + b^2 + c^2 < 2(ab + bc + ca) \qquad \ldots(iii)$$

$$\Rightarrow \qquad 1 < \frac{a^2+b^2+c^2}{ab+bc+ca} < 2$$

Adding 2(ab + bc + ca) throughout in (iii)

$$3(ab + bc + ca) < (a + b + c)^2 < 4(ab + bc + ca)$$

$$3 < \frac{(a+b+c)}{ab+bc+ca} < 4. \qquad \textit{Hence proveed}$$

Illustration :

If a, b, c d be four distinct positive quantities in H.P., then show that

(a) $a + d > b + c$ *(b)* $ad > bc$.

Sol. $\because$ *a, b, c, d are in H.P.*

(a) *Then A.M. > H.M.*

for first three terms $\therefore$ $\dfrac{a+c}{2} > b$ ...(i)

and for last three terms, $\dfrac{b+d}{2} > c$ ...(ii)

from (i) and (ii) $a + c + b + d > 2b + 2c$

$\Rightarrow$ $a + d > b + c$

(b) *Again G.M. > H.M.*

For first three terms, $\sqrt{ac} > b$

$\Rightarrow$ $ac > b^2$ ...(i)

and for last three terms $\sqrt{bd} > c$

$\Rightarrow$ $bd > c^2$ ...(ii)

from (i) and (ii) $(ac)(bd) > b^2c^2$

$\Rightarrow$ $ad > bc$.

Practice Problem

Practice Problem

Q.1 If p^{th} term of an HP is qr and q^{th} term is pr. Find its r^{th} terms.

Q.2 If 'a' is the A.M. of 'b' and 'c'; 'b' the G.M. of 'c' and 'a', then prove that 'c' is the H.M. of 'a' and 'b'.

Q.3 If the roots of the equation $a(b-c)x^2 + b(c-a)x + c(a-b) = 0$ are equal then prove that a, b, c are in H.P.

Q.4 If $a^x = b^y = c^z = d^w = ...$ and a, b, c, d are in G.P. then prove that x, y, z, w... are in H.P.

Q.5 Let $n \in N$, $n > 25$. Let A, G, H denote the arithmetic mean, geometric mean & harmonic mean of 25 & n. The least value of n for which $A, G, H \in \{25, 26, \ldots\ldots n\}$

Q.6 Let a_1, a_2, a_3, a_4 and a_5 be such that a_1, a_2 & a_3 are in A.P. and a_3, a_4 and a_5 are in H.P. then $\log a_1, \log a_3$ & $\log a_5$ are in

(A) G.P. (B) A.P. (C) H.P. (D) None

Q.7 Let Show that if $a, b, c \in R^+$ then $\dfrac{bc}{b+c} + \dfrac{ac}{a+c} + \dfrac{ab}{a+b} \le \dfrac{1}{2}(a+b+c)$

Q.8 If a, b, c, d be four distinct positive quantities in G.P., then show that

(a) $a + d > b + c$ (b) $c^{-1}d^{-1} + a^{-1}b^{-1} > 2(b^{-1}d^{-1} + a^{-1}c^{-1} - a^{-1}d^{-1})$

Answer key

Q.1	pq	Q.5	225	Q.6	B

Solved Examples

Q.1 If sum of n, 2n, 3n terms of an A.P. are S_1, S_2, S_3, respectively then find the value of $\frac{S_3}{S_2 - S_1}$.

Sol. Let a be the first term and d be the common difference of the given A.P. Then, sumof n terms is

$$S_1 = \frac{n}{2}[2a + (n-1)d]$$

Sum of 2n terms,

$$S_2 = \text{Sum of 2n terms} = \frac{2n}{2}[2a + (2n-1)d]$$

Sum of 3n terms,

$$S_3 = \frac{3n}{2}[2a + (3n-1)d]$$

Now,

$$S_2 - S_1 = \frac{2n}{2}[2a + (2n-1)d] - \frac{n}{2}[2a + (n-1)d]$$

$$= \frac{n}{2}[2\{2a + (2n-1)d\} - \{2a + (n-1)\}d]$$

$$= \frac{n}{2}[2a + (3n-1)d]$$

$\therefore \quad 3(S_2 - S_1) = \frac{3n}{2}[2a + (3n-1)d] = S_3$ [Using (3)]

$\Rightarrow \quad \frac{S_3}{S_2 - S_1} = 3.$ [Ans. 3]

Q.2 If the 10th, 15th, 25th terms of an A.P. are in G.P then find the common ratio of the G.P.

Sol. Let the first term and common ratio of the AP be a and d respectively.

$T_{10} = a + 9d$, $T_{15} = a + 14d$, $T_{25} = a + 24d$.

$$R = \frac{T_{15}}{T_{10}} = \frac{T_{25}}{T_{15}}$$

$R = \frac{a+14d}{a+9d} = \frac{a+24d}{a+14d} \Rightarrow \frac{a+14d-a-24d}{a+9d-a-14d} = 2.$ [Ans. 2]

Q.3 The sum of the squares of three distinct real numbers, which are in G.P. is S^2. If their sum is αS, if $\alpha^2 \in (a, b) - \{c\}$ then find the value of $ab + c$.

Sol. Let the numbers be ar, a, a/r such that $a\left(r+1+\frac{1}{r}\right) = \alpha S$

and $a^2\left(r^2+1+\frac{1}{r^2}\right) = S^2$

Put $r + \frac{1}{r} = t$ $\therefore$ $r^2 + \frac{1}{r^2} = r^2 - 2$

$\therefore$ $a(t+1) = \alpha S$ and $a^2(r^2 - 1) = S^2$

Eliminating S, we get $a^2(r^2 - 1) = \frac{a^2(r+1)^2}{\alpha^2}$

$\therefore$ $(t-1)\alpha^2 = (t+1)$

or $t = \frac{\alpha^2+1}{\alpha^2-1}$

Now $t = r + \frac{1}{r}$ $\therefore$ $r^2 - rt + 1 = 0$

Now $t = r + \frac{1}{r}$ $\therefore$ $r^2 - rt + 1 = 0$

For t to be real $r^2 - 4 > 0$ $\therefore$ $(t+2)(t-2) > 0$

$\therefore$ $t < -2$ or $t > 2$

Hence from (1), we get

$$\frac{\alpha^2+1}{\alpha^2-1} < -2 \quad \text{or} \quad \frac{\alpha^2+1}{\alpha^2-1} > 2$$

or $$\frac{\alpha^2+1}{\alpha^2-1} + 2 < 0 \quad \text{or} \quad \frac{\alpha^2+1}{\alpha^2-1} - 2 > 0$$

$(\alpha^2 - 1)$ is positive or negative

$$\therefore \quad \frac{3\left(\alpha^2 - \frac{1}{3}\right)(\alpha^2-1)}{(\alpha^2-1)^2} < 0$$

$$\therefore \quad \alpha^2 \in \left(\frac{1}{3}, 1\right) \cup (1, 3) \Rightarrow \alpha^2 \in \left(\frac{1}{3}, 3\right) - \{1\}$$

$\therefore$ $a = \frac{1}{3}$, $b = 3$, $c = 1$.

$ab + c = 2$. [Ans. 2]

Q.4 An A.P. and a H.P., have the same first term, the same last term, and the same number of terms; prove that the product of the rth term from the beginning in one series and the rth term from the end in the other is independent of r.

Sol. T_r of A.P. $= a + (r-1)d$

where $b = a + (n-1)d$

$\therefore$ T_r of A.P. $= a + (r-1)\dfrac{b-a}{n-1}$

$$= \frac{a(n-r)+(r-1)}{n-1} \qquad \ldots(i)$$

T_r from end of H.P. a, …!, b (n terms)

$= $ Reciprocal of T_r from end of A.P. $\dfrac{1}{a} \ldots\ldots \dfrac{1}{b}$

$= $ Reciprocal to T_r from begining of A.P. $\dfrac{1}{b} - \dfrac{1}{a}$ (n terms)

Replace a by $\dfrac{1}{b}$ and b by $\dfrac{1}{a}$ in (i) then take reciprocal.

or reciprocal of $\dfrac{\frac{1}{b}(n-r)+\frac{1}{a}(r-1)}{n-1} = \dfrac{ab(n-1)}{a(n-r)+b(r-1)}$ …(ii)

or reciprocal of $\dfrac{\quad}{n-1} = \dfrac{\quad}{a(n-r)+b(r-1)}$ …(ii)

Multiplying (i) and (ii), we get the product = ab, which is independent of r.

Q.5(a) If $A_1, A_2 : G_1, G_2$; and H_1, H_2 be two A.M.'s and G.M.'s and H.M.'s between two quantities, then prove that $\dfrac{G_1 G_2}{H_1 H_2} = \dfrac{A_1 + A_2}{H_1 + H_2}$.

(b) If $A_1, A_2 : G_1, G_2$; and H_1, H_2 be two A.M.'s and G.M.'s and H.M.'s between two quantities 'a' and 'b' then $A_1H_2 = A_2H_1 = G_1G_2 = ab$.

Sol.(a) Sum of n A.M.s = n × single A.M.

$$A_1 + A_2 = 2\left(\frac{a+b}{2}\right) = a + b$$

Product of n G.M.s = (single G.M.)^n

$$G_1G_2 = (\sqrt{ab})^2 = ab$$

$$\frac{1}{a}, \frac{1}{H_1}, \frac{1}{H_2}, \frac{1}{b} \text{ are in A.P.}$$

$$\therefore \quad \frac{1}{H_1} + \frac{1}{H_2} = \frac{1}{a} + \frac{1}{b} = \frac{a+b}{ab}$$

or $$\frac{H_1 + H_2}{H_1H_2} = \frac{A_1 + A_2}{G_1G_2}$$

(b) a, A_1, A_2, b are in A.P. ...(i)

a, H_1, H_2, b are in H.P.

$\therefore \quad \frac{1}{a}, \frac{1}{H_1}, \frac{1}{H_2}, \frac{1}{b}$ are in A.P.

Multiply by ab.

$\therefore \quad b, \frac{ab}{H_1}, \frac{ab}{H_2}, a$ are in A.P.

Take in reverse order.

or $\quad a, \frac{ab}{H_2}, \frac{ab}{H_1}, b$ are in A.P. ...(ii)

Compare (i) and (ii)

$\therefore \quad A_1 = \frac{ab}{H_2}$ and $A_2 = \frac{ab}{H_1}$

$\therefore \quad A_1H_2 = A_2H_1 = ab = G_1G_2$

Q.6 Let the sequence $a_1, a_2, \ldots\ldots a_n$ form an A.P. and let $a_1 = 0$, prove that

$$\frac{a_3}{a_2} + \frac{a_4}{a_3} + \frac{a_5}{a_4} + \ldots\ldots \frac{a_n}{a_{n-1}} - a_2\left(\frac{1}{a_2} + \frac{1}{a_3} + \ldots\ldots + \frac{1}{a_{n-2}}\right) = \frac{a_{n-1}}{a_2} + \frac{a_2}{a_{n-1}}.$$

Sol. Let d be the common difference of the given A.P.

Then since $a_1 = 0$, we have $a_2 = d$

$a_3 = 2d, \ldots\ldots a_n = (n-1)\,d.$

Hence L.H.S. $= \frac{2d}{d} + \frac{3d}{2d} + \frac{4d}{3d} + \ldots\ldots + \frac{(n-1)d}{(n-2)d} - d\left(\frac{1}{d} + \frac{1}{2d} + \frac{1}{3d} + \ldots\ldots + \frac{1}{(n-3)d}\right)$

$= (1+1) + \left(1+\frac{1}{2}\right) + \left(1+\frac{1}{3}\right) + \ldots\ldots + \left(1+\frac{1}{n-3}\right) + \left(1+\frac{1}{n-2}\right) - \left(1+\frac{1}{2}+\frac{1}{3}+\ldots\ldots+\frac{1}{n-3}\right)$

$= 1 + 1 + 1 + \ldots\ldots$ to $(n-2)$ terms $+ \frac{1}{n-2}$

$= (n-2) + \frac{1}{n-2} = \frac{a_{n-1}}{a_2} + \frac{a_2}{a_{n-1}}$

Q.7 Prove that if the sum of n terms of the following series

$$\frac{2n+1}{2n-1}+3\left(\frac{2n+1}{2n-1}\right)^2+5\left(\frac{2n+1}{2n-1}\right)^2+\ldots\ldots$$

is 36 then n = 4.

Sol. We put $x=\frac{2n+1}{2n-1}$ ...(i)

$\therefore \quad 1-x=-\frac{2}{2n-1}$ ...(ii)

or $\quad \frac{x}{1-x}=-\frac{2n+1}{2}$ ...(iii)

Let $\quad S = x + 3x^2 + 5x^3 + \ldots\ldots (2n-1)\, x^n$.

$\therefore \quad xS = \quad x^2 + 3x^3 + \ldots\ldots (2n-3)\, x^n + (2n-1)\, x^{n+1}$

$\therefore \quad S\,(1-x) = x + [2x^2 + 2x^3 + \ldots\ldots (n-1) \text{ terms}] - (2n-1)x^{n+1}$

$\therefore \quad S=\frac{x}{1-x}[1-2n+1+(2n+1)\,x^{n-1}-(2n+1)\,x^{n-1}]$

$=-\frac{2n+1}{2}(-2n)=n\,(2n+1)=36$, given

$\therefore \quad n = 4.$

$\therefore \quad n = 4.$

Q.8 If $\sum_{r=1}^{n} I(r) = n(2n^2+9n+13)$, then find the sum $\sum_{r=1}^{n}\sqrt{I(r)}$.

Sol. $S_n=\sum_{r=1}^{n} I(r) = n(2n^2+9n+13)$

$\Rightarrow \quad I(r) = S_r - S_{r-1}$
$= r(2r^2 + 9r + 13) - (r-1)\,(2\,(r-1)^2 + 9(r-1) + 13)$
$= 6r^2 + 12r + 6 = 6(r+1)^2$

$\Rightarrow \quad \sqrt{I(r)}=\sqrt{6}(r+1)$

$\Rightarrow \quad \sum_{r=1}^{n}\sqrt{I(r)}=\sqrt{6}\sum_{r=1}^{n}(r+1)$

$$=\sqrt{6}\left(\frac{n^2+3n}{2}\right)$$

$$=\sqrt{\frac{3}{2}}(n^2+3n)$$

Q.9 Find the sum to n terms of the series $1^2 - 2^2 + 3^2 - 4^2 + 5^2 - 6^2 + \ldots\ldots$.

Sol. Clearly, n^{th} term of the given series is negative or positive accordingly as n is even or odd, respectively.

(a) n is even :

$$1^2 - 2^2 + 3^2 - 4^2 + 5^2 - 6^2 + \ldots\ldots + (n-1)^2 - n^2$$
$$= (1^2 - 2^2) + (3^2 - 4^2) + (5^2 - 6^2) + \ldots\ldots + ((n-1)^2 - n^2)$$
$$= (1-2)(1+2) + (3-4)(3+4) + (5-6)(5+6) + \ldots\ldots + ((n-1)-(n))(n-1+n)$$
$$= -(1+2+3+4+\ldots\ldots+(n-1)+n)$$
$$= -\frac{n(n+1)}{2}$$

(b) n is odd :

$$(1^2 - 2^2) + (3^2 - 4^2) + \ldots\ldots + \{(n-2)^2 - (n-1)^2\} + n^2$$
$$= (1-2)(1+2) + (3-4)(3+4) + \ldots\ldots + [(n-2)-(n-1)][(n-2)+(n-1)] + n^2$$
$$= -(1+2+3+4+\ldots\ldots+(n-2)+(n-1)) + n^2$$
$$= -\frac{(n-1)(n-1+1)}{2} + n^2$$

$n(n+1)$

$$= \frac{n(n+1)}{2}$$

Q.10 Find the greatest value of $(a+x)^3 (a-x)^4$ for any real value of x numerically less than a.

Sol. Let $z = (a+x)^3 (a-x)^4$

$$= 3^3 \cdot 4^4 \left(\frac{a+x}{3}\right)^3 \left(\frac{a-x}{4}\right)^4 \quad \ldots (1)$$

z will be maximum, when $\left(\frac{a+x}{3}\right)^3 \left(\frac{a-x}{4}\right)^4$ is maximum but $\left(\frac{a+x}{3}\right)^3 \left(\frac{a-x}{4}\right)^4$ is product of $3 + 4 = 7$ factors.

The sum of which $= 3\left(\frac{a+x}{3}\right) + 4\left(\frac{a-x}{4}\right) = (a+x) + (a-x) = 2a.$

$\therefore$ $\left(\frac{a+x}{3}\right)^3 \left(\frac{a-x}{4}\right)^4$ will be maximum if all the factors are equal i.e., if $\frac{a+x}{3} = \frac{a-x}{4}$.

or $4a + 4x = 3a - 3x$ or $x = -\frac{a}{7}$

So from (1) maximum value of z

$$= 3^3 \cdot 4^4 \left[\frac{a-(a/7)}{3}\right]^3 \left[\frac{a-(a/7)}{4}\right]^4$$

$$= 3^3 \cdot 4^4 \left(\frac{6a}{3\times 7}\right)^3 \left(\frac{8a}{7\times 4}\right)^4$$

Q.11 Find the sum $\sum_{r=1}^{n} \frac{1}{r(r+1)(r+2)(r+3)}$

Sol. $\frac{1}{r(r+1)(r+2)(r+3)}$

$$= \frac{r+3-r}{3[r(r+1)(r+2)(r+3)]}$$

$$= \frac{r+3-r}{3[r(r+1)(r+2)(r+3)]}$$

$$= -\frac{1}{3}\left[\frac{1}{(r+1)(r+2)(r+3)} - \frac{1}{r(r+1)(r+2)}\right]$$

$$= -\frac{1}{3}[V(r) - V(r-1)]$$

$\Rightarrow \quad \sum_{r=1}^{n} \frac{1}{r(r+1)(r+2)(r+3)}$

$$= -\frac{1}{3}[V(n) - V(0)]$$

$$= \frac{1}{3}\left[\frac{1}{6} - \frac{1}{(n+1)(n+2)(n+3)}\right]$$

QUADRATIC EQUATION

1. INTRODUCTION :

1.1 Polynomial :

An expression of the type $P_n(x) = a_0x^n + a_1x^{n-1} + \ldots\ldots + a_n$ is called a polynomial of degree 'n', where all powers of x are non-negative integers and a_0 which is called **leading coefficient** of the polynomial should not be equal to zero.

⇒ If co-efficients $a_0, a_1, a_2 \ldots\ldots a_n$ are real then polynomial is called real polynomial and if co-efficients are in the form of $(a + ib)$ then it is called complex polynomial.

e.g., : $(2 + 3i)x^3 + 5x^2 + 6x + 3$ is called a complex polynomial.

If $n = 1$ then $P(x) = a_0x + a_1$ is called a linear polynomial.

If $n = 2$ then $P(x) = a_0x^2 + a_1x + a_2$ is called a quadratic polynomial.

If $n = 3$ then $P(x) = a_0x^3 + a_1x^2 + a_2x + a_3$ is called a cubic polynomial.

If $n = 4$ then $P(x) = a_0x^4 + a_1x^3 + a_2x^2 + a_3x + a_4$ is called a bi-quadratic polynomial.

⇒ $P_n(\alpha)$ means value of the polynomial $P_n(x)$ at $x = \alpha$.

If $P_n(\alpha) = 0$, then α is called as root or zero of the polynomial.

1.2 Remainder Theorem :

The remainder theorem states that if a polynomial $P(x)$ is divided by a linear function $x - k$, then the remainder is $P(k)$.

$$\frac{P(x)}{x-k} = Q(x) + \frac{R}{x-k} \text{ where } Q(x) \text{ is quotient and } R \text{ is remainder.}$$

⇒ $P(x) = Q(x)\,(x - k) + R$ at $x = k$, $P(k) = R$

1.3 Factor Theorem :

Let $P(x) = (x - k)\,Q(x) + R$

when $P(k) = 0$, $P(x) = (x - k)\,Q(x)$. Therefore, $P(x)$ is exactly divisible by $x - k$.

1.4 Quadratic Expression and Quadratic Equation :

A second degree expression in one variable contains the variable with an exponent of 2; but not higher power. Such expressions are called as quadratic expression.

⇒ e.g., : $y = ax^2 + bx + c$,

where a = leading coefficient & c = absolute term of quadratic polynomial.

⇒ If above is equated to zero called as quadratic equation.

e.g., : $ax^2 + bx + c = 0;\ a \neq 0$

⇒ If leading coefficient is 1 then polynomial is called **monic polynomial**.

Solving a quadratic equation means finding the values of x for which $ax^2 + bx + c$ vanishes and these values of x are also called the roots of quadratic equation.

1.5 Identity :

Let $ax^2 + bx + c = 0$ be a quadratic equation. Now, if this quadratic equation has more than two distinct roots then it becomes an identity and in this case $a = b = c = 0$.

Note: Identity is an equation which is true for all values of x.

Let us say α, β, γ are three distinct roots of the given quadratic equation. Then,

$ax^2 + bx + c = k\,(x - \alpha)\,(x - \beta)\,(x - \gamma)$, for some constant k.

⇒ $ax^2 + bx + c = k\,[x^3 - (\alpha + \beta + \gamma)\,x^2 + (\alpha\beta + \beta\gamma + \gamma\alpha)\,x - \alpha\beta\gamma]$

Comparing the co-efficient of x^3 on both sides, we get $k = 0$

and $k = 0 \Rightarrow a = 0, b = 0$ and $c = 0$

⇒ If a quadratic equation is satisfied by more than two distinct values of x, then all the co-efficients must be zero. And when all the co-efficients are zero, quadratic equation is true for all $x \in R$ and hence, it becomes an identity.

Illustration :

For what values of p, the equation $(p + 2)(p - 1)x^2 + (p - 1)\,(2p + 1)\,x + p^2 - 1 = 0$ has more than

Illustration :

For what values of p, the equation $(p + 2)(p - 1)x^2 + (p - 1)\,(2p + 1)\,x + p^2 - 1 = 0$ has more than two roots.

Sol. *$(p + 2)\,(p - 1)\,x^2 + (p - 1)\,(2p + 1)\,x + p^2 - 1 = 0$ will have more than two roots if all the co-efficients are zero.*

$\Rightarrow (p + 2)\,(p - 1) = 0 \Rightarrow p = -2, 1$

and $(p - 1)\,(2p + 1) = 0 \Rightarrow p = 1, p = \dfrac{-1}{2}$

and $p^2 - 1 = 0 \Rightarrow p = 1, -1$

$\therefore$ *All the co-efficients are zero when p = 1.* ***Ans.***

2. SOLUTION OF QUADRATIC EQUATION :

2.1 Factorization Method :

Let $ax^2 + bx + c = a(x - \alpha)\,(x - \beta) = 0$

Then $x = \alpha$ and $x = \beta$ will satisfy the given equation

Hence factorize the equation and equating each to zero gives roots of equation.

e.g. $3x^2 - 2x - 1 = 0 \equiv (x - 1)\,(3x + 1) = 0$

$$x = 1, -\frac{1}{3}.$$

2.2 Hindu Method (Sri Dharacharya Method) :

$ax^2 + bx + c = 0$ means we have to find to those values of x for which $ax^2 + bx + c = 0$.
Finding roots of $ax^2 + bx + c = 0$; $a \neq 0$; $a, b, c \in R$.

$$\Rightarrow \quad x^2 + \frac{b}{a}x + \frac{c}{a} = 0 \qquad \Rightarrow \quad \left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a} = \frac{b^2 - 4ac}{4a^2}$$

$$\Rightarrow \quad x + \frac{b}{2a} = \frac{\pm\sqrt{b^2 - 4ac}}{2a} \qquad \Rightarrow \quad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Hence $\alpha = \dfrac{-b + \sqrt{D}}{2a}$ and $\beta = \dfrac{-b - \sqrt{D}}{2a}$ where $D = b^2 - 4ac$

3. RELATION BETWEEN ROOTS AND COEFFICIENT :

$ax^2 + bx + c = 0$; $\quad a \neq 0$; $a, b, c \in R$

If α, β are the roots then $\quad \alpha + \beta = -\frac{b}{a}$; $\quad \alpha\beta = \frac{c}{a}$ and $\alpha - \beta = \pm\frac{\sqrt{D}}{a}$

$ax^2 + bx + c = 0$; $\quad a \neq 0$; $a, b, c \in R$

If α, β are the roots then $\quad \alpha + \beta = -\frac{b}{a}$; $\quad \alpha\beta = \frac{c}{a}$ and $\alpha - \beta = \pm\frac{\sqrt{D}}{a}$

4. FORMATION OF A QUADRATIC EQUATION WHEN ROOTS ARE GIVEN :

Let α and β be the given roots of a quadratic equation, then

$$(x - \alpha)(x - \beta) = 0$$
$$x^2 - x(\alpha + \beta) + \alpha\beta = 0$$

$x^2 - x$ (sum of the roots) + Product of the roots = 0

Note : Some Transformation in terms of $\alpha + \beta$ and $\alpha\beta$:

$$\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$$

$$\alpha^3 + \beta^3 = (\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta)$$

$$\alpha - \beta = \sqrt{(\alpha + \beta)^2 - 4\alpha\beta}$$

$$\alpha^4 + \beta^4 = [(\alpha + \beta)^2 - 2\alpha\beta]^2 - 2\alpha^2\beta^2$$

Illustration :

If the product of the roots of the quadratic equation $mx^2 - 2x + (2m - 1) = 0$ is 3, then the value of m

(A) 1 (B) 2 (C) – 1 (D) 3

Sol. Product of the roots $\frac{c}{a} = 3 = \frac{2m-1}{m}$

$\therefore \quad 3m - 2m = -1 \qquad \Rightarrow m = -1$ **Ans.**

Illustration :

If α, β are roots of the $ax^2 + bx + c = 0$, then the value of $\frac{1}{(a\alpha+b)^2}+\frac{1}{(a\beta+b)^2}$ is

(A) $\frac{b^2-2ac}{ac}$ (B) $\frac{2ac-b^2}{ac}$ (C) $\frac{b^2-2ac}{a^2c^2}$ (D) $\frac{b^2}{a^2c}$

Sol. Since α, β are the roots of the $ax^2 + bx + c$

then $a\alpha^2 + b\alpha + c = 0$

$\alpha(a\alpha + b) + c = 0$

Sol. Since α, β are the roots of the $ax^2 + bx + c$

then $a\alpha^2 + b\alpha + c = 0$

$\alpha(a\alpha + b) + c = 0$

$(a\alpha + b) = \frac{-c}{\alpha}$ ....(1)

Similarly

$(a\beta + b) = \frac{-c}{\beta}$ ....(2)

$$\therefore \quad \frac{1}{(a\alpha+b)^2}+\frac{1}{(a\beta+b)^2} = \frac{1}{(-c/\alpha)^2}+\frac{1}{(-c/\beta)^2}$$

$$\Rightarrow \quad \frac{\alpha^2}{c^2}+\frac{\beta^2}{c^2} = \frac{\alpha^2+\beta^2}{c^2} = \frac{(\alpha+\beta)^2-2\alpha\beta}{c^2} = \frac{\frac{b^2}{a^2}-\frac{2c}{a}}{c^2} = \frac{b^2-2ac}{a^2c^2}$$ **Ans.**

Illustration :

If the equation $(k - 2)x^2 - (k - 4)x - 2 = 0$ has difference of roots as 3 then the value of k is

(A) 1, 3 (B) 3, $\frac{3}{2}$ (C) 2, $\frac{3}{2}$ (D) $\frac{3}{2}$, 1

Sol. $(\alpha - \beta) = \sqrt{(\alpha+\beta)^2-4\alpha\beta}$

Now, $\alpha + \beta = \frac{(k-4)}{(k-2)}$, $\alpha\beta = \frac{-2}{k-2}$

$\therefore$ $(\alpha - \beta) = \sqrt{\left(\frac{k-4}{k-2}\right)^2 + \frac{8}{(k-2)}} = \frac{\sqrt{k^2+16-8k+8(k-2)}}{(k-2)}$

$3 = \frac{\sqrt{k^2+16-8k+8(k-2)}}{(k-2)}$

$3k - 6 = \pm k$

$k = 3, \frac{3}{2}$. **Ans.**

Illustration :

If the roots of the quadratic equation $x^2 + mx + n = 0$ are tan 30° and tan 15°, respectively, then find the value of $2 + n - m$.

Sol. *The equation $x^2 + mx + n = 0$ has roots tan 30° and tan 15°.*
Therefore

$\tan 30^o + \tan 15^o = -m$ ...(i)

$\tan 30^o \tan 15^o = n$ ...(ii)

Now, $\tan 45^o = \tan (30^o + 15^o)$ $\Rightarrow$ $1 = \frac{\tan 30^o + \tan 15^o}{1 - \tan 30^o \tan 15^o}$

$\tan 30^o \tan 15^o = n$ ...(ii)

Now, $\tan 45^o = \tan (30^o + 15^o)$ $\Rightarrow$ $1 = \frac{\tan 30^o + \tan 15^o}{1 - \tan 30^o \tan 15^o}$

$\Rightarrow$ $1 = \frac{-m}{1-n}$ [Using (i) and (ii)]

$\Rightarrow$ $1 - n = -m$ $\Rightarrow$ $n - m = 1$

$\Rightarrow$ $2 + n - m = 3$

Illustration :

If the difference between the roots of the equation $x^2 + ax + 1 = 0$ is less than $\sqrt{5}$, then find the set of possible values of a.

Sol. *If α, β are roots of $x^2 + ax + 1 = 0$, then*

$|\alpha - \beta| < \sqrt{5}$

$\Rightarrow$ $\left|\sqrt{(\alpha+\beta)^2 - 4\alpha\beta}\right| < \sqrt{5}$

$\Rightarrow$ $\left|\sqrt{a^2 - 4}\right| < \sqrt{5}$

$\Rightarrow$ $a^2 - 4 < 5$

$\Rightarrow$ $a^2 - 9 < 0$

$\therefore$ $a \in (-3, 3)$.

5. NATURE OF ROOTS :

Consider the quadratic equation

$$ax^2 + bx + c = 0$$

where a, b, $c \in R$ and $a \neq 0$.

Roots of the equation are given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Now, we observe that the roots depend upon the value of the quantity $b^2 - 4ac$. This quantity is generally denoted by D and is known as the **discriminant** of the quadratic equation which decides nature of the roots. We also observe the following results :

(i) If $D > 0 \Rightarrow$ roots are real and distinct.

(ii) If $D = 0 \Rightarrow$ roots are equal.

Note : From (i) and (ii) it is clear that for real roots of a quadratic equation D must be greater than or equal to zero. (i.e. $D \geq 0$)

(iii) If $D < 0 \Rightarrow$ roots are imaginary.

Important Note :

(iii) If $D < 0 \Rightarrow$ roots are imaginary.

Important Note :

(1) If co-efficients of the quadratic equation are rational then its irrational roots always occur in pair. If $p + \sqrt{q}$ is one of the roots then other root will be $p - \sqrt{q}$.

(2) If co-efficients of the quadratic equation are real then its imaginary roots always occur in complex conjugate pair. If $p + iq$ is one of the roots then other root will be $p - iq$.

6. ROOTS UNDER PARTICULAR CASES :

(i) Exactly one root is at infinity :

If exactly one root is ∞ and other root is finite, then co-efficient of x^2 must tend to zero and co-efficient of x must not be equal to zero.

Put $x = \frac{1}{y}$ in $ax^2 + bx + c = 0$, we get

$cy^2 + by + a = 0$ must have one root zero $\Rightarrow P = 0$ i.e. $\frac{a}{c} = 0$

Hence, $a = 0$ and $-\frac{b}{c} \neq 0 \Rightarrow b \neq 0$.

original equation becomes $bx + c = 0$

(ii) Both the roots at infinity :

When both roots of the equation are infinity then, co-efficient of x^2 and co-efficient of x both must tend to zero and c must not be equal to zero. The equation $cy^2 + by + a = 0$ must have both roots zero.

i.e. $-\dfrac{b}{c} = 0$ and $\dfrac{a}{c} = 0 \Rightarrow b = 0\ ;\ a = 0$ and $c \neq 0$.

In this case the equation becomes $y = c$.

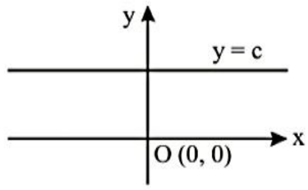


Illustration :

Find the value of P for which the equation $(P^3 - 3P^2 + 2P)x^2 + (P^3 - P)x + P^3 + 3P^2 + 2P = 0$

(i) *has exactly one root at infinity*

(ii) *has both the roots at infinity*

(iii) *becomes an identity*

Sol.

(i) ***Equation has exactly one root at infinity***

$a = 0,$ $\qquad b \neq 0$

$P = 0, 1, 2$ $\qquad P \neq \pm 1, 0$

hence $P = 2$

$a = 0,$ $\qquad b \neq 0$

$P = 0, 1, 2$ $\qquad P \neq \pm 1, 0$

hence $P = 2$

(ii) ***Equation has both the roots at infinity***

Sol. $a = 0,$ $\qquad b = 0$ $\qquad c \neq 0$

$P = 0, 1, 2$ $\qquad P = \pm 1, 0$ $\qquad P \neq -1, -2, 0$

hence $P = 1$

(iii) ***Equation becomes an identity***

Sol. $a = 0,$ $\qquad b = 0$ $\qquad c = 0$

$P = 0, 1, 2$ $\qquad P = \pm 1, 0$ $\qquad P = -1, -2, 0$

hence $P = 0$

Illustration :

If $a, b, c \in R$ *such that* $a + b + c = 0$ *and* $a \neq c$*, then prove that the roots of* $(b + c - a)x^2 + (c + a - b)x + (a + b - c) = 0$ *are real and distinct.*

Sol. *at* $x = 1,$ $\qquad f(x) = a + b + c = 0$

hence, $x = 1$ *is a root of the given equation.*

Product of roots $= \dfrac{a+b-c}{b+c-a} = \dfrac{-2c}{-2a} = \dfrac{c}{a}$

Since $c \neq a$ *hence other root is not unity.*

$\therefore$ *roots are real and distinct*

Illustration :

If $\cos\alpha$, $\sin\beta$, $\sin\alpha$ *are in G.P., then check the nature of roots of* $x^2 + 2\cot\beta \cdot x + 1 = 0$.

Sol. *We have* $\sin^2\beta = \cos\alpha \sin\alpha$

The discriminant of the given equation is

$$D = 4\cot^2\beta - 4$$

$$= 4\left[\frac{\cos^2\beta - \sin^2\beta}{\sin^2\beta}\right] = \frac{4(1-2\sin^2\beta)}{\sin^2\beta} = \frac{4(1-2\sin\alpha\cos\alpha)}{\sin^2\beta} = \left[\frac{2(\sin\alpha - \cos\alpha)}{\sin\beta}\right]^2 \geq 0$$

Illustration :

The roots of the quadratic equation $2x^2 - 7x + 4 = 0$ *are*

(A) Rational and different *(B) Rational and equation*

(C) Irrational and different *(D) Imaginary and different*

Sol. $b^2 - 4ac = 49 - 32 = 17 > 0$ *(not a perfect square)*

$\therefore$ *Its roots are irrational and different.*

Illustration :

The roots of the quadratic equation $x^2 - 2(a + b)x + 2(a^2 + b^2) = 0$ *are*

Illustration :

The roots of the quadratic equation $x^2 - 2(a + b)x + 2(a^2 + b^2) = 0$ *are*

(A) Rational and different *(B) Rational and equation*

(C) Irrational and different *(D) Imaginary and different*

Sol. $A = 1,\ B = -2(a + b),\ C = 2(a^2 + b^2)$

$$B^2 - 4AC = 1\,[2(a + b)]^2 - 4(1)\,(2a^2 + 2b^2)$$
$$= 4a^2 + 4b^2 + 8ab - 8a^2 - 8b^2$$
$$= -4a^2 - 4b^2 + 8ab$$
$$= -4\,(a - b)^2 < 0$$

So roots are imaginary and different.

Illustration :

The quadratic equation with rational coefficients whose one root is $2+\sqrt{3}$ *is*

(A) $x^2 - 4x + 1 = 0$ *(B)* $x^2 + 4x + 1 = 0$ *(C)* $x^2 + 4x - 1 = 0$ *(D)* $x^2 + 2x + 1 = 0$

Sol. *The required equation is*

$$x^2 - \left\{\left(2+\sqrt{3}\right)+\left(2-\sqrt{3}\right)\right\}x+\left(2+\sqrt{3}\right)\left(2-\sqrt{3}\right)=0$$

or $x^2 - 4x + 1 = 0$ **[Ans. 1]**

Practice Problem

Q.1 If α, β are roots of the equation $x^2 + px - q = 0$ and γ, δ are roots of $x^2 + px + r = 0$ then the value of $(\alpha - \gamma)(\alpha - \delta)$ is
(A) $p + r$ (B) $p - r$ (C) $q - r$ (D) $q + r$

Q.2 If α, β are roots of the equation $2x^2 - 35x + 2 = 0$, then the value of $(2\alpha - 35)^3 \cdot (2\beta - 35)^3$ is equal to
(A) 1 (B) 8 (C) 64 (D) none of these

Q.3 A certain polynomial P(x), $x \in R$ when divided by $x - a$, $x - b$ and $x - c$ leaves remainders a, b and c, respectively. Then find the remainder when P(x) is divided by $(x - a)(x - b)(x - c)$ where a, b, c are distinct.

Q.4 If $(a^2 - 1)x^2 + (a - 1)x + a^2 - 4a + 3 = 0$ be an identity in x, then find the value of a.

Q.5 For what value of m will the equation $x^2 - 2x(1 + 3m) + 7(3 + 2m) = 0$ have equal roots.

Q.6 If p, q and r are odd integers, then prove that roots of $px^2 + qx + r = 0$ cannot be rational.

Answer key

Answer key

Q.1	D	Q.2	C	Q.3	x
Q.4	1	Q.5	$m = 2;\ \frac{-10}{9}$		

7. QUADRATIC EXPRESSION AND ITS GRAPH :

In $y = ax^2 + bx + c$, if $a, b, c \in R$ and $a \neq 0$. Graph of quadratic takes the shape of a parabola. The parabola opens upward or downward according as $a > 0$ or $a < 0$ respectively.

Figure - (i)

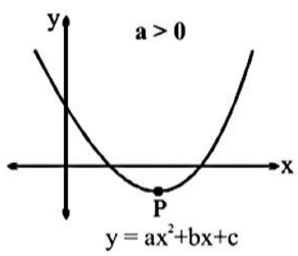
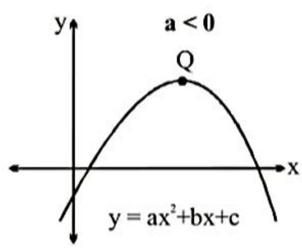


Figure - (ii)



The lowest point P in figure-(i) and highest point Q in figure-(ii) is called as vertex of parabola. Now for different values of a, b, c if graph $y = ax^2 + bx + c$ is plotted then following 6 different shapes are obtained.

Case-I : If $a > 0$ and $D > 0$

Then quatratic equation has two roots and graph cuts the x-axis at two distinct points.

(i) For $\alpha < x < \beta \Rightarrow$ y is negative.

(ii) For $x < \alpha$ or $x > \beta \Rightarrow$ y is positive.

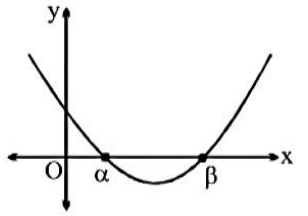
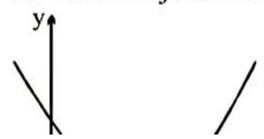
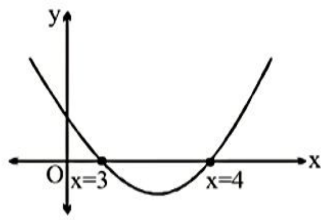


Illustration :

Draw graph of $y = x^2 - 7x + 12$ and find set of values of x where y is positive.





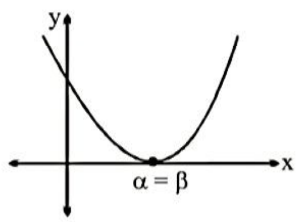
Sol. $y = x^2 - 7x + 12 = (x - 3)(x - 4)$.

Clearly, $y > 0$ *if* $x < 3$ *or* $x > 4$

i.e., $(-\infty, 3) \cup (4, \infty)$.

Case-II : If $a > 0$ and $D = 0$

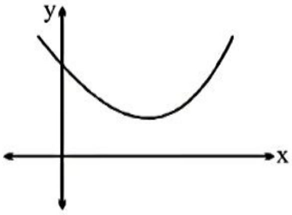
Then curve touches x-axis. Hence both zeroes of polynomial coincides.



In this type equation becomes $y = a(x - \alpha)^2$ and $y \geq 0$, for $x \in R$.

Case-III : If $a > 0$ and $D < 0$

Then curve completely lies above x-axis.



In this case imaginary roots appears and $y > 0$ for $x \in R$.

Illustration :

Find range of k for which graph of $y = x^2 - 3x + k$ lies completely above x-axis.

Sol. $D < 0$

$9 - 4k < 0$

$k > \frac{9}{4}$

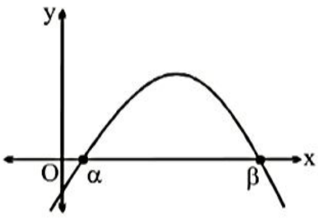
$\left(\frac{9}{4}, \infty\right)$. ***Ans.***

Case-IV : If $a < 0$ and $D > 0$

Then graph is downward and cuts the x-axis at two distinct points.

Case-IV : If $a < 0$ and $D > 0$

Then graph is downward and cuts the x-axis at two distinct points.



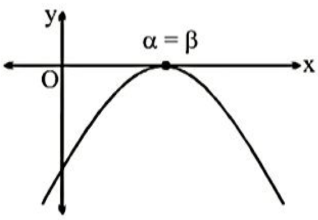
In this case

(a) $y > 0$, if $\alpha < x < \beta$

(b) $y < 0$, if $x < \alpha$ or $x > \beta$

Case-V : If $a < 0$ and $D = 0$

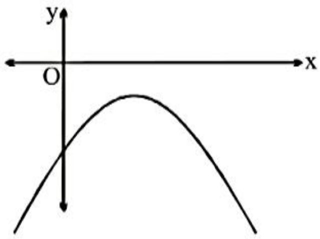
Then graph touches the x-axis from below.



In this case $x \in R$, $y \le 0$ for $x \in R$.

Case-VI : If $a < 0$ and $D < 0$

Then graph lies completely below the x-axis and $y < 0$ for $x \in R$.



✍ **Important Note**

(1) The quadratic expression $ax^2 + bx + c$, where $a, b, c \in R$ is positive $\forall\ x \in R$, if $a > 0$ and $D < 0$ (Case-III).

(2) The quadratic expression $ax^2 + bx + c$, where $a, b, c \in R$ is negative $\forall\ x \in R$, if $a < 0$ and $D < 0$ (Case-VI).

Illustration :

A quadratic equation with rational coefficient has one of its roots as $2\sin^2\left(\frac{\pi}{5}\right)$ *if the sum of the*

$D < 0$ (Case-VI).

Illustration :

A quadratic equation with rational coefficient has one of its roots as $2\sin^2\left(\frac{\pi}{5}\right)$ *if the sum of the roots of quadratic equation is S and product of roots is P. Then P = KS implies that the value of K equals________.*

Sol. *One root* $= 2\sin^2\frac{\pi}{5} = 1 - \cos\frac{\pi}{5} = 1 - \left(\frac{\sqrt{5}-1}{4}\right) = \frac{5-\sqrt{5}}{4}$

$\therefore$ *Other root* $= \frac{5+\sqrt{5}}{4}$ *[As coefficients are rational, roots will be conjugate surds.]*

$\therefore$ *Sum of roots* $= S = \frac{5}{2}$

Product of roots $= P = \frac{20}{16} = \frac{5}{4}$

$$k = \frac{P}{S} = \frac{5/4}{5/2} = \frac{1}{2}.$$ ***Ans.***

Illustration :

If $x = 3+\sqrt{5}$ *find the value of* $x^4 - 12x^3 + 44x^2 - 48x + 17$.

Sol. $x = 3 + \sqrt{5} \quad \Rightarrow \quad x - 3 = \sqrt{5}$

$\Rightarrow \quad (x-3)^2 = 5 \quad \Rightarrow \quad x^2 - 6x + 4 = 0$

Now, $x^4 - 12x^3 + 44x^2 - 48x + 17 = (x^2 - 6x + 4)(x^2 - 6x + 4) + 1$

we know that, dividend = (divisior) (quotient) + R

But $x^2 - 6x + 4 = 0$

$\Rightarrow \quad x^4 - 12x^3 + 44x^2 - 48x + 17 = 1.$ ***Ans.***

Illustration :

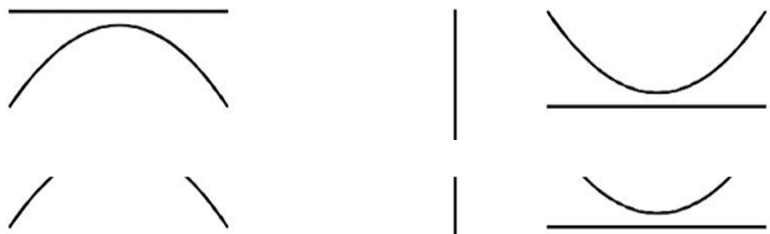
The quadratic equation $ax^2 + bx + c = 0$ *has no real root, then prove that*

$c(a + b + c) > 0.$

Sol. *As given equation has no real roots* $\Rightarrow D < 0$

$\therefore$ *Parabola either always lie above the x-axis or below the x-axis as shown*

$a < 0, D < 0$ | $a > 0, D < 0$



Here $f(x) = ax^2 + bx + c < 0 \ \forall x \in R$

$f(0) = c < 0$

$f(1) = a + b + c < 0$

Here $f(x) = ax^2 + bx + c > 0 \ \forall x \in R$

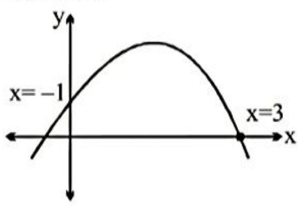
$f(0) = c > 0$

$f(1) = a + b + c > 0$

But in both the cases $c(a + b + c) > 0.$ ***Ans.***

Illustration :

Consider the graph of quadratic polynomial $y = ax^2 + bx + c$ *as shown below. Which of the following is/are correct.*



(A) $\dfrac{a-b+c}{abc} = 0$

(B) $abc(9a + 3b + c) < 0$

(C) $\dfrac{a+3b+9c}{abc} < 0$

(D) $ab(a - 3b + 9c) > 0$

Sol. *Clearly from given figure*

$a < 0, c > 0$

Also, $\dfrac{-b}{2a} > 0 \quad \Rightarrow \quad b > 0$

So, $abc < 0$

Also $f(-1) = a - b + c = 0, \qquad f(3) = 9a + 3b + c = 0$

Clearly $f\left(\dfrac{1}{3}\right) > 0 \quad \Rightarrow \dfrac{a}{9} + \dfrac{b}{3} + c \quad \Rightarrow \quad a + 3b + 9c > 0$

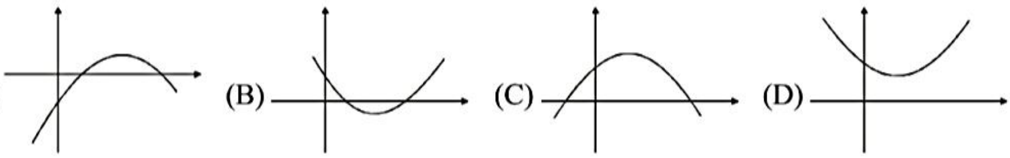
Also $f\left(\dfrac{-1}{3}\right) > 0 \quad \Rightarrow \dfrac{a}{9} - \dfrac{b}{3} + c > 0 \quad \Rightarrow \quad a - 3b + 9c > 0$

Now verify alternatives. ***Ans. A, C***

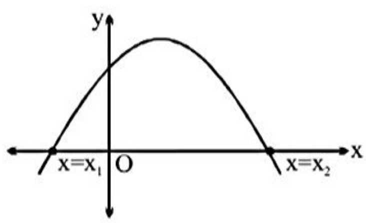
Practice Problem

Q.1 The best possible graph of the function $f(x) = Ax^2 + Bx + C$, where
$A = (\sec\theta - \cos\theta)(\text{cosec}\theta - \sin\theta)(\tan\theta + \cot\theta)$
$B = (\sin\theta + \text{cosec}\theta)^2 + (\cos\theta + \sec\theta)^2 - (\tan^2\theta + \cot^2\theta)$ and
$C = 12$, is represented by

$A = (\sec\theta - \cos\theta)(\text{cosec}\theta - \sin\theta)(\tan\theta + \cot\theta)$
$B = (\sin\theta + \text{cosec}\theta)^2 + (\cos\theta + \sec\theta)^2 - (\tan^2\theta + \cot^2\theta)$ and
$C = 12$, is represented by



Q.2 Consider the graph of quadratic trinomial $y = ax^2 + bx + c$ as shown below where x_1 and x_2 are roots of the equation $ax^2 + bx + c = 0$. Which of the following is/are correct?



(A) $a - b - c < 0$ (B) $bc < 0$
(C) $b > 0$ (D) b and c have the same sign different from a.

Q.3 The trinomial $ax^2 + bx + c$ has no real roots and $a + b + c < 0$. Comment on the sign of the number c.

Answer key

Q.1 B Q.2 A, C, D Q.3 $c < 0$

8. SOLVING QUADRATIC AND RATIONAL INEQUALITIES (WAVY CURVE METHOD) :

While solving such inequations following steps to be taken.

(i) Factorise given-expression into linear factors

(ii) Make the coefficient of x positive in all factors

(iii) Plot the points where given expression vanishes or undefined (denominator becomes zero) on number line in increasing order

(iv) Start the number line from right to left taking positive or negative value.

While solving rational inequalities different situtations arive.

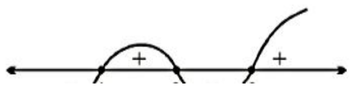
Type-1 : Inequalities involving non-repeated linear factors

Illustration :

$(x-1)(x-2)(x-3) \geq 0$

Sol. *The set of values for which given expression is* ≥ 0

$[1, 2] \cup [3, \infty)$



$(x-1)(x-2)(x-3) \geq 0$

Sol. *The set of values for which given expression is* ≥ 0

$[1, 2] \cup [3, \infty)$.

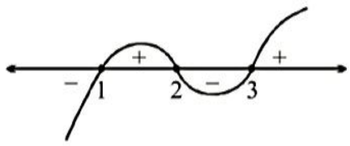
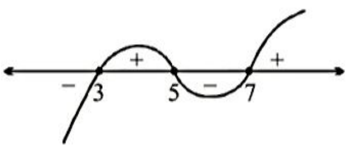


Illustration :

$(x-3)(x-5)(x-7) < 0$

Sol. *In this problem given expression < 0.*



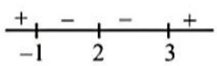
$\Rightarrow \quad (-\infty, 3) \cup (5, 7)$.

Type-2 : Quadratic inequality involving Repeated linear factos.

Illustration :

$(x+1)(x-3)(x-2)^2 \geq 0$.

Sol. $(x+1)(x-3)(x-2)^2 \geq 0$



$x \in (-\infty, -1] \cup \{2\} \cup [3, \infty)$. ***Ans.***

Illustration :

$$x(x+6)(x+2)^2(x-3) > 0$$

Sol. $x(x+6)(x+2)^2(x-3) > 0$

$\Rightarrow x \in (-6, 0) \cup (3, \infty) - \{-2\}$ ***Ans.***

Illustration :

$$(x-1)^2(x+1)^3(x-4) < 0$$

Sol. $(x-1)^2(x+1)^3(x-4) < 0$

$x \in (-1, 1) \cup (1, 4).$ ***Ans.***

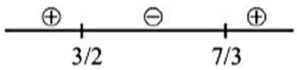
Type-3 : Quadratic / algebraic inequality of the type of $\frac{f(x)}{g(x)}$. (Rational inequality) involving modulus also.

Illustration :

$$\frac{2x-3}{3x-7} > 0$$

$$\frac{2x-3}{3x-7} > 0$$

Sol.



$$x \in \left(-\infty, \frac{3}{2}\right) \cup \left(\frac{7}{3}, \infty\right)$$ ***Ans.***

Illustration :

$$\frac{x^2-5x+12}{x^2-4x+5} > 3$$

Sol. *$x^2 - 4x + 5$ is always positive*

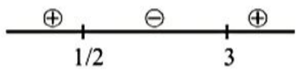
since $D = 16 - 20 = -4 < 0$

hence, we can cross multiply $x^2 - 4x + 5$ without changing the sign of inequality.

$x^2 - 5x + 12 > 3x^2 - 12x + 15$

$2x^2 - 7x + 3 < 0$

$(2x-1)(x-3) < 0$



$$x \in \left(\frac{1}{2}, 3\right)$$ ***Ans.***

Illustration :

$$\frac{x^2-5x+6}{x^2+x+1}<0$$

Sol. $$\frac{x^2-5x+6}{x^2+x+1}<0$$

As denominator is always positive

$\Rightarrow x^2 - 5x + 6 < 0 \Rightarrow (x-2)(x-3) < 0$

$\Rightarrow x \in (2, 3)$ ***Ans.***

Illustration :

$$\frac{(x-1)^2(x+1)^3}{x^4(x-2)}<0$$

Sol. $$\frac{(x-1)^2(x+1)^3}{x^4(x-2)}<0$$

$x \in (-1, 0) \cup (0, 1) \cup (1, 2)$ ***Ans.***

Illustration :

$x \in (-1, 0) \cup (0, 1) \cup (1, 2)$ ***Ans.***

Illustration :

$$\frac{x+1}{x-1} \geq \frac{x+5}{x+1}$$

Sol. $$\frac{x+1}{x-1} \geq \frac{x+5}{x+1} \Rightarrow \frac{x+1}{x-1}-\frac{x+5}{x+1} \geq 0 \Rightarrow \frac{(x+1)^2-(x-1)(x+5)}{(x-1)(x+1)} \geq 0$$

$$\Rightarrow \frac{-2x+6}{(x-1)(x+1)} \geq 0 \Rightarrow \frac{x-3}{(x-1)(x+1)} \leq 0$$

$\Rightarrow x \in (-\infty, -1) \cup (1, 3]$. ***Ans.***

Illustration :

$$\frac{2(x-4)}{(x-1)(x-7)} \geq \frac{1}{x-2}$$

Sol. $$\frac{2(x-4)}{(x-1)(x-7)} \geq \frac{1}{x-2}$$

$$\Rightarrow \quad \frac{2(x-4)}{(x-1)(x-7)}-\frac{1}{(x-2)} \geq 0$$

$$\Rightarrow \quad \frac{2(x-4)(x-2)-(x-1)(x-7)}{(x-1)(x-7)(x-2)} \geq 0$$

$$\Rightarrow \quad \frac{x^2-4x+9}{(x-1)(x-2)(x-7)} \geq 0$$

$\Rightarrow \quad x \in (1, 2) \cup (7, \infty)$ **Ans.**

Illustration :

$$\frac{x^2+6x-7}{|x+4|} < 0$$

Solve the inequality using method of interval.

Sol. $\dfrac{x^2+6x-7}{|x+4|} < 0$

$\Rightarrow \quad x^2 + 6x - 7 < 0 \Rightarrow (x + 7)(x - 1) < 0$

$\Rightarrow \quad x \in (-7, 1) - \{-4\}$ **Ans.**

Type-4 : Double inequality and biquadratic inequality.

$\Rightarrow \quad x^2 + 6x - 7 < 0 \Rightarrow (x + 7)(x - 1) < 0$

$\Rightarrow \quad x \in (-7, 1) - \{-4\}$ **Ans.**

Type-4 : Double inequality and biquadratic inequality.

Illustration :

$$1 < \frac{3x^2-7x+8}{x^2+1} \leq 2$$

Sol. $1 < \dfrac{3x^2-7x+8}{x^2+1} \leq 2$

Here, we need to make sure

$\Rightarrow \quad \dfrac{3x^2-7x+8}{x^2+1} > 1$ *and* $\dfrac{3x^2-7x+8}{x^2+1} \leq 2$

$\Rightarrow \quad 3x^2 - 7x + 7 > x^2 + 1$ *and* $3x^2 - 7x + 8 \leq 2x^2 + 2$

$\Rightarrow \quad 2x^2 - 7x + 7 > 0$ *and* $x^2 - 7x + 6 \leq 0$

Here, $a > 0$ *and* $D < 0$ *and* $(x - 1)(x - 6) \leq 0$

$\Rightarrow \quad x \in R$ *and* $x \in [1, 6]$

Taking intersection of both, we get $x \in [1, 6]$. **Ans.**

Practice Problem

Q.1 Solve the inequality -

(i) $x(x-2)(x+3) \geq 0$ (ii) $x(x-4)^2(x+6)^3(x-1) \leq 0$

Q.2 Solve the inequality $1 < \dfrac{x^2+3x+4}{x^2+4x+5} < 2$

Q.3 $\dfrac{(x+2)(x^2-2x+1)}{(4+3x-x^2)} \geq 0$

Q.4 $\dfrac{x^4-3x^3+2x^2}{x^2-x-30} > 0$

Q.5 Number of positive integral solution of $\dfrac{x^3(2x-3)^2(x-4)^6}{(x-3)^3(3x-8)^4} \leq 0$

(A) 1 (B) 2 (C) 3 (D) 4

Q.6 Find the set of values of a for which the quadratic polynomial

(i) $(a+4)x^2 - 2ax + 2a - 6 < 0 \quad \forall \ x \in R$

(ii) $(a-1)x^2 - (a+1)x + (a+1) > 0 \quad \forall \ x \in R$

Q.6 Find the set of values of a for which the quadratic polynomial

(i) $(a+4)x^2 - 2ax + 2a - 6 < 0 \quad \forall \ x \in R$

(ii) $(a-1)x^2 - (a+1)x + (a+1) > 0 \quad \forall \ x \in R$

Answer key

Q.1 (i) $[-3, 0] \cup [2, \infty)$; (ii) $[-6, 0] \cup [1, \infty)$ Q.2 $x \in (-\infty, -3)$

Q.3 $(-\infty, -2] \cup (-1, 4)$ Q.4 $(-\infty, -5] \cup (1, 2) \cup (6, \infty)$ Q.5 C

Q.6 (i) $(-\infty, -6)$; (ii) $(5/3, \infty)$

9. SYMMETRIC EXPRESSIONS :

The symmetric expressions of the roots α, β of an equation are those expressions in α and β, whice do not change by interchanging a and b. To find the value of such an expression, we generally express that in terms of $\alpha + \beta$ and $\alpha\beta$.

Some examples of symmetric expressions are

(i) $\alpha^2 + \beta^2$ (ii) $\alpha^2 + \alpha\beta + \beta^2$ (iii) $\dfrac{1}{\alpha} + \dfrac{1}{\beta}$ (iv) $\dfrac{\alpha}{\beta} + \dfrac{\beta}{\alpha}$

(v) $\alpha^2\beta + \beta^2\alpha$ (vi) $\left(\dfrac{\alpha}{\beta}\right)^2 + \left(\dfrac{\beta}{\alpha}\right)^2$ (vii) $\alpha^3 + \beta^3$ (viii) $\alpha^4 + \beta^4$

10. FORMATION OF QUADRATIC EQUATION WHOSE ROOTS ARE SYMMETRIC EXPRESSION OF α AND β :

Let α and β be the roots of a quadratic equation $ax^2 + bx + c = 0$ then finding another quadratic equation whose roots are $2\alpha + 3, 2\beta + 3$.

Suppose $2\alpha + 3 = y \Rightarrow \alpha = \dfrac{y-3}{2}$

Put the value of α in the given equation ($\because$ α is its roots) and get a quadratic in y.

$$\frac{a\,(y-3)^2}{4} + \frac{b\,(y-3)}{2} + c = 0$$

$$a\,(y-3)^2 + 2b\,(y-3) + 4c = 0$$

$$ay^2 + 2y\,(b-3a) + 9a - 6b + 4c = 0$$

Replace y by x and get the desired equation.

$$ax^2 + 2x\,(b-3a) + 9a - 6b + 4c = 0.$$

Note: If α, β are roots of the equation $ax^2 + bx + c = 0$ then the equation whose roots are

$$ay^2 + 2y\,(b-3a) + 9a - 6b + 4c = 0$$

Replace y by x and get the desired equation.

$$ax^2 + 2x\,(b-3a) + 9a - 6b + 4c = 0.$$

Note: If α, β are roots of the equation $ax^2 + bx + c = 0$ then the equation whose roots are

(i) $-\alpha, -\beta \Rightarrow ax^2 - bx + c = 0$ (Replace x by $-x$)

(ii) $\dfrac{1}{\alpha}, \dfrac{1}{\beta} \Rightarrow cx^2 = bx + a = 0$ $\left(\text{Replace x by } \dfrac{1}{x}\right)$

(iii) $\alpha^n, \beta^n, n \in N \Rightarrow a\left(x^{\frac{1}{n}}\right)^2 + b\left(x^{\frac{1}{n}}\right) + c = 0$ $\left(\text{Replace x by } x^{\frac{1}{n}}\right)$

(iv) $k\alpha, k\beta \Rightarrow ax^2 + kbx + k^2c = 0.$ $\left(\text{Replace x by } \dfrac{x}{k}\right)$

(v) $k + a, k + b \Rightarrow a(x-k)^2 + b\,(x-k) + c = 0$ (Replace x by $(x-k)$)

(vi) $\dfrac{\alpha}{k}, \dfrac{\beta}{k} \Rightarrow k^2ax^2 + kbx + c = 0$ (Replace x by kx)

(vii) $\alpha^{\frac{1}{n}}, \beta^{\frac{1}{n}}$; $n \in N \Rightarrow a(x^n)^2 + b(x^n) + c = 0$ (Replace x by x^n)

Illustration :

If α, β *are the root of a quadratic equation* $x^2 - 3x + 5 = 0$ *then the equation whose roots are* $(\alpha^2 - 3\alpha + 7)$ *and* $(\beta^2 - 3\beta + 7)$ *is*

(A) $x^2 + 4x + 1 = 0$ (B) $x^2 - 4x + 4 = 0$ (C) $x^2 - 4x - 1 = 0$ (D) $x^2 + 2x + 3 = 0$

Sol. *Since* α, β *are the roots of equation* $x^2 - 3x + 5 = 0$

So $\alpha^2 - 3\alpha + 5 = 0$

$\beta^2 - 3\beta + 5 = 0$

$\therefore$ $\alpha^2 - 3\alpha = -5$

$\beta^2 - 3\beta = -5$

putting in $(\alpha^2 - 3\alpha + 7)$ *and* $(\beta^2 - 3\beta + 7)$ ....(1)

$-5 + 7, -5 + 7$

$\therefore$ *2 and 2 are the roots*

$\therefore$ *the required equation is* $x^2 - 4x + 4 = 0.$ ***Ans.***

Illustration :

If α, β *are roots of the equation* $x^2 - 5x + 6 = 0$ *then the equation whose roots are* $\alpha + 3$ *and*

Illustration :

If α, β *are roots of the equation* $x^2 - 5x + 6 = 0$ *then the equation whose roots are* $\alpha + 3$ *and*
$\beta + 3$ *is*

(A) $x^2 - 11x + 30 = 0$ (B) $(x - 3)^2 - 5(x - 3) + 6 = 0$

(C) Both (1) and (2) (D) None

Sol. *Let* $\alpha + 3 = x$

$\therefore$ $\alpha = x - 3$ *(Replace x by x – 3)*

So the required equation is

$(x - 3)^2 - 5(x - 3) + 6 = 0$ ......(1)

$x^2 - 6x + 9 - 5x + 6 = 0$

$x^2 - 11x + 30 = 0$ ......(2) ***[Ans. C]***

Illustration :

If α, β *are roots of the equation* $2x^2 + x - 1 = 0$ *then the eqution whose roots are* $\frac{1}{\alpha}, \frac{1}{\beta}$ *will be*

(A) $x^2 + x - 2 = 0$ (B) $x^2 + 2x - 8 = 0$ (C) $x^2 - x - 2 = 0$ (D) None of these

Sol. *From the given equation*

$$\alpha + \beta = \frac{-1}{2},\ \alpha\beta = \frac{-1}{2}$$

The required equation is

$$x^2 - \left(\frac{1}{\alpha} + \frac{1}{\beta}\right)x + \frac{1}{\alpha\beta} = 0 \quad \Rightarrow x^2 - \left(\frac{\alpha+\beta}{\alpha\beta}\right)x + \frac{1}{\alpha\beta} = 0 \Rightarrow x^2 - \left(\frac{\frac{-1}{2}}{\frac{-1}{2}}\right)x + \frac{1}{\frac{-1}{2}} = 0$$

$\Rightarrow \quad x^2 - x - 2 = 0$ **[Ans. C]**

Short cut : Replace x by $\frac{1}{x} \Rightarrow \quad 2\left(\frac{1}{x}\right)^2 + \frac{1}{x} \quad 1 = 0 \Rightarrow x^2 - x - 2 = 0$

11. CONDITION OF COMMON ROOTS :

11.1 Condition for one common root :

Let $a_1x^2 + b_1x + c_1 = 0$ and $a_2x^2 + b_2x + c_2 = 0$ have a common root α.

Hence $a_1\alpha^2 + b_1\alpha + c_1 = 0$

$a_2\alpha^2 + b_2\alpha + c_2 = 0$

by cross multiplication

$$\frac{\alpha^2}{b_1c_2 - b_2c_1} = \frac{\alpha}{a_2c_1 - a_1c_2} = \frac{1}{a_1b_2 - a_2b_1}$$

by cross multiplication

$$\frac{\alpha^2}{b_1c_2 - b_2c_1} = \frac{\alpha}{a_2c_1 - a_1c_2} = \frac{1}{a_1b_2 - a_2b_1}$$

$\therefore \quad \alpha = \frac{b_1c_2 - b_2c_1}{a_2c_1 - a_1c_2} = \frac{a_2c_1 - a_1c_2}{a_1b_2 - a_2b_1}$ Which is the required condition.

This is also the condition that the two quadratic functions $a_1x^2 + b_1x\,y + c_1y^2$ and $a_2x^2 + b_2x\,y + c_2y^2$ may have a common factor.

11.2 Condition for both the common roots :

If both roots of the given equations are common then $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$.

Illustration :

Find the value of k for which the equations $3x^2 + 4kx + 2 = 0$ and $2x^2 + 3x - 2 = 0$ have a common root.

Sol. $2x^3 + 3x - 2 \quad \Rightarrow \quad x = -2$ *or* $\frac{1}{2}$

when $x = -2$ is common root.

$\Rightarrow \quad 12 - 8k + 2 = 0 \rightarrow 8k = 14 \quad \Rightarrow \quad k = \frac{7}{4}$

when $x = \frac{1}{2}$ *is a common root.*

$$\Rightarrow \quad \frac{3}{4} + 2k + 2 = 0 \Rightarrow 2k = -\frac{11}{4} \quad \Rightarrow \quad k = \frac{11}{8}$$

Illustration :

If the quadratic equation $x^2 + bx + c = 0$ *and* $x^2 + cx + b = 0$ $(b \neq c)$ *have a common root then prove that their uncommon roots are the roots of the equation* $x^2 + x + bc = 0$

Sol. $\left.\begin{array}{l} x^2 + bx + c = 0 \\ x^2 + cx + b = 0 \end{array}\right|$ *have common root*

$\Rightarrow \quad (b - c)x = b - c \quad \Rightarrow \quad x = 1$ *is the common root.*

Let $\quad$ *L be the root of* $x^2 + bx + c = 0$

$\therefore \quad$ *Product of roots* $= (1)(\alpha) = c \quad \Rightarrow \quad \alpha = C$

Similarly, Let β *be the other root of* $x^2 + cx + b = 0$

$\Rightarrow \quad \beta = b$

Sum of uncommon roots $= \alpha + \beta = b + c$

Product of uncommon roots $= \alpha\beta = bc$

Also, $\quad$ *1 is common root* $\Rightarrow$ *It must satisfy both the equations* $\Rightarrow 1 + bt + c = 0 \Rightarrow b + c = -1$

$\therefore \quad$ *Required equation is* $x^2 - (b + c)x + bc = 0$

Also, $\quad$ *1 is common root* $\Rightarrow$ *It must satisfy both the equations* $\Rightarrow 1 + bt + c = 0 \Rightarrow b + c = -1$

$\therefore \quad$ *Required equation is* $x^2 - (b + c)x + bc = 0$

$\Rightarrow \quad x^2 + x + bc = 0$

Illustration :

If $Q_1(x) = x^2 + (k - 29)x - k$ *and* $Q_2(x) = 2x^2 + (2k - 43)x + k$ *both are factors of a cubic polynomial P(x), then the largest value of k is*

(A) 0 $\qquad$ *(B) 33* $\qquad$ *(C) 23* $\qquad$ *(D) 30*

Sol. *Two quadratic polynomials can be a factor of cubic polynomial only when they have atleast one root common*

$\Rightarrow \quad x^2 + (b - 2a)x - k = 0 \qquad \ldots (1)$

and $\quad 2x^2 + (2k - 43)x + k = 0 \qquad \ldots (2)$ *Must have a common roots*

Multiple equation (1) by 2 and subtracting, we get

$$15x + 3k = 0 \Rightarrow x = \frac{-k}{5} \text{ is the common root}$$

and it must satisfy equation (1)

$$\Rightarrow \quad \frac{k^2}{25} + (k - 29)\left(-\frac{k}{5}\right) - k = 0$$

$$\Rightarrow \quad \left(-\frac{k}{5}\right)\left[-\frac{k}{5} + k - 29 + 5\right] = 0$$

$$\Rightarrow \quad k = 0 \text{ or } k = 30$$

Practice Problem

Q.1 If α and β are roots of $2x^2 - 7x + 6 = 0$, then the quadratic equation whose roots are $-\frac{2}{\alpha}, -\frac{2}{\beta}$ is

(A) $3x^2 + 7x + 4 = 0$ (B) $3x^2 - 7x + 4 = 0$

(C) $6x^2 + 7x + 2 = 0$ (D) $6x^2 - 7x + 2 = 0$

Q.2 If roots of quadratic equation $ax^2 + bx + c = 0$ are α and β then symmetric expression of its roots is

(A) $\frac{\alpha}{\beta} + \frac{\beta^2}{\alpha}$ (B) $\alpha^2\beta^{-2} + \alpha^{-2}\beta$ (C) $\alpha^2\beta + 2\alpha\beta^2$ (D) $\left(\alpha + \frac{1}{\alpha}\right)\left(\beta + \frac{1}{\beta}\right)$

Q.3 If $\alpha \neq \beta$ and $\alpha^2 = 5\alpha - 3$ and $\beta^2 = 5\beta - 3$, find the equation whose roots are α/β and β/α.

Q.4(a) Find the value of p and q if the equation $px^2 + 5x + 2 = 0$ and $3x^2 + 10x + q = 0$ have both roots is common.

(b) If the equation $x^2 - 4x + 5 = 0$ and $x^2 + ax + b = 0$ have a common root find a and b, where $a, b \in R$.

Q.5 If the equations $4x^2 \sin^2\theta - (4\sin\theta)x + 1 = 0$ and

common.

(b) If the equation $x^2 - 4x + 5 = 0$ and $x^2 + ax + b = 0$ have a common root find a and b, where $a, b \in R$.

Q.5 If the equations $4x^2 \sin^2\theta - (4\sin\theta)x + 1 = 0$ and $a^2(b^2 - c^2)x^2 + b^2(c^2 - a^2)x + c^2(a^2 - b^2) = 0$ have a common root and the 2nd equation has equal root find the possible values of θ in $(0, \pi)$.

Q.6 If the quadratic equations $x^2 + ax + 12 = 0$ and $x^2 + bx + 15 = 0$ and $x^2 + (a+b)x + 36 = 0$ have a common positive root find a and b and the root of the equation.

Q.7 If $a, b, c \in R$ and equations $ax^2 + bx + c = 0$ and $x^2 + 2x + 9 = 0$ have a common root, then find a : b : c.

Answer key

Q.1 A

Q.2 B

Q.3 $3x^2 - 19x + 3$

Q.4 (a) $p = \frac{3}{2}; q = 4$; (b) $a = -4, b = 5$

Q.5 $\theta = \frac{\pi}{6}$ or $\frac{5\pi}{6}$

Q.6 $a = -7,\ b = -8$; roots are (3, 4), (3, 5) and (3, 12).

Q.7 $a : b : c = 1 : 2 : 9$

12. MAXIMUM AND MINMUM VALUES OF QUADRATIC AND RATIONAL FUNCTIONS :

12.1 $y = ax^2 + bx + c$ attains its maximum value or minimum value at the point with abscissa $x = -\dfrac{b}{2a}$ according as $a < 0$ or $a > 0$.

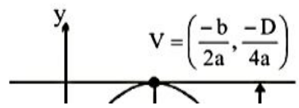
$$y = ax^2 + bx + c$$

$$y = a\left(x^2 + \frac{bx}{a} + \frac{c}{a}\right)$$

$$= a\left[x^2 + 2\cdot\frac{b}{2a}\cdot x + \frac{b^2}{4a^2} - \frac{b^2}{4a^2} + \frac{c}{a}\right]$$

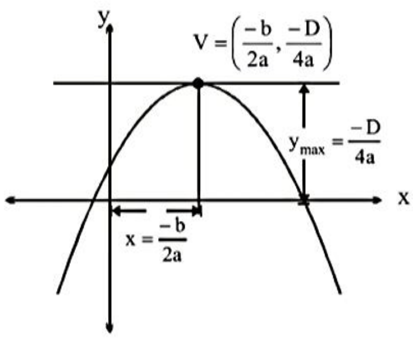
$$= a\left[\left(x + \frac{b}{2a}\right)^2 + \frac{c}{a} - \frac{b^2}{4a^2}\right]$$

$$= a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a}$$



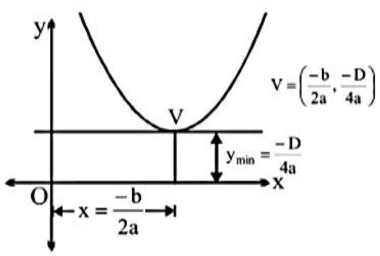
$$= a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a}$$

$$= a\left(x + \frac{b}{2a}\right)^2 - \frac{D}{4a}$$



Now, If $a < 0$, then $y_{max} = \dfrac{-D}{4a}$ and it occurs at $x = \dfrac{-b}{2a}$.

If $a > 0$, then $y_{min} = \dfrac{-D}{4a}$ and it occurs at $x = \dfrac{-b}{2a}$.



where $D = b^2 - 4ac$

Note:

Maximum or minimum value can also be obtained by making a perfect square and then taking an interpretation.

Illustration :

Find the maximum value of $f(x) = -3x^2 + 6x + 5$.

Sol. *Since,* $a < 0$

$$\therefore f(x)_{max} = \frac{-D}{4a} = -\frac{36-4(-3)5}{4(-3)} = \frac{36+60}{12} = 8$$

Aliternative method :

$f(x) = -3(x^2 - 2x + 1) + 5 + 3 = -3(x-1)^2 + 8$

Clearly $f(x)_{max} = 8$ *at* $x = 1$.

Illustration :

Let $P(x) = ax^2 + bx + 8$ *is a quadratic polynomial. If the minimum value of* $P(x)$ *is 6 when* $x = 2$, *find the values of a and b.*

Sol. $-\frac{b}{2a} = 2$

$4a = -b$ ...(i)

$P(2) = 4a + 2b + 8 = 6$

2a

$4a = -b$ ...(i)

$P(2) = 4a + 2b + 8 = 6$

$4a + 2b = -2$ ...(ii)

using (i) & (ii)

$b = -2, \quad a = \frac{1}{2}$

Illustration :

For $x \geq 0$, *what is the smallest possible value of the expression* $\log_{10}(x^3 - 4x^2 + x + 26) - \log_{10}(x + 2)$?

Sol. $y = \log_{10}(x^3 - 4x^2 + x + 26) - \log_{10}(x + 2)$

$x^3 - 4x^2 + x + 26 > 0, \qquad x + 2 > 0$

$(x + 2)(x^2 - 6x + 13) > 0$

$x + 2 > 0, \qquad x^2 - 6x + 13 > 0$

since $x^2 - 6x + 13$ *is always positive*

therefore $x > -2$

$$y = \log \frac{(x+2)(x^2-6x+13)}{(x+2)}$$

$= \log_{10}(x^2 - 6x + 13)$

$y = \log_{10}[(x-3)^2 + 4]$

$y_{min} = \log_{10} 4$

12.2 Range of functions expressed in the form of $\frac{P(x)}{Q(x)}$ where P(x) and Q(x) are either linear or quadratic polynomials.

TYPE-1 : $y = \frac{ax+b}{px+q}\ \frac{(\text{Linear})}{(\text{Linear})}$.

Illustration :

Find the range of the function. $\quad y = \frac{3x+2}{x-1}, \quad x \neq 1$

Sol. $y = \frac{3x-3+5}{x-1}$

$\Rightarrow \quad y = 3 + \frac{5}{x-1} \qquad \Rightarrow \quad y - 3 = \frac{5}{x-1}$

$\Rightarrow \quad x - 1 = \frac{5}{y-3} \qquad \Rightarrow \quad x = \frac{5}{y-3} + 1 = \frac{5+y-3}{y-3}$

$\Rightarrow \quad x - 1 = \frac{5}{y-3} \qquad \Rightarrow \quad x = \frac{5}{y-3} + 1 = \frac{5+y-3}{y-3}$

$\Rightarrow \quad x = \frac{y+2}{y-3}, \quad y \neq 3$

for $\quad y = 3 \quad$ *x is not defined*

$\therefore \quad$ *range is* $R - \{3\}$

TYPE-2 $\quad y = \frac{ax+b}{px^2+qx+r}\ \frac{(\text{linear})}{(\text{quadratic})}$

Illustration :

If x is real then find the range of the function $y = \frac{x+2}{x^2+3x+6}$

Sol. $y = \frac{x+2}{x^2+3x+6}$

$\Rightarrow \quad x^2y + 3xy + 6y = x + 2$

$\Rightarrow \quad x^2y + x(3y-1) + 6y - 2 = 0$

$\because \quad$ *x is real* $\quad \therefore \ D \geq 0$

$\Rightarrow \quad (3y-1)^2 - 4y\,(6y-2) \geq 0$

$\Rightarrow \quad 9y^2 - 6y + 1 - 24y^2 + 8y \geq 0$

$\Rightarrow \quad -15y^2 + 2y + 1 \geq 0$

$\Rightarrow \quad 15y^2 - 2y - 1 \leq 0$

$\Rightarrow \quad (5y + 1)(3y - 1) \leq 0$

$$\Rightarrow \quad y \in \left[-\frac{1}{5}, \frac{1}{3}\right]$$

TYPE-3 $\quad y = \dfrac{ax^2 + bx + c}{px^2 + qx + r}\left(\dfrac{\text{Quadratic}}{\text{Quadratic}}\right)$

Illustration :

If x is real then prove that $y = \dfrac{x^2 - 3x + 4}{x^2 + 3x + 4}$ *lies from* $\dfrac{1}{7}$ *to 7.*

Sol. $\dfrac{x^2 - 3x + 4}{x^2 + 3x + 4} = y$

$x^2(y - 1) + 3x(y + 1) + 4(y - 1) = 0 \qquad \ldots(i)$

For a quadratic equation, co-efficient of $x^2 \neq 0$

$\therefore \quad v \neq 1$

$x^2(y - 1) + 3x(y + 1) + 4(y - 1) = 0 \qquad \ldots(i)$

For a quadratic equation, co-efficient of $x^2 \neq 0$

$\therefore \quad y \neq 1$

$\because \quad$ *x is real* $\therefore D \geq 0$

$\Rightarrow \quad 9(y + 1)^2 - 16(y - 1)^2 \geq 0$

$\Rightarrow \quad -7y^2 + 50y - 7 \geq 0 \quad \Rightarrow \quad 7y^2 - 50y + 7 \leq 0$

$\Rightarrow \quad (7y - 1)(y - 7) \leq 0$

$\Rightarrow \quad y \in \left[\frac{1}{7}, 7\right]$ *but* $y = 1$ *is not included.*

If $\quad y = 1 \Rightarrow \dfrac{x^2 - 3x + 4}{x^2 + 3x + 4} = 1$

$\Rightarrow \quad 6x = 0 \quad \Rightarrow \quad x = 0$

$\therefore \quad y = 1$ *is also one of the values in the range.*

Hence, $y \in \left[\frac{1}{7}, 7\right]$

TYPE-4 $\quad y = \dfrac{ax^2 + bx + c}{px^2 + qx + r}\left(\dfrac{\text{Quadratic}}{\text{Quadratic}}\right)$

when P(x) and Q(x) has exactly one linear factor is common.

Illustration :

If x is real then find the range of the function $y = \dfrac{x^2 - 3x + 2}{x^2 + x - 6}$.

Sol. $y = \dfrac{x^2 - 3x + 2}{x^2 + x - 6}$

$\Rightarrow \quad \dfrac{(x-1)(x-2)}{(x+3)(x-2)}, \qquad x \neq 2$

$\Rightarrow \quad y = \dfrac{x-1}{x+3} \qquad \ldots(i)$

$\Rightarrow \quad xy + 3y = x - 1$

$\Rightarrow \quad x(1 - y) = 3y + 1 \qquad \Rightarrow \qquad x = \dfrac{3y+1}{1-y}, \qquad y \neq 1$

y is not defined at x = 2

$\therefore$ *on putting x = 2 in (i)*

$y = \dfrac{2-1}{2+3} = \dfrac{1}{5}$

Hence, range of the function is $R - \left\{\dfrac{1}{5}, 1\right\}$

$2+3 \quad 5$

Hence, range of the function is $R - \left\{\dfrac{1}{5}, 1\right\}$

TYPE-5 $\quad y = \dfrac{ax^2 + px + c}{px^2 + qx + r}$ when y takes all real values.

Illustration :

Prove that $y = \dfrac{(x+1)(x-2)}{x(x+3)}$ *can have any value in* $(-\infty, \infty)$ *for* $x \in R$

Sol. $y = \dfrac{(x+1)(x+2)}{x(x+3)}$

$\Rightarrow \quad x^2y + 3xy = x^2 - x - 2$

$\Rightarrow \quad x^2 (y - 1) + x (3y + 1) + 2 = 0$

$\because \quad$ *x is real* $\qquad \therefore \qquad D \geq 0$

$\Rightarrow \quad (3y + 1)^2 - 4.2 (y - 1) \geq 0$

$\Rightarrow \quad 9y^2 + 6y + 1 - 8y + 8 \geq 0$

$\Rightarrow \quad 9y^2 - 2y + 9 \geq 0$

$\Rightarrow \quad D < 0$

$\therefore \quad$ *It is true for all* $y \in R$

$\therefore \quad$ *Range of the given expression is R.*

Illustration :

Find all possible values of 'a' for which the expression $\dfrac{ax^2-7x+5}{5x^2-7x+a}$ *may be capable of all values, x being any real quantity.*

Sol. *Let* $y = \dfrac{ax^2-7x+5}{5x^2-7x+a}$

$\Rightarrow \quad 5x^2y - 7xy + ay = ax^2 - 7x + 5$

$\Rightarrow \quad x^2(5y - a) - 7x(y - 1) + ay - 5 = 0$

$\because \quad x$ is real $\quad \therefore \quad D \geq 0$

$\Rightarrow \quad 49(y - 1)^2 - 4(5y - a)(ay - 5) \geq 0$

$\Rightarrow \quad 49(y^2 - 2y + 1) - 4(5ay^2 - 25y - a^2y + 5a) \geq 0$

$\Rightarrow \quad y^2(49 - 20a) + 2y(1 + 2a^2) + 49 - 20a \geq 0$

Which is true for all $y \in R$

$\therefore \quad D \leq 0$ & *leady coefficient* $\quad 49 - 2a > 0$

$4(1 + 2a^2)^2 - 4(49 - 20a)^2 \leq 0$

$(1 + 2a^2 + 41 - 20a)(1 + 2a^2 - 49 + 20a) \leq 0$

$(a^2 - 10a + 25)(a^2 + 10a - 24) \leq 0$

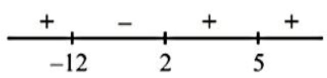
$\therefore \quad D \leq 0$ & *leady coefficient* $\quad 49 - 2a > 0$

$4(1 + 2a^2)^2 - 4(49 - 20a)^2 \leq 0$

$(1 + 2a^2 + 41 - 20a)(1 + 2a^2 - 49 + 20a) \leq 0$

$(a^2 - 10a + 25)(a^2 + 10a - 24) \leq 0$

$(a - 5)^2(a + 12)(a - 2) \leq 0$



$a \in [-12, 2] \cup \{5\}$

but $\quad a < \dfrac{49}{2}$

$\therefore \quad a \in [-12, 2]$

Now when $a = -12$

$$y = \frac{-12x^2-7x+5}{5x^2-7x-12} = \frac{-(12x^2+7x-5)}{5x^2-7x-12} = -\frac{(12x-5)(x+1)}{(5x-12)(x+1)}$$

Here $(x + 1)$ *is a common factor in numerator and denominator*

$\therefore \quad$ *y does not take all real numbers.*

Similarly for $a = 2$ *numerator and denominator contains a common linear factor and again y does not take all real numbers.*

Hence, $a \in (-12, 2)$

13. RESOLVING A GENERAL QUADRATIC EXPRESSION IN x AND y INTO TWO LINEAR FACTORS :

$$f(x, y) = ax^2 + 2bxy + by^2 + 2gx + 2fy + C$$

Writing the above equation as a quadrating equation is x,

$$ax^2 + 2x\,(hy + g) + by^2 + 2fy + C = 0$$

Solving for x, we get

$$x = \frac{-(hy+g) \pm \sqrt{(hy+g)^2 - a(by^2+2fy+c)}}{a}$$

$\Rightarrow \quad ax + hy + g = \pm\sqrt{y^2(h^2-ab)+2y(hg-af)+(g^2-ac)}$

Now f(x, y) can be writing as product of two linear factors only when quantity under radical sign is a perfect square.

As quantity under radical sign is a quadratic equation in y. Therefore, it will be perfect square only when D = 0

$\Rightarrow \quad (hg - af)^2 - (h^2 - ab)\,(g^2 - ac) = 0$

$\Rightarrow \quad abc + 2fgh - af^2 - bg^2 - ch^2 = 0$

which is the require condition

$\Rightarrow \quad (hg - af)^2 - (h^2 - ab)\,(g^2 - ac) = 0$

$\Rightarrow \quad abc + 2fgh - af^2 - bg^2 - ch^2 = 0$

which is the require condition

Illustration :

Prove that the expression $2x^2 + 3xy + y^2 + 2y + 3x + 1$ can be factorised into two linear factors. Find them.

Sol. $2x^2 + 3xy + y^2 + 2y + 3x + 1$

Comparing given equation with $ax^2 + 2hxy + by^2 + 2gx + 2fy + c$, we get

$$a = 2,\ h = \frac{3}{2},\ b = 1 g = \frac{3}{2},\ f = 1,\ c = 1$$

Clearly, $\quad abc + 2fgh - af^2 - bg^2 - ch^2 = 0$

$\Rightarrow$ *Given expression can be factorized into two linear factor*

To find the factors, form the quadratic in x

$$2x^2 + 3x\,(y + 1) + (y + 1)^2$$

$\Rightarrow \quad x = \dfrac{-3(y+1) \pm \sqrt{(y+1)^2}}{y} \qquad \Rightarrow \quad 4x = -3(y + 1) \pm (y + 1)$

$\Rightarrow \quad 4x = -2(y + 1) \qquad or \qquad 4x = -4(y + 1)$

$\Rightarrow \quad 2x + y + 1 = 0 \qquad or \qquad x + y + 1 = 0$

Illustration :

If the equation $x^2 + 16y^2 - 3x + 2 = 0$ is satisfied by real values of x and y then prove that $1 \le x \le 2$ and $-1/8 \le y \le 1/8$.

Sol. $x^2 - 3x + 16y^2 + 2 = 0$ As $x \in R \Rightarrow D \ge 0$

$\Rightarrow \quad 9 - 64y^2 - 8 \ge 0 \qquad \Rightarrow \quad 64y^2 - 1 \le 0$

$\Rightarrow \quad (8y - 1)(8y + 1) \le 0 \qquad \Rightarrow \quad y \in \left[-\frac{1}{8}, \frac{1}{8}\right]$

Again, $-(x^2 - 3x + 2) = 16y^2$

As R.H.S. $\ge 0 \qquad \Rightarrow \quad -(x^2 - 3x + 2) \ge 0$

$\Rightarrow \quad x^2 - 3x + 2 = 0$

$\Rightarrow \quad x \in [1, 2]$

Practice Problem

Q.1 If x is real, prove that the expression $y = \dfrac{x^2 + 2x - 11}{2(x-3)}$ can have all numerical values except which lie between 2 and 6.

Q.2 If x is real then find the range of $y = \dfrac{x^2 + 3x - 4}{x^2 + 7x + 12}$.

Q.3 Show that in the equation, $x^2 - 3xy + 2y^2 - 2x - 3y - 35 = 0$, for every real value of x there is a real

$x^2 + 7x + 12$

Q.3 Show that in the equation, $x^2 - 3xy + 2y^2 - 2x - 3y - 35 = 0$, for every real value of x there is a real
value of y, and for every value of y there is a real value of x.

Q.4 Find the range of the function $f(x) = 6^x + 3^x + 6^{-x} + 3^{-x} + 2$.

Answer key

Q.2 $R - \{1, 5\}$ **Q.4** $[4, \infty)$

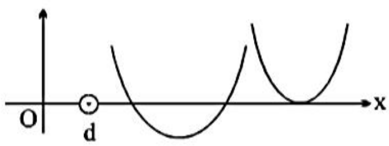
14. LOCATION OF ROOTS :

This article deals with an elegant approach of solving problems on quadratic equations when the roots are located / specified on the number line with variety of constraints :

Consider $f(x) = ax^2 + bx + c$ with $a > 0$.

TYPE–1 : Both roots of the quadratic equation are greater than a specified number say (d). The necessary and sufficient condition for this are :

(i) $a > 0$; (ii) $D \ge 0$; (iii) $f(d) > 0$; (iv) $-\dfrac{b}{2a} > d$



Note : If $a < 0$ then intercept accordingly.

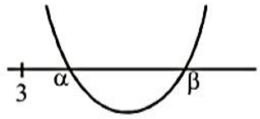
Illustration :

Find all the values of the parameter d for which both roots of the equation

$x^2 - 6dx + (2 - 2d + 9d^2) = 0$ *exceed the number 3.*

Sol. $x^2 - 6dx + (2 - 2d + 9d^2) = 0$

if both roots exceed 3



Conditions

(i) $D > 0$

$(6d)^2 - 4(2 - 2d + 9d^2) > 0$

$36d^2 - 8 + 8d - 36d^2 > 0$

$d > 1$

(ii) $f(3) > 0$

$9 - 18d + (2 - 2d + 9d^2) > 0$

$(9d - 11)(d - 1) > 0$

$d \in (-\infty, 1) \cup \left(\frac{11}{9}, \infty\right)$

$-b$

$9 - 18d + (2 - 2d + 9d^2) > 0$

$(9d - 11)(d - 1) > 0$

$d \in (-\infty, 1) \cup \left(\frac{11}{9}, \infty\right)$

(iii) $\frac{-b}{2a} > 3$

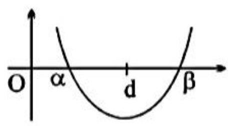
$-\left(\frac{-6d}{2}\right) > 3$

$d > 1$

Taking infersection of all the above three conditions we get

$d \in \left(\frac{11}{9}, \infty\right)$

TYPE–2: Both roots lie on either side of a fixed number say (d). Alternatively one root is greater than d and other less than d or d lies between the roots of the given equation.



Conditions for this

$$\left.\begin{array}{l}\text{(i) } a > 0 \\ \text{and (ii) } f(d) < 0\end{array}\right\} \text{ or } \begin{array}{ll}\text{(i)} & a < 0 \\ \text{(ii)} & f(d) > 0\end{array}$$

Note that no consideration for discriminant will be useful here.

Illustration :

Find the value of k for which one root of the equation of $x^2 - (k+1)x + k^2 + k - 8 = 0$ *exceed 2 and other is smaller than 2.*

Sol. $x^2 - (k+1)x + k^2 + k - 8 = 0$

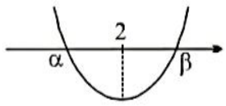
$a > 0$, hence $f(2) < 0$

$4 - (K+1)\,2 + k^2 + k - 8 < 0$

$k^2 - k - 6 < 0$

$(k-3)(k+2) < 0$

$k \in (-2, 3)$



TYPE–3 : Exactly one root lies in the interval (d, e) when d < e.

Conditions for this are :

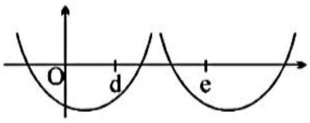
(i) $a \neq 0$;

(ii) $f(d) \cdot f(e) < 0$

(iii) An another case arises

when $f(d) \cdot f(e) = 0$

then we have to check end points



For f (d) = 0 i.e. one root is "d"

(iii) An another case arises

when $f(d) \cdot f(e) = 0$

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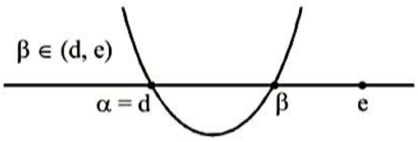
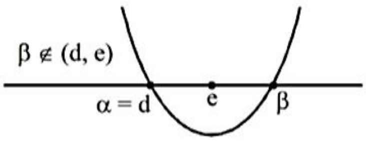
For $f(d) = 0$, i.e., one root is "d"





Check if other root lies between "d" and "e" or not. If yes then we will include that point otherwise we will exclude that point

similarly for $f(e) = 0$

we will check for the other root and find out if it lies between "d" and "e" or not.

Note : If $f(d)\,f(e) < 0$ then exactly one root lies in the interval (d, e) but not the converse.

Illustration :

Find the set of values of m for which exactly one root of the equation

$x^2 + mx + (m^2 + 6m) = 0$ *lie in* $(-2, 0)$ *[Ans.* $(-6, -2) \cup (-2, 0)$*]*

Sol. $x^2 + mx + (m^2 + 6m) = 0$

If exactly one root lies in $(-2, 0)$ *then* $f(-2)\,f(0) < 0$

$(m^2 + 4m + 4)(m^2 + 6m) < 0$

$m \in (-6, -2) \cup (-2, 0)$

We have to find out the conditions when one of the root is –2, or 0.

Case I : *if one root is –2*

then $f(-2) = 0$

$m = -2$

$x^2 - 2x - 8 = 0$

$x = 4, -2$, *no root lie in* $(-2, 0)$ *for* $m = -2$.

Case II : *If one root is zero.*

then $m = 0$, *or* -6

If $m = 0,$ $x^2 = 0$ *both the roots are zero and no root lies in* $(-2, 0)$

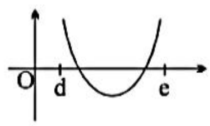
If $m = -6,$ $x = 0, 6$ *no root lies in* $(2, 0)$

Hence $m \in (-6, -2) \cup (-2, 0)$

TYPE–4 :

When both roots are confined between the number d and e (d < e). Conditions for this are

(i) $a > 0$; (ii) $D \geq 0$; (iii) $f(d) > 0$; (iv) $f(e) > 0$



$d < -\frac{b}{2a} < e$

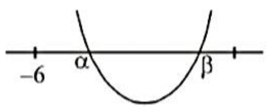
Illustration :

If α, β *are the roots of the quadratic equation*
$x^2 + 2(k-3)x + 9 = 0$ $(\alpha \neq \beta)$. *If* $\alpha, \beta \in (-6, 1)$ *then find the values of k.*

Sol. $x^2 + 2(k-3)x + 0 = 0$

$\alpha, \beta \in (-6, 1)$

Since heading coefficient is 1, hence



(i) $D \geq 0$

$4(k-3)^2 - 4 \times 9 \geq 0$

$(k-6)k \geq 0$

$k \in (-\infty, 0] \cup [6, \infty)$

(ii) $f(-6) > 0$

$36 - 12(k-3) + 9 > 0$

$36 + 36 - 12k + 9 > 0$

$12k < 81$

$k < \frac{27}{4}$

(iii) $f(1) > 0$

$1 + 2k - 3 + 9 > 0$

$2k > 6$

$k > 3$

(iv) $-6 < \frac{-b}{2a} < 1$

$-6 < 3 - k < 1$

$2 < k < 9$

Taking intersection of above four condition we get $k \in \left[6, \frac{27}{4}\right)$

TYPE–5 :

One root is greater than e and the other root is less than d.

Conditions are :

(i) f (d) < 0 and f (e) < 0 if (a > 0)

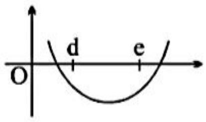
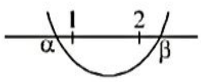


Illustration :

Illustration :

Find all the values of k for which one root of the quadratic equation $(k - 5)x^2 - 2kx + k - 4 = 0$ *is smaller than 1 and the other root exceed 2.*

Sol. $(k - 5)x^2 - 2kx + k - 4 = 0$



Case I *(i)* $k - 5 > 0$

$k > 5$

(ii) $f(1) < 0$

$(k - 5) - 2k + k - 4 < 0$

$-9 < 0$

(iii) $f(2) < 0$

$4(k - 5) - 4k + k - 4 < 0$

$k - 24 < 24$

$k < 24$

$k \in (5, 24)$

Case II : *(i)* $k - 5 < 0$

$k < 5$

(ii) $f(1) > 0$

$-9 > 0$ *that is not possible.*

Hence solution is $k \in (5, 24)$.

Practice Problem

Q.1 Let α be a real root of the quadratic equation $ax^2 + bx + c$ and β be a real root of the equation $-x^2 + bx + c = 0$. Show that there exists a root γ of the equation $\frac{a}{2}x^2 + bx + c = 0$ that lie between α and β. $(\alpha, \beta \neq 0)$.

Q.2 Find all the values of a for which both roots of the equation $x^2 + x + a = 0$ exceed the quantity a.

Q.3 Find the set of values of a for which zeroes of the quadratic polynomial $(a^2 + a + 1)\,x^2 + (a - 1)x + a^2$ are located on either side of 3.

Q.4 Find all possible values of a for which exactly one root of the quadratic equation $x^2 - (a + 1)x + 2a = 0$ lie in the interval (0, 3).

Q.5 If $x^2 + 2ax + a < 0 \ \forall\ x \in [1, 2]$, then find the values of a.

Answer key

Answer key

Q.2 $(-\infty, -2)$ Q.3 ϕ Q.4 $(-\infty, 0] \cup (6, \infty)]$ Q.5 $a \in \left(-\infty, -\frac{4}{5}\right)$

15. THEORY OF EQUATIONS :

Relation between roots and coefficients of polynomial equation :

15.1 For Quadratic Equation :

If α and β are roots of a quadratic equation $ax^2 + bx + c = 0$ then

$ax^2 + bx + c = a\,(x - \alpha)\,(x - \beta)$

$\Rightarrow\ ax^2 + bx + c = a\,[x^2 - x\,(\alpha + \beta) + \alpha\beta]$

Comparing co-efficients on both sides, we get

$$-a\,(\alpha + \beta) = b \Rightarrow \alpha + \beta = \frac{-b}{a}$$

$$\alpha + \beta = c \Rightarrow\ \alpha\beta = \frac{c}{a}$$

15.2 For Cubic Equation :

If α, β and γ are roots of a cubic equation $ax^3 + bx^2 + cx + d = 0$ then

$$ax^3 + bx^2 + cx + d = a(x - \alpha)(x - \beta)(x - \gamma)$$

$$ax^3 + bx^2 + cx + d = a[x^3 - (\Sigma \alpha) x^2 + (\Sigma \alpha\beta)x^2 - \alpha\beta\gamma]$$

Comparing co-efficients on both sides, we get

$$\therefore \quad \alpha + \beta + \gamma = -\frac{b}{a}$$

$$\alpha\beta + \beta\gamma + \gamma\alpha = \frac{c}{a}$$

and $\quad \alpha\beta\gamma = -\frac{d}{a}$

15.3 For Bi-quadratic Equation :

If α, β, γ and δ are roots of a bi-quadratic equation $ax^4 + bx^3 + cx^2 + dx + e = 0$ then

$$ax^4 + bx^3 + cx^2 + dx + e = a(x - \alpha)(x - \beta)(x - \gamma)(x - \delta)$$

$$ax^4 + bx^3 + cx^2 + dx + e = a[x^4 - (\Sigma \alpha) x^3 + (\Sigma \alpha\beta)x^2 - (\Sigma \alpha\beta\gamma)x + \alpha\beta\gamma\delta]$$

Comparing co-efficients on both sides, we get

$$\therefore \quad \alpha + \beta + \gamma + \delta = \frac{-b}{a}$$

$$\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta = \frac{c}{a}$$

a

$$\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta = \frac{c}{a}$$

$$\alpha\beta\gamma + \beta\gamma\delta + \gamma\delta\alpha + \delta\alpha\beta = \frac{-d}{a}$$

and $\quad \alpha\beta\gamma\delta = \frac{e}{a}$

NOTE:

A polynomial equations of degree odd with real coefficient must have at least one real root as imaginary roots always occur in pair of conjugates.

Illustration :

Find the

(i) sum of the squares and

(ii) sum of the cubes of the roots of the cubic equation, $x^3 - px^2 + qx - r = 0$ α, β, γ

Sol. *Given* $x^3 - px^2 + qx - r = 0$

Let the root be α, β, γ

$\alpha + \beta + \gamma = p, \qquad \Sigma\alpha\beta = q$

$\alpha\beta\gamma = r$

(i) $\alpha^2 + \beta^2 + \gamma^2 = (\alpha + \beta + \gamma)^2 - 2\Sigma\alpha\beta$

$= p^2 - 2q$

(ii) $\alpha^3 + \beta^3 + \gamma^3 = 3\alpha\beta\gamma + (\alpha + \beta + \gamma)\,(\alpha^2 + \beta^2 + \gamma^2 - \alpha\beta - \beta\gamma - \gamma\alpha)$

$= 3\alpha\beta\gamma + (\alpha + \beta + \gamma)\,[(\alpha + \beta + \gamma)^2 - 3\Sigma\alpha\beta]$

$= 3r + p[p^2 - 3q]$

Illustration :

Solve the cubic $4x^3 + 16x^2 - 9x - 36 = 0$, the sum of its two roots being equal to zero.

Sol. Given $4x^3 + 16x^2 - 9x - 36 = 0$

Let the roots of equation be $\alpha_1 - \alpha_1\beta$

$$\alpha - \alpha + \beta = \frac{-16}{4}$$

$$\beta = -4$$

Product of roots $\alpha\,(-\alpha)\,(\beta) = \dfrac{-(-36)}{4}$

$$\alpha^2 = \frac{9}{4} \qquad \Rightarrow \qquad \alpha = \pm\frac{3}{2}$$

Product of roots $\alpha\,(-\alpha)\,(\beta) = \dfrac{-(-36)}{4}$

$$\alpha^2 = \frac{9}{4} \qquad \Rightarrow \qquad \alpha = \pm\frac{3}{2}$$

roots are $-\dfrac{3}{2}, \dfrac{3}{2}, -4.$

Illustration :

If a, b, c are the roots of cubic $x^3 - x^2 + 1 = 0$ then find the value of $a^{-2} + b^{-2} + c^{-2}$.

Sol. $x^2 - x^2 + 1 = 0$ if a, b, c are roots

then $a + b + c = 1$, $ab + bc + ca = 0$, $abc = 1$

$$\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} = \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)^2 - 2\left(\frac{1}{ab} + \frac{1}{bc} + \frac{1}{ca}\right)$$

$$= \left(\frac{ab + bc + ca}{abc}\right)^2 - \left(\frac{a + b + c}{abc}\right)$$

$$= 0 - 2\left(\frac{1}{-1}\right) = 2$$

Illustration :

If a polynomial is defined as $P(x) = 2x^5 + ax^4 + bx^3 + cx^2 + dx + e$ such that $P(0) = 4$, $P(1) = 5$, $P(2) = 8$, $P(3) = 13$ and $P(4) = 20$. Find the value of $P(5)$

Sol. *Consider the polynomial*

$$P(x) = Q(x) + x^2 + 4$$

$$P(0) = Q(0) + 4 = 4$$

$$Q(0) = 0$$

$$P(1) = Q(1) + 5 \quad \Rightarrow \quad Q(1) = 0$$

Similarly $Q(2) = 0$, $Q(3) = 0$ and $Q(4) = 0$

hence, $Q(x) = 2x\,(x-1)\,(x-2)\,(x-3)\,(x-4)$

$$\therefore \quad P(x) = 2x\,(x-1)\,(x-2)\,(x-3)\,(x-4) + x^2 + 4$$

$$P(5) = 2\cdot5\cdot4\cdot3\cdot2\cdot1 + 25 + 4 = 269$$

Practice Problem

Q.1 $\alpha, \beta, \gamma, \delta$ are the roots of the equation $\tan\left(\frac{\pi}{4} + x\right) = 3\tan 3x$ no two of which have equal tangents, find

Practice Problem

Q.1 $\alpha, \beta, \gamma, \delta$ are the roots of the equation $\tan\left(\frac{\pi}{4} + x\right) = 3\tan 3x$ no two of which have equal tangents, find the value of $\tan\alpha + \tan\beta + \tan\gamma + \tan\delta$.

Q.2 Find the cubic each of whose roots is greater by unity than a root of the equation

$$x^3 - 5x^2 + 6x - 3 = 0.$$

Q.3 Form a cubic whose roots are the cubes of the roots of $x^3 + 3x^2 + 2 = 0$.

Q.4 The length of the sides of a triangle are the 3 distinct roots of the equation $4x^3 - 24x^2 + 47x - 30 = 0$, If the area of triangle is Δ, find the value of 100Δ.

Answer key

Q.1 Zero   Q.2 $y^3 - 8y^2 + 19y - 1 = 0$ Q.3 $y^3 + 33y^2 + 12y + 8 = 0$

Q.4 150 where $\Delta = 3/2$

16. LOG INEQUALITIES :

(1) For $a > 1$, If $\log_a x > \log_a y$, then $\Rightarrow$ $x > y$ that is if base is greater than unity the inequality remains unchanged when log is removed.

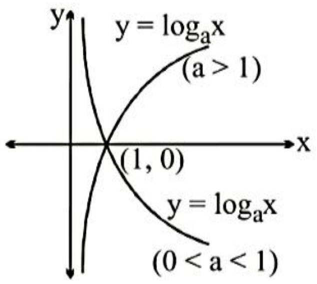
(2) For $0 < a < 1$. If $\log_a x > \log_a y$, then $\Rightarrow$ $x < y$ that is if base is less than unity then inequality sign is reversed when log is removed.

(3) If $a > 1$. $\log_a x < P \Rightarrow x < a^P$.

(4) If $a > 1$. $\log_a x > P \Rightarrow x > a^P$.

(5) If $a < 1$. $\log_a x < P \Rightarrow x > a^P$.

(6) If $a < 1$. $\log_a x > P \Rightarrow x < a^P$.



$\Rightarrow$ If base is less than unity then value of log x decreases as x-increases.

$\Rightarrow$ If base is greater than unity then value of log x increases as x increases.

To solve log inequality when base is constant :

(1) First define log i.e. find the condition that base is positive, expression inside log is positive and base is not

To solve log inequality when base is constant :

(1) First define log i.e. find the condition that base is positive, expression inside log is positive and base is not equal to unity. Term it as initial condition

(2) Check weather base is greater than unity or less than unity.

(3) If base is greater than unity then remove the log without changing the inequality and if base is less than unity then reverse the inequality when log is removed.

(4) Solve the inequality and term it as final condition.

(5) Take the intersection of initial and final conditions.

Illustration :

$$\log_{\frac{1}{3}}(5x-1) > 0$$

Sol. *First define logarithm*

$$5x - 1 > 0$$

$$5x > 1$$

$$x > \frac{1}{5} \rightarrow \quad \textit{(initial condition)}$$

Now as base is less than unity then on removing log, inequality will be reversed $\log_{\frac{1}{3}}(5x-1) > 0$

$$5x - 1 < \left(\frac{1}{3}\right)^0$$

$$5x - 1 < 1$$

$$x < \frac{2}{5} \rightarrow \quad \textit{(final condition)}$$

Taking intersection of initial and final condition, we get $x \in \left(\frac{1}{5}, \frac{2}{5}\right)$. ***Ans.***

Illustration :

$\log_{0.5}(x^2 - 5x + 6) > -1$

Sol. *Expression inside log should be positive.*

$x^2 - 5x + 6 > 0$

$(x - 2)(x - 3) > 0$

$x < 2$ *or* $x > 3$

$x \in (-\infty, 2) \cup (3, \infty) \rightarrow$ *(initial condition)*

Now, $x^2 - 5x + 6 < (0.5)^{-1}$ *(inequality is reversed since base is less than one.)*

$x^2 - 5x + 6 < 2$

$x \in (-\infty, 2) \cup (3, \infty) \rightarrow$ *(initial condition)*

Now, $x^2 - 5x + 6 < (0.5)^{-1}$ *(inequality is reversed since base is less than one.)*

$x^2 - 5x + 6 < 2$

$x^2 - 5x + 4 < 0$

$(x - 1)(x - 4) < 0$

$1 < x < 4 \rightarrow$ *(final condition)*

Taking intersection of initial condition and final condition we get $x \in (1, 2) \cup (3, 4)$. ***Ans.***

Illustration :

$\log^2_{0.5} x + \log_{0.5}x - 2 \le 0$

Sol. *Put* $\log_{0.5}x = t$

$t^2 + t - 2 \le 0$

$(t + 2)(t - 1) \le 0$

$-2 \le t \le 1$

$-2 \le \log_{0.5}x \le 1$

$\Rightarrow \quad x > 0 \rightarrow$ *(initial condition)*

Now, since base is less than unity hence inequality will be reversed when log is removed.

$(0.5)^{-2} \ge x \ge (0.5)^1$

$4 \ge x \ge 0.5$

$x \in [0.5, 4] \rightarrow$ *(final condition)*

Taking intersection of initial and final condition, we get $x \in [0.5, 4]$. ***Ans.***

To solve log inequality when base is also variable :

(1) Define log i.e. find the condition that expression inside log is positive, base is positive and base is not equal to unity. Term it as initial condition.

(2) Take the case I, when base is greater than unity call it condition "I".

(3) Solve the inequality as per case I that is remove the log without changing the inequality and term it as condition I(a).

(4) Take the intersection of condition I, condition I(a) and initial condition and term it as condition A.

(5) Take the case II, when base is less than unity, term it as condition II.

(6) Solve the inequality as per case II i.e. reverse the inequality on removing the log and term it as condition II(a).

(7) Take the intersection of condition II, condition II(a) and initial condition. Call it condition B.

(8) Take the union of condition formed in step 4 and condition formed in step 7 that is find the union of "A" and "B".

Illustration :

Illustration :

$$log_x(x^3 - x^2 - 2x) < 3$$

Sol. *Defining log*

$$x^3 - x^2 - 2x > 0$$

$$x (x^2 - x - 2) > 0$$

$$x (x - 2) (x + 1) > 0$$

$$x \in (-1, 0) \cup (2, \infty)$$

Base > 0

$$x > 0$$

$$x \in (2, \infty) \quad \rightarrow \quad (\text{initial condition})$$

Case I : *when base is greater than 1.*

$$x > 1 \quad \text{condition } I$$

$$x^3 - x^2 - 2x < x^3$$

$$x^2 + 2x > 0$$

$$x (x + 2) > 0$$

$$x \in (-\infty, -2) \cup (0, \infty) \quad \rightarrow \quad \text{condition } I(a)$$

Taking intersection of initial condition, condition I and condition I(a)
we get $x \in (2, \infty) \rightarrow$ *(condition "A").*

Case II : *when base is less than 1*

$0 < x < 1$ condition II

$\log_x(x^3 - x^2 - 2x) < 3$

$x^3 - x^2 - 2x > x^3$

$x^2 + 2x < 0$

$x(x + 2) < 0$

$-2 < x < 0$ condition II (a)

Taking intersection of initial condition, condition II and condition II(a)
we get $x \in \phi \rightarrow$ *(condition "B").*
Taking union of condition "A" and condition "B" we get $x \in (2, \infty)$*]*

Illustration :

$\log_{2x}(x^2 - 5x + 6) < 1$

Sol. $x^2 - 5x + 6 > 0$

$\log_{2x}(x^2 - 5x + 6) < 1$

Sol. $x^2 - 5x + 6 > 0$

$(x - 2)(x - 3) > 0$

$x \in (-\infty, 2) \cup (3, \infty)$

$2x > 0$

$x > 0$

$x \in (0, 2) \cup (3, \infty) \rightarrow$ *(initial condition)*

Case I $1 < 2x$

$\frac{1}{2} < x \rightarrow$ *(condition I)*

Since base is greater than 1.
inequality remains unchanged

$x^2 - 5x + 6 < (2x)^1$

$x^2 - 7x + 6 < 0$

$(x - 6)(x - 1) < 0$

$1 < x < 6 \rightarrow$ condition I(a)

taking intersection of initial condition, condition I and condition I(a),
we get $x \in (1, 2) \cup (3, 6) \rightarrow$ *condition (A)*

Case II $\quad 0 < 2x < 1$

$$0 < x < \frac{1}{2} \rightarrow \text{(condition II)}$$

$$x^2 - 5x + 6 > 2x$$
$$x^2 - 7x + 6 > 0$$
$$(x - 6)(x - 1) > 0$$
$$x \in (-\infty, 1) \cup (6, \infty) \rightarrow \quad \text{(condition II(a)}$$

Taking intersection of initial condition, condition II and condition II(a),

we get $x \in \left(0, \frac{1}{2}\right) \rightarrow$ *condition B*

Taking union of condition "A" and condition "B", we get $x \in \left(0, \frac{1}{2}\right) \cup (1, 2) \cup (3, 6)$. ***Ans.***

Practice Problem

Q.1 $\quad \log_{0.5}\left(\log_6 \frac{x^2 + x}{x + 4}\right) < 0.$

Q.2 $\quad \log_{x^2} \frac{4x - 5}{|x - 2|} \geq \frac{1}{2}.$

Practice Problem

Q.1 $\quad \log_{0.5}\left(\log_6 \frac{x^2 + x}{x + 4}\right) < 0.$

Q.2 $\quad \log_{x^2} \frac{4x - 5}{|x - 2|} \geq \frac{1}{2}.$

Q.3 $\quad 2\log_5 x - \log_x 125 < 1.$

Q.4 $\quad \log_{x^2}(2 + x) < 1.$

Q.5 $\quad \left(\frac{1}{2}\right)^{\log_2(x^2 - 1)} > 1.$

Answer key

Q.1 $\quad x \in (-4, -3) \cup (8, \infty)$

Q.2 $\quad x \in [\sqrt{6} - 1, 2) \cup (2, 5].$

Q.3 $\quad x \in \left(0, \frac{1}{5}\right) \cup \left(1, \sqrt{125}\right)$

Q.4 $\quad x \in (-2, -1) \cup (2, \infty)$

Q.5 $\quad x \in (-\sqrt{2}, -1) \cup (1, \sqrt{2})$

Solved Examples

Q.1 Find the values of 'x' for which the inequality $-1 \le \dfrac{3x^2 - 7x + 8}{x^2 + 1} \le 1$, is satisfied.

Sol. $-1 \le \dfrac{3x^2 - 7x + 8}{x^2 + 1} \le 1$

$\Rightarrow \quad \dfrac{3x^2 - 7x + 8}{x^2 + 1} \ge -1 \quad$ and $\quad \dfrac{3x^2 - 7x + 8}{x^2 + 1} \le 1$

$\Rightarrow \quad 4x^2 - 7x + 9 \ge 0 \quad$ and $\quad 2x^2 - 7x + 7 \le 0$

$\Rightarrow \quad x \in R \quad$ and $\quad x \in \phi$

Taking $x \in \phi$.

Q.2 $(x^2 + 3x + 1)(x^2 + 3x - 3) \ge 5$

Sol. $(x^2 + 3x + 1)(x^2 + 3x - 3) \ge 5$

Let $x^2 + 3x = t$

$\Rightarrow \quad (t + 1)(t - 3) \ge 5 \Rightarrow t^2 - 2t - 8 \ge 0$

$\Rightarrow \quad (t - 4)(t + 2) \ge 0$

$\Rightarrow \quad t \in (-\infty, -2] \cup [4, \infty)$

$\Rightarrow \quad x^2 + 3x \le -2 \quad$ or $\quad x^2 + 3x \ge 4$

$\Rightarrow \quad x^2 + 3x + 2 \le 0 \quad$ or $\quad x^2 + 3x \ge 4$

$\Rightarrow \quad (x + 1)(x + 2) \le 0 \quad$ or $\quad (x + 4)(x - 1) \ge 0$

$\Rightarrow \quad x \in [-2, -1] \quad$ or $\quad x \in (-\infty, -4] \cup [1, \infty)$

$\Rightarrow \quad x^2 + 3x + 2 \le 0 \quad$ or $\quad x^2 + 3x \ge 4$

$\Rightarrow \quad (x + 1)(x + 2) \le 0 \quad$ or $\quad (x + 4)(x - 1) \ge 0$

$\Rightarrow \quad x \in [-2, -1] \quad$ or $\quad x \in (-\infty, -4] \cup [1, \infty)$

Taking union

$x \in (-\infty, -4] \cup [-2, -1] \cup [1, \infty)$. Ans.

Q.3 If both the roots of the equation $x^2 - 6ax + 2 - 2a + 9a^2 = 0$ exceed 3, then

(A) $a < 1$ (B) $a > \dfrac{11}{9}$ (C) $a > \dfrac{3}{2}$ (D) $a < \dfrac{5}{2}$

Sol. The quadratic equation $f(x) = x^2 - 6ax + 2 - 2a + 9a^2 = 0$ ......(1)

will have real roots if $D = 36a^2 - 4(2 - 2a + 9a^2) \ge 0$

$\Rightarrow \quad -8(1 - a) \ge 0 \quad$ or $\quad a \ge 1$ ......(2)

The roots of (1) will exceed 3 if

$$\frac{-b}{2a} = -\left(\frac{-6a}{2}\right) = 3a > 3 \text{ or } a > 1 \qquad ......(3)$$

and $\quad f(3) = 9 - 18a + 2 - 2a + 9a^2 > 0$

$\Rightarrow \quad 9a^2 - 20a + 11 > 0 \quad \Rightarrow (9a - 11)(a - 1) > 0 \quad \Rightarrow \left(a - \dfrac{11}{9}\right)(a - 1) > 0$

$\Rightarrow \quad a < 1 \text{ or } a > \dfrac{11}{9}$ ......(4)

Thus (2), (3) and (4) will hold simultaneously if $a > \dfrac{11}{9}$. Ans.

Q.4 If one root of the equation $x^2 + ax + b = 0$ is also a root of $x^2 + mx + n = 0$, show that its other root is a root of $x^2 + (2a - m)x + a^2 - am + n = 0$.

Sol. Let α be a root of the equation $x^2 + ax + b = 0$ which is also a root of $x^2 + mx + n = 0$.

Let β be the other root of $x^2 + ax + b = 0$, then $\alpha + \beta = -a$.

We have $\alpha = -a - \beta$

Since a is a root of $x^2 + mx + n = 0$, we get

$$(-a - \beta)^2 + m(-a - \beta) + n = 0$$

or $\quad \beta^2 + 2a\beta + a^2 - ma - m\beta + n = 0$

or $\quad \beta^2 + (2a - m)\beta + a^2 - ma + n = 0$

Thus, β is a root of $x^2 + (2a - m)x + a^2 - ma + n = 0$. Ans.

Q.5 Let a, b, c be positive integers and consider all the quadratic equations of the form $ax^2 - bx + c = 0$ which have two distinct real roots in the open interval $0 < x < 1$. Find the least positive integer a for which such a quadratic equation exists.

Sol. Let α, β be two distinct real roots of $f(x) = ax^2 - bx + c = 0$ lying in $(0, 1)$.

Then $f(x) = a(x - \alpha)(x - \beta)$

Now, $\quad f(0)\, f(1) = a^2 \alpha (1 - \alpha) \beta (1 - \beta)$

Now, $\quad f(0)\, f(1) = a^2 \alpha (1 - \alpha) \beta (1 - \beta)$

As $\quad 0 < \alpha < 1,\ 0 < \alpha(1 - \alpha) \le \frac{1}{4}$, with equality holding for $\alpha = \frac{1}{2}$.

Since $\quad 0 < \alpha, \beta < 1$ and $\alpha \ne \beta$, $0 < \alpha(1 - \alpha)\beta(1 - \beta) < \frac{1}{16}$

$$\Rightarrow \quad 0 < f(0)\, f(1) < \frac{a^2}{16} \qquad ....(1)$$

As a, b, c are positive integers.

$$f(0)\, f(1) = c(a - b + c) \ge 1 \qquad [\because f(0)\, f(1) > 0] \qquad ....(2)$$

(1) and (2) imply $\frac{a^2}{16} > 1$, that is $a \ge 5$. Since the roots of $f(x) = 0$ are real and distinct, its discriminant

$= b^2 - 4ac > 0$.

$$\Rightarrow \quad b^2 > 4ac \ge 20 \qquad [\because c \ge 1]$$

Hence, the minimum possible value of b is 5. Let us try the least values of a, b and c, that is $a = 5$, $b = 5$ and $c = 1$. It is easy to check that $5x^2 - 5x + 1 = 0$ has two distinct real roots lying between 0 and 1. Thus, the least positive integral value of a is 5. Ans.

Q.6 For what values of the parameter m is the inequality $\left|\dfrac{x^2+mx+1}{x^2+x+1}\right|<3$ satisfied for all real values of x?

Sol. The inequality is equivalent to $-3<\dfrac{x^2+mx+1}{x^2+x+1}<3$

Since, $x^2+x+1=\left(x+\dfrac{1}{2}\right)^2+\dfrac{3}{4}>0$, we have

$$-3\,(x^2+x+1)<x^2+mx+1<3\,(x^2+x+1)$$

Thus, for each $x \in R$

$\therefore \quad 4x^2+(m+3)\,x+4>0$ ....(1)

and $\quad 2x^2-(m-3)\,x+2>0$ ....(2)

Since, the co-efficient of x^2 in L.H.S. of (1) = 4 > 0, the inequality (1) will be valid if

$(m+3)^2-64<0$, i.e. if $(m+11)\,(m-5)<0$

or $\quad -11<m<5$ ....(3)

Since, the co-efficient of x^2 in L.H.S. of (2) = 2 > 0, the inequality (2) will be valid if

$(m-3)^2-16<0$ i.e. if $(m+1)\,(m-7)<0$

or $\quad -1<m<7$ ....(4)

The conditions (3) and (4) will hold simultaneously if $-1<m<5$. **Ans.**

or $\quad -1<m<7$ ....(4)

The conditions (3) and (4) will hold simultaneously if $-1<m<5$. **Ans.**

Q.7 For what real values of a is one of the equation $(2a+1)x^2-ax+a-2=0$ greater and the other is smaller than unity?

Sol. The quadratic equation $(2a+1)x^2-ax+(a-2)=0$ ....(1)

will have real roots if $D=a^2-4\,(2a+1)\,(a-2)\geq 0$

$\Rightarrow \quad a^2-4\,(2a^2-3a-2)\geq 0$

$\Rightarrow \quad a^2-4\,(2a^2-3a-2)\geq 0 \quad \Rightarrow \quad 7a^2-12a-8\leq 0$

$$\Rightarrow \quad a^2-\frac{12a}{7}-\frac{8}{7}\leq 0 \quad \Rightarrow \quad \left(a-\frac{6}{7}\right)^2\leq\frac{8}{7}+\frac{36}{49}=\frac{92}{49}$$

$$\Rightarrow \quad \frac{6}{7}-\frac{2\sqrt{23}}{7}\leq a\leq\frac{6}{7}+\frac{2\sqrt{23}}{7}$$

Next, if α_1, α_2 are the roots of the given equation, then the desired condition is satisfied if and only if $(1-\alpha_1)\,(1-\alpha_2)<0$

$$\Rightarrow \quad 1-(\alpha_1+\alpha_2)+\alpha_1\,\alpha_2<0 \quad \Rightarrow \quad 1-\frac{a}{2a+1}+\frac{a-2}{2a+1}<0$$

$$\Rightarrow \quad \frac{2a+1-a+a-2}{2a+1}<0 \Rightarrow \frac{2a-1}{2a+1}<0 \quad \Rightarrow \quad \frac{(2a-1)\,(2a+1)}{(2a+1)^2}<0$$

$$\Rightarrow \quad \left(a+\frac{1}{2}\right)\left(a-\frac{1}{2}\right) < 0 \qquad \Rightarrow \quad \frac{-1}{2} < a < \frac{1}{2}$$

But $\quad \frac{6}{7}-\frac{2\sqrt{23}}{7} < \frac{-1}{2} \qquad$ and $\frac{1}{2} < \frac{6}{7}+\frac{2\sqrt{23}}{7}$

Therefore, the required values of a lie in the range $\frac{-1}{2} < a < \frac{1}{2}$. Ans.

Q.8 Solve th equation $2^{|x+2|} - |2^{x+1} - 1| = 2^{x+1} + 1$.

Sol.

Case I: Let $x + 2 \ge 0$. In this case $|x+2| = x+2$ and the equation becomes

$$2^{x+2} - 2^{x+1} - 1 - |2^{x+1} - 1| = 0$$

$$\Rightarrow \quad 2^{x+1}(2-1) - 1 - |2^{x+1} - 1| = 0$$

$$\Rightarrow \quad 2^{x+1} - 1 = |2^{x+1} - 1|$$

$$\Rightarrow \quad 2^{x+1} - 1 \ge 0 \text{ or } x + 1 \ge 0 \text{ or } x \ge -1$$

Case II: Let $x + 2 < 0$. In this case $|x+2| = -x-2$ and the equation becomes

$$2^{-(x+2)} - |2^{x+1} - 1| = 2^{x+1} + 1 \qquad \text{......(1)}$$

Put $\quad 2^{x+1} = y$. Then (1) becomes

$$\frac{1}{2y} - |y-1| = y+1$$

$$\frac{1}{2y} - |y-1| = y+1$$

or $\quad 2y^2 + 2y + |y-1|2y = 1 \qquad$ ......(2)

If $y \ge 1$, then L.H.S. of (2) > 1. If $y < 1$, then (2) becomes $2y^2 + 2y - 2y(y-1) - 1 = 0$

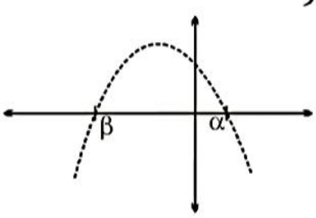
$$\Rightarrow \quad 4y = 1 \text{ or } y = \frac{1}{4}$$

Also $2^{x+1} = 2^{-2}$ or $x = -3$. Ans.

Q.9 If in the quadratic equation $ax^2 + bx + c = 0$, $a < 0$, $b < 0$, $c > 0$ also α and β are its's roots where $\alpha > \beta$

(A) $|\alpha| > |\beta|$ (B) $|a| = |\beta|$ (C) $|\beta| > |\alpha|$ (D) $|\beta| > |\alpha| \ge \alpha > \beta$

Sol. $\left.\begin{array}{l} \alpha+\beta = -\frac{b}{a} = \text{negative} \\ \alpha\beta = \frac{c}{a} = \text{negative} \end{array}\right\}$ both roots are of opposite sign and root in greater magnitude is negative.



Q.10 If $a \in R_+$ and the roots of the equation $ax^2 - 3x + c = 0$, are two consecutive odd positive inters, then

(A) $a \in (1, \infty)$ (B) $a \in (1, 4)$ (C) $a \in \left(0, \frac{3}{4}\right]$ (D) $a \in (0, \infty)$

Sol. Let α and $\alpha + 2$ be two consecutive odd positive integers

$\therefore$ $a\alpha^2 - 3\alpha + c = 0,\ a(\alpha+2)^2 - 3(\alpha+2) + c = 0$

$\Rightarrow$ $(a\alpha^2 - 3\alpha + c) + 4a\alpha + 4a - 6 = 0$

$\Rightarrow$ $4a\alpha + 4a - 6 = 0$, since $a\alpha^2 - 3\alpha + c = 0$

$\Rightarrow$ $3 = 2a(1 + \alpha)$ where $\alpha \geq 1$

$\Rightarrow$ $\frac{3}{2a} - 1 = \alpha \Rightarrow \frac{3}{2a} - 1 \geq 1 \Rightarrow \frac{3}{2a} \geq 2$

$\Rightarrow$ $a \leq \frac{3}{4}$. Ans.

Q.11 For what real values of a does the range of the function $y = \frac{x-1}{a - x^2 + 1}$ not contain any values belonging to the interval $[-1, -1/3]$?

$a - x^2 + 1$

to the interval $[-1, -1/3]$?

Sol. $y = \frac{x-1}{a - x^2 + 1}$

$ay - x^2y + y = x - 1$

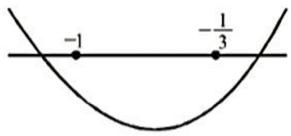
$x^2y + x - (1 + ay + y) = 0$

$\because$ x is real $\therefore$ $D \geq 0$

$1 + 4y(1 + ay + y) \geq 0$

$4y^2(a + 1) + 4y + 1 \geq 0$

Case-I $a + 1 > 0$



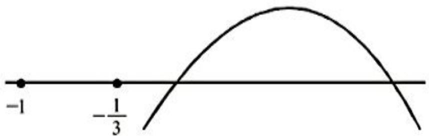
$f(-1) < 0$ and $f\left(\frac{-1}{3}\right) < 0$

$a < \frac{-1}{4}$ and $a < \frac{-1}{4}$

but $a > -1$ $\therefore$ $a \in \left(-1, \frac{-1}{4}\right)$

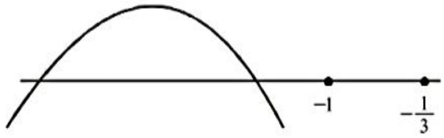
Case-II $\quad a + 1 = 0 \quad \Rightarrow \quad a = -1$

$$4y + 1 \geq 0 \quad \Rightarrow \quad y \geq \frac{-1}{4}$$



Case-III $\quad a + 1 < 0 \quad \Rightarrow \quad a < -1$

$$f(-1) < 0 \quad \text{and} \quad f\left(-\frac{1}{3}\right) < 0$$



$$\Rightarrow \quad a < -\frac{1}{4} \qquad a \in (-\infty, -1)$$

Finally, from Case-I, II and II

$$a \in \left(-\infty, -\frac{1}{4}\right) \qquad \text{Ans.}$$

Q.12 For what real values of a do the roots of $x^2 - 2x - a^2 + 1 = 0$ lie between the roots $x^2 - 2(a + 1)x + a(a - 1) = 0$.

Sol. As $y = x^2 - 2x - a^2 + 1$ and $y = x^2 - 2(a + 1)x + a(a - 1)$ are upward opening parabolas, the roots α, β of $x^2 - 2x - a^2 + 1 = 0$ will lie between the roots of $f(x) = x^2 - 2(a + 1)x + a(a - 1) = 0$ if $f(\alpha) < 0$ and $f(\beta) < 0$.

α, β of $x^2 - 2x - a^2 + 1 = 0$ will lie between the roots of $f(x) = x^2 - 2(a + 1)x + a(a - 1) = 0$ if $f(\alpha) < 0$ and $f(\beta) < 0$.

The roots of $x^2 - 2x - a^2 + 1 = 0 \qquad$ ....(1)

are $\quad x = 1 \pm a$

and the root x of $x^2 - 2(a + 1)x + a(a - 1) = 0$

will be real and distinct if $4(a + 1)^2 - 4a(a - 1) > 0$

i.e. if $\quad 3a + 1 > 0$ or $a > \frac{-1}{3}$

We have $f(1 + a)$ and $f(1 - a) < 0$

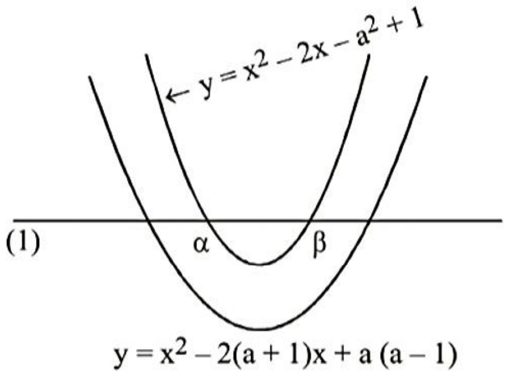
$\Rightarrow \quad (1 + a)^2 - 2(a + 1)(1 + a) + a(a - 1) < 0$

and $\quad (1 - a)^2 - 2(a + 1)(1 - a) + a(a - 1) < 0$

$\Rightarrow \quad a > \frac{-1}{3}$ and $4a^2 - 3a - 1 < 0$

$\Rightarrow \quad a > \frac{-1}{3}$ and $(4a + 1)(a - 1) < 0$

$\Rightarrow \quad a > \frac{-1}{3}$ and $\frac{-1}{4} < a < 1$



Thus, $\frac{-1}{4} < a < 1. \qquad$ Ans.

Q.13 For what real values of the parameter a does the equation $x^4 + 2ax^3 + x^2 + 2ax + 1 = 0$ have at least two distinct negative roots?

Sol. The equation (1) does not have 0 as a root.

Dividing (1) by x^2, we can rewrite it as

$$x^2 + \frac{1}{x^2} + 2a\,c + 1 = 0 \qquad \Rightarrow \qquad \left(x + \frac{1}{x}\right)^2 + 2a\left(x + \frac{1}{x}\right) - 1 = 0 \qquad ......(2)$$

Put $x + \frac{1}{x} = y$. Then we can write (2) as

$$y^2 + 2ay - 1 = 0 \qquad ......(3)$$

Since, the discriminant of (3) is $4a^2 + 4$, which is positive, (3) has two distinct real roots.

These roots are given by

$$y = \frac{-2a \pm \sqrt{4a^2 + 4}}{2} = -a \pm \sqrt{a^2 + 1}$$

Since, $-a + \sqrt{a^2+1} > 0$, the equation

2

Since, $-a + \sqrt{a^2+1} > 0$, the equation

$$x + \frac{1}{x} = -a + \sqrt{a^2 + 1}$$

has either positive roots or non-real complex roots. Since $-a - \sqrt{a^2+1} < 0$, both the roots of the equation

$$x + \frac{1}{x} = -a - \sqrt{a^2 + 1} \qquad ....(4)$$

We either negative or non-real complex. In case (4) has negative roots, we can rewrite (4) as

$$\left(\sqrt{-x} - \frac{1}{\sqrt{-x}}\right)^2 = a - 2 + \sqrt{a^2 + 1}$$

or $\left(\sqrt{-x} - \frac{1}{\sqrt{-x}}\right)^2 > 0$, we must have, $a - 2 + \sqrt{a^2+1} > 0$ i.e. $\sqrt{a^2+1} > |2 - a|$.

Since, $a^2 + 1 = (a+2)^2 + 4a - 3$, $\sqrt{a^2+1} > |2-a|$ if and only if $(a-3) > 0$ that is, if and only if $a > \frac{3}{4}$.

Q.14 Find all real roots of the equation $\sqrt{x^2-p}+2\sqrt{x^2-1}=x$ where p is a real parameter.

Sol. As $\sqrt{x^2-p} \geq 0,\ 2\sqrt{x^2-1} \geq 0$, we get from (1) that

$$x = \sqrt{x^2-p}+2\sqrt{x^2-1} \geq 0$$

Therefore, all the roots of (1) are non-negative. If $p < 0$, then $\sqrt{x^2-p}+2\sqrt{x^2-1} > |x| \geq x$

$$\Rightarrow \quad \sqrt{x^2-p}+2\sqrt{x^2-1} > |x| \geq x$$

Thus, the equation (1) has no solution for $p < 0$. In other for (1) to have a solution we must $p \geq 0$. We rewriter (1) as

$$2\sqrt{x^2-1} = x - \sqrt{x^2-p}$$

and square it to obtain

$$4(x^2-1) = x^2 + x^2 - p - 2x\sqrt{x^2-p}$$

$$\Rightarrow \quad 2x^2 + p - 4 = -2x\sqrt{x^2-p}$$

Squaring again, we get

$$4x^4 + 4x^2(p-4) + (p-4)^2 = 4x^2(x^2-p)$$

Squaring again, we get

$$4x^4 + 4x^2(p-4) + (p-4)^2 = 4x^2(x^2-p)$$

$$\Rightarrow \quad x^2(8p-16) + (p-4)^2 = 0$$

$$\Rightarrow \quad x^2 = \frac{(p-4)^2}{8(2-p)}$$

As $x^2 > 0$, we get $2 - p > 0 \Rightarrow 0 \leq p < 2$

For $0 \leq p < 2$, we get

$$x = \frac{|p-4|}{2\sqrt{2}\sqrt{2=p}}$$

Putting this value of x in (1), we get

$$\sqrt{\frac{(p-4)^2}{8(2-p)}-p}+2\sqrt{\frac{(p-4)^2}{8(2-p)}-1} = \frac{p-4}{2\sqrt{2}\sqrt{2-p}}$$

$$\Rightarrow \sqrt{p^2-8p+16+8p^2}+2\sqrt{p^2-8p+16-16+8p} = 4-p$$

$$\Rightarrow \sqrt{(3p-4)^2}+2\sqrt{p^2} = 4-p$$

$$\Rightarrow |3p-4| = -(3p-4) \Rightarrow 3p-4 \leq 0$$

Thus, $p \leq \frac{4}{3}$. hence, $0 \leq p \leq \frac{4}{3}$, and for this value of p, $x = \frac{p-4}{2\sqrt{2}\sqrt{2-p}}$. Ans.

Q.15 Find all the values of the parameter a for which exactly one root of the equation $e^a x^2 - e^{2a} x + e^a - 1 = 0$ lies in the interval (1, 2).

Sol. Let $f(x) = e^a x^2 - e^{2a} x + e^a - 1$.

Note that $y = f(x)$ is an upward opening parabola. Exactly one root of (1) will lie in the interval (1, 2) if $f(1)\,f(2) < 0$.

We have $\quad f(1) = e^a - e^{2a} + e^a - 1 = -(e^a - 1)^2$

Note that $\quad f(1) < 0$ if $a \neq 0$

Thus, we must have $f(2) > 0$.

$\Rightarrow \quad f(2) = e^a (2^2) - e^{2a} (2) + e^a - 1 > 0$

$\Rightarrow \quad e^a$ lies between the roots of $2y^2 - 5y + 1 = 0$

i.e. between $\dfrac{5-\sqrt{17}}{4}$ and c

Now, $\quad \dfrac{5-\sqrt{17}}{4} < e^a < \dfrac{5-\sqrt{17}}{4}$

$$\Rightarrow \quad \log_e\left(\frac{5-\sqrt{17}}{4}\right) < a < \log_e\left(\frac{5+\sqrt{17}}{4}\right)$$

As $a \neq 0$, we get

As $a \neq 0$, we get

$$a \in \left(\log_e\left(\frac{5-\sqrt{17}}{4}\right), 0\right) \cup \left(0, \log_e\left(\frac{5+\sqrt{17}}{4}\right)\right). \qquad \text{Ans.}$$

Q.16 Let x, y, z be real variables satisfying the equations $x + y + z = 6$ and $xy + yz + zx = 7$. Find the range in which variables can lie.

Sol. Eliminating z from the two given equations, we get $xy + (y + x)\{6 - (x + y)\} = 7$

$\Rightarrow \quad -(x + y)^2 + xy + 6(x + y) - 7 = 0$

$\Rightarrow \quad y^2 + y(x - 6) + x^2 - 6x + 7 = 0$

As y is real, we get $(x - 6)^2 - 4(x^2 - 6x + 7) \geq 0$

$\Rightarrow \quad x^2 - 12x + 36 - 4x^2 + 24x - 28 \geq 0$

$\Rightarrow \quad 3x^2 - 12x - 8 \leq 0$

Thus, x must lie between the roots of $3x^2 - 12x - 8 = 0$, that is, between

$$\frac{12-\sqrt{144+96}}{6} \leq x \leq \frac{12+\sqrt{144+96}}{6}$$

$$\Rightarrow 2 - \frac{2\sqrt{15}}{3} \leq x \leq 2 + \frac{2\sqrt{15}}{3}.$$

Hence, x, y, z $\left[2-\dfrac{2}{3}\sqrt{15}, 2+\dfrac{2}{3}\sqrt{15}\right]$. $\qquad$ Ans.

TRIGONOMETRIC EQUATION

1. INTRODUCTION :

An equation involving one or more trigonometrical ratios of unknown angle is called a trigonometric equation. e.g., $\cos x = \frac{1}{2}$; $\sin^2 x - 4\cos x = 1$.

The value of an unknown angle which satisfies the given trigonometric equation is called a solution or root of the equation. For example $2\sin\theta = \sqrt{3}$, Clearly $\theta = 60°$ and $120°$ are solutions of the equation between $0°$ and $360°$.

Now suppose $\tan\theta = 1$, then there will be many possible vlaues of θ, so our main ojective is to write down all the solution in short form. Since all trigonometric functions are periodic and therefore solution of all trigonometrical equation can be generalized with the help of periodicity of trigonometrical function.

1.1. Principal solution of a Trigonometric Equation :

1.1. Principal solution of a Trigonometric Equation :

The solutions of a trigonometric equation $\sin\theta = \frac{1}{2}$ lying in the interval $[0, 2\pi]$ are $\frac{\pi}{6}$ and $\frac{5\pi}{6}$. Thus principal solutions of $\sin\theta = \frac{1}{2}$ will be $\frac{\pi}{6}$ and $\frac{5\pi}{6}$.

1.2 General solution of a Trigonometric Equation :

(i) If $\sin\theta = \sin\alpha \Rightarrow \theta = n\pi + (-1)^n\alpha$, where $\alpha \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, $n \in I$.

(ii) If $\cos\theta = \cos\alpha \Rightarrow \theta = 2n\pi \pm \alpha$, where $\alpha \in [0, \pi]$, $n \in I$.

(iii) If $\tan\theta = \tan\alpha \Rightarrow \theta = n\pi + \alpha$, where $\alpha \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, $n \in I$.

(iv) If $\sin^2\theta = \sin^2\alpha \Rightarrow \theta = n\pi \pm \alpha$.

(v) If $\cos^2\theta = \cos^2\alpha \Rightarrow \theta = n\pi \pm \alpha$.

(vi) If $\tan^2\theta = \tan^2\alpha \Rightarrow \theta = n\pi \pm \alpha$.

Note : α is called the principal angle.

Proof :

(i) $\sin\theta = \sin\alpha \Rightarrow \sin\theta - \sin\alpha = 0 \Rightarrow 2\cos\left(\frac{\theta+\alpha}{2}\right)\sin\left(\frac{\theta-\alpha}{2}\right) = 0$

$\Rightarrow \quad \cos\left(\frac{\theta+\alpha}{2}\right) = 0$ or $\sin\left(\frac{\theta-\alpha}{2}\right) = 0$

$\Rightarrow \quad \frac{\theta+\alpha}{2} = (2m+1)\frac{\pi}{2}$ or $\frac{\theta-\alpha}{2} = m\pi$, where $m \in I$.

$\Rightarrow \quad \theta = (2m+1)\pi - \alpha$ or $\theta = 2m\pi + \alpha$, where $m \in I$.

$\Rightarrow \quad \theta = (2m+1)\pi + (-1)^{2m+1}\alpha$ or $\theta = 2m\pi + (-1)^{2m}\alpha$

$\Rightarrow \quad \theta = n\pi + (-1)^n\alpha,\ n \in I$.

(ii) $\cos\theta = \cos\alpha \;\Rightarrow\; \cos\alpha - \cos\theta = 0 \;\Rightarrow\; 2\sin\left(\frac{\alpha+\theta}{2}\right)\sin\left(\frac{\theta-\alpha}{2}\right) = 0$

$\Rightarrow \quad \sin\frac{\theta+\alpha}{2} = 0$ or $\sin\frac{\theta-\alpha}{2} = 0$, $\frac{\theta+\alpha}{2} = n\pi$ or $\frac{\theta-\alpha}{2} = n\pi$

$\Rightarrow \quad \theta = 2n\pi - \alpha$, or $\theta = 2n\pi + \alpha \;\; n \in I$

$\Rightarrow \quad \theta = 2n\pi \pm \alpha$.

2 2 2 2

$\Rightarrow \quad \theta = 2n\pi - \alpha$, or $\theta = 2n\pi + \alpha \;\; n \in I$

$\Rightarrow \quad \theta = 2n\pi \pm \alpha$.

(iii) $\tan\theta = \tan\alpha \Rightarrow \frac{\sin\theta}{\cos\theta} = \frac{\sin\alpha}{\cos\alpha} \Rightarrow \sin\theta\cos\theta - \cos\theta.\sin\alpha = 0$

$\Rightarrow \quad \sin(\theta-\alpha) = 0 \Rightarrow \theta - \alpha = n\pi$

$\Rightarrow \quad \theta = n\pi + \alpha$, where $n \in I$

(iv) $\sin^2\theta = \sin^2\alpha$

$\sin^2\theta - \sin^2\alpha = \sin(\theta+\alpha)\sin(\theta-\alpha) = 0$

$\sin(\theta+\alpha) = 0$ or $\sin(\theta-\alpha) = 0$

$\theta + \alpha = n\pi$ or $\theta - \alpha = n\pi,\ n \in I$

$\theta = n\pi \pm \alpha,\ n \in I$

(v) $\cos^2\theta = \cos^2\alpha \Rightarrow 1 - \sin^2\theta = 1 - \sin^2\alpha \Rightarrow \sin^2\theta - \sin^2\alpha$

$\theta = n\pi \pm \alpha,\ n \in I$

(vi) $\tan^2\theta = \tan^2\alpha \Rightarrow \tan\theta = \pm\tan\alpha = \tan(\pm\alpha)$

$\Rightarrow \quad \theta = n\pi \pm \alpha$, where $n \in I$

Remember :

Trigonometric equation withtheir general solution

Trigonometrical equation		General Solution
$\sin\theta = 0$	$\Rightarrow$	$\theta = n\pi$
$\cos\theta = 0$	$\Rightarrow$	$\theta = (2n+1)\pi/2$
$\tan\theta = 0$	$\Rightarrow$	$\theta = n\pi$
$\sin\theta = 1$	$\Rightarrow$	$\theta = 2n\pi + \pi/2$
$\sin\theta = \sin\alpha$	$\Rightarrow$	$\theta = n\pi + (-1)^n\alpha$
$\cos\theta = \cos\alpha$	$\Rightarrow$	$\theta = 2n\pi \pm \alpha$
$\tan\theta = \tan\alpha$	$\Rightarrow$	$\theta = n\pi + \alpha$
$\sin^2\theta = \sin^2\alpha$	$\Rightarrow$	$\theta = n\pi \pm \alpha$
$\tan^2\theta = \tan^2\alpha$	$\Rightarrow$	$\theta = n\pi \pm \alpha$
$\cos^2\theta = \cos^2\alpha$	$\Rightarrow$	$\theta = n\pi \pm \alpha$
$\left.\begin{array}{l}\sin\theta = \sin\alpha \\ \cos\theta = \cos\alpha\end{array}\right\|$	$\Rightarrow$	$\theta = 2n\pi + \alpha$
$\left.\begin{array}{l}\sin\theta = \sin\alpha \\ \tan\theta = \tan\alpha\end{array}\right\|$	$\Rightarrow$	$\theta = 2n\pi + \alpha$
$\left.\begin{array}{l}\tan\theta = \tan\alpha \\ \cos\theta = \cos\alpha\end{array}\right\|$	$\Rightarrow$	$\theta = 2n\pi + \alpha$

$\cos\theta = \cos\alpha$

Illustration :

Solve $4\cos^2\theta - 4\sin\theta - 1 = 0,\ 0 \le \theta \le 2\pi.$

Sol. $4(1 - \sin^2\theta) - 4\sin\theta - 1 = 0 \Rightarrow 4\sin^2\theta + 4\sin\theta - 3 = 0$

$$\Rightarrow \quad \sin\theta = \frac{1}{2}, -\frac{3}{2} \text{ (Not possible)}$$

$$\sin\theta = \frac{1}{2} = \sin\frac{\pi}{6} = \sin\left(\pi - \frac{\pi}{6}\right) = \frac{5\pi}{6}$$

$$\theta = \frac{\pi}{6}, \frac{5\pi}{6}$$

Thus we have two principal solutions.

Illustration :

Find the general value of θ satisfying both $\sin\theta = -\frac{1}{2}$ and $\tan\theta = \frac{1}{\sqrt{3}}$ simultaneously.

Sol. $\sin\theta < 0,\ \tan\theta > 0$ so θ is in third quadrant.

Now, $\sin\theta = -\frac{1}{2} \Rightarrow \theta = \frac{7\pi}{6}, \frac{11\pi}{6}$

$\tan\theta = \frac{1}{\sqrt{3}} \Rightarrow \theta = \frac{\pi}{6}, \frac{7\pi}{6}$

Common solution $= \frac{7\pi}{6}$

So general solution $\theta = 2n\pi + \frac{7\pi}{6}$.

Illustration :

Solve : $\sqrt{3}\sec 2\theta = 2$.

Sol. $\cos 2\theta = \frac{\sqrt{3}}{2} = \cos\frac{\pi}{6} \Rightarrow 2\theta = 2n\pi \pm \frac{\pi}{6}$

π

Sol. $\cos 2\theta = \frac{\sqrt{3}}{2} = \cos\frac{\pi}{6} \Rightarrow 2\theta = 2n\pi \pm \frac{\pi}{6}$

$\theta = n\pi \pm \frac{\pi}{12}, n \in I$

Illustration :

Solve : $\tan(3\theta) = -1$.

Sol. $\tan 3\theta = \tan\left(-\frac{\pi}{4}\right)$

$\Rightarrow \quad 3\theta = n\pi + \left(-\frac{\pi}{4}\right) \Rightarrow \theta = \frac{n\pi}{3} - \frac{\pi}{12}, n \in I$

Illustration :

Solve $7\cos^2\theta + 3\sin^2\theta = 4$

Sol. $7(1-\sin^2\theta) + 3\sin^2\theta = 4, \quad 4\sin^2\theta = 3$

$\sin^2\theta = \frac{3}{4} = \left(\frac{\sqrt{3}}{2}\right)^2 = \sin^2\frac{\pi}{3}$

$\theta = n\pi \pm \frac{\pi}{3}, n \in I$

2. TYPES OF TRIGONOMETRIC EQUATION :

2.1 TYPE-I :

Solution of trigonometric equation by factorization or equation which are expressed in quadratic form or which can be expressed in quadratic form :

Illustration :

Solve $(2\sin x - \cos x)(1 + \cos x) = \sin^2 x$ *in* $[0, 2\pi]$.

Sol. $(1 + \cos x)(2\sin x - \cos x) - (1 - \cos^2 x) = 0$

$(1 + \cos x)[(2\sin x - \cos x) - (1 - \cos x)] = 0, (1 + \cos x)(2\sin x - 1) = 0$

$\sin x = \frac{1}{2}, \cos x = -1.$

$x = \frac{\pi}{6}, \frac{5\pi}{6}, \pi.$

$x = \frac{\pi}{6}, \frac{5\pi}{6}, \pi.$

Illustration :

Find the general solutions of equation $1 - \cos x \cot x = \cot x - \cos x$.

Sol. $(1 + \cos x)(1 - \cot x) = 0 \quad \Rightarrow \quad \cos x = -1$ *or* $\cot x = 1$

$\Rightarrow \quad (2n + 1)\pi, \quad n\pi + \frac{\pi}{4}, \ n \in I$

Illustration :

Find the general solutions of equation $3\cos^2 x - 10\cos x + 3 = 0$.

Sol. $(3\cos x - 1)(\cos x - 3) = 0 \ \Rightarrow \cos x = \frac{1}{3}, 3$ *but* $\cos x \neq 3$

$\therefore \quad \cos x = \frac{1}{3} \Rightarrow x = 2n\pi \pm \cos^{-1}\left(\frac{1}{3}\right)$

Illustration :

Find the general solutions of equation $2\sin^2 2x + 6\sin^2 x = 5$.

Sol. $8\sin^2 x \cdot \cos^2 x + 6\sin^2 x = 5.$

$\Rightarrow \quad 8\sin^4 x - 14\sin^2 x + 5 = 0 \quad \Rightarrow \quad (2\sin^2 x - 1)(4\sin^2 x - 5) = 0$

$$\Rightarrow \quad \sin^2 x = \frac{1}{2}, \frac{5}{4} \Rightarrow \sin^2 x = \frac{1}{2} \Rightarrow n\pi \pm \frac{\pi}{4}, n \in I$$

Illustration :

Find the general solutions of equation $(1 - \tan\theta)(1 + \sin 2\theta) = 1 + \tan\theta$.

Sol. $$(1 - \tan\theta)\left(1 + \frac{2\tan\theta}{1+\tan^2\theta}\right) = (1 + \tan\theta) \quad \Rightarrow \quad \frac{(1-\tan\theta)(1+\tan\theta)^2}{1+\tan^2\theta} = (1 + \tan\theta)$$

$$\therefore \quad 1 + \tan\theta = 0 \text{ or } \frac{(1-\tan^2\theta)}{1+\tan^2\theta} = 1$$

$$\tan\theta = -1 \text{ or } \cos 2\theta = 1$$

$$\tan\theta = -1 \text{ or } \cos 2\theta = 1$$

$$\theta = n\pi - \frac{\pi}{4} \quad \text{or} \quad 2\theta = 2n\pi$$

$$\therefore \quad \theta = n\pi - \frac{\pi}{4}, n\pi, \quad n \in I$$

Practice Problem

Q.1 Find the most general values of θ satisfying

(i) $\tan^2\theta = \frac{1}{3}$ (ii) $\sec^2\theta = \frac{4}{3}$ (iii) $2\cot^2\theta = \operatorname{cosec}^2\theta$

Q.2 What is the most general value of θ satisfying both the equations $\cos\theta = -\frac{1}{\sqrt{2}}$ and $\tan\theta = 1$.

Q.3 Solve : $3\sin^2 x - \sin x \cos x - 4\cos^2 x = 0$

Q.4 Find the number of solution of the equation $16^{\sin^2 x} + 16^{\cos^2 x} = 10$, where $0 \le x \le 2\pi$.

Q.5 Find the general solution of $\cos 4x + 6 = 7\cos 2x$, and also find the sum of all the solution in $[0, 3\pi]$.

Answer key

Q.1 (i) $n\pi \pm \frac{\pi}{6}$ (ii) $n\pi \pm \frac{\pi}{6}$ (iii) $n\pi \pm \frac{\pi}{4}$ Q.2 $2n\pi + \frac{5\pi}{4}$

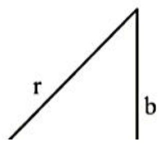
Q.3 $n\pi - \frac{\pi}{4}$, $n\pi + \alpha$, where $\tan \alpha = \frac{4}{3}$. Q.4 8 Q.5 3π

2.2 TYPE-II :

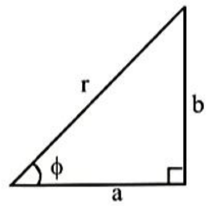
Solving trigonometric equations by introducing an Auxiliary argument. Equation of the form of $a \cos \theta + b \sin \theta = c$

To solve equation, we convert the equation to the form $\cos \theta = \cos \alpha$ or $\sin \theta = \sin \alpha$ etc.

For this let us suppose that $\left.\begin{matrix} a = r\cos\phi \\ b = r\sin\phi \end{matrix}\right\} \Rightarrow \left\{\begin{matrix} r = \sqrt{a^2+b^2} \\ \text{and } \tan\phi = \frac{b}{a}, \end{matrix}\right.$



For this let us suppose that $\left.\begin{matrix} a = r\cos\phi \\ b = r\sin\phi \end{matrix}\right\} \Rightarrow \left\{\begin{matrix} r = \sqrt{a^2+b^2} \\ \text{and } \tan\phi = \frac{b}{a}, \end{matrix}\right.$



Substituting these values in the equation $a \cos \theta + b \sin \theta = c$, we have

$r \cos \phi \cos \theta + r \sin \phi \sin \theta = c$

$\Rightarrow \quad r \cos (\theta - \phi) = c$

$\Rightarrow \quad \cos (\theta - \phi) = \frac{c}{r} = \frac{c}{\sqrt{a^2+b^2}} = \cos \beta$ (suppose)

$\Rightarrow \quad \theta - \phi = 2n\pi \pm \beta$

$\Rightarrow \quad \theta = 2n\pi + \phi \pm \beta, n \in Z$

Here ϕ and β are known as a, b and c are given.

Hence, we can solve the equation of this type by putting.

$a = r \cos \phi$ and $b = r \sin \phi$ provided $\left|\frac{c}{\sqrt{a^2+b^2}}\right| \le 1$ [$\because$ $\cos \beta$ lies between -1 and 1]

or $\frac{|c|}{\sqrt{a^2+b^2}} \le 1$ or $|c| \le \sqrt{a^2+b^2}$

Illustration :

Find general value of $\sin x + \cos x = \sqrt{2}$.

Sol. $\frac{1}{\sqrt{2}}\sin x + \frac{1}{\sqrt{2}}\cos x = 1, \qquad \sin x \sin\left(\frac{\pi}{4}\right) + \cos x \cos\left(\frac{\pi}{4}\right) = 1$

$\cos\left(x - \frac{\pi}{4}\right) = 1, \; x - \frac{\pi}{4} = 2n\pi, \qquad x = 2n\pi + \frac{\pi}{4}$

Illustration :

Find general value of $\sqrt{3}\cos x + \sin x = 2$.

Sol. $\frac{\sqrt{3}}{2}\cos x + \frac{1}{2}\sin x = 1, \qquad \cos x \cos\frac{\pi}{6} + \sin x \sin\left(\frac{\pi}{6}\right) = 1.$

$\cos\left(x - \frac{\pi}{6}\right) = 1 \qquad \text{so} \qquad x - \frac{\pi}{6} = 2n\pi, \; x = 2n\pi + \frac{\pi}{6}, \; n \in I.$

$\cos\left(x - \frac{\pi}{6}\right) = 1 \qquad \text{so} \qquad x - \frac{\pi}{6} = 2n\pi, \; x = 2n\pi + \frac{\pi}{6}, \; n \in I.$

Illustration :

Find the general solutions of equation $\sin x + \cos x = \frac{3}{2}$

Sol. $\frac{1}{\sqrt{2}}\sin x + \frac{1}{\sqrt{2}}\cos x = \frac{3}{2\sqrt{2}}$

$\sin\left(x + \frac{\pi}{4}\right) = \frac{3}{2\sqrt{2}} > 1$ *so no solution.*

Illustration :

Find the general solutions of equation $(\sec x - 1) = (\sqrt{2} - 1)\tan x$.

Sol. $\sec x + \tan x = \sqrt{2}\tan x + 1$

$\sec x - \tan x = \frac{1}{\sqrt{2}\tan x + 1}$

$$2\tan x = (\sqrt{2}\tan x + 1) - \frac{1}{(\sqrt{2}\tan x + 1)} = \frac{(\sqrt{2}\tan x + 1)^2 - 1}{(\sqrt{2}\tan x + 1)}$$

$$2\tan x = \frac{\sqrt{2}\tan x(\sqrt{2}\tan x + 2)}{(\sqrt{2}\tan x + 1)}$$

$\Rightarrow \quad \tan x = 0,\ x = m\pi,\ m \in I.$

or $\quad 2\sqrt{2}\tan x + 2 = 2\tan x + 2\sqrt{2}$

$\Rightarrow \quad \tan x\left(2\sqrt{2} - 2\right) = 2\sqrt{2} - 2,\ \ \tan x = 1$

$\Rightarrow \quad x = n\pi + \frac{\pi}{4},\ n \in I.$

Practice Problem

Q.1 Find the general solution of the equation $4\cos x + 3\sin x = 5$.

Q.2 If $1 + \sin^3 x + \cos^3 x = \frac{3}{2}\sin 2x$, then find general solution of x.

Q.3 Solve : $\sqrt{3}\cos\theta - 3\sin\theta = 4\sin 2\theta\cos 3\theta$.

Q.4 Solve : $3\sqrt{3}\sin^3 x + \cos^3 x + 3\sqrt{3}\sin x + \cos x = 1$

Q.3 Solve : $\sqrt{3}\cos\theta - 3\sin\theta = 4\sin 2\theta\cos 3\theta$.

Q.4 Solve : $3\sqrt{3}\sin^3 x + \cos^3 x + 3\sqrt{3}\sin x + \cos x = 1$.

Answer key

Q.1 $2n\pi + \alpha$, where $\tan\alpha = \frac{3}{4}$

Q.2 $(2n+1)\pi$, $(4n-1)\frac{\pi}{2}$

Q.3 $\frac{n\pi}{3} + \frac{\pi}{18}$, $\frac{-n\pi}{2} + \frac{\pi}{6}$

Q.4 $2n\pi$, $(3n-1)\frac{2\pi}{3}$

2.3 TYPE-III :

Solving equations by tranforming a sum of trigonometric functions into a product.

Illustration :

General solution of $\sin x + \sin 5x = \sin 2x + \sin 4x$ is

(A) $\frac{n\pi}{3}$ (B) $\frac{2n\pi}{3}$ (C) $2n\pi$ (D) $n\pi$

Sol. $2 \sin 3x \cos 2x = 2\sin 3x \cos x$

$\Rightarrow \quad \sin 3x = 0$ or $\cos 2x = \cos x$

$\Rightarrow \quad 3x = n\pi$ or $2x = 2n\pi \pm x$

$\therefore \quad \Rightarrow \quad x = \dfrac{n\pi}{3}, 2n\pi, \dfrac{2n\pi}{3}$

$\Rightarrow \quad x = \dfrac{n\pi}{3}$

2.4 TYPE-IV :

Solving equations by tranforming a product of trigonometric functions into a sum.

Illustration :

Number of solutions of the trigonometric equation in $[0, \pi]$, $\sin 3\theta = 4\sin\theta \cdot \sin 2\theta \cdot \sin 4\theta$.

(A) 4 (B) 6 (C) 8 (D) 10

Sol. $\sin 3\theta = 4 \sin \theta \sin (3\theta - \theta) \sin(3\theta + \theta) = 4 \sin\theta (\sin^2 3\theta - \sin^2\theta)$

$\Rightarrow \quad \sin 3\theta + 4 \sin^3\theta = 4 \sin \theta \sin^2 3\theta$

$\Rightarrow \quad 3 \sin \theta = 4 \sin \theta \sin^2 3\theta \quad \Rightarrow \quad \sin\theta = 0$ or $\sin^2 3\theta = \dfrac{3}{4}$

$\Rightarrow \quad \sin 3\theta + 4 \sin^3\theta = 4 \sin \theta \sin^2 3\theta$

$\Rightarrow \quad 3 \sin \theta = 4 \sin \theta \sin^2 3\theta \quad \Rightarrow \quad \sin\theta = 0$ or $\sin^2 3\theta = \dfrac{3}{4}$

$\Rightarrow \quad \theta = n\pi$ or $3\theta = n\pi \pm \dfrac{\pi}{3} \quad \Rightarrow \quad \theta = n\pi$ or $\theta = \dfrac{n\pi}{3} \pm \dfrac{\pi}{9}, n \in I$

$\Rightarrow \quad \theta = 0, \pi, \dfrac{\pi}{9}, \dfrac{2\pi}{9}, \dfrac{4\pi}{9}, \dfrac{5\pi}{9}, \dfrac{7\pi}{9}, \dfrac{8\pi}{9}$

Illustration :

Find the number of solutions of the equations,

$\sin x + 2 \sin 2x = 3 + \sin 3x$ in $[0, \pi]$

(A) No solution (B) Infinite solution (C) Exactly one solution (D) Two solutions

Sol. We have,

$\Rightarrow \quad \sin 3x - \sin x - 2 \sin 2x + 3 = 0$

$\Rightarrow \quad 2 \cos 2x \cdot \sin x - 4 \sin x \cdot \cos x + 3 = 0$

$\Rightarrow \quad \sin x (2 \cos 2x - 4 \cos x) + 3 = 0$

$\Rightarrow \quad \sin x (4 \cos^2 x - 4 \cos x - 2) + 3 = 0$

$\Rightarrow \quad \sin x (4 \cos^2 x - 4 \cos x + 1) + 3 - 3 \sin x = 0$

$\Rightarrow \quad \sin x (2 \cos x - 1)^2 + 3 (1 - \sin x) = 0 \qquad ...(i)$

since $x \in [0, \pi]$, $\therefore$ $\sin x \geq 0$ *and* $1 - \sin x \geq 0$

$\therefore$ *each part in equation (i) must be zero.*

i.e. $\sin x\,(2\cos x - 1)^2 = 0$ *and* $3\,(1 - \sin x) = 0$

from the second equation of system we have

$\sin x = 1 \Rightarrow \cos x = 0$ *hence* $\sin x\,(2\cos^2 x - 1)^2 \neq 0$

$\therefore$ *not a single solution of the second equation is a solution of the first.*

Hence the original equation has no real solution.

Illustration :

Find the number of solution of the equation in $[0, 2\pi]$, $\tan(5\pi\cos\alpha) = \cot(5\pi\sin\alpha)$

Sol. $5\pi\cos\alpha = n\pi + \left(\frac{\pi}{2} - 5\pi\sin\alpha\right)$

$\Rightarrow \sin\alpha + \cos\alpha = \frac{(2n+1)}{10}$ *as* $-\sqrt{2} \leq \sin\alpha + \cos\alpha \leq \sqrt{2}$

$n = 0, \pm 1, \pm 2, \pm 3, \pm 4, \pm 5, \pm 6, -7$

For each value of n, we get two values of $\alpha \in [0, 2\pi]$

$\therefore$ $2 \times 14 = 28$ *solutions.*

Practice Problem

Q.1 Find the general values of x satisfying $\cos^2 x + \cos^2 2x + \cos^2 3x + \cos^2 4x = 2$.

Q.2 $\operatorname{cosec} x - \operatorname{cosec} 2x = \operatorname{cosec} 4x$. Find general vlaues of x ?

Q.3 Solve $\cos x \cos 2x \cos 3x = \frac{1}{4}$

Q.4 Solve $\cos\theta + \cos 3\theta - 2\cos 2\theta = 0$

Q.5 Solve $\sec 4\theta - \sec 2\theta = 2$.

Answer key

Q.1 $(2m+1)\frac{\pi}{10}, (2n+1)\frac{\pi}{2}, (2k+1)\frac{\pi}{4}$ Q.2 $(2m-1)\frac{\pi}{7}, m \neq 7k - 3, k \in I$

Q.3 $m\pi \pm \frac{\pi}{3}, (2n+1)\frac{\pi}{8}$ Q.4 $(2n+1)\frac{\pi}{4}$ Q.5 $(2m+1)\frac{\pi}{2}, (2n+1)\frac{\pi}{10}$

2.5 TYPE-V :

Solving equations by a change of variable or by substitution method :

(i) Equations of the form $P(\sin x \pm \cos x, \sin x.\cos x) = 0$, where P(x, z) is a polynominal, can be solved by the change.$\cos x \pm \sin x = t$

$\Rightarrow$ $1 \pm 2\sin x.\cos x = t^2$. Consider the equation ; $\sin x + \cos x = 1 + \sin x . \cos x$.

(ii) Many equations can be solved by introducing a new variable e.g. consider the equation $\sin^4 2x + \cos^4 2x = \sin 2x. \cos 2x$.

Illustration :

Find general value of x satisfying the equaiton $\sin^4 2x + \cos^4 2x = \sin 2x \cos 2x$.

Sol. *Let* $\sin 2x . \cos 2x = y$.

$\Rightarrow \quad \sin^4 2x + \cos^4 2x = \sin 2x. \cos 2x$

$\Rightarrow \quad (\sin^2 2x + \cos^2 2x) - 2 \sin^2 2x \cos^2 2x = \sin 2x \cos 2x$

$\Rightarrow \quad 1 - 2y^2 = y \Rightarrow 2y^2 + y - 1 = 0$

$$\Rightarrow \quad y = -1, \frac{1}{2}$$

$$\Rightarrow \quad \sin 2x . \cos 2x = y = -1, \frac{1}{2}$$

$$\Rightarrow \quad \sin 2x . \cos 2x = y = -1, \frac{1}{2}$$

$\Rightarrow \quad \sin 4x = -2$ *(rejected)*, 1

$$\Rightarrow \quad \sin 4x = 1 \qquad \Rightarrow \quad 4x = 2n\pi + \frac{\pi}{2}, n \in I$$

$$\Rightarrow \quad x = (4n + 1)\frac{\pi}{8}$$

Illustration :

Solve the equation : $\sin x + \cos x - 2\sqrt{2} \sin x \cos x = 0$.

Sol. *Let* $(\sin x + \cos x) = t$ *and using the equation*

$$\sin x \cdot \cos x = \frac{t^2 - 1}{2}, \text{ we get}$$

$$t - 2\sqrt{2}\left(\frac{t^2 - 1}{2}\right) = 0$$

$$\Rightarrow \quad \sqrt{2}\, t^2 - t - \sqrt{2} = 0$$

The numbers $t_1 = \sqrt{2}$, $t_2 = -\frac{1}{\sqrt{2}}$ *are roots of this quadritic equation.*

Thus the solution of the given equation reduces to the solution of two trigonometric equations :

$$\sin x + \cos x = \sqrt{2}$$

or $$\frac{1}{\sqrt{2}}\sin x + \frac{1}{\sqrt{2}}\cos x = 1$$

or $$\sin x \cdot \cos\frac{\pi}{4} + \sin\frac{\pi}{4}\cos x = 1$$

$$\Rightarrow \quad \sin\left(x+\frac{\pi}{4}\right) = 1$$

$$\Rightarrow \quad x + \frac{\pi}{4} = (4n+1)\frac{\pi}{2}$$

$$\Rightarrow \quad x = 2n\pi + \frac{\pi}{4}$$

$$\sin x + \cos x = -\frac{1}{\sqrt{2}}$$

$$\frac{1}{\sqrt{2}}\sin x + \frac{1}{\sqrt{2}}\cos x = -\frac{1}{2}$$

$$\sin x\cos\frac{\pi}{4} + \sin\frac{\pi}{4}\cos x = -\frac{1}{2}$$

$$\sin\left(x+\frac{\pi}{4}\right) = -\frac{1}{2}$$

$$x + \frac{\pi}{4} = n\pi + (-1)^n\cdot\left(-\frac{\pi}{6}\right)$$

$$x = n\pi + (-1)^{n+1}\frac{\pi}{6} - \left(\frac{\pi}{4}\right)$$

Illustration :

Solve the equation : $\sin^{10} x + \cos^{10} x = \frac{29}{16}\cos^4 2x.$

Sol. *Using half-angle formulae we can represent the given equation in the form;*

Solve the equation : $\sin^{10} x + \cos^{10} x = \frac{29}{16}\cos^4 2x.$

Sol. *Using half-angle formulae we can represent the given equation in the form;*

$$\left(\frac{1-\cos 2x}{2}\right)^5 + \left(\frac{1+\cos 2x}{2}\right)^5 = \frac{29}{16}\cos^4 2x$$

Put $\cos 2x = t$,

$$\left(\frac{1-t}{2}\right)^5 + \left(\frac{1+t}{2}\right)^5 = \frac{29}{16}t^4$$

$$\Rightarrow \quad 24t^4 - 10t^2 - 1 = 0$$

$$\Rightarrow \quad (12t^2+1)(2t^2-1) = 0$$

whose only real root is, $t^2 = \frac{1}{2}$.

$$\therefore \quad \cos^2 2x = \frac{1}{2}$$

$$\Rightarrow \quad 1 + \cos 4x = 1$$

$$\Rightarrow \quad \cos 4x = 0$$

$$\Rightarrow \quad 4x = (2n+1)\frac{\pi}{2}$$

$$\Rightarrow \quad x = \frac{n\pi}{4} + \frac{\pi}{8};\ n \in \text{Integer}$$

Illustration :

Solve : $\tan\theta + \tan 2\theta + \tan 3\theta = 0$

Sol. $(\tan\theta + \tan 2\theta) + \left(\dfrac{\tan\theta + \tan 2\theta}{1 - \tan\theta\tan 2\theta}\right) = 0$

$\Rightarrow \quad (\tan\theta + \tan 2\theta)\,(2 - \tan\theta\ \tan 2\theta) = 0$

$\Rightarrow \quad \tan 2\theta + \tan\theta = 0 \quad or \quad \tan\theta\tan 2\theta = 2$

when $\tan 2\theta + \tan\theta = 0$

$\tan 2\theta = -\tan\theta = \tan(-\theta)$

$2\theta = n\pi - \theta \quad \Rightarrow \quad \theta = \dfrac{n\pi}{3}$

when $\tan\theta\tan 2\theta = 2$

$\therefore \quad \tan\theta\ \dfrac{2\tan\theta}{1 - \tan^2\theta} = 2$

$\Rightarrow \quad \tan^2\theta = 1 - \tan^2\theta$

$\Rightarrow \quad \tan^2\theta = \dfrac{1}{2}$

$\Rightarrow \quad \tan^2\theta = \dfrac{1}{2}$

$\Rightarrow \quad \theta = n\pi \pm \tan^{-1}\left(\dfrac{1}{2}\right)$

Illustration :

Solve for x : $\sin 3\alpha = 4\sin\alpha\sin(x + \alpha)\sin(x - \alpha)$

Sol. $\sin 3\alpha = 4\sin\alpha\,(\sin^2 x - \sin^2\alpha)$

$\Rightarrow \quad \sin 3\alpha + 4\sin^3\alpha = 4\sin\alpha - \sin^2 x$

$\Rightarrow \quad 3\sin\alpha = 4\sin\alpha\sin^2 x$

$\Rightarrow \quad \sin\alpha = 0 \quad or \quad \sin^2 x = \dfrac{3}{4}$

$\Rightarrow \quad \alpha = n\pi\ or\ x = n\pi \pm \dfrac{\pi}{3}$

Illustration :

Solve for x : $\tan\theta + \tan 2\theta + \sqrt{3}\tan\theta\tan 2\theta = \sqrt{3}$.

Sol. $\tan\theta + \tan 2\theta = \sqrt{3}\,(1 - \tan\theta\tan 2\theta)$

$\Rightarrow \quad \tan 3\theta = \sqrt{3}$

$\therefore \quad 3\theta = n\pi + \dfrac{\pi}{3}$

$\theta = \dfrac{n\pi}{3} + \dfrac{\pi}{9}$

Illustration :

Solve the equation : $4\cot 2\theta = \cot^2\theta - \tan^2\theta$

Sol. $\dfrac{4(1-\tan^2\theta)}{2\tan\theta} = \dfrac{(1-\tan^2\theta)(1+\tan^2\theta)}{\tan^2\theta}$

$\Rightarrow \quad 1 - \tan^2\theta = 0 \quad or \quad 2\tan\theta = 1 + \tan^2\theta$

$\Rightarrow \quad \tan^2\theta = 1 \quad or \quad (\tan\theta - 1)^2 = 0$

Sol. $\dfrac{}{2\tan\theta} = \dfrac{}{\tan^2\theta}$

$\Rightarrow \quad 1 - \tan^2\theta = 0 \quad or \quad 2\tan\theta = 1 + \tan^2\theta$

$\Rightarrow \quad \tan^2\theta = 1 \quad or \quad (\tan\theta - 1)^2 = 0$

$\Rightarrow \quad \tan^2\theta = 1 \quad \Rightarrow \quad \theta = n\pi \pm \dfrac{\pi}{4}$

Illustration :

Find the general solution of equation $\cos 2\theta = (\sqrt{2}+1)\left(\cos\theta - \dfrac{1}{\sqrt{2}}\right)$

Sol. $\Rightarrow \quad (2\cos^2\theta - 1) = (\sqrt{2} + 1)\,\dfrac{(\sqrt{2}\cos\theta - 1)}{\sqrt{2}}$

$\Rightarrow \quad \sqrt{2}\cos\theta - 1 = 0 \quad or \quad \sqrt{2}\cos\theta + 1 = \dfrac{\sqrt{2}+1}{\sqrt{2}}$

$\Rightarrow \quad \cos\theta = \dfrac{1}{\sqrt{2}} \quad or \quad \cos\theta = \dfrac{1}{2}$

$\Rightarrow \quad \theta = 2n\pi \pm \dfrac{\pi}{4} \quad or \quad \theta = 2n\pi \pm \dfrac{\pi}{3}$

Practice Problem

Q.1 Prove that the equation $x^3 - 2x + 1 = 0$ is satisfied by putting for x, either of the values.

$\sqrt{2}\sin 45^\circ$, 2 sin 18° and 2 sin 234°.

Q.2 If $\sin(\pi\cos\theta) = \cos(\pi\sin\theta)$, prove that $\cos\left(\theta \pm \frac{\pi}{4}\right) = \frac{1}{2\sqrt{2}}$

Q.3 If $\sin(\pi\cot\theta) = \cos(\pi\tan\theta)$, prove that either cosec 2θ or cot 2θ is equal to $n + \frac{1}{4}$, where n is a positive or negative integer.

2.6 TYPE-VI :

Solving equations with the use of boundedness of the function.

Solving equations with the use of boundedness of the function.

Remember :-

$-1 \le \sin x \le 1,\ -1 \le \cos x \le 1,\ \tan x \in R,\ \cot x \in R.$

$|\operatorname{cosec} x| \ge 1,\ |\sec x| \ge 1.$

Illustration :

Solve for x : cos x + cos 2x + cos 3x = 3.

Sol. $\cos x = 1$ and $\cos 2x = 1$ and $\cos 3x = 1$

when $\cos x = 1 \Rightarrow x = 2n\pi,\ n \in I$

when $\cos 2x = 1 \Rightarrow x = \frac{2n\pi}{2} = n\pi,\ n \in I$

when $\cos 3x = 1 \Rightarrow 3x = 2n\pi$

$\Rightarrow x = \frac{2n\pi}{3},\ n \in I$

$2n\pi,\ n \in I$ *Ans.*

Illustration :

Solve for x : $\sin^3 x - \cos^3 x = 1 + \sin x \cos x$.

Sol. $(\sin x - \cos x)(\sin^2 x + \cos^2 x + \sin x \cos x) = 1 + \sin x \cos x$

$\Rightarrow \quad (\sin x - \cos x)(1 + \sin x \cos x) = 1 + \sin x \cos x$

$\Rightarrow \quad \sin x - \cos x = 1 \qquad or \qquad 1 + \sin x \cos x = 0$

$\Rightarrow \quad \cos\left(x + \frac{\pi}{4}\right) = \frac{-1}{\sqrt{2}}$ | $2 \sin x \cos x = -2$

$\Rightarrow \quad x + \frac{\pi}{4} = 2n\pi \pm \frac{3\pi}{4}$ | $\sin 2x = -2$

$\Rightarrow \quad x = 2n\pi \pm \frac{3\pi}{4} - \frac{\pi}{4}$ | $\Rightarrow$ No solution.

$\Rightarrow \quad x = (2n-1)\pi,\ 2n\pi + \frac{\pi}{2}$

Illustration :

Illustration :

Solve for x : $\sin x\left(\cos\frac{x}{4} - 2\sin x\right) + \left(1 + \sin\frac{x}{4} - 2\cos x\right)\cos x = 0$

Sol. $\sin\left(\frac{5x}{4}\right) + \cos x = 2$

$\Rightarrow \quad \sin\left(\frac{5x}{4}\right) = 1 \quad and \quad \cos x = 1$

$\Rightarrow \quad \frac{5x}{4} = (4n+1)\frac{\pi}{2}$ | $x = 2m\pi$

$\Rightarrow \quad x = (4n+1)\frac{2\pi}{5}$ | $x = 0, \pm 2\pi, \pm 4\pi$

Period of the given equation is 8π.

$\therefore \quad$ consider $\quad x \in [0, 8\pi)$

$\Rightarrow \quad x = \frac{2\pi}{5}, 2\pi, \frac{18\pi}{5}, \frac{26\pi}{5}, \frac{34\pi}{5}$

common solution $= 2\pi$

$\therefore \quad 8n\pi + 2\pi = 2\pi(4n+1) \qquad$ Ans.

Illustration :

If $x, y \in [0, 2\pi]$ then find the total number of order pairs (x, y) satisfying the equation $\sin x \cos y = 1$.

Sol. *We have $\sin x \cos y = 1$*

$\Rightarrow \quad \sin x = 1, \cos y = 1 \quad$ *or* $\quad \sin x = -1, \cos y = -1$

If $\quad \sin x = 1, \cos y = 1 \quad \Rightarrow \quad x = \dfrac{\pi}{2}, y = 0, 2\pi$

If $\quad \sin x = -1, \cos y = -1 \quad \Rightarrow \quad x = \dfrac{3\pi}{2}, y = \pi$

Then the possible ordered pairs are $\left(\dfrac{\pi}{2}, 0\right), \left(\dfrac{\pi}{2}, 2\pi\right), \left(\dfrac{3\pi}{2}, \pi\right)$.

Practice Problem

Practice Problem

Q.1 Solve : $\cos^{50} x - \sin^{50} x = 1$.

Q.2 Solve : $\sin^2 x + \cos^2 y = 2\sec^2 z$ for x, y, z.

Q.3 Solve : $1 + \sin x \sin^2 \dfrac{x}{2} = 0$

Q.4 Solve : $12 \sin x + 5 \cos x = 2y^2 - 8y + 21$, to get the values of x and y.

Q.5 Solve for x and y : $1 - 2x - x^2 = \tan^2 (x + y) + \cot^2 (x + y)$

Answer key

Q.1 $n\pi$ Q.2 $x = n\pi + \dfrac{\pi}{2}, y = m\pi, z = k\pi$. Q.3 $x \in \phi$

Q.4 $2n\pi + \alpha$, where $\tan \alpha = \dfrac{12}{5}$ Q.5 $x = -1, y = n\pi \pm \dfrac{\pi}{4} + 1$

2.7 TYPE-VII :

Solution of trigonometric equation of the form $f(x) = \sqrt{\phi(x)}$.

(i) $f(x) \geq 0, \phi(x) \geq 0$ (ii) $f^2(x) = \phi(x)$

Illustration :

Solve for x, $\sqrt{(1-\cos x)} = \sin x$.

Sol. $\sin x \geq 0$ and $1 - \cos x \geq 0$ ...(i)

$\therefore \quad 1 - \cos x = \sin^2 x = 1 - \cos^2 x$

$\Rightarrow \quad (1 - \cos x)\,[1 - (1 + \cos x)] = 0 \quad \Rightarrow \quad \cos x = 0, 1$

when $\cos x = 0, \quad x = 2n\pi \pm \dfrac{\pi}{2}$ but $\sin x \geq 0 \Rightarrow x = 2n\pi + \dfrac{\pi}{2}$

when $\cos x = 1, \quad x = 2n\pi$

when $\cos x = 0, \quad x = 2n\pi \pm \dfrac{\pi}{2}$ but $\sin x \geq 0 \Rightarrow x = 2n\pi + \dfrac{\pi}{2}$

when $\cos x = 1, \quad x = 2n\pi$

Both are satisfying in equality (i)

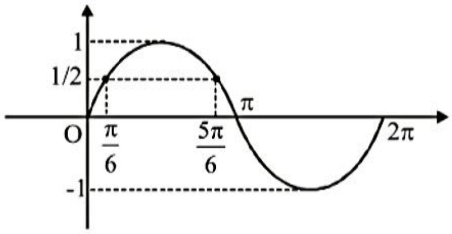
3 TRIGONOMETRIC INEQUALITIES AND SYSTEM OF INEQUALITY:

To solve the trigonometric inequalities of the type $f(x) \leq a$ or $f(x) \geq a$, where f(x) is some trigonometric ratio we take following steps

1. Draw the graph of f(x) in an interval length equal to the fundamental period of f(x).
2. Draw the line $y = a$.
3. Take the portion of the graph for which the inequality is satisfied.
4. To generalise, add $p \cdot n, n \in I$ and in the final solution where p is the fundamental period of f(x).

Illustration :

Solve : $\sin x > +\frac{1}{2}$.



Sol. $\sin x = \frac{1}{2}, x = \frac{\pi}{6}, \frac{5\pi}{6}$

$\sin x > \frac{1}{2}$ *for* $\frac{\pi}{6} < x < \frac{5\pi}{6}$ *fundamental period of* $\sin x$ *is* 2π. *So adding* $2n\pi$ *on both sides,*

$2n\pi + \frac{\pi}{6} < x < 2n\pi + \frac{5\pi}{6}$.

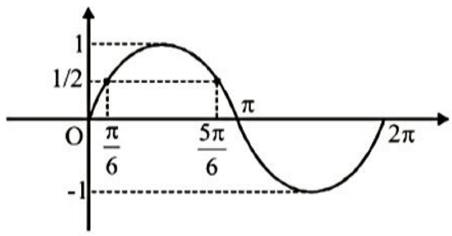
Illustration :

Solve : $2\sin^2\theta - \sin\theta \geq 0$, *where* $\frac{\pi}{2} \leq \theta \leq \frac{3\pi}{2}$.

Sol. $\sin\theta\,(2\sin\theta - 1) \geq 0$ *which is possible. only where*

Sol. $\sin\theta\,(2\sin\theta - 1) \geq 0$ *which is possible. only where*

$\sin\theta \geq \frac{1}{2}$ *or* $\sin\theta \leq 0$



$\sin\theta \geq \frac{1}{2} \quad \Rightarrow \quad \frac{\pi}{2} \leq \theta \leq \frac{5\pi}{6}$

$\sin\theta \leq 0 \quad \Rightarrow \quad \pi \leq \theta \leq \frac{3\pi}{2}$

$\theta \in \left[\frac{\pi}{2}, \frac{5\pi}{6}\right] \cup \left[\pi, \frac{3\pi}{2}\right]$

Illustration :

Solve : $\sin\theta + \sqrt{3}\cos\theta \geq 1, -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$

Sol. $\frac{1}{2}\sin\theta + \frac{\sqrt{3}}{2}\cos\theta \geq \frac{1}{2}$

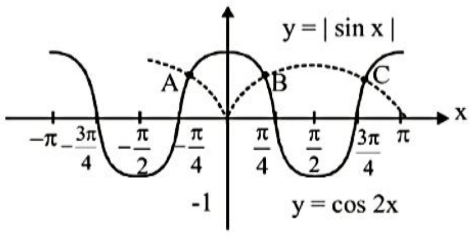
$\sin\left(\theta + \frac{\pi}{3}\right) \geq \frac{1}{2}$

$\frac{\pi}{6} \leq \theta + \frac{\pi}{3} \leq \frac{5\pi}{6} \quad \Rightarrow \quad -\frac{\pi}{6} \leq \theta \leq \frac{\pi}{2}$

Illustration :

Solve : $\cos 2x > |\sin x|,\ x \in \left(-\frac{\pi}{2}, \pi\right)$

Sol.



For points B and C

$\cos 2x = \sin x,\ 2\sin^2 x + \sin x - 1 = 0,\ \sin x = -1,\ \frac{1}{2}$

$\sin x = \frac{1}{2} \quad \Rightarrow \quad x = \frac{\pi}{6}, \frac{5\pi}{6}.$

Since $\cos 2x$ *and* $|\sin x|$ *are even fucntion so x-coordinate of point A* $= -\frac{\pi}{6}$

From graph $\Rightarrow x \in \left(-\frac{\pi}{6}, \frac{\pi}{6}\right) \cup \left(\frac{5\pi}{6}, \pi\right)$

From graph $\Rightarrow x \in \left(-\frac{\pi}{6}, \frac{\pi}{6}\right) \cup \left(\frac{5\pi}{6}, \pi\right)$

Practice Problem

Q.1 Find the general solution of the following inequations

(i) $\sin x > 0$ (ii) $\sin x > \frac{1}{2}$ (iii) $\sin x \le \frac{1}{2}$

(iv) $\log_2\left(\sin\frac{x}{2}\right) < -1$ (v) $\cos x < -\frac{1}{2}$ (vi) $\tan x > 0$

Q.2 $\tan^2 x - \left(\sqrt{3}+1\right)\tan x + \sqrt{3} < 0.$

Q.3 Solve the inequality, $\sin x \ge \cos 2x.$

Q.4 $\sqrt{5 - 2\sin x} \ge 6\sin x - 1.$

Answer key

Q.1 (i) $\bigcup_{n\in I} (2n\pi, 2n\pi+\pi)$

(ii) $\bigcup_{n\in I} \left(2n\pi+\frac{\pi}{6}, 2n\pi+\frac{5\pi}{6}\right)$

(iii) $\bigcup_{n\in I} \left[2n\pi, 2n\pi+\frac{\pi}{6}\right] \bigcup_{n\in I} \left[2n\pi+\frac{5\pi}{6}, 2n\pi+2\pi\right]$

(iv) $\bigcup_{n\in I} \left(4n\pi, 4n\pi+\frac{\pi}{3}\right) \bigcup_{n\in I} \left(4n\pi+\frac{5\pi}{3}, 4n\pi+2\pi\right)$

(v) $\bigcup_{n\in I} \left(2n\pi+\frac{2\pi}{3}, 2n\pi+\frac{4\pi}{3}\right)$

(vi) $\bigcup_{n\in I} \left(n\pi, n\pi+\frac{\pi}{2}\right)$

(v) $\bigcup_{n\in I} \left(2n\pi+\frac{2\pi}{3}, 2n\pi+\frac{4\pi}{3}\right)$

(vi) $\bigcup_{n\in I} \left(n\pi, n\pi+\frac{\pi}{2}\right)$

Q.2 $\bigcup_{n\in I} \left(n\pi+\frac{\pi}{4}, n\pi+\frac{\pi}{3}\right)$

Q.3 $\bigcup_{n\in I} \left[2n\pi+\frac{\pi}{6}, 2n\pi+\frac{5\pi}{6}\right], 2n\pi-\frac{\pi}{2}.$

Q.4 $\bigcup_{n\in I} \left[2n\pi-\frac{7\pi}{6}, 2n\pi+\frac{\pi}{6}\right]$

Solved Examples

Q.1 If $\cos p\,\theta + \cos q\,\theta = 0$, then the different vlaues of θ are in A.P., where the common difference is

(A) $\frac{\pi}{p+q}$ (B) $\frac{\pi}{p-q}$ (C) $\frac{2\pi}{p \pm q}$ (D) $\frac{3\pi}{p \pm q}$

Sol. $\cos p\theta = -\cos q\,\theta = \cos(\pi - q\,\theta)$

$p\theta = 2n\pi \pm (\pi - q\theta)$, $(p \pm q)\,\theta = (2n \pm 1)\pi$

$$\theta = \frac{(2n \pm 1)\pi}{(p \pm q)} = \frac{r\pi}{(p \pm q)}, \text{ where } r = -3, -1, 1, 3, \ldots$$

$\Rightarrow$ $\theta = \ldots, \frac{-3\pi}{(p \pm q)}, \frac{-\pi}{(p \pm q)}, \frac{\pi}{(p \pm q)}, \frac{3\pi}{(p \pm q)}, \ldots$

So common difference $= \frac{2\pi}{(p \pm q)}$

Q.2 : If $3\tan^2\theta - 2\sin\theta = 0$, then θ is equal to

$n\pi$ π π π

Q.2 : If $3\tan^2\theta - 2\sin\theta = 0$, then θ is equal to

(A) $\frac{n\pi}{4}$ (B) $n\pi + (-1)^n \frac{\pi}{6}$ (C) $n\pi + (-1)^n \frac{\pi}{3}$ (D) $n\pi + \frac{\pi}{3}$

Sol. $\frac{3\sin^2\theta}{\cos^2\theta} - 2\sin\theta = 0$, $\cos\theta \neq 0$

$\Rightarrow$ $3\sin^2\theta - 2\sin\theta(\cos^2\theta) = 0$, $3\sin^2\theta - 2\sin\theta(1 - \sin^2\theta) = 0$

$\Rightarrow$ $\sin\theta(2\sin^2\theta + 3\sin\theta - 2) = 0$

$\Rightarrow$ $\sin\theta(2\sin\theta - 1)(\sin\theta + 2) = 0 \Rightarrow \sin\theta = 0, \frac{1}{2}, -2$ (rejected)

$\Rightarrow$ $\theta = n\pi, n\pi + (-1)^n \frac{\pi}{6}$

Q.3 If $0 \le x \le 2\pi$, then the number of solutions of equation $3(\sin x + \cos x) - 2(\sin^3 x + \cos^3 x) = 8$ is

(A) 0 (B) 1 (C) 2 (D) 4

Sol. $3(\sin x + \cos x) - 2(\sin x + \cos x)(\sin^2 x + \cos^2 x - \sin x \cos x) = 8$

$\Rightarrow$ $(\sin x + \cos x)[3 - 2 + 2\sin x \cos x] = 8$

$\Rightarrow$ $(\sin x + \cos x)[1 + 2\sin x \cos x] = 8$

$\Rightarrow$ $(\sin x + \cos x)^3 = 8$, $\sin x + \cos x = 2$

No solution

Q.4 If $\frac{1}{6}\sin\theta, \cos\theta, \tan\theta$ are in G.P. then θ is equal to $(n \in I)$.

(A) $2n\pi \pm \frac{\pi}{3}$ (B) $2n\pi \pm \frac{\pi}{6}$ (C) $n\pi + (-1)^n \frac{\pi}{3}$ (D) $n\pi + \frac{\pi}{3}$

Sol. $\cos^2\theta = \frac{1}{6}\sin\theta\tan\theta \quad \Rightarrow \quad 6\cos^3\theta = 1 - \cos^2\theta$

$\Rightarrow \quad 6\cos^3\theta + \cos^2\theta - 1 = 0 \quad \Rightarrow \quad (2\cos\theta - 1)(3\cos^2\theta + 2\cos\theta + 1) = 0$

$\Rightarrow \quad \cos\theta = \frac{1}{2} \quad \Rightarrow \quad \theta = 2n\pi \pm \frac{\pi}{3}, \; n \in I$

Q.5 If $\sin x + \cos x = \sqrt{\left(y + \frac{1}{y}\right)}$, $x \in [0, \pi]$ then

(A) $x = \frac{\pi}{4}, y = 1$ (B) $y = 0$ (C) $y = 2$ (D) $x = \frac{3\pi}{4}$

Sol. $\sqrt{y + \frac{1}{y}} \geq \sqrt{2}$ assuming $y > 0$

Sol. $\sqrt{y + \frac{1}{y}} \geq \sqrt{2}$ assuming $y > 0$

But $|\sin x + \cos x| \leq \sqrt{2}$ so $y = 1$ & $x = \frac{\pi}{4}$

Q.6 Let $\theta \in [0, 4\pi]$ satisfying the equation $(\sin\theta + 2)(\sin\theta + 3)(\sin\theta + 4) = 6$. If the sum of all the values of θ is of the term $k\pi$ then value of k is

(A) 6 (B) 5 (C) 4 (D) 2

Sol. since, L.H.S. ≥ 6 and R.H.S. $= 6$, so equality holds

only if $\sin\theta = -1 \quad \Rightarrow \quad \theta = \frac{3\pi}{2}, \frac{7\pi}{2}$

$\therefore \quad \text{sum} = 5\pi \quad \Rightarrow \quad k = 5$

Q.7 For $n \in I$, the general solution of the equation $\left(\sqrt{3} - 1\right)\sin\theta + \left(\sqrt{3} + 1\right)\cos\theta = 2$ is

(A) $\theta = 2n\pi \pm \frac{\pi}{4} + \frac{\pi}{12}$ (B) $\theta = n\pi + (-1)^n\frac{\pi}{4} + \frac{\pi}{12}$

(C) $\theta = 2n\pi \pm \frac{\pi}{4}$ (D) $\theta = n\pi + (-1)^n\frac{\pi}{4} - \frac{\pi}{12}$

Sol. $$\frac{(\sqrt{3}-1)}{2\sqrt{2}}\sin\theta + \frac{(\sqrt{3}+1)}{2\sqrt{2}}\cos\theta = \frac{1}{\sqrt{2}}$$

$\Rightarrow$ $$\sin\frac{\pi}{12}\sin\theta + \cos\frac{\pi}{12}\cos\theta = \cos\frac{\pi}{4}$$

$$\cos\left(\theta - \frac{\pi}{12}\right) = \cos\frac{\pi}{4}, \qquad \theta - \frac{\pi}{12} = 2n\pi \pm \frac{\pi}{4}$$

$$\theta = 2n\pi \pm \frac{\pi}{4} + \frac{\pi}{12}$$

Q.8 If $\tan\left(\frac{p\pi}{4}\right) = \cot\left(\frac{q\pi}{4}\right)$ and $n \in I$, then

(A) $p + q = 0$ (B) $p + q = 2n + 1$ (C) $p + q = 2n$ (D) $p + q = 2\,(2n + 1)$

Sol. $$\tan\left(\frac{p\pi}{4}\right) = \tan\left(\frac{\pi}{2} - \frac{q\pi}{4}\right)$$

$\Rightarrow$ $$\frac{p\pi}{4} = n\pi + \frac{\pi}{2} - \frac{q\pi}{4} \quad \Rightarrow \quad \frac{(p+q)}{4} = n + \frac{1}{2}$$

$\Rightarrow$ $$\frac{p\pi}{4} = n\pi + \frac{\pi}{2} - \frac{q\pi}{4} \quad \Rightarrow \quad \frac{(p+q)}{4} = n + \frac{1}{2}$$

$\Rightarrow$ $(p + q) = 2(2n + 1)$

Q.9 The number of solution of the equation $\sin x + \sin 2x + \sin 3x = \cos x + \cos 2x + \cos 3x$, $0 \le x \le 2\pi$ is

(A) 7 (B) 5 (C) 4 (D) 6

Sol. $(\sin x + \sin 3x) + \sin 2x = (\cos x + \cos 3x) + \cos 2x$

$\Rightarrow$ $2 \sin 2x \cos x + \sin 2x = 2 \cos 2x \cos x + \cos 2x$

$\Rightarrow$ $\sin 2x\,(2 \cos x + 1) = \cos 2x\,(2 \cos x + 1)$

$\Rightarrow$ $\cos x = -\frac{1}{2}, \quad \tan 2x = 1$

$$x = 2n\pi \pm \frac{2\pi}{3}, 2x = n\pi + \frac{\pi}{4}$$

$\Rightarrow$ $$x = 2n\pi \pm \frac{2\pi}{3}, \frac{n\pi}{2} + \frac{\pi}{8}$$

$0 \le x \le 2\pi$

$$x = \frac{\pi}{8}, \frac{5\pi}{8}, \frac{2\pi}{3}, \frac{9\pi}{8}, \frac{4\pi}{3}, \frac{13\pi}{8}$$

Q.10 The general solution of the equation $\sin 3\alpha = 4 \sin \alpha \sin (x+\alpha) \sin (x-\alpha)$ is $(\alpha \neq n\pi)$.

(A) $n\pi \pm \frac{\pi}{4}\ \forall\ n \in I$ (B) $n\pi \pm \frac{\pi}{3}\ \forall\ n \in I$

(C) $n\pi \pm \frac{\pi}{9}\ \forall\ n \in I$ (D) $n\pi \pm \frac{\pi}{12}\ \forall\ n \in I$

Sol. $\sin 3\alpha = 4 \sin \alpha (\sin^2 x - \sin^2 \alpha)$

$\Rightarrow$ $3 \sin \alpha - 4 \sin^3 \alpha = 4 \sin \alpha \sin^2 x - 4 \sin^3 \alpha$

$\Rightarrow$ $3 \sin \alpha = 4 \sin \alpha \sin^2 x$

If $\sin \alpha \neq 0$ $\quad \sin^2 x = \frac{3}{4}, \quad \sin x = \pm\frac{\sqrt{3}}{2}$

$x = n\pi \pm \frac{\pi}{3}\ \forall\ n \in I$

If $\sin \alpha = 0$; i.e. $\alpha = n\pi$, then equation becomes an identiy.

Q.11 Find the general solution of $\sin^2 x + \frac{1}{4}\sin^2 3x = \sin x \sin^2 3x$

Sol. $\sin^2 x - \sin x \sin^2 3x + \frac{1}{4}\sin^2 3x = 0$

Sol. $\sin^2 x - \sin x \sin^2 3x + \frac{1}{4}\sin^2 3x = 0$

$\Rightarrow$ $\left(\sin x - \frac{1}{2}\sin^2 3x\right)^2 + \frac{1}{4}\sin^2 3x\,(1-\sin^2 3x) = 0$

$\Rightarrow$ $\left(\sin x - \frac{1}{2}\sin^2 3x\right)^2 + \frac{1}{4}\sin^2 3x \cos^2 3x = 0$

$\Rightarrow$ $\left(\sin x - \frac{1}{2}\sin^2 3x\right)^2 + \frac{1}{16}\sin^2 6x = 0$

$\Rightarrow$ $\sin x - \frac{1}{2}\sin^2 3x = 0$ and $\sin 6x = 0$

$\Rightarrow$ $2 \sin x = \sin^2 3x$ and $\sin 6x = 0$ $\Rightarrow$ $x = \frac{k\pi}{6} = k \in I.$

$$\sin^2\left(3\left(\frac{k\pi}{6}\right)\right) = \sin^2\left(\frac{k\pi}{2}\right) = \begin{cases} 1, & \text{if k is odd} \\ 0, & \text{if k is even} \end{cases}$$

$\Rightarrow$ $\sin x = 0$ or $\frac{1}{2}$

$\Rightarrow$ $x = n\pi$ or $x = n\pi + (-1)^n \frac{\pi}{6}, \quad n \in I$

Q.12 Find the general solution of $\tan\left(\frac{\pi}{2}\cos\theta\right)=\cot\left(\frac{\pi}{2}\sin\theta\right)$

Sol. $\tan\left(\frac{\pi}{2}\cos\theta\right)=\tan\left(\frac{\pi}{2}-\frac{\pi}{2}\sin\theta\right)$

$\Rightarrow \quad \frac{\pi}{2}\cos\theta = n\pi+\frac{\pi}{2}-\frac{\pi}{2}\sin\theta \quad \Rightarrow \quad \sin\theta+\cos\theta=(2n+1)$

$\Rightarrow \quad \sqrt{2}\cos\left(\theta-\frac{\pi}{4}\right)=(2n+1) \quad \Rightarrow \quad n=0,-1$ are the only possibility

$\Rightarrow \quad \cos\left(\theta-\frac{\pi}{4}\right)=\pm\frac{1}{\sqrt{2}}$, when $\cos\left(\theta-\frac{\pi}{4}\right)=\frac{1}{\sqrt{2}}=\cos\frac{\pi}{4}$

$\Rightarrow \quad \theta-\frac{\pi}{4}=2n\pi\pm\frac{\pi}{4}, \quad \theta=2n\pi \quad$ or $\quad 2n\pi+\frac{\pi}{2}$

when $\cos\left(\theta-\frac{\pi}{4}\right)=-\frac{1}{\sqrt{2}}=\cos\left(\frac{3\pi}{4}\right)$, $\theta-\frac{\pi}{4}=2n\pi\pm\frac{3\pi}{4}$

$\theta=(2n+1)\pi \quad$ or $\quad 2n\pi-\frac{\pi}{2}$

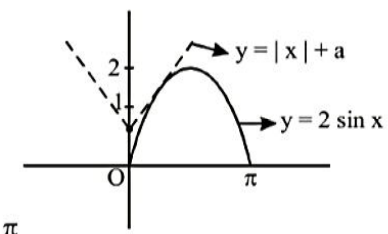
$\theta=(2n+1)\pi \quad$ or $\quad 2n\pi-\frac{\pi}{2}$

So $\quad \theta=2n\pi,(2n+1)\pi,\left(2n\pm\frac{1}{2}\right)\pi$

$\theta=m\pi,\left(2n\pm\frac{1}{2}\right)\pi \quad$ Ans.

Q.13 Prove that the equation $2\sin x=|x|+a$ has no solution for $a\in\left(\frac{3\sqrt{3}-\pi}{3},\infty\right)$

Sol. The equation $2\sin x=|x|+a$ will have a solution so long as the line $y=|x|+a$ intersects or at least touches the curve, $y=2\sin x$. So $\frac{dy}{dx}=2\cos x=1$.



$\Rightarrow \quad \cos x=\frac{1}{2} \quad \Rightarrow \quad x=\frac{\pi}{3}$

Hence, solution will not be possible if $\quad x+a>2\sin x$ at $x=\frac{\pi}{3}$

$\frac{\pi}{3}+a>2\cdot\frac{\sqrt{3}}{2} \Rightarrow a>\frac{3\sqrt{3}-\pi}{3}$

Q.14 Find the smallest positive root of the equation $\sqrt{\sin(1-x)} = \sqrt{\cos x}$?

Sol. $\sin(1-x) \geq 0$ and $\cos x \geq 0$

$\sin(1-x) = \cos x, \qquad \cos\left(\frac{\pi}{2} - (1-x)\right) = \cos x$

$\Rightarrow \quad \frac{\pi}{2} - 1 + x = 2n\pi \pm x \quad \Rightarrow \quad x = \dfrac{2n\pi - \frac{\pi}{2} + 1}{2}$

For $n = 2$, $\quad x = \frac{7\pi}{4} + \frac{1}{2}$ which is the smallest positive root of the given equation.

Q.15 Solve the equation $\tan^4 x + \tan^4 y + 2\cot^2 x \cot^2 y = 3 + \sin^2(x+y)$ for the values of x and y.

Sol. $\tan^4 x + \tan^4 y + 2\cot^2 x \cot^2 y - 2 = 1 + \sin^2(x+y)$

$(\tan^2 x - \tan^2 y)^2 + 2(\tan x \tan y - \cot x \cot y)^2 = -1 + \sin^2(x+y)$

L.H.S. ≥ 0 and R.H.S. $\leq 0 \quad \Rightarrow \quad$ L.H.S. = R.H.S. = 0

Sol. $\tan^4 x + \tan^4 y + 2\cot^2 x \cot^2 y - 2 = 1 + \sin^2(x+y)$

$(\tan^2 x - \tan^2 y)^2 + 2(\tan x \tan y - \cot x \cot y)^2 = -1 + \sin^2(x+y)$

L.H.S. ≥ 0 and R.H.S. $\leq 0 \quad \Rightarrow \quad$ L.H.S. = R.H.S. = 0

$\Rightarrow \quad \tan^2 x = \tan^2 y \quad$ and $\quad \tan x \tan y = \cot x \cot y$

$\Rightarrow \quad \tan^2 x \tan^2 y = 1 \quad$ and $\quad \sin^2(x+y) = 1$

$\Rightarrow \quad \tan^2 x = \tan^2 y = 1$

$x = n\pi \pm \frac{\pi}{4},\ y = n\pi \pm \frac{\pi}{4},\ x + y = 2n\pi \pm \frac{\pi}{2}$

So, $\quad x = y = n\pi \pm \frac{\pi}{4}$. $\qquad$ Ans.

Q.16 Solve : $\sin x + \sin y = \sin(x+y)$ and $|x| + |y| = 1$

Sol. $2\sin\left(\frac{x+y}{2}\right)\left[\cos\left(\frac{x-y}{2}\right) - \cos\left(\frac{x+y}{2}\right)\right] = 0$

$\Rightarrow \quad 4\sin\left(\frac{x+y}{2}\right)\sin\frac{x}{2} \cdot \sin\frac{y}{2} = 0$

a. $\sin\left(\frac{x+y}{2}\right)=0 \quad \Rightarrow x+y=2n\pi \quad \Rightarrow \quad x+y=0, n\in I$

b. $\sin\frac{x}{2}=0 \quad \Rightarrow x=2m\pi, m\in I \quad \Rightarrow \quad x=0$

c. $\sin\frac{y}{2}=0 \quad \Rightarrow y=2p\pi, p\in I \quad \Rightarrow \quad y=0$

Now, In $|x|+|y|=1$, if $x=0$, then $|y|=1 \quad \Rightarrow \quad y=\pm 1$

If $y=0$, then $|x|=1 \quad \Rightarrow \quad x=\pm 1$

If $y=-x$, then $|x|+|-x|=1 \Rightarrow x=\pm\frac{1}{2}$ and $y=\mp\frac{1}{2}$

Solutions are $(0, 1), (0, -1), (1, 0), (-1, 0), \left(\frac{1}{2}, \frac{-1}{2}\right)$ and $\left(\frac{-1}{2}, \frac{1}{2}\right)$.

Q.17 Solve the inequality $\sin^4\left(\frac{x}{3}\right)+\cos^4\left(\frac{x}{3}\right)>\frac{1}{2}$

Q.17 Solve the inequality $\sin^4\left(\frac{x}{3}\right)+\cos^4\left(\frac{x}{3}\right)>\frac{1}{2}$

Sol. $1-2\sin^2\left(\frac{x}{3}\right)\cos^2\left(\frac{x}{3}\right)>\frac{1}{2}$

$\Rightarrow 1-\frac{1}{2}\sin^2\left(\frac{2x}{3}\right)>\frac{1}{2}$

$\Rightarrow \quad \sin^2\left(\frac{2x}{3}\right)<1$

which is always true except when $\sin^2\left(\frac{2x}{3}\right)=1$

$\frac{2x}{3}=n\pi\pm\frac{\pi}{2} \quad$ or $\quad x=\frac{3n\pi}{2}\pm\frac{3\pi}{4}$

So, solution of x is $R\sim\{x : x=\frac{3n\pi}{2}\pm\frac{3\pi}{4}, n\in I\}$.

Q.18 Solve $3\tan 2x - 4\tan 3x = \tan^2 3x \tan 2x$.

Sol. $3(\tan 2x - \tan 3x) = \tan 3x(1 + \tan 3x \tan 2x)$

$$\Rightarrow 3\left(\frac{\tan 2x - \tan 3x}{1+\tan 3x \tan 2x}\right) = \tan 3x \Rightarrow -3\tan(3x - 2x) = \tan 3x$$

$$\Rightarrow -3\tan x = \frac{3\tan x - \tan^3 x}{1 - 3\tan^2 x} \quad \Rightarrow -3 = \frac{3 - \tan^2 x}{1 - 3\tan^2 x}$$

or $\tan x = 0$ so $x = n\pi$, $\tan^2 x = \frac{3}{5}$

$x = n\pi \pm \tan^{-1}\sqrt{\frac{3}{5}}$, $n \in I$

$x = n\pi$, $n \in I$. Ans.

Q.19 Find the general solution of the equation $\sin^{100}x - \cos^{100}x = 1$.

Sol. $\sin^{100}x = 1 + \cos^{100}x$ L.H.S. ≤ 1, R.H.S. ≥ 1

So, L.H.S. = R.H.S. = 1

$x = n\pi$, $n \in I$. Ans.

Q.19 Find the general solution of the equation $\sin^{100}x - \cos^{100}x = 1$.

Sol. $\sin^{100}x = 1 + \cos^{100}x$ L.H.S. ≤ 1, R.H.S. ≥ 1

So, L.H.S. = R.H.S. = 1

$\cos^{100}x = 0$, $\sin^{100}x = 1$; $x = n\pi \pm \frac{\pi}{2}$. Ans.]

Q.20 Find the set of all x in $(-\pi, \pi)$ satisfying $|4\sin x - 1| < \sqrt{5}$.

Sol. $-\sqrt{5} < 4\sin x - 1 < \sqrt{5}$, $\frac{-(\sqrt{5}-1)}{4} < \sin x < \frac{(\sqrt{5}-1)}{4}$

$$\Rightarrow \quad -\sin\left(\frac{\pi}{10}\right) < \sin x < \cos\left(\frac{2\pi}{10}\right) \Rightarrow \sin\left(\frac{-\pi}{10}\right) < \sin x < \sin\left(\frac{3\pi}{10}\right)$$

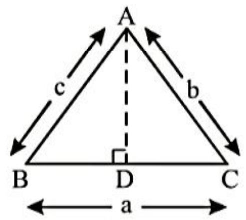
$$x \in \left(\frac{-\pi}{10}, \frac{3\pi}{10}\right)$$

SOLUTIONS OF TRIANGLE

1. INTRODUCTION :

In any triangle, the three sides and the three angles are often called the elements of the triangle. When three elements of a triangle are given, the process of calculating its other three elements is called solution of the Triangle.

In any triangle ABC, the side BC, opposite to the angle A, is denoted by a; the sides CA and AB opposite to the angle B and C respectively are denoted by b & c.



In any triangle ABC

(i) $A + B + C = 180^\circ$, $A, B, C > 0$

(ii) $a + b > c$, $b + c > a$, $c + a > b$

(iii) $|a - b| < c$, $|b - c| < a$, $|c - a| < b$.

(iv) $a, b, c > 0$

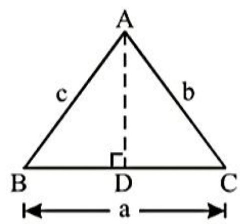
1.1 SINE RULE :

In any triangle ABC, $\boxed{\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}}$ i.e., the sines of the angle are proportional to the opposite sides.

Proof :-

Consider the acute angle triangle ABC

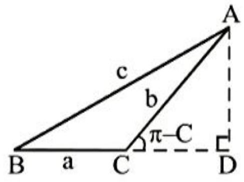
Draw AD perpendicular to the opposite side BC



In the triangle ABD, we have $\frac{AD}{AB} = \sin B$, so that $AD = c \sin B$

In the triangle ACD, we have $\frac{AD}{AC}$

Consider the obtuse angle ΔABC



In a ΔABD, $\quad \sin B = \dfrac{AD}{AB} = \dfrac{AD}{c}$

In a ΔACD, $\quad \sin(\pi - C) = \dfrac{AD}{AC} = \dfrac{AD}{b}$

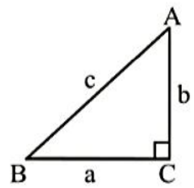
so, $\quad c \sin B = b \sin(\pi - C) = b \sin C$

In a similar manner, by drawing a perpendicular from B upon CA, we have

$$\frac{c}{\sin C} = \frac{a}{\sin A}$$

Consider right angled triangle $\quad C = 90^\circ$

If one of the angles C, be a right angle as in the figure, we have



$$\sin C = 1, \qquad \sin A = \frac{a}{c} \quad \text{and} \quad \sin B = \frac{b}{c}$$

So, $\quad \dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C} \qquad$ [Because $\sin C = 1$]

Hence for any type of triangle $\quad \dfrac{a}{\quad} = \dfrac{b}{\quad} = \dfrac{c}{\quad}$

Hence for any type of triangle, $\quad \dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$

Illustration :

Prove that $a \cos \dfrac{B-C}{2} = (b + c) \sin\left(\dfrac{A}{2}\right)$.

Sol. $\quad \dfrac{b+c}{a} = \dfrac{\sin B + \sin C}{\sin A} = \dfrac{2 \sin\left(\dfrac{B+C}{2}\right)\cos\left(\dfrac{B-C}{2}\right)}{2 \sin\dfrac{A}{2} \cos\dfrac{A}{2}}$

$$= \frac{\cos\dfrac{A}{2}\cos\left(\dfrac{B-C}{2}\right)}{\sin\dfrac{A}{2}\cos\dfrac{A}{2}} = \frac{\cos\left(\dfrac{B-C}{2}\right)}{\sin\dfrac{A}{2}}$$

so $\quad a \cos\left(\dfrac{B-C}{2}\right) = (b + c) \sin \dfrac{A}{2}$

Illustration :

Prove that $\frac{a-b}{a+b}=\tan\frac{A-B}{2}\cdot\cot\frac{A+B}{2}$.

Sol. $\frac{a-b}{a+b}=\frac{\sin A-\sin B}{\sin A+\sin B}=\frac{2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)}{2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)}$

$$=\tan\left(\frac{A-B}{2}\right)\cot\left(\frac{A+B}{2}\right)$$

so $\frac{a-b}{a+b}=\tan\left(\frac{A-B}{2}\right)\cot\left(\frac{A+B}{2}\right)$

Illustration :

Prove that $\frac{a^2\sin(B-C)}{\sin B+\sin C}+\frac{b^2\sin(C-A)}{\sin C+\sin A}+\frac{c^2\sin(A-B)}{\sin A+\sin B}=0$.

Sol. $\frac{a^2\sin(B-C)}{\sin B+\sin C}=K^2\frac{\sin^2 A\sin(B-C)}{\sin B+\sin C}$ $\quad\because\quad \frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}=K$ *(constant)*

$$=K^2\frac{\sin A\sin(B+C)\sin(B-C)}{\sin B+\sin C}=K^2\sin A\frac{(\sin^2 B-\sin^2 C)}{\sin B+\sin C}$$

$$=K^2\sin A\,(\sin B-\sin C)$$

Similarly $\frac{b^2\sin(C-A)}{\sin C-\sin A}=K^2\sin B\,(\sin C-\sin A)$, $\frac{c^2\sin(A-B)}{\sin A-\sin B}=K^2\sin C\,(\sin A-\sin B)$

so sum of three terms = 0

Illustration :

Prove that $\frac{1+\cos(A-B)\cos C}{1+\cos(A-C)\sin B}=\frac{a^2+b^2}{a^2+c^2}$

Sol. $\frac{1+\cos(A-B)\cos C}{1+\cos(A-C)\sin B}=\frac{1-\cos(A-B)\cos(A+B)}{1-\cos(A-C)\cos(A+C)}$

$$=\frac{1-(\cos^2 A-\sin^2 B)}{1-(\cos^2 A-\sin^2 C)}=\frac{\sin^2 A+\sin^2 B}{\sin^2 A+\sin^2 C}=\frac{a^2+b^2}{a^2+c^2}$$

Illustration :

In any trianlge, if $\frac{a^2-b^2}{a^2+b^2}=\frac{sin(A-B)}{sin(A+B)}$. *Then prove that the triangle is either right angled or Isosceles.*

Sol. $\frac{a^2-b^2}{a^2+b^2}=\frac{sin^2 A-sin^2 B}{sin^2 A+sin^2 B}=\frac{sin(A+B)sin(A-B)}{sin^2 A+sin^2 B}$

So, $\frac{sin(A+B)sin(A-B)}{sin^2 A+sin^2 B}=\frac{sin(A-B)}{sin(A+B)}$

$\Rightarrow \quad sin\,(A-B)=0 \quad or \quad \frac{sin(A+B)}{sin^2 A+sin^2 B}=\frac{1}{sin(A+B)} \quad or \quad \frac{sin C}{sin^2 A+sin^2 B}=\frac{1}{sin C}$

$\Rightarrow$ *so* $A = B$ *or* $sin^2 C = sin^2 A + sin^2 B$, $\quad c^2 = a^2 + b^2$

so either isosceles or right angled triangle.

Illustration :

If $A = 75^o$, $B = 45^o$, *then prove that* $b + c\sqrt{2} = 2a$.

If $A = 75^o$, $B = 45^o$, *then prove that* $b + c\sqrt{2} = 2a$.

Sol. $A = 75^o, \quad B = 45^o \quad \Rightarrow \quad C = 60^o$

$\frac{a}{sin 75^o}=\frac{b}{sin 45^o}=\frac{c}{sin 60^o}$ *so* $b+\sqrt{2}\,c = \frac{sin 45^o}{sin 75^o}a+\sqrt{2}\frac{sin 60^o}{sin 75^o}a$

$$b + c\sqrt{2} = \frac{\frac{1}{\sqrt{2}}}{\frac{\sqrt{3}+1}{2\sqrt{2}}}a+\sqrt{2}\frac{\frac{\sqrt{3}}{2}}{\frac{\sqrt{3}+1}{2\sqrt{2}}}a=\frac{2}{\sqrt{3}+1}a+\frac{2\sqrt{3}a}{\sqrt{3}+1}=2a$$

Illustration :

If $\frac{cos A}{a}=\frac{cos B}{b}=\frac{cos C}{c}$ *and the side* $a = 2$, *then find the area of the triangle?*

Sol. $\frac{cos A}{a}=\frac{cos B}{b}=\frac{cos C}{c}, \qquad \frac{cos A}{sin A}=\frac{cos B}{sin B}=\frac{cos C}{sin C}$

so $\quad cot A = cot B = cot C \Rightarrow$ *equilateral triangle.*

$$Area = \frac{\sqrt{3}}{4}a^2=\frac{\sqrt{3}}{4}(2)^2=\sqrt{3}$$

Practice Problem

Q.1 In a triangle ABC, if $\cos^2 A + \cos^2 B - \cos^2 C = 1$, then identify the type of the triangle?

Q.2 If angles A, B and C of a triangle ABC are in A.P. and if $\frac{b}{c} = \frac{\sqrt{3}}{\sqrt{2}}$, then find angle A.

Q.3 Prove that $b^2 \cos 2A - a^2 \cos 2B = b^2 - a^2$.

Q.4 Prove that $\frac{a\sin(B-C)}{b^2-c^2} = \frac{b\sin(C-A)}{c^2-a^2} = \frac{c\sin(A-B)}{a^2-b^2}$

Q.5 If in any triangle angles are in the ratio 1 : 2 : 3, then prove that the corresponding sides are as $1 : \sqrt{3} : 2$.

Q.6 If in a triangle ABC, $a \sin A = b \sin B$, then prove that the triangle is an isosceles triangle.

Answer key

Q.1 Right angle triangle Q.2 75º

1.2 COSINE RULE :

In a ΔABC, we have $\boxed{\cos A = \frac{b^2+c^2-a^2}{2bc},\ \cos B = \frac{c^2+a^2-b^2}{2ca} \text{ and } \cos C = \frac{a^2+b^2-c^2}{2ab}}$

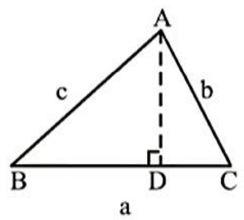
where a, b & c are sides and A, B & C are angle of the triangle.

Proof : Consider the acute angle ΔABC,

By geometry, we have

$$AB^2 = BC^2 + CA^2 - 2BC \cdot CD \qquad \ldots(i)$$

But $\frac{CD}{CA} = \cos C$, so that $CD = b \cos C$.



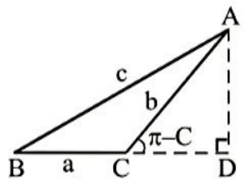
Hence (i) becomes $c^2 = a^2 + b^2 - 2ab \cos C$.

$$2ab \cos C = a^2 + b^2 - c^2, \text{ so } \cos C = \frac{a^2+b^2-c^2}{2ab}$$

Consider the obtuse angle ΔABC,

By geometry, we have $\qquad AB^2 = BC^2 + CA^2 + 2BC \cdot CD \qquad$ ...(ii)

But $\qquad \dfrac{CD}{CA} = \cos(\angle ACD) = \cos(180^\circ - C) = -\cos C$



so $\qquad CD = -b \cos C$

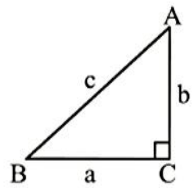
so equation (ii) becomes

$$c^2 = a^2 + b^2 + 2a(-b \cos C) = a^2 + b^2 - 2ab \cos C.$$

so once again, $\cos C = \dfrac{a^2 + b^2 - c^2}{2ab}$

Consider the right angle ΔABC,

If $\angle C = 90^\circ$, $\quad$ then $a^2 + b^2 = c^2$



so $\cos C = \dfrac{a^2 + b^2 + c^2}{2ab} = \dfrac{c^2 - a^2}{2ab} = 0$, we know that $\cos 90^\circ = 0$

so here also our formula is valid. So we can say in any type of triangle ABC, $\cos C = \dfrac{a^2 + b^2 - c^2}{2ab}$

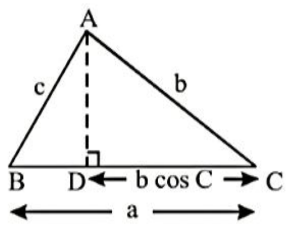
similarly cos B, cos A can be proved.

Note :- There is another way to prove cosine law consider the triangle as shown in figure.

$$AD = AC \sin C, = b \sin C, CD = AC \cos C = b \cos C$$

so $\qquad BD = BC - CD = a - b \cos C.$

ΔADB is a right angle triangle so



$$AB^2 = AD^2 + BD^2, \qquad c^2 = (b \sin C)^2 + (a - b \cos C)^2$$

$$c^2 = b^2 \sin^2 C + a^2 + b^2 \cos^2 C - 2ab \cos C = a^2 + b^2 - 2ab \cos C$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

Note :- (a) $\quad$ If the three sides of a triangle are known, we can find all the angles by using cosine rule.

(b) $\quad$ If in ΔABC, $a > b > c$, then $\angle A > \angle B > \angle C$ $\quad$ or $\quad$ vice-versa.

Illustration :

In any triangle ABC, prove that a (b cos C – c cos B) = $b^2 - c^2$.

Sol. $a(b\cos C - c\cos B) = ab\cos C - ac\cos B$

$$= ab\frac{(a^2+b^2-c^2)}{2ab} - ac\frac{(a^2+c^2-b^2)}{2ac}$$

$$= \frac{a^2+b^2-c^2-a^2-c^2+b^2}{2} = b^2 - c^2$$

Illustration :

If $a = \sqrt{3}$, $b = \frac{1}{2}\left(\sqrt{6}+\sqrt{2}\right)$, $c = \sqrt{2}$, then find $\angle A$?

Sol. $$\cos A = \frac{b^2+c^2-a^2}{2bc} = \frac{\frac{1}{4}\left(8+4\sqrt{3}\right)+2-3}{\sqrt{12}+\sqrt{4}} = \frac{\left(1+\sqrt{3}\right)}{2\left(1+\sqrt{3}\right)} = \frac{1}{2}$$

$$A = \frac{\pi}{3}$$

Illustration :

If the angle A, B, C of a triangle are in A.P. and sides a, b, c are in G.P., then prove that a^2, b^2, c^2 are in A.P.

Sol. *Given,* $2B = A + C = \pi - B \Rightarrow B = \frac{\pi}{3}$

Also, a, b, c are in G.P. $\Rightarrow b^2 = ac$

Now, $\cos B = \cos 60^o = \frac{1}{2} = \frac{c^2+a^2-b^2}{2ac} \Rightarrow ca = c^2 + a^2 - b^2$

$\Rightarrow b^2 = c^2 + a^2 - b^2 \Rightarrow 2b^2 = a^2 + c^2 \Rightarrow a^2, b^2, c^2$ are in A.P.

Illustration :

If in a triangle ABC, $\angle C = 60^o$, then prove that $\frac{1}{a+c} + \frac{1}{b+c} = \frac{3}{a+b+c}$.

Sol. *By the cosine formula, we have*

$c^2 = a^2 + b^2 - 2ab\cos C, \Rightarrow c^2 = a^2 + b^2 - 2ab\cos 60^o = a^2 + b^2 - ab$

Now, $$\frac{1}{a+c} + \frac{1}{b+c} - \frac{3}{a+b+c}$$

$$= \left[\frac{(b+c)(a+b+c)+(a+c)(a+b+c)-3(a+c)(b+c)}{(a+b)(b+c)(a+b+c)}\right]$$

$$= \frac{(a^2+b^2-ab)-c^2}{(a+b)(b+c)(a+b+c)} = 0$$

so $$\frac{1}{a+c} + \frac{1}{b+c} = \frac{3}{a+b+c}$$

Illustration :

If the sides of a triangle are $(x^2 + x + 1)$, $(x^2 - 1)$ & $(2x + 1)$, then find the largest angle?

Sol. Let $a = x^2 + x + 1, \quad b = 2x + 1, \quad c = x^2 - 1$

$a > 0 \Rightarrow x \in R$

$b > 0 \Rightarrow x > -\frac{1}{2}$

$c > 0 \Rightarrow x < -1 \text{ or } x > 1$

So, $x \in (1, \infty)$

$a - b = x^2 - x > 0 \Rightarrow a > b,$

$a - c = x + 2 > 0 \Rightarrow a > c$

So angle $\angle A$ is the largest angle

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} = \frac{(2x+1)^2 + (x^2-1)^2 - (x^2+x+1)^2}{2(2x+1)(x^2-1)}$$

$$= \frac{-\left(2x^3 + x^2 - 2x - 1\right)}{2(2x^3 + x^2 - 2x - 1)} = \frac{-1}{2}$$

$$\therefore \quad \angle A = \frac{2\pi}{3}$$

Practice Problem

Q.1 In a triangle, the angles A, B, C are in A.P. show that $2\cos\frac{1}{2}(A - C) = \frac{a + c}{\sqrt{a^2 - ac + c^2}}$.

Q.2 If the sides of a triangle are a, b, $\sqrt{(a^2 + ab + b^2)}$, then find the greatest angle?

Q.3 If $a\cos A = b\cos B$, then prove that the triangle is isosceles or right angled.

Q.4 If in a triangle ABC, $\frac{\sin A}{4} = \frac{\sin B}{5} = \frac{\sin C}{6}$, then the value of $\cos A + \cos B + \cos C$?

Q.5 The sides of a triangle are 8 cm, 10 cm and 12 cm. Prove that the greatest angle is double the smallest angle.

Q.6 In a triangle ABC, if $\frac{b + c}{11} = \frac{c + a}{11} = \frac{a + b}{13}$. then prove that $\frac{\cos A}{7} = \frac{\cos B}{19} = \frac{\cos C}{25}$

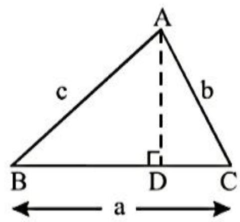
Answer key

Q.2 $\dfrac{2\pi}{3}$ Q.4 $\dfrac{69}{48}$

1.3 PROJECTION FORMULA :

In any triangle with usual noations, $\boxed{\begin{array}{l} a = b\cos C + c\cos B \\ b = c\cos A + a\cos C \\ c = a\cos B + b\cos A \end{array}}$

Consider the acute angle ΔABC



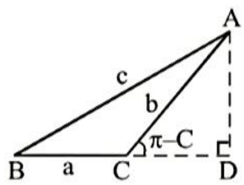
$\dfrac{BD}{BA} = \cos B$ so that $BD = c\cos B$

an $\dfrac{CD}{CA} = \cos C$ so that $BD = b\cos C$

Hence, $a = BC = BD + DC = c\cos B + b\cos C$

Hence, $a = BC = BD + DC = c\cos B + b\cos C$

Consider the obtuse angle ΔABC

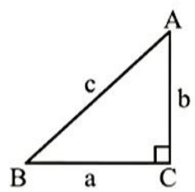


$\dfrac{BD}{BA} = \cos B,$ so that $BD = c\cos B$ and $\dfrac{CD}{CA} = \cos ACD$

$\dfrac{CD}{CA} = \cos(180° - C) = -\cos C,$ $CD = -b\cos C$

Hence, in the case, $a = BC - BD = c\cos B - (-b\cos C)$

$a = c\cos B + b\cos C$



Consider the right angle ΔABC, $\angle C = 90°$

then $a = c\cos B + 0 = c\cos B + b\cos 90°$

so that in each case

$$a = b\cos C + c\cos B, \quad b = c\cos A + a\cos C, \quad c = a\cos B + b\cos A$$

Illustration :

In any triangle prove that $(b + c)\cos A + (c - a)\cos B + (a + b)\cos C = a + b + c$.

Sol. *L.H.S.* $= (b\cos A + a\cos B) + (c\cos A + a\cos C) + (b\cos C + c\cos B)$

$= a + b + c.$

Illustration :

In any triangle $a\cos^2\frac{C}{2} + c\cos^2\frac{A}{2} = \frac{3b}{2}$, *then find the relation between the sides of the triangle?*

Sol. $a\cos^2\frac{C}{2} + c\cos^2\frac{A}{2} = \frac{3b}{2} \Rightarrow \quad a(1+\cos C) + c(1+\cos A) = 3b$

$a + c + a\cos C + c\cos A = 3b, \qquad a + c + b = 3b, \qquad a + c = 2b$

so, *a, b, c are in A.P.*

Practice Problem

Q.1 Prove that $a^3\cos(B-C) + b^3\cos(C-A) + c^3\cos(A-B) = 3abc$.

Q.2 Prove that $a(b^2+c^2)\cos A + b(c^2+a^2)\cos B + c(a^2+b^2)\cos C = 3abc$.

Q.3 In a ΔABC, prove that $c\cos(A-\alpha) + a\cos(C+\alpha) = b\cos\alpha$.

Q.4 Prove that $\frac{\cos C + \cos A}{c+a} + \frac{\cos B}{b} = \frac{1}{b}$.

1.4 TANGENT RULE (NAPIER ANALOGY) :

This rule is used when two sides and included angle are known.

$$\boxed{\begin{aligned} \tan\left(\frac{B-C}{2}\right) &= \frac{b-c}{b+c}\cot\left(\frac{A}{2}\right) \\ \tan\left(\frac{C-A}{2}\right) &= \frac{c-a}{c+a}\cot\left(\frac{B}{2}\right) \\ \tan\left(\frac{A-B}{2}\right) &= \frac{a-b}{a+b}\cot\left(\frac{C}{2}\right) \end{aligned}}$$

In any triangle, we have $\frac{b}{c} = \frac{\sin B}{\sin C}$

$$\therefore \qquad \frac{b-c}{b+c} = \frac{\sin B - \sin C}{\sin B + \sin C} = \frac{2\cos\left(\frac{B+C}{2}\right)\sin\left(\frac{B-C}{2}\right)}{2\sin\left(\frac{B+C}{2}\right)\cos\left(\frac{B-C}{2}\right)}$$

$$= \frac{\tan\left(\frac{B-C}{2}\right)}{\tan\left(\frac{B+C}{2}\right)} = \frac{\tan\left(\frac{B-C}{2}\right)}{\cot\left(\frac{A}{2}\right)}$$

$$\tan\left(\frac{B-C}{2}\right) = \frac{b-c}{b+c}\cdot\cot\frac{A}{2}$$

Illustration :

In any triangle ABC, if $b = \sqrt{3}$, $c = 1$ and $A = 30^o$. Find the value of a, c, A & C?

Sol. *We have* $$\tan\left(\frac{B-C}{2}\right) = \left(\frac{b-c}{b+c}\right)\cot\left(\frac{A}{2}\right)$$

$$\tan\left(\frac{B-C}{2}\right) = \left(\frac{b-c}{b+c}\right)\cot\frac{A}{2} = \frac{(\sqrt{3}-1)}{(\sqrt{3}+1)}\cdot\cot 15^o$$

$$= \frac{(\sqrt{3}-1)}{(\sqrt{3}+1)}\cdot\frac{(\sqrt{3}+1)}{(\sqrt{3}-1)} \qquad \text{as } \tan 15^o = \frac{\sqrt{3}-1}{\sqrt{3}+1}$$

$$\frac{B-C}{2} = 45^o, \quad B - C = 90^o,$$

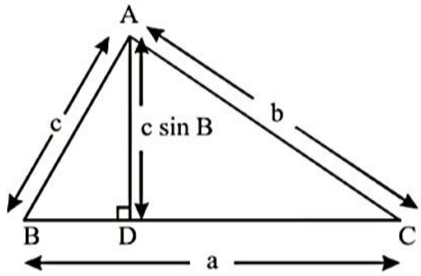
$A + B + C = 180^o$, so $B + C = 150^o$, $B = 120^o$, $C = 30^o$

Since $A = C$, *we have* $a = c = 1.$

1.5 AREA OF TRIANGLE :

1.5 AREA OF TRIANGLE :

$$\boxed{\Delta = \frac{1}{2}bc\sin A = \frac{1}{2}ca\sin B = \frac{1}{2}ab\sin C = \sqrt{s(s-a)(s-b)(s-c)}}$$



Different formulae for area of triangle are as follows :

$$AD = c \sin B$$

Area of triangle ABC is $\Delta = \frac{1}{2}(BC)(AD)$

$$\Delta = \frac{1}{2}(a)(c \sin B) \qquad = \frac{1}{2}ac\sin B$$

$AD = b \sin C$, $\qquad \Delta = \frac{1}{2}(a)(b \sin C) = \frac{1}{2}ab\sin C.$

$$\Delta = \frac{1}{2}\ ac\sin B = \frac{1}{2}ab\sin C = \frac{1}{2}bc\sin A$$

From our tenth class knowledge, we know area of triangle with sides a, b & c is denoted by Δ

$$\Delta = \sqrt{s(s-a)(s-b)(s-c)} \qquad \text{where} \quad s = \frac{a+b+c}{2}.$$

Illustration :

Prove that $\dfrac{(a+b+c)(b+c-a)(c+a-b)(a+b-c)}{4b^2c^2} = sin^2 A$

Sol. $\dfrac{(a+b+c)(b+c-a)(c+a-b)(a+b-c)}{4b^2c^2}$

$$= \frac{(a+b+c)(b+c-a)(c+a-b)(a+b-c)}{4b^2c^2}$$

$$= \frac{4s(s-a)(s-b)(s-c)}{b^2c^2} = \frac{4\Delta^2}{b^2c^2} = \frac{4}{b^2c^2}\left(\frac{1}{2}bc\ \sin A\right)^2$$

$$= sin^2 A$$

Illustration :

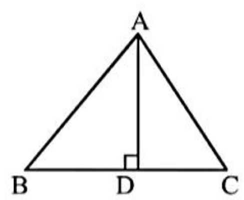
If the sides of a triangle are 17, 25, 28, then find the greatest length of the altitude.

If the sides of a triangle are 17, 25, 28, then find the greatest length of the altitude.

Sol. *Note that from geometry the greatest altitude is perpendicular to the shortest side.*

Let $a = 17$, $b = 25$, $c = 28$

Now $\Delta = \dfrac{1}{2}AD\ (BC) \Rightarrow AD = \dfrac{2\Delta}{17}$



where $\Delta = \sqrt{s(s-a)(s-b)(s-c)} = 210$

$$AD = \frac{2\times 210}{17} = \frac{420}{17}$$

Practice Problem

Q.1 If $c^2 = a^2 + b^2$, then prove that $4s\,(s-a)\,(s-b)\,(s-c) = a^2b^2$.

Q.2 In ΔABC, find the value of $\dfrac{b^2 \sin 2C + c^2 \sin 2B}{\Delta}$?

Answer key

Q.2 4

1.6 HALF ANGLE FORMULA :

1.6.1 Sine of half the angles in terms of the sides :-

$$\boxed{\begin{aligned} \sin\frac{A}{2} &= \sqrt{\frac{(s-b)(s-c)}{bc}} \\ \sin\frac{B}{2} &= \sqrt{\frac{(s-c)(s-a)}{ca}} \\ \sin\frac{C}{2} &= \sqrt{\frac{(s-a)(s-b)}{ab}} \end{aligned}}$$

Proof:

In any triangle ABC, we know $\cos A = \dfrac{b^2+c^2-a^2}{2bc}$

But cos A can be wirtten as $\cos A = 1 - 2\sin^2\dfrac{A}{2}$

Hence, $2\sin^2\dfrac{A}{2} = 1 - \cos A = 1 - \dfrac{(b^2+c^2-a^2)}{2bc}$

$$= \frac{2bc-(b^2+c^2)+a^2}{2b^2} = \frac{a^2-(b^2+c^2-2bc)}{2bc}$$

$$= \frac{a^2-(b-c)^2}{2bc} = \frac{(a+b-c)(a-b+c)}{2bc}$$

Let 2s stand for a + b + c

so $s = \dfrac{a+b+c}{2} =$ semiperimeter.

$a + b - c = a + b + c - 2c = 2s - 2c = 2\,(s - c)$

$a - b + c = a + b + c - 2b = 2s - 2c = 2\,(s - b)$

so $2\sin^2\dfrac{A}{2} = \dfrac{2(s-c)2(s-b)}{2bc}$, $\sin\dfrac{A}{2} = \sqrt{\dfrac{(s-b)(s-c)}{bc}}$

Similarly $\sin\dfrac{B}{2} = \sqrt{\dfrac{(s-c)(s-a)}{ca}}$ and $\sin\dfrac{C}{2} = \sqrt{\dfrac{(s-a)(s-b)}{ab}}$

1.6.2 The cosines of half the angles in terms of the sides :

$$\boxed{\begin{aligned}\cos\frac{A}{2}&=\sqrt{\frac{s(s-a)}{bc}}\\ \cos\frac{B}{2}&=\sqrt{\frac{s(s-b)}{ac}}\\ \cos\frac{C}{2}&=\sqrt{\frac{s(s-c)}{ab}}\end{aligned}}$$

Proof :

We know $\cos A = 2\cos^2\frac{A}{2} - 1,\qquad 2\cos^2\frac{A}{2} = 1 + \cos A.$

$$2\cos^2\frac{A}{2} = 1 + \frac{b^2+c^2-a^2}{2bc} = \frac{b^2+c^2+2bc-a^2}{2bc} = \frac{(b+c)^2-a^2}{2bc}$$

$$2\cos^2\frac{A}{2} = \frac{(b+c+a)(b+c-a)}{2bc}$$

$$a+b+c=2s \qquad a+b+c-2a = 2s-2a = 2(s-a)$$

$$2\cos^2\frac{A}{2} = \frac{2s(2)(s-a)}{2bc} = \frac{2s(s-a)}{bc}$$

$$2\cos^2\frac{A}{2} = \frac{2s(2)(s-a)}{2bc} = \frac{2s(s-a)}{bc}$$

$$\cos\frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}$$ Similarly

$$\cos\frac{B}{2} = \sqrt{\frac{s(s-b)}{ac}}$$

$$\cos\frac{C}{2} = \sqrt{\frac{s(s-c)}{ab}}$$

1.6.3 The tangent of half the angles in terms of the sides :

$$\boxed{\begin{aligned}\tan\frac{A}{2}&=\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}=\frac{\Delta}{s(s-a)}\\ \tan\frac{B}{2}&=\sqrt{\frac{(s-c)(s-a)}{s(s-b)}}=\frac{\Delta}{s(s-b)}\\ \tan\frac{C}{2}&=\sqrt{\frac{(s-a)(s-b)}{s(s-c)}}=\frac{\Delta}{s(s-c)}\end{aligned}}$$

Proof :

$$\text{since,} \quad \tan\frac{A}{2} = \frac{\sin\frac{A}{2}}{\cos\frac{A}{2}}$$

$$\tan\frac{A}{2} = \frac{\sqrt{\frac{(s-b)(s-c)}{bc}}}{\sqrt{\frac{s(s-a)}{bc}}} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}} \qquad \text{Similarly}$$

$$\tan\frac{B}{2} = \sqrt{\frac{(s-c)(s-a)}{s(s-b)}}, \qquad \tan\frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{s(s-c)}}$$

Note :-

Since, in a triangle, A is always less than 180º, so $\frac{A}{2}$ is always less then 90º. Therefore, the sine, consine and tangent of $\frac{A}{2}$ (half angle) are therefore always positive.

Since, in a triangle, A is always less than 180º, so $\frac{A}{2}$ is always less then 90º. Therefore, the sine, consine and tangent of $\frac{A}{2}$ (half angle) are therefore always positive.

1.6.4 The sine of any angle of triangle in terms of the sides:

We know $\quad \sin A = 2\sin\frac{A}{2}\cos\frac{A}{2}$

But by the previous discussion

$$\sin\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}} \quad \text{and} \quad \cos\frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}$$

So $\quad \sin A = \sqrt{\frac{(s-b)(s-c)}{bc}}\sqrt{\frac{s(s-a)}{bc}}$

$$= \frac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)} = \frac{2}{bc}\Delta$$

Illustration :

In any triangle prove that $(a+b+c)\left(\tan\frac{A}{2}+\tan\frac{B}{2}\right)=2c\cot\frac{C}{2}$

Sol. L.H.S. $=(2s)\left[\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}+\sqrt{\frac{(s-a)(s-c)}{s(s-b)}}\right]$

$$=2s\sqrt{\frac{(s-c)}{s}}\left[\sqrt{\frac{s-b}{s-a}}+\sqrt{\frac{s-a}{s-b}}\right]$$

$$=2\sqrt{s(s-c)}\left[\frac{s-b+s-a}{\sqrt{(s-a)(s-b)}}\right]=\frac{2\sqrt{s(s-c)}\,c}{\sqrt{(s-a)(s-b)}}$$

$$=\frac{2c}{\tan\frac{c}{2}}=2c\cot\frac{C}{2}$$

$$=\frac{2c}{\tan\frac{c}{2}}=2c\cot\frac{C}{2}$$

Illustration :

If the sides of a triangle be in arithmetic progression, prove that the contangents of half the angles are also in arithmetic progression.

Sol. $a+c=2b$ *(given)*

we have to prove that $\cot\frac{A}{2}+\cot\frac{C}{2}=2\cot\frac{B}{2}$

so $\sqrt{\frac{s(s-a)}{(s-b)(s-c)}}+\sqrt{\frac{s(s-c)}{(s-a)(s-c)}}=2\sqrt{\frac{s(s-b)}{(s-c)(s-a)}}$

Multiplying both side by $\sqrt{\frac{(s-a)(s-b)(s-c)}{s}}$

$(s-a)+(s-c)=2(s-b)$

$a+c=2b$ *(which is the given relation)*

Illustration :

If $\cos\dfrac{A}{2} = \sqrt{\dfrac{b+c}{2c}}$, *then prove that* $a^2 + b^2 = c^2$.

Sol. $\cos\dfrac{A}{2} = \sqrt{\dfrac{b+c}{2a}}$

$\Rightarrow \quad \dfrac{s(s-a)}{bc} = \dfrac{b+c}{2c}$ *(squaring)*

$\Rightarrow \quad 2s\,(2s - 2a) = 2b\,(b + c)$

$\Rightarrow \quad (a + b + c)\,(b + c - a) = 2b^2 + 2bc$

$\Rightarrow \quad (b + c)^2 - a^2 = 2b^2 + 2bc$

$\Rightarrow \quad b^2 + c^2 + 2bc - a^2 = 2b^2 + 2bc$

$\Rightarrow \quad c^2 = a^2 + b^2$

Illustration :

Illustration :

If in a triangle ABC, if $\Delta = a^2 - (b - c)^2$, *then find the value of tan A?*

Sol. $\Delta = (a + b - c)\,(a - b + c) = (a + b + c - 2c)\,(a + b + c - 2b)$

$\Delta = (2s - 2c)\,(2s - 2b)$

$\Rightarrow \quad \Delta^2 = [2\,(s - b)\,2\,(s - c)]^2$

$\Rightarrow \quad s\,(s - a)\,(s - b)\,(s - c) = 16\,(s - b)^2\,(s - c)^2$

$\Rightarrow \quad \dfrac{(s-b)(s-c)}{s(s-a)} = \dfrac{1}{16}$

$\Rightarrow \quad \tan^2\dfrac{A}{2} = \dfrac{1}{16}$

$\Rightarrow \quad \tan\dfrac{A}{2} = \dfrac{1}{4}$

$\Rightarrow \quad \tan A = \dfrac{2\tan\dfrac{A}{2}}{1-\tan^2\dfrac{A}{2}} = \dfrac{2\cdot\dfrac{1}{4}}{1-\dfrac{1}{16}} = \dfrac{8}{15}$

Practice Problem

Q.1 In any triangle ABC, prove that $(b + c - a)\left(\cot\frac{B}{2} + \cot\frac{C}{2}\right) = 2a\cot\frac{A}{2}$

Q.2 In any triangle, if $\tan\frac{A}{2} = \frac{5}{6}$ and $\tan\frac{B}{2} = \frac{20}{37}$. Find $\tan\frac{C}{2}$ and prove that in this triangle $a + c = 2b$.

Q.3 If a, b and c be in A.P. Prove that $\cos A \cot\frac{A}{2}$, $\cos B \cot\frac{B}{2}$ and $\cos C \cot\frac{C}{2}$ are in A.P.

Q.4 If a, b and c are in H.P. Prove that $\sin^2\frac{A}{2}$, $\sin^2\frac{B}{2}$ and $\sin^2\frac{C}{2}$ are also in H.P.

Q.5 If a^2, b^2 and c^2 be in A.P. Prove that cot A, cot B and cot C are in A.P. also.

Answer key

Q.2 122

Q.2 $\frac{122}{205}$

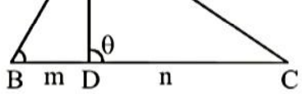
1.7 m - n THEOREM :

Let D be a point on the side BC of a ΔABC, such that BD : DC = m : n and $\angle ADC = \theta$, $\angle BAD = \alpha$ and $\angle DAC = \beta$. Prove that

(a) $(m + n)\cot\theta = m\cot\alpha - n\cot\beta$

(b) $(m + n)\cot\theta = n\cot B - m\cot C$.

Proof :



(a) Given $\frac{BD}{DC} = \frac{m}{n}$ and $\angle ADC = \theta$

$\angle ADB = 180° - \theta$, $\angle BAD = \alpha$ and $\angle DAC = \beta$.

So $\angle ABD = \theta - \alpha = B$, $\quad C = 180° - (\theta + \beta)$

From ΔABD, $\quad \frac{BD}{\sin\alpha} = \frac{AD}{\sin(\theta - \alpha)}$ ...(i)

From ΔADC, $\quad \frac{DC}{\sin\beta} = \frac{AD}{\sin(\theta + \beta)}$ ...(ii)

Dividing equation (i) by (ii)

$$\frac{BD}{DC}\cdot\frac{\sin\beta}{\sin\alpha}=\frac{\sin(\theta+\beta)}{\sin(\theta-\alpha)} \quad\Rightarrow\quad \frac{m\sin\beta}{n\sin\alpha}=\frac{\sin(\theta+\beta)}{\sin(\theta-\alpha)} \qquad \ldots\text{(iii)}$$

$$\frac{m\sin\beta}{n\sin\alpha}=\frac{\sin\theta\cos\beta+\cos\theta\sin\beta}{\sin\theta\cos\alpha-\cos\theta\sin\alpha}$$

$\Rightarrow$ $m\sin\beta\ (\sin\theta\cos\alpha-\cos\theta\sin\alpha)=n\sin\alpha\,(\sin\theta\cos\beta+\cos\theta+\sin\beta)$

Now dividing both sides by $\sin\alpha\sin\beta\sin\theta$

$\Rightarrow$ $m\cot\alpha-m\cot\theta=n\cot\beta+n\cot\theta$

$\Rightarrow$ $(m+n)\cot\theta=m\cot\alpha-n\cot\beta.$

(b) We have, $\angle CAD=180^\circ-(\theta+C)$

$\angle ABC=B,\quad \angle ACD=C,\quad \angle BAD=(\theta-B)$

Putting these values in equation (iii) we get

$m\sin(\theta+C)\sin B=n\sin C\sin(\theta-B)$

$m(\sin\theta\cos C+\cos\theta\sin C)\sin B=n\sin C(\sin\theta\cos B-\cos\theta\sin B)$

dividing both sides by $\sin\theta\sin B\sin C$

$\Rightarrow$ $m(\cot C+\cot\theta)=n(\cot B-\cot\theta)$

$\therefore$ $(m+n)\cot\theta=n\cot B-m\cot C$

Illustration :

In a triangle ABC, $\angle ABC=45^o$. Point 'D' is on BC so that 2BD = CD and $\angle DAB=15^o$. $\angle ACB$ in degree equals.

(A) 30^o *(B) 60^o* *(C) 75^o* *(D) 90^o*

Sol. *Applying m-n theorem, in ΔABC*

$(BD+DC)\cot 60^o=CD\cot 45^o-BD\cot C$

$\Rightarrow$ $3\cot 60^o=2\cot 45^o-\cot C$

$\Rightarrow$ $\cot C=2-\sqrt{3}$

$\Rightarrow$ $\boxed{C=75^o}$ *Ans.*

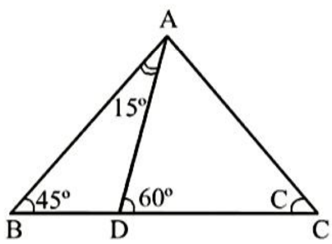


Illustration :

In a triangle ABC altitude AD, $\angle BAC = 45^o$, DB = 3 and CD = 2. The area of the ΔABC is?

(A) 6   (B) 15   (C) $\frac{15}{4}$ (D) 12

Sol. Let $\angle BAD = \alpha$

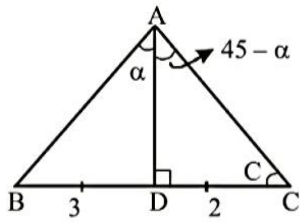
Applying (m – n) theorem, in ΔABC

$(3 + 2) \cot 90^o = 3 \cot \alpha - 2 \cot (45^o - \alpha)$

$\Rightarrow \quad 0 = \frac{3}{\tan\alpha} - \frac{(1+\tan\alpha)}{(1-\tan\alpha)}$

$\Rightarrow \quad 3 - 3 \tan \alpha = 2 \tan \alpha + 2 \tan^2 \alpha$

$\Rightarrow \quad 2 \tan^2 \alpha + 5 \tan \alpha - 3 = 0$



$\Rightarrow \quad \tan \alpha = \frac{1}{2}, -3$

$\tan \alpha = \frac{1}{2} \qquad [\alpha \in (0, 45^o), \tan \alpha \in (0, 1)]$

$\tan \alpha = \frac{1}{2} \qquad [\alpha \in (0, 45^o), \tan \alpha \in (0, 1)]$

$\Delta ABD, \tan \alpha = \frac{3}{AD} = \frac{1}{2} \qquad \Rightarrow \qquad AD = 6$

Area of $\Delta ABC = \frac{1}{2} \times 5 \times 6 = 15$ units

Illustration :

If the median of a triangle ABC through A is perpendicular is AB then $\frac{\tan A}{\tan B}$ has the value equal to

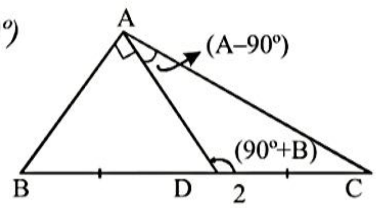
(A) $\frac{1}{2}$ (B) 2   (C) –2   (D) $-\frac{1}{2}$

Sol. Applying (m – n) theorem, $\quad \Delta ABC$

$(BD + CD) \cot (90^o + B) = BD \cot 90^o - CD \cot (A - 90^o)$

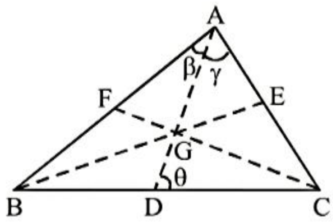
$\Rightarrow \quad -2 \tan B = 0 + \tan A$

$\Rightarrow \quad \frac{\tan A}{\tan B} = -2 \qquad$ *Ans.*



1.8 CENTROID AND MEDIANS OF ANY TRIANGLE :

If ABC be any triangle, and D, E and F respectively the middle points of BC, CA and AB the lines AD, BE and CF are called the medians of the triangle.



$$\boxed{\begin{aligned} AD &= \frac{1}{2}\sqrt{(2b^2+2c^2-a^2)} \\ BE &= \frac{1}{2}\sqrt{(2c^2+2a^2-b^2)} \\ CF &= \frac{1}{2}\sqrt{(2a^2+2b^2-c^2)} \end{aligned}}$$

From geometry, we know that the medians meet in a common point G, such that

$$AG = \frac{2}{3}AD, \quad BG = \frac{2}{3}BE \quad \text{and} \quad CG = \frac{2}{3}CF$$

The point 'G' is called the centroid of the triangle.

1.8.1 Length of the Medians :

$$AD^2 = AC^2 + CD^2 - 2ACCD \cos C = b^2 + \frac{a^2}{4} - 2b\frac{a}{2} \cos C$$

$$AD^2 = b^2 + \frac{a^2}{4} - ab \cos C.$$

and $\quad c^2 = b^2 + a^2 - 2ab \cos C$

so $\quad 2AD^2 - c^2 = b^2 - \frac{a^2}{2} \quad$ so that $\quad AD = \frac{1}{2}\sqrt{(2b^2+2c^2-a^2)}$

we can also write, $\quad AD = \frac{1}{2}\sqrt{(b^2+c^2+b^2+c^2-a^2)}$

$$AD = \frac{1}{2}\sqrt{(b^2+c^2+2bc\cos A)} \quad \text{similarly}$$

$$BE = \frac{1}{2}\sqrt{(2c^2+2a^2-b^2)} \quad \text{and} \quad CF = \frac{1}{2}\sqrt{(2a^2+2b^2-c^2)}$$

1.8.2 Angles that the Median AD makes with the sides :

Let $\angle BAD = \beta$ and $\angle CAD = \gamma$, we have

$$\frac{\sin\gamma}{\sin C} = \frac{DC}{AD} = \frac{a}{2x}, \sin\gamma = \frac{a}{2x}\sin C, \text{ where } AD = x \text{ (say)}$$

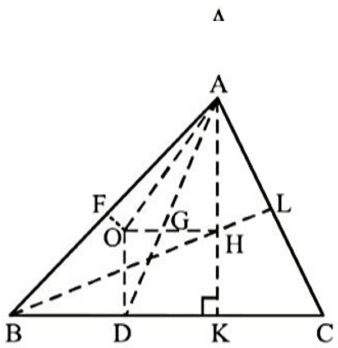
$$x = AD = \frac{1}{2}\sqrt{(2b^2 + 2c^2 - a^2)}, \ \sin\gamma = \frac{a\sin C}{\sqrt{(2b^2 + 2c^2 - a^2)}}$$

Similarly, $\sin\beta = \dfrac{a\sin B}{\sqrt{(2b^2 + 2c^2 - a^2)}}$

Again, if the $\angle ADC = \theta$, we have

$$\frac{\sin\theta}{\sin C} = \frac{AC}{AD} = \frac{b}{x}, \sin\theta = \frac{b}{x}\sin C = \frac{2b\sin C}{\sqrt{(2b^2 + 2c^2 - a^2)}}$$

Note : The centroid lies on the line segment joining the circumcentre to the orthocentre and divides the line segment in the ratio 1 : 2.



Let O and H be the circumcentre and orthocentre respectively.
Draw OD and HK perpendicular to BC.
Let AD and OH meet in G. By geometry ΔAGP and ΔOGD are similar

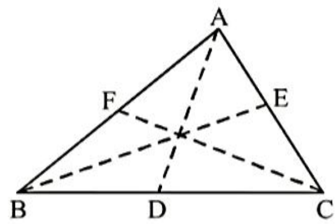
$$\frac{OG}{GP} = \frac{AG}{GD} = \frac{2}{1}$$

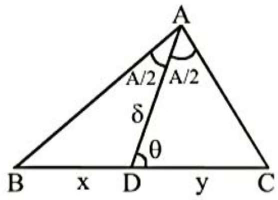
The centroid therefore lies on the line segment joining the circumcentre to the orthocentre and divides it in the ratio 1 : 2.

1.9 BISECTORS OF THE ANGLES :

If AD bisects the angle A and divide the base into portions x and y, we have by geometry. The length of bisectors will be as follows :

$$\boxed{\begin{aligned} AD &= \frac{2bc}{(b+c)}\cos\frac{A}{2} \\ BE &= \frac{2ca}{(c+a)}\cos\frac{B}{2} \\ CF &= \frac{2ab}{(a+b)}\cos\frac{C}{2} \end{aligned}}$$





$$\frac{x}{y} = \frac{AB}{AC} = \frac{c}{b}, \quad \frac{x}{c} = \frac{y}{b} \quad \Rightarrow \quad \frac{x+y}{b+c} = \frac{a}{b+c} \qquad \ldots(i)$$

giving x and y

Also, if δ be the length of AD and θ the angle it makes with BC, we have

$$\Delta ABD + \Delta ACD = \Delta ABC, \qquad \frac{1}{2}c\,\delta\sin\frac{A}{2} + \frac{1}{2}b\,\delta\sin\frac{A}{2} = \frac{1}{2}bc\sin A$$

$$\delta = \frac{bc}{b+c}\cdot\frac{\sin A}{\sin\frac{A}{2}} = \frac{2bc}{(b+c)}\cos\frac{A}{2}$$

$$\theta = \angle DAB + B = \frac{A}{2} + B.$$

Thus, we have the length of bisector and its inclination to BC.

1.10 CIRCUM CIRCLE :

To find the magnitude of R, the radius of the circum circle of any triangle ABC.

$$\boxed{2R = \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}} \qquad \boxed{R = \frac{abc}{4\Delta}}$$

Proof : Consider any triangle ABC as shown in three figure

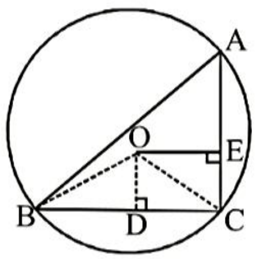


Figure (1)

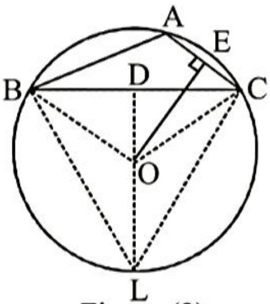


Figure (2)

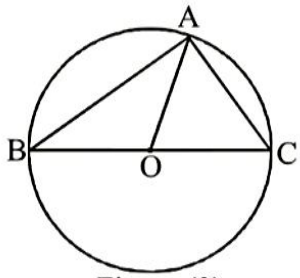


Figure (3)

Bisecting the two sides BC and CA in D and E respectivley and draw DO and EO perpendicular to BC and CA.

By geometry, O is the centre of the circumcircle. Join OB and OC.

The point O may either lie within the triangle as in figure (1) or without it as in figure (2) or upon one of the sides as in figure (3).

Taking the first figure, the two triangles BOD and COD are equal in all respects, so that

$$\angle BOD = \angle COD, \therefore \angle BOD = \frac{1}{2}(2\angle BAC) = \angle BAC = A$$

Also, $BD = BO\sin(BOD) = BO\sin A = R\sin A$ [as $R = BO$]

$$\frac{a}{2} = R\sin A$$

If A be obtuse, as in figure (2), we have

$$\angle BOD = \frac{1}{2}\angle BOC = \angle BLC = 180^\circ - A$$

$\sin(\angle BOD) = \sin(180^\circ - A) = \sin A$

and $\quad R = \dfrac{a}{2\sin A}$

If A be right angle as in figure (3) we have

$$R = OA = OC = \frac{a}{2} = \frac{a}{2\sin A} \text{ as } \sin A = \sin 90^\circ = 1$$

$$R = OA = OC = \frac{a}{2} = \frac{a}{2\sin A} \text{ as } \sin A = \sin 90^\circ = 1$$

so in all the three cases, we have

$$R = \frac{a}{2\sin A} = \frac{b}{2\sin B} = \frac{c}{2\sin C}$$

as we know $\quad \Delta = \dfrac{1}{2}bc\sin A$

As $\quad \sin A = \dfrac{a}{2R}$

$$\therefore \quad \Delta = \frac{1}{2}bc\frac{a}{2R} = \frac{abc}{4R}$$

$$\boxed{R = \frac{abc}{4\Delta}}$$

Note :

(a) In case of acute angle triangle, circumcentre lies within the triangle.

(b) In case of obtuse angle triangle, circumcentre lies outside the triangle.

(c) In case of right angle triangle, circumcentre lies on the mid point of hypotenuse.

1.11 INCIRCLE :

Radius of incircle $\boxed{r = \dfrac{\Delta}{s} = (s-a)\tan\dfrac{A}{2} = (s-b)\tan\dfrac{B}{2} = (s-c)\tan\dfrac{C}{2}}$

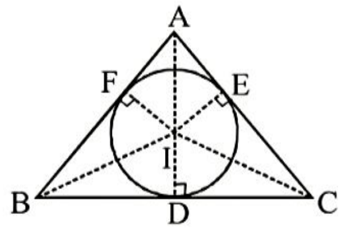
$$\boxed{r = 4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}$$

1.11.1 The value of r, the radius of the incircle of the triangle ABC :

Proof : Consider the triangle ABC as shown in figure. Bisect the two angles B and C by the two lines BI and CI meeting in I.

By geometry, I is the centre of the incircle, join IA, and draw ID, IE and IF perpendicular to the three sides.

The ID = IE = IF = r



We have Area of $\Delta IBC = \dfrac{1}{2}(ID)(BC) = \dfrac{1}{2}ra$

Area of $\Delta ICA = \dfrac{1}{2}(IE)(AC) = \dfrac{1}{2}r.b$, Area of $\Delta IAB = \dfrac{1}{2}(IF)(AB) = \dfrac{1}{2}r.c$

Hence by addition, we have

$$\frac{1}{2}r(a+b+c) = \text{Sum of Areas of } \Delta IBC, \Delta IAC, \Delta IBA$$

$$= \Delta ABC = \Delta$$

$$r = \frac{\Delta}{\frac{a+b+c}{2}} = \frac{\Delta}{s}$$

1.11.2 The same 'r' can be expressed in a different way also :

Consider the same figure ABC as shown above. The angles IBD and IDB are respectively equal to the angles IBF and IFB, so the two triangle IDB and IFB are equal in all respects.

Hence,	BD = BF,	so that	2BD = BD + BF
so also,	AE = AF,	so that	2AE = AE + AF
and	CE = CD,	so that	2CE = CE + CD

Hence, by addition, we have

$$2BD + 2AE + 2CE = (BD + CD) + (BF + AF) + (AE + CE)$$

$\therefore \quad 2BD + 2b = a + b + c = 2s,$

Hence, $\quad BD = s - b = BF, \quad$ similarly $\quad CE = s - c, \quad AF = s - a$

$$\tan IBD = \frac{ID}{BD} = \frac{r}{(s-b)} = \tan\frac{B}{2}$$

$$r = (s-b)\tan\frac{B}{2} = (s-a)\tan\frac{A}{2} = (s-c)\tan\frac{C}{2}$$

1.11.3 A third value of r may be found as follows :

1.11.3 A third value of r may be found as follows :

$$a = BD + CD = ID \cot IBD + ID \cot ICD$$

$$= r\cot\frac{B}{2} + r\cot\frac{C}{2} = r\left[\frac{\cos\frac{B}{2}}{\sin\frac{B}{2}} + \frac{\cos\frac{C}{2}}{\sin\frac{C}{2}}\right]$$

$$a = r\frac{\sin\left(\frac{B+C}{2}\right)}{\sin\frac{B}{2}\sin\frac{C}{2}} = r\frac{\cos\frac{A}{2}}{\sin\frac{B}{2}\sin\frac{C}{2}}$$

$$r = a\frac{\sin\frac{B}{2}\sin\frac{C}{2}}{\cos\frac{A}{2}} \qquad \left[\text{as} \quad \frac{a}{\sin A} = 2R\right]$$

As $\quad a = 2R\sin A = 2R\,2\sin\frac{A}{2}\cos\frac{A}{2}$

$$\therefore \quad \boxed{r = 4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}$$

1.12 ESCRIBED CIRCLE :

1.12.1 To find the value of r_1, the radius of the escribed circle opposite the angle A of the triangle ABC :

$$\boxed{\begin{aligned} r_1 &= \frac{\Delta}{(s-a)} = s\tan\frac{A}{2} = 4R\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2} \\ r_2 &= \frac{\Delta}{(s-b)} = s\tan\frac{B}{2} = 4R\cos\frac{A}{2}\sin\frac{B}{2}\cos\frac{C}{2} \\ r_3 &= \frac{\Delta}{(s-c)} = s\tan\frac{C}{2} = 4R\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2} \end{aligned}}$$

Proof : Produce AB and AC to L and M. Bisect the angles CBL and BCM by the lines BI_1 and CI_1 and let these lines meet in I_1.

Draw I_1D_1, I_1E_1 and I_1F_1 perpendicular to these lines respectively.

The two triangles I_1D_1B and I_1F_1B are equal in all respect, so that $I_1F_1 = I_1D_1$ similarly $I_1E_1 = I_1D_1$.

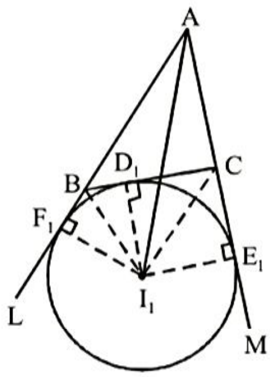
The three perpendicular I_1D_1, I_1E_1 and I_1F_1 being equal, the point I_1 is the centre of the required circle.

The three perpendicular I_1D_1, I_1E_1 and I_1F_1 being equal, the point I_1 is the centre of the required circle.

Now, the area ABI_1C is equal to the triangles ABC and I_1BC, it is also equal to the sum of the triangle I_1BA and I_1CA.

Hence, $\Delta ABC + \Delta I_1BC = \Delta I_1CA + \Delta I_1AB$

$$\therefore \quad \Delta + \frac{1}{2}(I_1D_1)(BC) = \frac{1}{2}(I_1E_1)(CA) + \frac{1}{2}(I_1F_1)(AB)$$



$$\therefore \quad \Delta + \frac{1}{2}r_1 a = \frac{1}{2}r_1 \cdot b + \frac{1}{2}r_1 c$$

$$\Delta = \frac{r_1}{2}(b+c-a) = \frac{r_1}{2}(a+b+c-2a) = r_1(s-a)$$

$$r_1 = \frac{\Delta}{(s-a)} \qquad \text{similarly } r_2 = \frac{\Delta}{(s-b)},\ r_3 = \frac{\Delta}{(s-c)}$$

1.12.2 A Second Value of r_1 can be obtained :

Since AE_1 and AF_1 are tangents, $AE_1 = AF_1$

Similarly, $BF_1 = BD_1$ and $CE_1 = CD_1$

$\therefore \quad 2AE_1 = AE_1 + AF_1 = AB + BF_1 + AC + CE_1$

$= AB + BD_1 + AC + CD_1$

$= AB + AC + BC = 2s$

$\therefore \quad AE_1 = s = AF_1$

Also $BD_1 = BF_1 = AF_1 - AB = s - c$

similarly $CD_1 = CE_1 = AE_1 - AC = s - b \quad \therefore \quad I_1E_1 = AE_1 \tan(I_1AE_1)$

so $r_1 = s \tan \dfrac{A}{2}$

1.12.3 A third value of r_1 may be obtained :

For, since I_1C bisects the angle BCE_1, we have

$$\angle I_1CD_1 = \frac{1}{2}(180^\circ - C) = 90^\circ - \frac{C}{2}$$

so, $\angle I_1BD_1 = 90^\circ - \dfrac{B}{2}$

$\therefore \quad a = BC = BD_1 + D_1C = I_1D_1 \cot I_1BD_1 + I_1D_1 \cot I_1CD_1$

$$= r_1\left(\tan\frac{B}{2} + \tan\frac{C}{2}\right) = r_1\left(\frac{\sin\frac{B}{2}}{\cos\frac{B}{2}} + \frac{\sin\frac{C}{2}}{\cos\frac{C}{2}}\right) = r_1\,\frac{\sin\left(\frac{B+C}{2}\right)}{\cos\frac{B}{2}\cos\frac{C}{2}}$$

$$\Rightarrow \quad a\cos\frac{B}{2}\cos\frac{C}{2} = r_1\cos\frac{A}{2}$$

$$r_1 = \frac{a\cos\frac{B}{2}\cos\frac{C}{2}}{\cos\frac{A}{2}} \text{ as } a = 2R\sin A = 4R\sin\frac{A}{2}\cos\frac{A}{2}$$

$$\therefore \quad r_1 = 4R\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$$

Similarly, $r_2 = 4R\sin\dfrac{B}{2}\cos\dfrac{C}{2}\cos\dfrac{A}{2}$

$$r_3 = 4R\sin\frac{C}{2}\cos\frac{A}{2}\cos\frac{B}{2}$$

Important results regarding r_1, r_2 and r_3.

Given r_1, r_2, and r_3

(i) semiperimeter $= s = \sqrt{(r_1r_2 + r_2r_3 + r_3r_1)} = \sqrt{\Sigma r_1r_2}$ (ii) $\Delta = \dfrac{r_1r_2r_3}{\sqrt{\Sigma r_1r_2}}$

(iii) $r = \dfrac{r_1r_2r_3}{\Sigma r_1r_2}$ (iv) $R = \dfrac{(r_1+r_2)(r_2+r_3)(r_3+r_1)}{4\,\Sigma\, r_1r_2}$

(v) $a = \dfrac{r_1(r_2+r_3)}{\sqrt{\Sigma\, r_1r_2}}, b = \dfrac{r_2(r_3+r_1)}{\sqrt{\Sigma\, r_1r_2}}, c = \dfrac{r_3(r_1+r_2)}{\sqrt{\Sigma\, r_1r_2}}$ (vi) $\sin A = \dfrac{2r_1\sqrt{\Sigma r_2r_3}}{(r_1+r_2)(r_1+r_3)}$

Illustration :

If in a triangle, $r_1 = r_2 + r_3 + r$, prove that the triangle is right angled.

Sol. $r_1 = r_2 + r_3 + r \Rightarrow r_1 - r = r_2 + r_3$

$$\Rightarrow \quad \frac{\Delta}{(s-a)} - \frac{\Delta}{s} = \frac{\Delta}{(s-b)} + \frac{\Delta}{(s-c)}$$

$$\Rightarrow \quad \frac{\Delta a}{s(s-a)} = \frac{\Delta(2s-b-c)}{(s-b)(s-c)}$$

$$\Rightarrow \quad s(s-a) = (s-b)(s-c)$$

$$\Rightarrow \quad s^2 - sa = s^2 - (b+c)s + bc$$

$\Rightarrow \quad (b + c - a)s = bc,$ $\quad (b + c - a)\, 2s = 2bc$

$(b + c - a)(b + c + a) = 2bc,$ $\quad (b + c)^2 - a^2 = 2bc,$ $\quad b^2 + c^2 - a^2 = 0$

$b^2 + c^2 = a^2$, *so the triangle is right angled.*

Illustration :

Prove that $\cos A + \cos B + \cos C = 1 + \dfrac{r}{R}$.

Sol. $\cos A + \cos B + \cos C = 1 + 4\sin\dfrac{A}{2}\sin\dfrac{B}{2}\sin\dfrac{C}{2}$

$$= 1 + \frac{4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}{R} = 1 + \frac{r}{R}$$

Illustration :

Prove that $\dfrac{a\cos A + b\cos B + c\cos C}{a+b+c} = \dfrac{r}{R}$

Sol. We have

$a \cos A + b \cos B + c \cos C = 2R \sin A \cos A + 2R \sin B \cos B + 2R \sin C \cos C$

$= R (\sin 2A + \sin B + \sin 2C)$

$= 4R \sin A \sin B \sin C$

$$a + b + c = 2R (\sin A + \sin B + \sin C) = 2R \cdot 4 \cos\frac{A}{2} \cos\frac{B}{2} \cos\frac{C}{2}$$

$$\Rightarrow \quad \text{so L.H.S.} = \frac{4R\sin A\sin B\sin C}{8R\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}} = \frac{4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}{R} = \frac{r}{R}$$

Illustration :

Prove that $r_1 + r_2 + r_3 - r = 4R.$

Sol. $r_1 + r_2 + r_3 - r = \dfrac{\Delta}{s-a} + \dfrac{\Delta}{s-b} + \dfrac{\Delta}{s-c} - \dfrac{\Delta}{s}$

$$= \Delta\left[\frac{(s-b)+(s-a)}{(s-a)(s-b)} + \frac{s-s+c}{3(s-c)}\right] = \Delta\left[\frac{c}{(s-a)(s-b)} + \frac{c}{s(s-c)}\right]$$

$$= \Delta c\ \frac{[s(s-c)+(s-a)(s-b)]}{s(s-a)(s-b)(s-c)} = \frac{\Delta c}{\Delta^2} = [2s^2 - s\,(a + b + c) + ab]$$

$$= \frac{c}{\Delta}[2s^2 - 2s^2 + ab] = \frac{abc}{\Delta} = 4R$$

Practice Problem

Prove that in a ΔABC,

Q.1 $\dfrac{1}{r^2} + \dfrac{1}{r_1^2} + \dfrac{1}{r_2^2} + \dfrac{1}{r_3^2} = \dfrac{a^2+b^2+c^2}{\Delta^2}$

Q.2 $\dfrac{1}{bc} + \dfrac{1}{ca} + \dfrac{1}{ab} = \dfrac{1}{2Rr}$

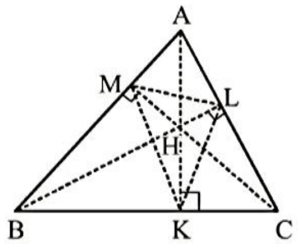
Q.3 $(r_1 - r)\,(r_2 - r)\,(r_3 - r) = 4Rr^2$

Q.4 $a\,(r\,r_1 + r_2\,r_3) = b\,(r\,r_2 + r_3\,r_1) = c\,(r\,r_3 + r_1\,r_2)$

Q.5 $\dfrac{1}{r_1} + \dfrac{1}{r_2} + \dfrac{1}{r_3} = \dfrac{1}{r}$

1.13 ORTHOCENTRE AND PEDAL TRIANGLE OF ANY TRIANGLE :

Let ABC be any triangle and let AK, BL and CM to be perpendiculars from A, B and C upon the opposite sides of the triangle. It can be easily shown from geometry, that these three perpendiculars meet in a common point H. This point H is called the orthocentre of the triangle. The triangle KLM, which is formed by joining the feet of these perpendicular, is called the pedal triangle of ABC.



1.13.1 Distances of the orthocentre of the angular points of the triangle :

Consider an acute angle triangle ABC.

We have, $HK = KB \tan(HBK) = KB \tan(90^\circ - C) = KB \cot C$

$$= AB \cos B \cot C = AB \cos B \frac{\cos C}{\sin C} = \frac{AB}{\sin C} \cos B \cos C$$

$$HK = \frac{c}{\sin C} \cos B \cos C = 2R \cos B \cos C.$$

Again, $AH = AL \sec(KAC) = c \cos A \sec(90^\circ - C)$

$$= c \cos A \operatorname{cosec} C = \frac{c}{\sin C} \cos A = 2R \cos A$$

Similarly $BH = 2R \cos B$ and $CH = 2R \cos C$.

The distances of the orthocentre from the angular point are therefore, $2R \cos A$, $2R \cos B$ and $2R \cos C$. Its distance from the sides a, b, c are $2R \cos B \cos C$, $2R \cos C \cos A$ and $2R \cos A \cos B$ respectively.

1.13.2 The sides and angles to the pedal triangle :

Consider an acute angle triangle ABC:

Since the angles HKC and HLC are right angles, the points H, L, C and K lie on a circle.

$\angle HKL = \angle HCL = 90^\circ - A$ Similarly.

H, K, B, M lie on a circle, and therefore,

$\angle HKM = \angle HBM = 90^\circ - A$

Hence $\angle MKL = 180^\circ - 2A$ = the supplement of 2A.

$\angle KLM = 180^\circ - 2B$, $\angle LMK = 180^\circ - 2C$.

Again, from the triangle ALM, we have

$$\frac{LM}{\sin A} = \frac{AL}{\sin(AML)} = \frac{AB\cos A}{\cos(HML)} = \frac{c\cos A}{\cos(HAL)} = \frac{c\cos A}{\sin C}$$

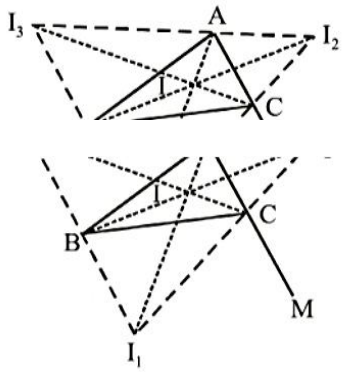
$$LM = \frac{c}{\sin C}\cos A \sin A = \frac{c}{\sin C}\sin A \cos A = a\cos A$$

so $\quad LM = a\cos A$, similarly $MK = b\cos B$, $KL = c\cos C$

The sides of the pedal triangle therefore a cos A, b cos B and c cos C; also its angles are the supplements of twice the angles of the triangle.

1.13.3 Excentric Triangle :

Let I be the centre of the incircle and I_1, I_2 and I_3 the centres of the escribed circles. Which are opposite to A, B and C respectively. IC bisects the angle ACB and I_1C bisects the angle



respectively. IC bisects the angle ACB and I_1C bisects the angle BCM.

$$\therefore \quad \angle ICI_1 = \angle ICB + \angle I_1CB = \frac{1}{2}\angle ACB + \frac{1}{2}\angle MCB$$

$$= \frac{1}{2}(\angle ACB + \angle MCB) = \frac{1}{2}(180^\circ) = 90^\circ$$

= A right angle.

Similarly, $\angle ICI_2$ is a right angle. Hence I_1CI_2 is a straight line to which IC is perpendicular. So, I_2AI_3 is a straight line to which IA is perpendiculars and I_3BI_1 is a straight line to which IB is perpendicular.

Also, since IA and I_1A both bisect the angle BAC, the three points A, I and I_1 are in a striaght line. Similarly, B I I_2 and C I I_3 are straight lines. Hence, I_1 I_2 I_3 is a triangle, which is such that A, B and C are the feet of the perpendiculars drawn from its vertices upon the opposite sides and such that I is the intersection of these perpendicular. i.e. ABC is its pedal triangle and I is its orthocentre. The triangle $I_1I_2I_3$ is often called the excentric triangle.

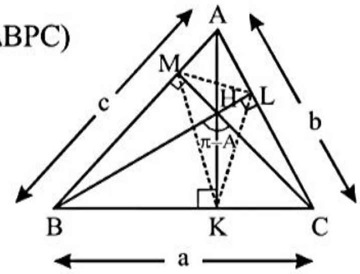
1.13.4 Prove that circumradii of ΔHBC, ΔHCA and ΔHAB and ΔABC are equal :

In ΔBHC, $\angle BHC = \pi - A$

In ΔBMC, $\angle B + \angle BCM + 90^\circ = 180^\circ$

$\angle BCM = 90^\circ - \angle B,\qquad \angle KHC = 90^\circ - (90^\circ - \angle B) = \angle B$

similarly $\angle KHB = \angle C$, so $\angle BHC = \angle B + \angle C = 180^\circ - \angle A$



In ΔBHC, $\dfrac{a}{\sin(\pi - A)} = 2R'$ (where R' is the circumcircle of ΔBPC)

$\dfrac{a}{\sin A} = 2R'$ But we know $\dfrac{a}{\sin A} = 2R$

so $2R' = 2R \Rightarrow R' = R$. So circum radii of ΔHBC, ΔHCA and ΔHAB and ΔABC are equal.

Note :- Radius of the circle circumscribing a pedal triangle in $\dfrac{R}{2}$.

Proof: $\dfrac{ML}{\sin(MKL)} = 2R' = \dfrac{a\cos A}{\sin(180^\circ - 2A)} = \dfrac{2R\sin A\cos A}{\sin 2A} = R$

Proof: $\dfrac{}{\sin(MKL)} = 2R' = \dfrac{}{\sin(180^\circ - 2A)} = \dfrac{}{\sin 2A} = R$

$$\boxed{R' = \frac{R}{2}}$$

1.13.5 Distance between the circumcentre and the orthocentre :

$$\boxed{OH = R\sqrt{(1 - 8\cos A\cos B\cos C)}}$$

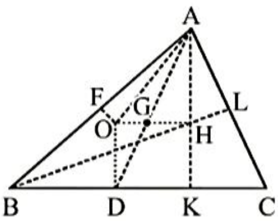
If OF be perpendicular to AB, we have

$\angle OAF = 90^\circ - \angle AOF = 90^\circ - C$

Also, $\angle OAH = A - \angle OAF - \angle HAL$

$= A - 2\,(90^\circ - C) = A + 2C - 180^\circ$

$= A + 2C - (A + B + C) = C - B$



Also, $OA = R,\qquad HA = 2R\cos A$

$OH^2 = OA^2 + HA^2 - 2\,OA\,HA\cos OAP = R^2 + 4R^2\cos^2 A - 4R^2\cos A\cos(C - B)$

$= R^2 + 4R^2\cos A\,[\cos A - \cos(C - B)]$

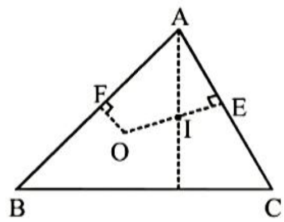
$= R^2 - 4R^2\cos A\,[\cos(B + C) + \cos(C - B)]$

$OH^2 = R^2 - 8R^2\cos A\cos B\cos C$

$OH = R\sqrt{(1 - 8\cos A\cos B\cos C)}$

1.13.6 Distance between the cirumcentre and incentre :

$$\boxed{OI = \sqrt{R^2 - 2Rr}}$$



Let O be the circumcentre and OF be perpendicular to AB.
Let I be the incentre and IE be perpendicular to AC.
Then, as in the previous article,

$$\angle OAF = 90^\circ - C$$

$$\therefore \quad \angle OAI = \angle IAF - \angle OAF = \frac{A}{2} - (90^\circ - C)$$

$$= \frac{A}{2} + C - \frac{A+B+C}{2} = \frac{C-B}{2}$$

Also, $$AI = \frac{IE}{\sin\frac{A}{2}} = \frac{r}{\sin\frac{A}{2}} = \frac{4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}{\sin\frac{A}{2}}$$

so $$AI = 4R\sin\frac{B}{2}\sin\frac{C}{2}$$

$$\therefore \quad OI^2 = OA^2 + AI^2 - 2OA \cdot AI\cos(OAI)$$

$$OI^2 = R^2 + 16R^2\sin^2\frac{B}{2}\sin^2\frac{C}{2} - 8R^2\sin\frac{B}{2}\sin\frac{C}{2}\cos\left(\frac{C-B}{2}\right)$$

$$OI^2 = R^2 + 16R^2\sin^2\frac{B}{2}\sin^2\frac{C}{2} - 8R^2\sin\frac{B}{2}\sin\frac{C}{2}\cos\left(\frac{C-B}{2}\right)$$

$$\frac{OI^2}{R^2} = 1 + 16\sin^2\frac{B}{2}\sin^2\frac{C}{2} - 8\sin\frac{B}{2}\sin\frac{C}{2}\left[\cos\frac{B}{2}\cos\frac{C}{2} + \sin\frac{B}{2}\sin\frac{C}{2}\right]$$

$$= 1 - 8\sin\frac{B}{2}\sin\frac{C}{2}\left[\cos\frac{B}{2}\cos\frac{C}{2} - \sin\frac{B}{2}\sin\frac{C}{2}\right]$$

$$= 1 - 8\sin\frac{B}{2}\sin\frac{C}{2}\cos\left(\frac{B+C}{2}\right) = 1 - 8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$$

$$OI = R\sqrt{\left(1 - 8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\right)}$$

we can write this in another form also.

$$OI^2 = R^2 - 8R^2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$$

$$= R^2 - 2R\left(4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\right)$$

$$= R^2 - 2Rr \qquad \text{as} \qquad r = 4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}.$$

1.14 CYCLIC QUADRILATERAL AND REGULAR POLYGON :

1.14.1 Polygon :

(i) Sum of interior angles of a polygon = $(n-2) \times \pi$. where $n \geq 2$ and n denotes number of sides of a polygon.

(ii) Sum of exterior angles of a polygon is 2π.

(iii) **Convex polygon :** If the highest interior angle is less than 180° then it is called convex polygon.

(iv) **Concave polygon :** Highest interior angle is more than 180° then it is concave polygon.

1.14.2 Cyclic Quadrilateral :

A cyclic quadrilateral is a quadrilateral which can be inscribed by a circle.

Note : The sum of the opposite angles of a cyclic quadrilateral is 180°.

In a cyclic quadrilateral sum of the products of the opposite sides is equal to the product of the diagonals.

Regular Polygon

A regular polygon is a polygon which has all its sides as well as its angles equal. If the polygon has n-

$(n-2)\pi$

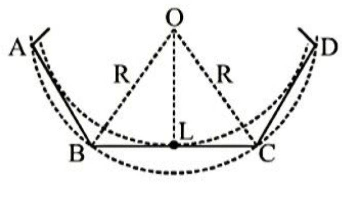
A regular polygon is a polygon which has all its sides as well as its angles equal. If the polygon has n-

sides, sum of its internal angles is $(n-2)\pi$ and each angle is $\dfrac{(n-2)\pi}{n}$.

Note : In the regular polygon, the centroid, the circumcentre and the in-centre are the same.

To find the perimeter (P) and Area(A) of a regular polygon inscribed in a circle of radius 'R'.

Let AB, BC and CD be three successive sides of the polygon and O be the centre of both the incircle and the circumcircle of the polygon.



$$\angle BOC = \frac{2\pi}{n}, \text{ so } \angle BOL = \frac{1}{2}\left(\frac{2\pi}{n}\right) = \frac{\pi}{n}$$

If 'a' be the side of the polygon, we have

$$a = BC = 2BL = 2R \sin(\angle BOL) = 2R \sin\left(\frac{\pi}{n}\right)$$

So, $\quad R = \dfrac{a}{2} \operatorname{cosec}\left(\dfrac{\pi}{n}\right)$

Again $\quad a = 2BL = 2OL \tan(\angle BOL),\ OL = \dfrac{a}{2\tan\left(\frac{\pi}{n}\right)} = \dfrac{a}{2}\cot\left(\dfrac{\pi}{n}\right) \Rightarrow r = \dfrac{a}{2}\cot\left(\dfrac{\pi}{n}\right)$

where R : Radius of circle circumscribing the polygon = OB = OC
r : Radius of circle inscribed in the polygon = OL

Perimeter $P = nBC = n(2BL) = 2n\,R\sin(\angle BOL) = 2n\,R\sin\left(\frac{\pi}{n}\right) = 2nR\sin\left(\frac{\pi}{n}\right)$

Area $A = n$ Area of $\Delta BOC = n\,\frac{1}{2}\,R\cdot R\cdot\sin(\angle BOC) = \frac{nR^2}{2}\sin\left(\frac{2\pi}{n}\right)$

Illustration :

Prove that the area of a regular polygon of 2n sides inscribed in a circle is the geometric mean of the areas of the inscribed and circumscribed polygons of n-sides.

Sol. *Let 'a' be the radius of the circle.*

Then S_1 = *Area of regular polygon of n-sides inscribed in the circle* $= \frac{1}{2}na^2\sin\left(\frac{2\pi}{n}\right)$

S_2 = *Area of regular polygon of n-sides circumscribing the circle* $= na^2\tan\left(\frac{\pi}{n}\right)$

S_3 = *Area of regular polygon of 2n-sides inscribed in the circle* $= na^2\sin\left(\frac{\pi}{n}\right)$

S_3 = *Area of regular polygon of 2n-sides inscribed in the circle* $= na^2\sin\left(\frac{\pi}{n}\right)$

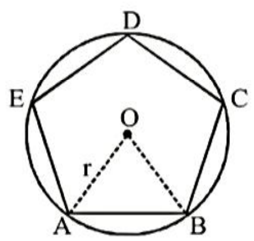
$\therefore$ *Geometric mean of* S_1 *and* $S_2 = \sqrt{\frac{1}{2}na^2\,2\sin\left(\frac{\pi}{n}\right)\cos\left(\frac{\pi}{n}\right)na^2\tan\left(\frac{\pi}{n}\right)} = na^2\sin\left(\frac{\pi}{n}\right) = S_3$

Illustration :

If the area of the circle is A_1 and the area of the regular pentagon inscribed in the circle is A_2. then find the ratio $\frac{A_1}{A_2}$?

Sol. *In ΔOAB,*

$OA = OB = r$ *and* $\angle AOB = \frac{360°}{5} = 72°$



$\therefore$ *Area of* $\Delta AOB = \frac{1}{2}(r)(r)\sin 72° = \frac{r^2}{2}\cos 18°$

A_2 = *Area of pentagon* $= \frac{5r^2}{2}\cos 18°$

A_1 = *Area of circle* $= \pi r^2$ *so* $\frac{A_1}{A_2} = \frac{2\pi}{5}\sec\left(\frac{\pi}{10}\right)$. ***Ans.***

Illustration :

The length of each side of a regular dodecagen is 20 cm. Find (1) The radius of its inscribed circle (2) The radius of its circumscribing circle (3) its area ?

Sol. *The angle subtended by a side at the centre of the polygon* $= \frac{360°}{12} = 30°$.

Hence, $\frac{20}{2} = r \tan 15° = R \sin 15°$

So, $r = 10 \cot 15° = 10\left(2+\sqrt{3}\right)$ cm, $R = \frac{10}{\sin 15°} = 10\left(\sqrt{6}+\sqrt{2}\right)$ cm

Area $= \frac{1}{2} \times 20 \times r \times 12 = 10 \times r \times 12 = 1200\left(2+\sqrt{3}\right)$ cm. ***Ans.***

Practice Problem

Q.1 Find the difference between the areas of a regular octagon and a regular hexagon if the perimeter of each is 24 cm.

Q.1 Find the difference between the areas of a regular octagon and a regular hexagon if the perimeter of each is 24 cm.

Q.2 If an equilateral triangle and a regular hexagon have the same perimeter, prove that their areas are as 2 : 3.

Q.3 Prove that the sum of the radii of the circles, which are respectively inscribed in and circumscribed about a regular polygon of n-sides is $\frac{a}{2} \cot \frac{\pi}{2n}$, where a is side of the polygon.

Q.4 Of two regular polygon of n-sides, one circumscribes and other is inscribed in a given circle. Prove that the perimeters of the circumscribing polygon, the circle, and the inscribed polygon are in the ratio $\sec\frac{\pi}{n} : \frac{\pi}{n} \operatorname{cosec}\frac{\pi}{n} : 1$ and that the areas of the polygons are in the ratio $\cos^2\frac{\pi}{n} : 1$.

Q.5 Given that the area of a polygon of n-sides circumscribed about a circle is to the area of the circumscribed polygon of 2n sides 3 : 2, find n ?

Answer key

Q.1 1.8866 cm

Q.5 5

1.15 SOLUTION OF TRIANGLES (Ambiguous cases) :

When three elements of a triangle are known, the other three elements can be evaluated. This process is called solution of triangles. Note following points

(i) If the three sides a, b, c are given, angle A is a obtained from

$$\tan\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}} \text{ or } \cos A = \frac{b^2+c^2-a^2}{2bc}$$

B and C can be obtained similarly.

(ii) If two sides b and c and the included angle A are given, then

$$\tan\left(\frac{B-C}{2}\right) = \frac{b-c}{b+c}\cot\frac{A}{2} \quad \text{gives} \quad \frac{B-C}{2} \quad \text{also} \quad \frac{B+C}{2} = 90^\circ - \frac{A}{2},$$

so that B and C can be evaluated. The third side is given by

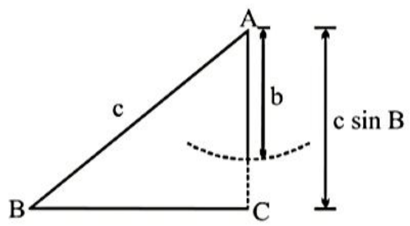
$$a = \frac{b\sin A}{\sin B} \text{ or } a^2 = b^2 + c^2 - 2bc\cos A.$$

(iii) If two sides b and c and the angle B (opposite to side b) are given, then $\sin C = \frac{c}{b}\sin B$,

(iii) If two sides b and c and the angle B (opposite to side b) are given, then $\sin C = \frac{c}{b}\sin B$,

$A = 180^\circ - (B + C)$ and $a = \frac{b\sin A}{\sin B}$ give the remaining elements.

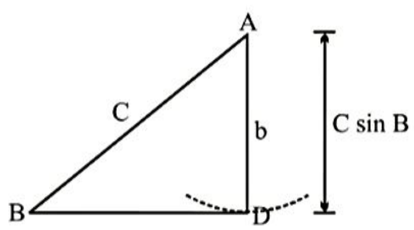
Here in this segment there are many cases of possibility of triangle. We will study them one by one.



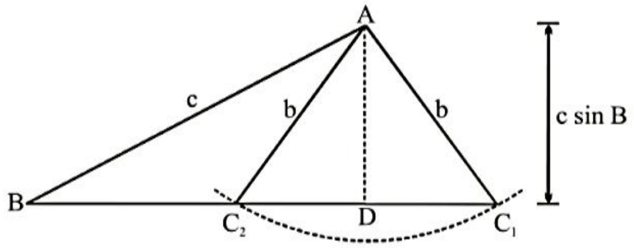
Case-I : $b < c\sin B$,

We draw the side c and angle B. Such kind of triangle is not possible.

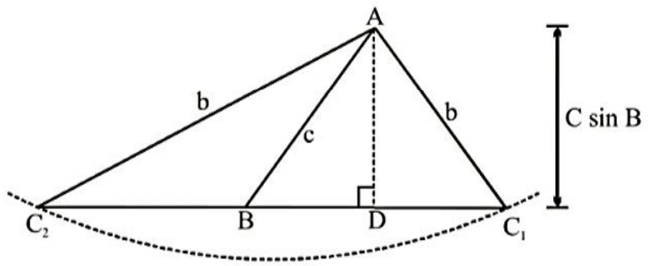
Case-II : $b = c\sin B$ and B is an acute angle, then there is only one triangle possible.



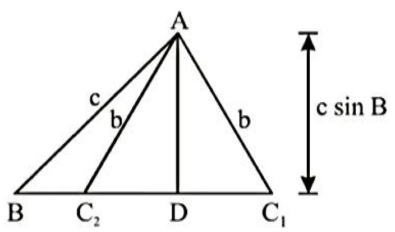
Case-III : If $b > c \sin B$, $b < c$ and B is an acute angle, then there are two values of angle C.



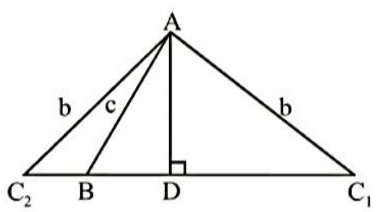
Case-IV : $b > c \sin B$, $c < b$ and B is an acute angle, then there is only one triangle possible.



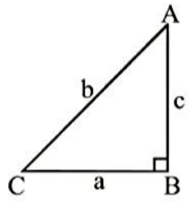
Case-V : $b > c \sin B$, $c > b$ and B is an obtuse angle. For any choice of point C, b will be greater than c which is a contradiction as $c > b$ (given). So there is no triangle possible. Because B is obtuse .



Case-VI: $b > c \sin B$, $c < b$ and B is an obtuse angle. We can see that the circle with A as centre and b as radius will cut the line only in one point. So, one triangle is possible.



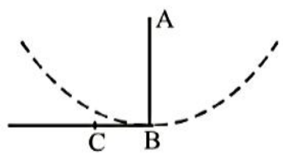
Case-VII: $b > c$ and $B = 90°$



Again the circle with A as centre and b as radius will cut the line only in one point. So only one triangle is possible.

Case-VIII: $b \le c$ and $B = 90°$

The circle with A as centre and b as radius will not cut the line in any point. So, no triangle is possible. Point 'C' will coincide with point 'B'



Alternative method : By applying the cosine rule, we have

$$\cos B = \frac{a^2 + c^2 - b^2}{2ac}.$$

$\Rightarrow \quad a^2 - (2c\cos B)\,a + c^2 - b^2 = 0$

$\Rightarrow \quad a = c\cos B \pm \sqrt{\left[(c\cos B)^2 - (c^2 - b^2)\right]}$

$\Rightarrow \quad a = c\cos B \pm \sqrt{\left[b^2 - (c\sin B)^2\right]}$

This equation leads to the following cases :

Case-I : If $b < c\sin B$, no such triangle is possible.

Case-II: Let $b = c\sin B$. There are further following two cases :

(a) B is an obtuse angle $\Rightarrow$ cos B is negative. There exists no solution triangle.

(b) B is an acute angle $\Rightarrow$ cos B is positive. There exists only one such triangle.

Case-III: Let $b > c \sin B$. There are further following two cases :

(a) B is an acute angle $\Rightarrow$ cos B is positive. In this cases two values of a will exists if and only if

$c \cos B > \sqrt{\left(b^2 - (c \sin B)^2\right)}$ or $c > b$ $\Rightarrow$ two such triangles are possible.

If $c < b$, only one such triangle is possible.

(b) B is an obtuse angle $\Rightarrow$ cos B is negative. In this case, triangle will exists if and only if

$\sqrt{\left(b^2 - (c \sin B)^2\right)} > |\cos B|$ $\Rightarrow$ $b > c$. So in this case only one triangle is possible.

If $b < c$ there exists no such triangle.

Illustration :

In a triangle ABC, the sides b, c and angle B are given such that a has two values a_1 and a_2. Then prove that $|a_1 - a_2| = \sqrt{(b^2 - c^2 \sin^2 B)}$.

Sol. $\cos B = \dfrac{a^2 + c^2 - b^2}{2ac} \Rightarrow a^2 - 2ac \cos B + c^2 - b^2 = 0$

$\Rightarrow \quad a_1 + a_2 = 2c \cos B, \quad a_1 a_2 = c^2 - b^2$

$\Rightarrow \quad a_1 + a_2 = 2c \cos B, \quad a_1 a_2 = c^2 - b^2$

$\Rightarrow \quad (a_1 - a_2)^2 = (a_1 + a_2)^2 - 4a_1 a_2 = 4c^2 \cos^2 B - 4(c^2 - b^2)$

$= 4b^2 - 4c^2 \sin^2 B = 4(b^2 - c^2 \sin^2 B)$

$\Rightarrow \quad |a_1 - a_2| = 2\sqrt{(b^2 - c^2 \sin^2 B)}$.

Illustration :

In a ΔABC, a, c and A are given and b_1, b_2 are two values of the third side b such that $b_2 = 2b_1$. Then prove that $\sin A = \sqrt{\left(\dfrac{9a^2 - c^2}{8c^2}\right)}$.

Sol. $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc} \Rightarrow b^2 - 2bc \cos A + (c^2 - a^2) = 0$

$b_1 + b_2 = 2c \cos A, \quad b_1 b_2 = c^2 - a^2$

$\Rightarrow \quad 3b_1 = 2 \cos A, \qquad 2b_1^2 = c^2 - a^2$

$$2\left(\frac{2c \cos A}{3}\right)^2 = c^2 - a^2 \Rightarrow 8c^2 (1 - \sin^2 A) = 9c^2 - 9a^2$$

$$\sin A = \sqrt{\left(\frac{(9a^2 - c^2)}{8c^2}\right)}.$$

Practice Problem

Q.1 If in a triangle ABC, $a = \left(1+\sqrt{3}\right)$ cm, $b = 2$ cm, and $\angle C = 60°$, then find the other two angles and the third side ?

Q.2 If $A = 30°$, $a = 7$, $b = 8$ in ΔABC, then find the number of triangles that can be constructed.

Q.3 If $b = 3$, $c = 4$ and $B = \frac{\pi}{3}$ in ΔABC, then find the number of triangles that can be constructed.

Answer key

Q.1 $A = 75°$, $B = 45°$, $c = \sqrt{6}$ cm Q.2 2 Q.3 0

Solved Examples

Q.1 In ΔABC, if $\frac{\sin A}{c \sin B} + \frac{\sin B}{c} + \frac{\sin C}{b} = \frac{c}{ab} + \frac{b}{ac} + \frac{a}{bc}$, then the value of angle A, is

(All symbols used have their usual meaning in a triangle.)

(A) 120° (B) 90° (C) 60° (D) 30°

Sol. We have

$$\frac{\sin A}{c \sin B} + \frac{\sin B}{c} + \frac{\sin C}{b} = \frac{c}{ab} + \frac{b}{ac} + \frac{a}{bc}$$

$$\Rightarrow \quad \frac{a}{cb} + \frac{b \sin B + c \sin C}{bc} = \frac{c^2 + b^2}{abc} + \frac{a}{bc}$$

$$\Rightarrow \quad a\,(b \sin B + c \sin C) = b^2 + c^2$$

$$\Rightarrow \quad a\left(b \cdot \frac{b}{2R} + c\frac{c}{2R}\right) = b^2 + c^2$$

$$\Rightarrow \quad \frac{a(b^2 + c^2)}{2R} = b^2 + c^2$$

$$\Rightarrow \quad \frac{a(b^2 + c^2)}{2R} = b^2 + c^2$$

$$\Rightarrow \quad a = 2R$$

$\Rightarrow$ ΔABC is a right angle triangle, $\angle A = 90^\circ$.

Q.2 In a triangle ABC, $3 \sin A + 4 \cos B = 6$ and $3 \cos A + 4 \sin B = 1$, then $\angle C$ can be

(A) 30° (B) 60° (C) 90° (D) 150°

Sol. Given

$$3 \sin A + 4 \cos B = 6 \qquad \ldots(i)$$

$$3 \cos A + 4 \sin B = 1 \qquad \ldots(ii)$$

Squaring and adding equation (i) & (ii)

$$(3 \sin A + 4 \cos B)^2 + (3 \cos A + 4 \sin B)^2 = 36 + 1$$

$$\Rightarrow \quad 9 + 16 + 24\,(\sin A \cos B + \cos A \sin B) = 37$$

$$\Rightarrow \quad \sin(A + B) = \frac{1}{2}$$

$\Rightarrow$ $A + B = 30^\circ$ or 150°

when $A + B = 30^\circ$ then $(3 \sin A + 4 \cos B) < 3 \sin 30^\circ + 4 \cos 30^\circ < 6$

so $A + B = 150^\circ$

$\therefore$ $\angle C = 30^\circ$

Q.3 In a triangle ABC, angle A is greater than B, if the measures of angles A and B satisfy the equation $3 \sin x - 4 \sin^3 x = k$, $0 < k < 1$, then measure of angle C is

(A) $\frac{\pi}{3}$ (B) $\frac{\pi}{2}$ (C) $\frac{2\pi}{3}$ (D) $\frac{5\pi}{6}$

Sol. We have $3 \sin x - 4 \sin^3 x = k$

$\Rightarrow \quad \sin 3x = k$

$\Rightarrow \quad \sin 3A = k, \sin 3B = k$

$\Rightarrow \quad \sin 3A - \sin 3B = 0$

$$\Rightarrow \quad 2\sin\left(\frac{3A-3B}{2}\right)\cos\left(\frac{3A+3B}{2}\right) = 0$$

$$\Rightarrow \quad \cos\frac{3}{2}(A+B) = 0 \qquad \because A > B$$

$$\Rightarrow \quad \frac{3}{2}(A+B) = \frac{\pi}{2}$$

$$\Rightarrow \quad A + B = \frac{\pi}{3}$$

$$\Rightarrow \quad C = \frac{2\pi}{3} \qquad \text{Ans.}$$

Q.4 If in a triangle ABC, $\sin A = \sin^2 B$ and $2 \cos^2 A = 3 \cos^2 B$, then the ΔABC is

(A) right angled (B) obtuse angled (D) isosceles (D) equilateral

Sol. $\sin A = \sin^2 B$ ...(i)

$2 \cos^2 A = 3 \cos^2 B$ ...(ii)

$\Rightarrow \quad 2(1 - \sin^2 A) = 3(1 - \sin A)$

$$\Rightarrow \quad \sin A = 1, \frac{1}{2}$$

But $\sin A \neq 1$ from equation (i)

$$\therefore \quad \sin A = \frac{1}{2} \quad \Rightarrow \quad A = 30^\circ \quad \text{or} \quad 150^\circ$$

$$\sin^2 A = \frac{1}{2} \quad \Rightarrow \quad B = 45^\circ \quad \text{or} \quad 135^\circ$$

in each can triangle is obtuse angled.

Q.5 In a triangle with sides a, b and c, a semicircle touching the sides AC and CB is inscribed whose diameter lies on AB. Then, the radius of the semicircle is

(A) $a/2$ (B) Δ / s

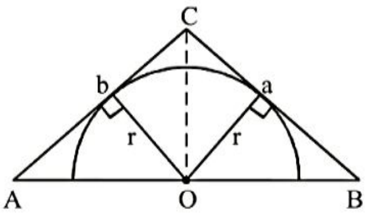
(C) $\dfrac{2\Delta}{a+b}$ (D) $\dfrac{2abc}{(s)(a+b)}\cos\dfrac{A}{2}\cos\dfrac{B}{2}\cos\dfrac{C}{2}$

Sol. Let radius of semicircle is r.

Area of ΔACB = Area of ΔAOC + Area of ΔBOC

$$\Rightarrow \quad \Delta = \frac{1}{2}ar + \frac{1}{2}br$$

$$\Rightarrow \quad r = \left(\frac{2\Delta}{a+b}\right)$$



$$\frac{2abc}{s(a+b)}\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2} = \frac{2abc}{s(a+b)}\sqrt{\frac{s(s-a)}{bc}}\sqrt{\frac{s(s-b)}{ca}}\sqrt{\frac{s(s-c)}{ab}}$$

$$= 2abc \sqrt{s(b-a)(s-b)(s-c)} = 2\Delta =$$

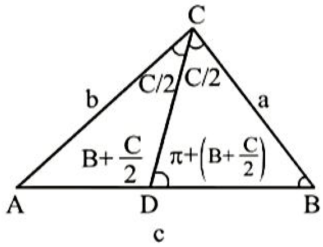
$$= \frac{2abc}{(a+b)}\frac{\sqrt{s(b-a)(s-b)(s-c)}}{abc} = \frac{2\Delta}{a+b} = r$$

Q.6 If in a triangle ABC, CD is the angular bisector of the angle ACB then CD is equal to

(A) $\dfrac{a+b}{2ab}\cos\dfrac{C}{2}$ (B) $\dfrac{a+b}{ab}\cos\dfrac{C}{2}$ (C) $\dfrac{2ab}{a+b}\cos\dfrac{C}{2}$ (D) $\dfrac{b\sin A}{\sin\left(B+\dfrac{C}{2}\right)}$

Sol. We know that $CD = \dfrac{2ab}{a+b}\cos\dfrac{C}{2}$

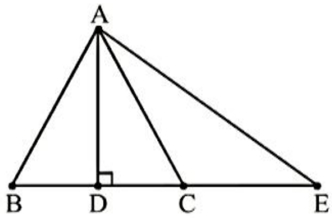
$\therefore \quad \angle ADC = B + \dfrac{C}{2}$



Applying SINE Rule in ΔACD

$$\frac{CD}{\sin A} = \frac{b}{\sin\left(B+\dfrac{C}{2}\right)} \Rightarrow CD = \frac{b\sin A}{\sin\left(B+\dfrac{C}{2}\right)}$$

Q.7 In triangle ABC, $|AB| = |AC|$. Points D and E lie on ray BC such that $|BD| = |DC|$ and $|BE| > |CE|$. Suppose that $\tan \angle EAC$. $\tan \angle EAD$ and $\tan \angle EAB$ form geometric progression, and that $\cot \angle DAE$, $\cot \angle CAE$ and $\cot \angle DAB$ from an arithmetic progression. If $|AE| = 10$, then which of the following is/are true.



(A) $\angle DEA = \dfrac{\pi}{4}$ (B) $\cot \angle DAC = 3$

(C) $\cot \angle CAE = 2$ (D) Area of the triangle ABC $= \dfrac{50}{3}$

Sol. Let $\angle EAD = \alpha$

and $\angle BAD = \angle DAC = \beta$

$\therefore$ $\angle EAC = \alpha - \beta$

$\tan \angle EAC$, $\tan \angle EAD$ and $\tan \angle EAB$ form a geometric progression.

$\Rightarrow$ $\tan^2 \alpha = \tan(\alpha - \beta) \cdot \tan(\alpha + \beta)$

$$\Rightarrow \quad \tan^2 \alpha = \left(\frac{\tan\alpha - \tan\beta}{1 + \tan\alpha \tan\beta}\right)\left(\frac{\tan\alpha + \tan\beta^3}{1 - \tan\alpha\tan\beta}\right) \Rightarrow \tan^2\alpha = \frac{\tan^2\alpha - \tan^2\beta}{1 - \tan^2\alpha \ \tan^2\beta}$$

$\Rightarrow$ $\tan^2 \alpha - \tan^4 \alpha \tan^2 \beta = \tan^2 \alpha - \tan^2 \beta$

$\Rightarrow$ $\tan \alpha = 1$ $\Rightarrow$ $\alpha = 45^\circ$

$\Rightarrow$ ΔADE is an isosceles triangle

$\Rightarrow$ $\tan^2 \alpha - \tan^4 \alpha \tan^2 \beta = \tan^2 \alpha - \tan^2 \beta$

$\Rightarrow$ $\tan \alpha = 1$ $\Rightarrow$ $\alpha = 45^\circ$

$\Rightarrow$ ΔADE is an isosceles triangle.

$AD = DE = \dfrac{AE}{\sqrt{2}} = 5\sqrt{2}$

ΔACD, $DC = AD \tan \beta$

Areaof $\Delta ABC = AD \cdot CD = AD^2 \tan \beta$

$\cot \angle DAE$, $\cot \angle CAE$, $\cot \angle DAB$ are in AP

$\therefore$ $2\cot(45^\circ - \beta) = \cot 45^\circ + \cot \beta$

$$\frac{2(\cot 45^\circ \cot\beta + 1)}{\cot 45^\circ - \cot\beta} = 1 + \cot\beta$$

By solving $\cot \beta = 3$ $\therefore$ Area of $\Delta ABC = \left(\dfrac{50}{3}\right)$ unit^2

Q.8 If the angles A, B, C of a triangle are in A.P. and sides a, b, c are in G.P. then a^2, b^2, c^2 are in

(A) A.P. (B) H.P. (C) G.P. (D) None of these

Sol. $2B = A + C$, $B = 60^\circ$

$$\cos B = \frac{a^2 + c^2 - b^2}{2ac} = \frac{1}{2}, \qquad a^2 + c^2 - b^2 = ac$$

as $ac = b^2$ so $a^2 + c^2 - b^2 = b^2$, $a^2 + c^2 = 2b^2$

Q.9 If $\frac{r}{r_1} = \frac{r_2}{r_3}$ then

(A) A = 90° (B) B = 90° (C) C = 90° (D) None of these

Sol. $\frac{r}{r_1} = \frac{r_2}{r_3} \Rightarrow r\,r_3 = r_1\,r_2 \Rightarrow \frac{\Delta}{s}\frac{\Delta}{(s-c)} = \frac{\Delta}{(s-a)}\frac{\Delta}{(s-b)}$

$\Rightarrow \quad (s-a)(s-b) = s(s-c)$

$$\frac{(s-a)(s-b)}{s(s-c)} = 1, \qquad \tan^2\frac{c}{2} = 1, \qquad \tan\frac{c}{2} = 1$$

c = 90°

Q.10 The ratio of the area of a regular polygon of n-sides inscribed in a circle to that of the the polygon of same number of sides circumscribing the same circle is 3 : 4. Then the value of n is?

(A) 6 (B) 4 (C) 8 (D) 12

Sol. Let 'a' be the radius of the circle, then the ratio of the area of the regular polygen of n sides inscribed and circumscribing the same circle is

$$\frac{s_1}{s_2} = \frac{\frac{1}{2}na^2\sin\left(\frac{2\pi}{n}\right)}{na^2\tan\left(\frac{\pi}{n}\right)} = \frac{3}{4} \quad \Rightarrow \quad \cos^2\left(\frac{\pi}{n}\right) = \frac{3}{4}$$

$$\cos\frac{\pi}{n} = \frac{\sqrt{3}}{2} \quad \text{so} \quad \frac{\pi}{n} = \frac{\pi}{6} \quad \text{Ans. } n = 6$$

Q.11 If in a ΔABC, $\sin^3 A + \sin^3 B + \sin^3 C = 3 \sin A \sin B \sin C$. Find the value of determinant.

$$\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$$

Sol. $\sin^3 A + \sin^3 B + \sin^3 C = 3 \sin A \sin B \sin C$

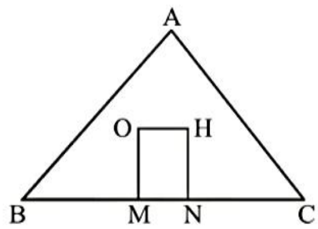
$\Rightarrow \quad \sin A = \sin B = \sin C$

$\Rightarrow \quad a = b = c$

$$\therefore \quad \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = 0$$

Q.12 In a scalene acute ΔABC, it is known that line joining circumcentre and orthocentre is parallel to BC. Prove that the angle $A \in \left(\frac{\pi}{3}, \frac{\pi}{2}\right)$.

Sol. Distance of circumcentre (O) from BC = Distance of orthocentre (H) from BC



$OM = HN$

$R \cos A = 2R \cos B \cos C$

$\Rightarrow \quad \cos A = 2 \cos B \cos C = \cos (B + C) + \cos (B - C)$

$\Rightarrow \quad \cos A = -\cos A + \cos (B - C)$

$\Rightarrow \quad \cos (B - C) = 2 \cos A$

$\therefore \quad 0 < \cos (B - C) < 1 \quad \Rightarrow \quad 0 < 2 \cos A < 1$

$\Rightarrow \quad \cos A \in \left(0, \frac{1}{2}\right)$

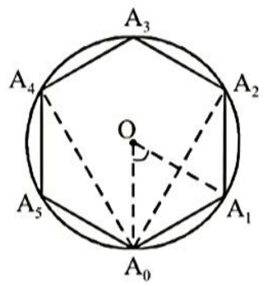
$\therefore \quad A \in \left(\frac{\pi}{3}, \frac{\pi}{2}\right)$

$\therefore \quad A \in \left(\frac{\pi}{3}, \frac{\pi}{2}\right)$

Q.13 If A_0, A_1, A_2, A_3, A_4 and A_5 be the consecutive vertices of a regular hexagon inscribed in a unit circle. Then find the product of length of (A_0A_1), (A_0A_2) and (A_0A_4).

Sol. ΔOA_0A_1

$\angle A_0OA_1 = \frac{2\pi}{6} = \frac{\pi}{3}$



$\cos\frac{\pi}{3} = \frac{1^2 + 1^2 - (A_0A_1)^2}{2 \cdot 1 \cdot 1} = \frac{1}{2}$

$A_0A_1 = 1$

$\Delta A_0OA_2 \qquad \cos\frac{2\pi}{3} = \frac{1^2 + 1^2 - (A_0A_2)^2}{2 \cdot 1 \cdot 1} = \frac{-1}{2}, A_0A_1 = \sqrt{3}$

Similarly $\quad A_0A_4 = \sqrt{3}$

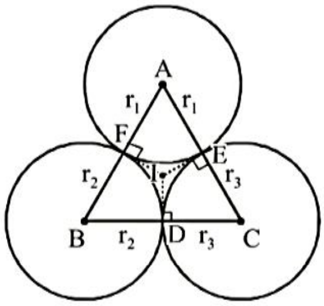
$\therefore \quad (A_0A_1)(A_0A_2)(A_0A_4) = 3$

Q.14 Three circles with radius r_1, r_2, r_3 touch one another externally. The tangents at their points of contact meet at a point whose distance from a point of contact is 2. The the value of $\left(\dfrac{r_1\, r_2\, r_3}{r_1 + r_2 + r_3}\right)$ is equal to

Sol. $a = r_2 + r_3$, $\quad b = r_3 + r_1$, $\quad\quad c = r_1 + r_2$

We have given ID = IE = IF = 2

$$2 = \frac{\text{Area of } \Delta ABC}{\text{semi perimeter of } \Delta ABC}$$



$$\Delta = \sqrt{s(s-a)(s-b)(s-c)} = \sqrt{(r_1 + r_2 + r_3) r_1 r_2 r_3}$$

$$2 = \frac{\sqrt{r_1 r_2 r_3 (r_1 + r_2 + r_3)}}{(r_1 + r_2 + r_3)} = \sqrt{\frac{r_1 r_2 r_3}{r_1 + r_2 + r_3}}$$

$\Rightarrow \quad \dfrac{r_1 r_2 r_3}{r_1 + r_2 + r_3} = 4 \quad\quad$ Ans.

Q.15 If in the triangle ABC, O is the circumcentre and R is the circumradius and R_1, R_2, R_3 are the circumradii

Q.15 If in the triangle ABC, O is the circumcentre and R is the circumradius and R_1, R_2, R_3 are the circumradii of the triangles OBC, OCA and OAB respectively, then prove that $\dfrac{a}{R_1} + \dfrac{b}{R_2} + \dfrac{c}{R_3} = \dfrac{abc}{R^3}$.

Sol. ΔBOC, using SINE rule

$$2R_1 = \frac{a}{\sin 2A}$$

$\therefore \quad \dfrac{a}{R_1} = 2 \sin 2A$

Similarly $\quad \dfrac{b}{R_2} = 2 \sin 2B, \quad\quad \dfrac{c}{R_3} = 2 \sin^2 C$

$\therefore \quad \dfrac{a}{R_1} + \dfrac{b}{R_2} + \dfrac{c}{R_3} = 2\,(\sin 2A + \sin 2B + \sin 2C)$

$$= 2\,(4 \sin A \sin B \sin C)$$

$$= 8 \cdot \frac{a}{2R} \cdot \frac{b}{2R} \cdot \frac{c}{2R} = \frac{abc}{R^3}$$

Q.16 If x, y and z are respectively the distances of the vertices of the ΔABC from its orthocentre then prove that

(i) $\frac{a}{x}+\frac{b}{y}+\frac{c}{z}=\frac{abc}{xyz}$ (ii) $x+y+z=2(R+r)$

Sol. $x = 2R\cos A$
$y = 2R\cos B$
$z = 2R\cos C$

(i) $\frac{a}{x}=\frac{2R\sin A}{2R\cos A}=\tan A$

similarly $\frac{b}{y}=\tan B,\quad \frac{c}{z}=\tan C$

$\therefore\quad \frac{a}{x}+\frac{b}{y}+\frac{c}{z}=\tan A+\tan B+\tan C=\tan A\tan B\tan C$

$$=\frac{a}{x}+\frac{b}{y}+\frac{c}{z}=\frac{abc}{xyz}$$

(ii) $x+y+z=2R(\cos A+\cos B+\cos C)=2R\left(1+4\sin\left(\frac{A}{2}\right)\sin\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right)\right)$

$$=2\left(R+4R\sin\left(\frac{A}{2}\right)\sin\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right)\right)$$
$$=2(R+r)$$

Q.17 If I be the in centre of ΔABC, then prove that $IA\cdot IB\cdot IC=abc\,\tan\left(\frac{A}{2}\right)\tan\left(\frac{B}{2}\right)\tan\left(\frac{C}{2}\right)$.

Sol. $IA=r\,\text{cosec}\,\frac{A}{2},\qquad IB=r\,\text{cosec}\,\frac{B}{2},\qquad IC=r\,\text{cosec}\,\frac{C}{2}$

$$IA\cdot IB\cdot IC=\frac{r^3}{\sin\left(\frac{A}{2}\right)\cdot\sin\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right)}=\frac{\left(4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\right)^3}{\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}$$

$$=64R^3\sin^2\frac{A}{2}\sin^2\frac{B}{2}\sin^2\frac{C}{2}$$

$$=64R^3\left(\frac{abc}{4\Delta}\right)^3\cdot\frac{(s-b)(s-c)}{bc}\cdot\frac{(s-c)(s-a)}{ca}\cdot\frac{(s-a)(s-b)}{ab}$$

$$=\frac{abc}{\Delta^3}\cdot\left(\frac{\Delta^2}{r}\right)^2=\frac{abc\Delta}{s^2}$$

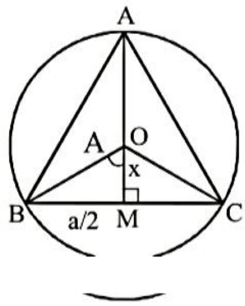
$$abc \tan\left(\frac{A}{2}\right)\tan\left(\frac{B}{2}\right)\tan\left(\frac{C}{2}\right) = abc\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}\sqrt{\frac{(s-c)(s-a)}{s(s-b)}}\sqrt{\frac{(s-a)(s-c)}{s(s-c)}}$$

$$= \frac{abc}{s^2}\sqrt{s(s-a)(s-b)(s-c)}$$

$$= \left(\frac{abc\Delta}{s^2}\right)$$

Q.18 If x, y, z are respectively be the perpendicular from the circumcentre to the sides of ΔABC then prove that $\frac{a}{x}+\frac{b}{y}+\frac{c}{z}=\frac{abc}{4xyz}$.

Sol. ΔBOM



$$\tan A = \frac{a/2}{x} = \frac{a}{2x}$$

similarly $\tan(B) = \frac{a}{2y}$

$\tan(C) = \frac{a}{2z}$

in a triangle $\tan A + \tan B + \tan C = \tan A \tan B \tan C$

$$\Rightarrow \quad \frac{a}{2x}+\frac{b}{2y}+\frac{c}{2z}=\frac{a}{2x}\frac{b}{2y}\cdot\frac{c}{2z}$$

$$\Rightarrow \quad \frac{a}{x}+\frac{b}{y}+\frac{c}{z}=\frac{abc}{4xyz}$$

Q.19 If two times the square of the diameter of the circumcircle of a triangle is equal to the sum of the squares of its sides then prove that the triangle is right angled.

Sol. In ΔABC

We have to prove that $8R^2 = a^2 + b^2 + c^2$

using SINE Rule

$$8R^2 = (2R\sin A)^2 + (2R\sin B)^2 + (2R\sin C)^2$$

$$\Rightarrow \quad 2 = \sin^2 A + \sin^2 B + \sin^2 C$$

$$\Rightarrow \quad 2 = \frac{1-\cos 2A}{2}+\frac{1-\cot 2B}{2}+\frac{1-\cos 2C}{2}$$

$$\Rightarrow \quad 4 = 3 - [-1 - 4\cos A\cos B\cos C]$$

$\Rightarrow \quad \cos A \cos B \cos C = 0$

$\Rightarrow \quad$ one of A, B or C must be 90º

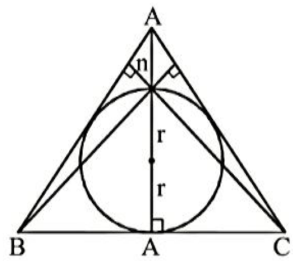
$\Rightarrow \quad$ right angle triangle.

Q.20 In an isosceles ΔABC, if the altitudes intersect on the inscribed circle then find the secant of the vertical angle 'A'.

Sol. $2r$ = distance of orthocentre from BC

$2r = 2R \cos B \cos C$

$2\left(4R \sin\frac{A}{2} \sin\frac{B}{2} \sin\frac{C}{2}\right) = 2R \cos B \cos C$



$\Rightarrow \quad (\cos A + \cos B + \cos C - 1) = \cos B \cos C$

$\because \quad B = C \qquad \therefore \quad B = \left(\frac{\pi - A}{2}\right)$

$\cos A + 2 \cos B - 1 = \cos^2 B$

$\cos A + 2 \cos B - 1 = \cos^2 B$

$\Rightarrow \quad \cos A + 2 \cos\left(\frac{\pi - A}{2}\right) - 1 = \cos^2\left(\frac{\pi - A}{2}\right)$

$\Rightarrow \quad 1 - 2 \sin^2\frac{A}{2} + 2 \sin\frac{A}{2} - 1 = \sin^2\frac{A}{2}$

$\Rightarrow \quad \sin\frac{A}{2} = \frac{2}{3}$

$\Rightarrow \quad \cos A = 1 - 2 \sin^2\frac{A}{2} = 1 - 2\left(\frac{2}{3}\right)^2 = \frac{1}{9} \qquad \therefore \quad \sec A = 9 \qquad$ Ans.

BINOMIAL THEOREM

1.1 BINOMIAL EXPRESSION :

An algebric expression consisting of two different terms is called a binomial expression.
e.g. (1) $x + y$ (2) $x^3 + y^3$
But $(x + nx)$ is not a binomial, it is called a monomial.

1.2 BINOMIAL THEOREM :

In elementary algebra, the binomial theorem describes the algebric expansion of powers of a binomial expression.According to the theorem it is possible to expand the powers $(x + y)^n$ into a sum involving terms of the form ax^by^c, where exponents b and c are non-negative integers with $b + c = n$ and the coefficient 'a' of each term is a specific positive integer depending on n and b.
This theorem was given by Newton.

Binomial Theorem $\Rightarrow$ $(x + y)^n$
- If $n \in N$ (Form a finite series)
- Any index $n \notin N$

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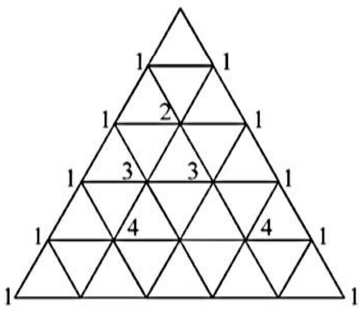
Binomial Theorem $\Rightarrow$ $(x + y)^n$
- If $n \in N$ (Form a finite series)
- Any index $n \notin N$
(Form an infinite series)

1.3 HISTORICAL DEVELOPMENT :

Earlier people used to multiply the brackets to expand the given binomial of known index.
Then came the Pascal triangle



Note that

(a) The powers of x go down until it reaches reaches zero, starting value is n.

(b) The power of y goes up from zero untill it reaches n.

(c) The n^{th} row of the Pascals triangle will be the coefficients of the expanded binomial.

1.4 STATEMENT OF THE THEOREM :

$(x+y)^n = {}^nC_0x^n + {}^nC_1x^{n-1}y + {}^nC_2\,x^{n-2}\,y^2 + \ldots\ldots\ldots + {}^nC_rx^{n-r}\cdot y^r + \ldots\ldots + {}^nC_n\,x^{n-n}\,y^n.$

We observe $\quad T_1 = {}^nC_0\,x^n$

$T_2 = {}^nC_1\,x^{n-1}\cdot y^1$

$\Rightarrow$ General term in the expansion of $(x+y)^n$ is

$T_{r+1} = {}^nC_r\,x^{n-r}\cdot y^r$

Where ${}^nC_r$ is called as combinatorial or binomial coefficient also denoted by $\binom{n}{r}$.

Also, $\quad (x+y)^n = \sum_{r=0}^{n} {}^nC_r\,x^{n-r}\cdot y^r$

Also, $\quad (x+y)^n = \sum_{r=0}^{\cdot\cdot} {}^nC_r\,x^{n-r}\cdot y^r$

Proof-1:

Binomial theorem for a positive integral index is a special case of a very general result called symmetric product which states that

$$(x+y_1)(x+y_2)\ldots\ldots(x+y_n) = x^n + S_1x^{n-1} + S_2\,x^{n-2} + S_3\,x^{n-3} + \ldots\ldots + S_n.$$

where S_r ($r = 1, 2, 3, \ldots\ldots, n$) denotes sum of product of quantities $y_1, y_2, \ldots\ldots\ldots, y_n$ taken r at a time.

Thus, $\quad S_1 = y_1 + y_2 + \ldots\ldots\ldots y_r$ (${}^nC_1$ terms)

$S_2 = y_1y_2 + y_2y_3 + \ldots\ldots\ldots$ (${}^nC_2$ terms)

$S_3 = y_1\,y_2\,y_3 + y_2\,y_3\,y_4 \ldots\ldots\ldots$ (${}^nC_3$ terms)

and so on

Now put

$$y_1 = y_2 = y_3 \ldots\ldots = y_n = y$$

We get

$S_1 = y + y + y \ldots\ldots\ldots + y$ (${}^nC_1$ terms) $= {}^nC_1y$

$S_2 = y^2 + y^2 + \ldots\ldots\ldots + y^2$ (${}^nC_2$ terms) $= {}^nC_2\,y^2$

and so on

In this special case

$$(x+y)^n = x^n + {}^nC_1\,x^{n-1}\cdot y + {}^nC_2\,x^{n-2}\,y^2 + \ldots\ldots\ldots + {}^nC_n\,y^n.$$

Proof-2: Combinatorial Proof

The coefficient of xy^2 in $(x + y)^3 = (x + y)(x + y)(x + y)$.

$= x\,x\,x + x\,x\,y + x\,y\,x + x\,y\,y + y\,x\,x + y\,x\,y + y\,y\,x + y\,y\,y$

$= x^3 + 3x^2y + 3xy^2 + y^3$

Equals ${}^3C_2 = 3$ because there are three x, y strings of length 3 with exactly two y's namely x y y, y x y, y y x.

Corresponding to the three 2 element subsets of {1, 2, 3} namely {2, 3}, {1, 3}, {1, 2}

where each subset specifies the position of y in corresponding string.

Similarly in $(x + y)^n$

1.5 IMPORTANT POINTS OF EXPANSION :

(1) Number of terms in expansion of $(x + y)^n$ is $n + 1$ i.e., one more than index.

or

By begger's method n coins and 2 beggars

$\therefore \quad {}^{n+1}C_1 \Rightarrow (n + 1)$ times.

Illustration :

$\therefore \quad {}^{n+1}C_1 \Rightarrow (n + 1)$ times.

Illustration :

Find number of term is the expansion of $(x + y + z + w)^{10}$.

Sol. *Coin = 10, beggers = 4*

${}^{13}C_3$ *terms.* ***Ans.***

(2) Sum of indices of x and y in each term in the expansion of $(x + y)^n$ is n.

(3) New expansions by $(x + y)^n$

(a) We have $(x + y)^n = {}^nC_0x^n + {}^nC_1\, x^{n-1}y + {}^nC_2\, x^{n-2}\, y^2 + \ldots\ldots + {}^nC_n\, y^n$ ....(1)

i.e., $$(x + y)^n = \sum_{r=0}^{r=n} {}^nC_r\, x^{n-r} \cdot y^r.$$

(b) Replace y by – y

$(x - y)^n = {}^nC_0\, x^n - {}^nC_1\, x^{n-1}y + {}^nC_2\, x^{n-2}\, y^2 + \ldots\ldots + {}^nC_r\, x^{n-r} \cdot y^r(-1)^r + \ldots\ldots + {}^nC_n(-1)^n \cdot y^n$

i.e., $$(x - y)^n = \sum_{r=0}^{r=n} {}^nC_r\,(-1)^r\, x^{n-r} \cdot y^r.$$

(c) Now replace x by 1 and y by x in Ist

$(1+x)^n = {}^nC_0 + {}^nC_1x + {}^nC_2x^2 + {}^nC_3x^3 + ... + {}^nC_rx^r + ... + {}^nC_n\, y^n.$

Also, i.e., $\quad (1+x)^n = \sum_{r=0}^{r=n} {}^nC_r\, x^r$.

(d) $(1-x)^n = {}^nC_0 - {}^nC_1\, x + {}^nC_2x^2 + ...+ {}^nC_r\,(-1)^r\, x^r + ... + {}^nC_n\,(-1)^n\, x^n.$

i.e., $\quad (1-x)^n = \sum_{r=0}^{r=n} (-1)^r\ {}^nC_r\, x^r$.

(e) Also remember

$(1+x)^n + (1-x)^n = 2\,[{}^nC_0 + {}^nC_2x^2 + ...]$

and $\quad (1+x)^n - (1-x)^n = 2\,[{}^nC_1 + {}^nC_3\, x^3 + {}^nC_5\, x^5 + ...]$

$\therefore$ Coefficient of x^r in the expansion of $(1+x)^n$ is ${}^nC_r$.

$T_{r+1} = {}^nC_r\, x^r$

$\therefore$ coefficient of $(r+1)^{th}$ term = coefficient of $x^r = {}^nC_r$ in the expansion of $(1+x)^n$.

For e.g., Find the coefficient of x^6 in $(1+3x+3x^2+x^3)^{15} = [(1+x)^3]^{15} = (1+x)^{45}$

$\therefore$ $T_{r+1} = {}^{45}C_rx^r$

$\therefore$ $r = 6.$

$\therefore$ coefficient is ${}^{45}C_6$.

For e.g., Find the coefficient of x^6 in $(1+3x+3x^2+x^3)^{15} = [(1+x)^3]^{15} = (1+x)^{45}$

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$\therefore$ $r = 6.$

$\therefore$ coefficient is ${}^{45}C_6$.

Illustration :

Find the value of $\left(1+\sqrt{2}\right)^7 + \left(1-\sqrt{2}\right)^7 = ?$

Sol. $\left(1+\sqrt{2}\right)^7 + \left(1-\sqrt{2}\right)^7 = 2\left[{}^7C_0 1 + {}^7C_2\left(\sqrt{2}\right)^2 + ... + {}^7C_6\left(\sqrt{2}\right)^6\right] = 476.$

Illustration :

Find the value of $\left(\sqrt{3}+3\right)^5 - \left(\sqrt{3}-3\right)^5 = ?$

Sol. $\left(\sqrt{3}+3\right)^5 - \left(\sqrt{3}-3\right)^5 = \left(\sqrt{3}\right)^5\left[\left(1+\sqrt{3}\right)^5 - \left(1-\sqrt{3}\right)^5\right]$

$= 2\cdot 9\cdot\sqrt{3}\left[{}^5C_1\left(\sqrt{3}\right) + {}^5C_3\left(\sqrt{3}\right)^3 + {}^5C_5\left(\sqrt{3}\right)^5\right] = 2376.$

Illustration :

Find sum of series upto n terms $\sum_{r=0}^{n}(-1)^r\ {}^nC_r\left[\left(\frac{1}{2}\right)^r+\frac{3^r}{2^{2r}}+\frac{7^r}{2^{3r}}+\ldots\ldots\text{upto } m \text{ terms}\right]$

Sol. $T_1 = \sum_{r=0}^{n}{}^nC_r(-1)^r\left(\frac{1}{2}\right)^r = \left(1-\frac{1}{2}\right)^n = \frac{1}{2^n}$

$$T_2 = \sum_{r=0}^{n}\left(\frac{3}{4}\right)^r(-1)^r\ {}^nC_r = \left(1-\frac{3}{4}\right)^n = \left(\frac{1}{4}\right)^n$$

$\vdots$

$T_1 + T_2 + T_3 + \ldots\ldots\ldots$ *upto m terms*

$\left(\frac{1}{2}\right)^n+\left(\frac{1}{2}\right)^{2n}+\left(\frac{1}{2}\right)^{3n}\ldots\ldots\ldots$ *m terms*

$= \frac{1}{2^n}\left(\frac{1-\left(\frac{1}{2^n}\right)^m}{1}\right) = \frac{2^{mn}-1}{(\quad)}.$

$$= \frac{1}{2^n}\left(\frac{1-\left(\frac{1}{2^n}\right)^m}{1-\frac{1}{2^n}}\right) = \frac{2^{mn}-1}{\left(2^n-1\right)\ 2^{mn}}.$$

---

(4) ${}^nC_0, {}^nC_1, {}^nC_2, \ldots\ldots\ldots\ldots, {}^nC_n$ are called binomial coefficient or combinatorial coefficients and may be simply written as $C_0, C_1, C_2, \ldots\ldots\ldots\ldots, C_n$.

$(x + y)^n = {}^nC_0\, x^n\, y^0 + {}^nC_1\, x^{n-1}\, y + {}^nC_2\, x^{n-2}\, y^2 + \ldots\ldots.. + {}^nC_0\, x^{n-n}\, y^n$.

Find the sum of all the combinatorial coefficient.

i.e., ${}^nC_0 + {}^nC_1 + {}^nC_2 + \ldots\ldots\ldots + {}^nC_n = 2^n$.

Put $x = 1$ and $y = 1$ to get sum of all the binomial coefficient.

$(x + 2y)^n$ find the sum of all the coefficients.

$(x + 2y)^2 = x^2 + 4xy + 4y^2$

$\therefore$ Sum of all coefficient $= 1 + 4 + 4 = 9$.

We can also get it by putting $x = y = 1$

$(1 + 2)^2 = 9$

$\therefore$ Sum of all binomial coefficients in

$(x + y)^n = 2^n \rightarrow$ In this case sum of coefficient = sum of binomial coefficient.

(5) Binomial coefficients of the term equidistant from beginning and end are equal.

$$(2x + 3y)^2 = {}^2C_0\,(2x)^2 + {}^2C_1\,(2x)^1\,(3y)^1 + {}^2C_2\,(2x)^0\,(3y)^2.$$

Now coefficient of 1st and last term is same ${}^2C_0 = {}^2C_2$.

(6) ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$

(7) $\left(\dfrac{n+1}{r+1}\right) {}^nC_r = {}^{n+1}C_{r+1}$

(8) Consecutive binomial coefficient.

$$\frac{{}^nC_r}{{}^nC_{r-1}} = \frac{n-r+1}{r} \text{ i.e., } \frac{{}^5C_4}{{}^5C_3} = \frac{5-4+1}{4} = \frac{1}{2}.$$

Illustration :

Illustration :

If three consecutive coefficients in the expansion of $(1 + x)^n$ be 165,330 and 462. Find number of terms in the expansion of $(1 + x)^n$.

Sol. $\dfrac{{}^nC_r}{{}^nC_{r-1}} = \dfrac{n-r+1}{r}.$

Let three consecutive coefficients of terms are ${}^nC_{r-1}$, ${}^nC_r$, ${}^nC_{r+1}$

$$\frac{{}^nC_r}{{}^nC_{r-1}} = \frac{n-r+1}{r} = \frac{330}{165} = 2. \qquad ....(1)$$

$$\frac{{}^nC_{r+1}}{{}^nC_r} = \frac{n-(r+1)+1}{r+1} = \frac{n-r}{r+1}$$

$$\frac{n-r}{r+1} = \frac{462}{330} = \frac{n-r}{r+1} = \frac{7}{5}$$

$n = 11$ and $r = 4.$

Practice Problem

Q.1 If a, b, c, d are the coefficient of any four consecutive terms in the expansion of $(1+x)^n$, $n \in N$ prove that $\frac{a}{a+b}+\frac{c}{c+d}=\frac{2b}{b+c}$.

Q.2 $\left(x+\sqrt{2}\right)^4+\left(x-\sqrt{2}\right)^4$.

Q.3 If $(1 + x + x^2)^n = a_0 + a_1x + a_2x^2 + \ldots\ldots\ldots + a_{2n}x^{2n}$.
Find (a) $a_0 + a_1 + a_2 + \ldots\ldots\ldots + a_{2n}$
(b) $a_0 - a_1 + a_2 - a_3 + \ldots\ldots\ldots + a_{2n}$.

Answer key

Q.2 $2x^4 + 24x^2 + 8$ Q.3 (a) 3^n (b) 1

2.1 IMPORTANT TERMS IN BINOMIAL :

(A) General term
(B) Term independent of x

2.1 IMPORTANT TERMS IN BINOMIAL :

(A) General term
(B) Term independent of x.
(C) Middle term

(A) GENERAL TERM :

$(T_{r+1})^{th}$ term is called as general term in $(x + y)^n$ and general term is given by

$$\boxed{T_{r+1} = {}^nC_r x^{n-r}.y^r}$$

Illustration :

Find fourth term in the expansion of $\left(2x-\frac{y}{2}\right)^7$.

Sol. $T_{r+1} = {}^7C_r\,(2x)^{7-r}\cdot\left(-\frac{y}{2}\right)^r$

Put $r = 3$ *to get* 4^{th} *term*

$T_4 = -\,{}^7C_3\,(2x)^4\cdot\left(\frac{y}{2}\right)^3 = -\,70x^4y^3$.

Illustration :

Find term involving x^3 & x^4 *in* $\left(2x^2 - \dfrac{1}{3x}\right)^6$

Sol. *Let* T_{r+1} *involving* x^3 *and* x^4 *then*

$$T_{r+1} = {}^6C_r\,(2x^2)^{6-r}\left(\frac{-1}{3x}\right)^r.$$

$$= {}^6C_r\,(2)^{6-r}\cdot\left(\frac{-1}{3}\right)^r x^{12-2r}\cdot\left(\frac{1}{x}\right)^r.$$

$$= {}^6C_r\,2^{6-r}\left(\frac{-1}{3}\right)^r x^{12-3r}$$

$\Rightarrow$ *Coeffiicent of* $x^3 \Rightarrow 12 - 3r = 3.$

$r = 3$ $\quad$ 4^{th} *term involve* x^3 *in expansion.*

$\Rightarrow$ *Coeff. of* $x^4 \Rightarrow 4 = 12 - 3r$

$r \neq \dfrac{8}{3}$. *So there is no term exist.*

Important : If r is non integral value then there is no term exist.

Illustration :

Important : If r is non integral value then there is no term exist.

Illustration :

Prove that coefficient of x^{50} *in* $(1 + x^2)^{25}(1 + x^{25})(1 + x^{40})(1 + x^{47})$ *is equal to* $1 + {}^{25}C_5$.

Sol. *As we are intersected in coefficient of* x^{50}*, we shall ignore all terms with exponent more than 50.*

So $(1 + x^2)^{25}(1 + x^{25})(1 + x^{40})(1 + x^{47})$

$= (1 + {}^{25}C_1 x^2 + \ldots\ldots + {}^{25}C_{25} x^{50}) \times (1 + x^{25} + x^{40} + x^{45} + x^{47})$

$= {}^{25}C_{25} + {}^{25}C_5 = 1 + {}^{25}C_5.$

(B) TERM INDEPENDENT OF x :

It means term containing x^0.

Illustration :

Find term independent of x in $\left(x^2 + \dfrac{1}{x^2} - 2\right)^{10}$.

Sol. $\left(x^2 + \dfrac{1}{x^2} - 2\right)^{10} = \left(x - \dfrac{1}{x}\right)^{20}$

$\Rightarrow$ $T_{r+1} = {}^{20}C_r\,x^{20-r}(-1)^r\dfrac{1}{x^r} = {}^{20}C_r\,x^{20-2r}(-1)^r$

$\Rightarrow$ $20 - 2r = 0\,;\ r = 10$

$\Rightarrow$ 11^{th} *term is independent of x.*

(C) MIDDLE TERM :

Let T_m is middle term is expansion $(x + y)^n$ then

Case I : If n is odd, then number of terms will be even so there is two middle terms

$$\left(\frac{n+1}{2}\right)^{th} \text{ and } \left(\frac{n+3}{2}\right)^{th}.$$

Case II : If n is even, then number of terms will be odd so only one term is middle term $\left(\frac{n}{2}+1\right)^{th}$.

Note : Binomial coefficient of middle term is greatest.

Illustration :

Find coefficient of t^8 in the expansion of $(1 + 2t^2 - t^3)^9$.

Sol. $T_{r+1} = {}^9C_r\, t^{2r}\, (2 - t)^r$

Now coefficient of t^8 appears in $r = 3$, $r = 4$.

$r = 3 \Rightarrow {}^9C_3\, t^6\, (2 - t)^3 = {}^9C_3 \times t^6 \times {}^3C_p\, 2^{3-p}\, (-t)^p$

Put $p = 2 \Rightarrow {}^9C_3 \times {}^3C_2 \times 2^1 \times t^8 = 84 \times 6 \times t^8$

Similarly when $r = 4$

$^9C_4\, t^8\, (2 - t)^4 = {}^9C_4 \times t^8 \times {}^4C_p \times {}^4C_p\, 2^{4-p}\, (-t)^p$

$r = 3 \Rightarrow {}^9C_3\, t^6\, (2 - t)^3 = {}^9C_3 \times t^6 \times {}^3C_p\, 2^{3-p}\, (-t)^p$

Put $p = 2 \Rightarrow {}^9C_3 \times {}^3C_2 \times 2^1 \times t^8 = 84 \times 6 \times t^8$

Similarly when $r = 4$

$^9C_4\, t^8\, (2 - t)^4 = {}^9C_4 \times t^8 \times {}^4C_p \times {}^4C_p\, 2^{4-p}\, (-t)^p$

Put $p = 0 \Rightarrow {}^9C_4 \times {}^4C_0 \times 2^4 \times t^8 = 126 \times 16 \times t^8$

Hence coefficient of t^8 in the expansion of $(1 + 2t^2 - t^3)^9$ is $84 \times 6 + 126 \times 16$

$= 504 + 2016 = 2520$. ***Ans.***

Alternative:

We have

$$\left(\left(1+2t^2\right)-t^3\right)^9 = {}^9C_0\left(1+2t^2\right)^9 - {}^9C_1\left(1+2t^2\right)^8\cdot t^3 + {}^9C_2\left(1+2t^2\right)^7\cdot t^6$$

$$- {}^9C_3\left(1+2t^2\right)^6\cdot t^9 + \ldots\ldots - {}^9C_9\left(t^3\right)^9$$

$\therefore$ *Coefficient of t^8 in the expansion of $(1 + 2t^2 - t^3)^9$*

$= {}^9C_0$ *(coefficient of t^8 in $(1 + 2t^2)^9$)* $- {}^9C_1\left(\text{coefficient of } t^3 \text{ in } \left(1+2t^2\right)^7\right)$

$$+ {}^9C_2\left(\text{coefficient of } t^2 \text{ in } \left(1+2t^2\right)^7\right)$$

$$= {}^9C_0\cdot{}^9C_4\, 2^4 - 0 + {}^9C_2\cdot{}^7C_1\cdot 2 = \frac{9\cdot 8\cdot 7\cdot 6}{4\cdot 3\cdot 2\cdot 1}\cdot 16 + \frac{9\cdot 8}{2\cdot 1}\cdot 7\cdot 2$$

$= 9\cdot 8\cdot 7\cdot 5 = 2520.$ ***Ans.***

Illustration :

Prove that ${}^{4}C_{4} \cdot {}^{100}C_{6} + {}^{4}C_{3} \cdot {}^{100}C_{7} + {}^{4}C_{2} \cdot {}^{100}C_{8} + {}^{4}C_{1} \cdot {}^{100}C_{9} + {}^{4}C_{0} \cdot {}^{100}C_{10} = {}^{104}C_{10}.$

Sol. $\underbrace{{}^{100}C_{6} + {}^{100}C_{7}} + 3\left({}^{100}C_{7} + {}^{100}C_{8}\right) + 3\left({}^{100}C_{8} + {}^{100}C_{9}\right) + {}^{100}C_{9} + {}^{100}C_{10}$

$= {}^{101}C_{7} + 3\ {}^{101}C_{8} + 3\ {}^{101}C_{9} + {}^{101}C_{10}$

$= \underbrace{{}^{101}C_{7} + {}^{101}C_{8}} + 2\left({}^{101}C_{8} + {}^{101}C_{9}\right) + {}^{101}C_{9} + {}^{100}C_{10}$

$= {}^{102}C_{8} + 2 \cdot {}^{102}C_{9} + {}^{102}C_{10} = {}^{102}C_{8} + {}^{102}C_{9} + {}^{102}C_{9} + {}^{102}C_{10}$
$= {}^{103}C_{9} + {}^{103}C_{10} = {}^{104}C_{10} \equiv {}^{104}C_{94}.$

Alternative : Proof by PNC

${}^{4}C_{4} \cdot {}^{100}C_{6} + {}^{4}C_{3} \cdot {}^{100}C_{7} + {}^{4}C_{2} \cdot {}^{100}C_{8} + {}^{4}C_{1} \cdot {}^{100}C_{9} + {}^{4}C_{0} \cdot {}^{100}C_{10}$
Out of 104 students of which 100 are boys and 4 are girls, we have to select 10 students.
This can be done in ${}^{104}C_{10} = {}^{104}C_{94}$

Out of 104 students of which 100 are boys and 4 are girls, we have to select 10 students.
This can be done in ${}^{104}C_{10} = {}^{104}C_{94}$
Hence, $x + y = 114$ *or* 198 ***Ans.***

Illustration :

Prove that middle term in the expansion of $(1 + x)^{2n}$ *is* $\dfrac{1 \cdot 3 \cdot 5 \,.......\, (2n-1) \cdot 2^{n}}{n\,!} \cdot x^{n}$.

Sol. $T_{\frac{2n}{2}+1} = T_{n+1} = {}^{2n}C_{n}x^{n} = \dfrac{(2n)!}{n!\,(n!)}x^{n} = \dfrac{2^{n}(n!)\left(1 \cdot 3 \,........\, \cdot 2n-1\right)}{n!\ n!}x^{n}.$

Illustration :

If sum of coefficient in the expansion of $(2 + 3cx + c^{2}x^{2})^{12}$ *is zero. Find c.*

Sol. *Put* $x = 1$ *in the expansion, then*
$2 + 3c + c^{2} = 0$
$c = -2,\ c = 1.$ ***Ans.***

Illustration :

Find coeffiicnet of x^{15} *in* $(x - x^2)^{10}$.

Sol. $T_{r+1} = {}^{10}C_r\,(x)^{10-r}\,(-1)^r\,(x^2)^r = {}^{10}C_r\,(-1)^r\,(x)^{10+r}$

$10 + r = 15 : r = 5$ *i.e.,* 6^{th} *term*

$T_6 = {}^{10}C_5\,(-1)^5 \cdot x^{15} = -\,{}^{10}C_5\,x^{15}$. ***Ans.***

Illustration :

Find coefficient term independent of x in the expansion of $\left(x^2 + \frac{1}{x}\right)^{12}$.

Sol. $T_{r+1} = {}^{12}C_r\,(x^2)^{12-r}\left(\frac{1}{x}\right)^r = {}^{12}C_r\,(x)^{24-3r}$

$r = 5$ *and* 9^{th} *term* $\Rightarrow T_9$. ***Ans.***

Illustration :

Illustration :

Find last four terms in the expansion of $(x + 2x^2)^8$.

Sol. *Above is equivalent to first four terms of* $(2x^2 + x)^8$

$T_9 = {}^8C_0\,(2x^2)^8 = 256\,x^{16}$

$T_8 = {}^8C_1\,(2x^2)^7\,x = 1024\,x^{15}$

$T_7 = {}^8C_2\,(2x^2)^6\,x^2 = 1792\,x^{14}$

$T_6 = {}^8C_3\,(2x^2)^5\,x^3 = 1792\,x^{13}$. ***Ans.***

Illustration :

In $\left(2x^2 - \frac{1}{3x}\right)^{11}$ *find term involving* x^6 *also find term independent of* x.

Sol. $T_{r+1} = {}^{11}C_r\,(2^{11-r})\,(x^2)^{11-r}\left(\frac{-1}{3}\right)^r \cdot \left(\frac{1}{x}\right)^r$

$= {}^{11}C_r \cdot (2)^{11-r}\,(-3)^{-r}\,(x)^{22-3r}$

$22 - 3r = 6 \Rightarrow t \neq \frac{16}{3}$ *and*

For x^6 $\quad 22 - 3r = 6 \Rightarrow r = \frac{16}{3}$ $\quad$ (which is not possible)

For independent of x $22 - 3r = 0 \Rightarrow r = \frac{22}{3}$ $\quad$ (which is not possible)

Both do not exist. **Ans.**

Illustration :

In the expansion of $\left(4^{\frac{1}{3}} + \frac{1}{6^{\frac{1}{4}}}\right)^{20}$ find number of rational terms

Sol. $T_{r+1} = {}^{20}C_r \cdot (4)^{\frac{20-r}{3}} \cdot (6)^{\frac{-r}{4}} = {}^{20}C_r \cdot 2^{\frac{40-2r}{3}} \cdot \frac{1}{6^{(r/4)}} = {}^{20}C_r\, (2)^{\left\{\frac{160-11r}{12}\right\}} (3)^{\left\{-\frac{r}{4}\right\}}$ $\quad [0 \le r \le 20]$

$\therefore r = 8, 20.$

$\therefore$ There are only two rational terms (T_9 and T_{21}) **Ans.**

$\therefore r = 8, 20.$

$\therefore$ There are only two rational terms (T_9 and T_{21}) **Ans.**

Illustration :

Find greatest value of term independent of x in $\left(x\sin\alpha + \frac{\cos\alpha}{x}\right)^{10}$ $\alpha \in R$.

Sol. $T_{r+1} = {}^{10}C_r\, (x)^{10-r} \cdot (\sin\alpha)^{10-r} (\cos\alpha)^r \left(\frac{1}{x}\right)^r$

$\Rightarrow \quad T_{r+1} = {}^{10}C_r\, (x)^{10-2r} (\sin\alpha)^{10-r} (\cos\alpha)^r$

$\Rightarrow \quad 10 - 2r = 0\ ;\ r = 5.$

$\Rightarrow \quad {}^{10}C_5\, (\sin\alpha)^5 \cdot (\cos\alpha)^5$

$\Rightarrow \quad T_{r+1} = {}^{10}C_5\, \frac{(\sin 2\alpha)^5}{32}$

$\therefore \quad$ Greatest value of $T_{r+1} = \frac{{}^{10}C_5}{32}.$ $\quad [\sin 2\alpha = 1]$ $\quad$ **Ans.**

Illustration :

The term independent of x in the expansion of $\left(\frac{x+1}{x^{2/3}-x^{1/3}+1}-\frac{x-1}{x-x^{1/2}}\right)^{10}$ *is*

(A) $T_5 = 210$ (B) $T_5 = -210$ (C) $T_4 = 180$ (D) $T_4 = -180$

Sol. *We have*

$$\left(\frac{x+1}{x^{2/3}-x^{1/3}+1}-\frac{x-1}{x-x^{1/2}}\right)^{10} = \left(\frac{(x^{1/3})^3+1^3}{x^{2/3}-x^{1/3}+1}-\frac{(x^{1/2})^2-1^2}{x^{1/2}(x^{1/2}-1)}\right)^{10}$$

$$= \left((x^{1/3}+1)-\frac{x^{1/2}-1}{x^{1/2}}\right)^{10} = (x^{1/3}+x^{-1/2})^{10}$$

We have,

$$T_{r+1} = {}^{10}C_r\,(x^{1/3})^{10-r}\,(-x^{-1/2})^r \qquad \text{........(1)}$$

$$= {}^{10}C_r\,x^{\frac{10-r}{3}-\frac{r}{2}}(-1)^r$$

It will be independent of x, if

$$= {}^{10}C_r\,x^{\frac{10-r}{3}-\frac{r}{2}}(-1)^r$$

It will be independent of x, if

$$\frac{10-r}{3}-\frac{r}{2} = 0.$$

$\Rightarrow \quad 20 - 2r - 3r = 0 \quad \Rightarrow \quad r = 4$

Putting r = 4 in (1), we get

$$T_5 = {}^{10}C_4\,(-1)^4 \qquad = {}^{10}C_4 = 210.$$

Hence (a) is correct answer. ***Ans.***

Illustration :

The ninth term in the expansion of $\left\{3^{\log_3\sqrt{25^{x-1}+7}}+3^{-1/8\log_3(5^{x-1}+1)}\right\}^{10}$ *is equal to 180, then x is*

(A) 2 (B) 1 (C) 4 (D) None of these

Sol. *We have,*

$$\left\{3^{\log_3\sqrt{25^{x-1}+7}}+3^{-1/8\log_3(5^{x-1}+1)}\right\}^{10} = \left[\sqrt{25^{x-1}+7}+(5^{x-1}+1)^{-1/8}\right]^{10} \quad \{\because a^{\log_a N} = N\}$$

Here, $T_9 = 180 \Rightarrow {}^{10}C_8\left\{\sqrt{25^{x-1}+7}\right\}^{10-8}\left\{(5^{x-1}+1)^{-1/8}\right\} = 180$

$\Rightarrow \quad {}^{10}C_8\,(25^{x-1} + 7)\,(5^{x-1} + 1)^{-1} = 180 \qquad \Rightarrow \quad 45\dfrac{\left(25^{x-1}+7\right)}{5^{x-1}+1} = 180$

$\Rightarrow \quad \dfrac{25^{x-1}+7}{5^{x-1}+1} = 4 \quad \Rightarrow \quad \dfrac{y^2+7}{y+1} = 4$, where $y = 5^{x-1}$

$\Rightarrow \quad y^2 - 4y + 3 = 0 \quad \Rightarrow \quad y = 3, 1 \qquad \Rightarrow 5^{x-1} = 3$ or $5^{x-1} = 1 \quad \Rightarrow 5^x = 15$ or $5^x = 5$.

Hence (b) is correct answer. ***Ans.***

Illustration :

The coefficient of x^{50} *in th expansion*

$(1 + x)^{1000} + 2x\,(1 + x)^{999} + 3x^2\,(1 + x)^{998} + \ldots. + 1001x^{1000}$ *is*

(A) ${}^{1002}C_{50}$ (B) ${}^{1002}C_{51}$ (C) ${}^{1005}C_{50}$ (D) ${}^{1005}C_{48}$

Sol. Let $S = (1 + x)^{1000} + 2x\,(1 + x)^{999} + 3x^2\,(1 + x)^{998} + \ldots. + 1001x^{1000}$ ... (1)

xS x^{1001}

Sol. Let $S = (1 + x)^{1000} + 2x\,(1 + x)^{999} + 3x^2\,(1 + x)^{998} + \ldots. + 1001x^{1000}$ ... (1)

$\Rightarrow \quad \dfrac{xS}{1+x} = x\,(1 + x)^{999} + 2x^2\,(1 + x)^{998} + \ldots\ldots. + 1000x^{1000} + 1001\,\dfrac{x^{1001}}{1+x}.$ ... (2)

After subtraction we get

$\Rightarrow \quad \dfrac{S}{1+x} = (1 + x)^{100} + x\,(1 + x)^{999} + x^2\,(1 + x)^{998} + \ldots\ldots. + x^{1000} - 1001\,\dfrac{x^{1001}}{1+x}.$

$$\Rightarrow \quad \frac{S}{1+x} = (1 + x)^{1000}\left(\frac{1-\left(\dfrac{x}{1+x}\right)^{1001}}{1-\dfrac{x}{1+x}}\right) - 1001\,\frac{x^{1001}}{1+x}$$

$\Rightarrow \quad S = (1 + x)^{1002} - x^{1001}\,(1 + x) - 1001\,x^{1001}$

$S = (1 + x)^{1002} - 1002x^{1001} - x^{1002}$

Coefficient of x^{50} *in* S = *coefficient of* x^{50} *in* $(1 + x)^{1002} = {}^{1002}C_{50}$. ***Ans.***

Illustration :

Find coefficient of x^3 *in the expansion of* $(1 - x + x^2)^5$.

Sol. $(1 - x + x^2)^5 = \{1 + x(x - 1)\}^5 = {}^5C_0 + {}^5C_1 x(x-1) + {}^5C_2 x^2 (x-1)^2 + {}^5C_3 x^3 (x-1)^3 + \ldots\ldots\ldots\ldots$

Coefficient of $x^3 = -2\,{}^5C_2 - {}^5C_3 = -20$. **Ans.**

Illustration :

Find coefficient of x^4 *in the expansion of* $(1 + x + x^2 + x^3)^{11}$.

Sol. $(1 + x + x^2 + x^3)^{11} = (1 + x)^{11}\,(1 + x^2)^{11}$

$= (1 + {}^{11}C_1 x + {}^{11}C_2 x^2 + {}^{11}C_2 x^3 + {}^{11}C_3 x + {}^{11}C_4 x^4 + \ldots\ldots..) \times (1 + {}^{11}C_1 x^2 + {}^{11}C_2 x^4 + \ldots\ldots\ldots.)$

(The terms which gives x^4 *are)*

$= {}^{11}C_2 + ({}^{11}C_2 \times {}^{11}C_1) + {}^{11}C_4 = 55 + 605 + 330 = 990$. **Ans.**

Illustration :

If $(1 + x + x^2)^n = a_0 + a_1x + a_2x^2 + \ldots\ldots.. + a_{2n}x^{2n}$ $(n \in N)$, *then*

(a) $a_0 + a_1 + a_2 + a_3 + \ldots\ldots.. + a_{2n}$

(b) $a_0 - a_1 + a_2 - a_3 + \ldots\ldots.. + a_{2n}$

(c) $a_1 + a_3 + a_5 + \ldots\ldots.. + a_{2n-1}$

(d) $a_0 + a_2 + a_4 + \ldots\ldots.. + a_{2n}$

(e) $a_0^2 - a_1^2 + a_2^2 - a_3^2 \ldots\ldots.. + a_{2n}^2$

(f) $a_0a_1 - a_1a_2 + a_2a_3 + \ldots\ldots..$

(g) $a_0a_2 - a_1a_3 + a_2a_4 \ldots\ldots..$

Sol.

(a) $a_0 + a_1 + a_2 + a_3 + \ldots\ldots.. + a_{2n} = 3^n$ *put* $x = 1$

(b) $a_0 - a_1 + a_2 - a_3 + \ldots\ldots.. + a_{2n} = 1$ *put* $x = -1$

(c) *From (a) and (b)* $a_1 + a_3 + a_5 + \ldots\ldots.. + a_{2n-1} = \dfrac{3^n - 1}{2}$

(d) *From (a) and (b)* $a_0 + a_2 + a_4 + \ldots\ldots.. + a_{2n} = \dfrac{3^n + 1}{2}$

(e) $(1 + x + x^2)^n = a_0 + a_1x + a_2x^2 + \ldots\ldots.. + a_{2n}x^{2n}$ ...(i)

$$\left(1 - \frac{1}{x} + \frac{1}{x^2}\right)^n = a_0 - \frac{a_1}{x} + \frac{a_2}{x^2} + \ldots\ldots + \frac{a_{2n}}{x^{2n}}$$

$$\Rightarrow \frac{1}{x^{2n}}(x^2 - x + 1)^n = a_0 - \frac{a_1}{x} + \frac{a_2}{x^2} + \ldots\ldots + \frac{a_{2n}}{x^{2n}} \qquad \ldots(ii)$$

From (i) & (ii)

$$(1 + x + x^2)^n \left(1-\frac{1}{x}+\frac{1}{x^2}\right)^n = (a_0 + a_1x + a_2x^2 + \ldots\ldots + a_{2n}x^{2n})\left(a_0 - \frac{a_1}{x}+\frac{a_2}{x^2}+\ldots\ldots+\frac{a_{2n}}{x^{2n}}\right)$$

$$\Rightarrow \frac{1}{x^{2n}}(x^4 + x^2 + 1)^n = (a_0 + a_1x + a_2x^2 + \ldots\ldots + a_{2n}x^{2n})\left(a_0 - \frac{a_1}{x}+\frac{a_2}{x^2}+\ldots\ldots+\frac{a_{2n}}{x^{2n}}\right) \ldots(iii)$$

Compairing constant terms on both sides.

$$a_0^2 - a_1^2 + a_2^2 - a_3^2 \ldots\ldots + a_{2n}^2 = a_n$$

(f) *Compairing coefficient of x in equation (iii) on both sides.*

$$a_0a_1 - a_1a_2 + a_2a_3 + \ldots\ldots = 0$$

(g) *Compairing coefficient of x^2 in equation (iii) on both sides.*

$$a_0a_2 - a_1a_3 + a_2a_4 \ldots\ldots = a_{n+1} \text{ or } a_{n-1}.$$

Practice Problem

Practice Problem

Q.1 Find coefficient of x^{20} in the expansion of $(1+x^2)^{40}\left(x^2+2+\frac{1}{x^2}\right)^{-5}$.

Q.2 Find term independent of x in $\left[\sqrt{\frac{x}{3}}+\sqrt{\frac{3}{2x^2}}\right]^{10}$.

Q.3 Find the ratio of coefficients of x^{10} in $(1-x^2)^{10}$ and term independent of x in $\left(x-\frac{2}{x}\right)^{10}$

Answer key

Q.1 ${}^{30}C_{25}$ Q.2 No term Q.3 $\frac{1}{32}$

3.1 NUMERICALLY GREATEST TERM IN $(x + y)^n$:

T_{r+1} term is said to be numerically greatest for a given value of x, y provided $T_{r+1} \geq T_r$ and

$$T_{r+1} \geq T_{r+2} \quad \Rightarrow \quad \frac{T_{r+1}}{T_r} \geq 1 \text{ as well as } \frac{T_{r+1}}{T_{r+2}} \geq 1$$

$$\frac{T_{r+1}}{T_r} = \frac{{}^nC_r(x)^{n-r}y^r}{{}^nC_{r-1}(x)^{n-r+1}y^{r-1}} = \frac{(n-r+1)}{r} \quad \underbrace{\left|\left(\frac{y}{x}\right)\right|}_{\substack{\text{Numerically} \\ \text{greatest term} \\ \text{is required}}}$$

Illustration :

Find the greatest term in the expansion of $(7 - 5x)^{11}$ where $x = 2/3$.

Sol. *Since we have to find numerically greatest term,*

$\therefore$ *greatest term in $(7 - 5x)^{11}$ = greatest term in $(7 + 5x)^{11}$*

Let r th term be the greatest term in the expansion of $(7 + 5x)^{11}$

Now, $t_r = {}^{11}C_{r-1}\,(7)^{11-r+1}\,(5x)^{r-1} = {}^{11}C_{r-1}\,7^{12-r}\,(5x)^{r-1}$ ...(i)

and $(r + 1)$*th term,* $t_{r+1} = {}^{11}C_r\,(7)^{11-r}\,(5x)^r$ ...(ii)

$$\therefore \quad \frac{t_r}{t_{r+1}} = \frac{{}^{11}C_{r-1}}{{}^{11}C_r} \cdot \frac{7^{12-r}(5x)^{r-1}}{7^{11-r}(5x)^r}$$

$$\text{or} \quad \frac{t_r}{t_{r+1}} = \frac{21r}{(12-r)10} \geq 1 \qquad ...(iii)$$

$$\text{or} \quad 21r \geq 120 - 10r \quad \text{or} \quad r \geq 3\frac{27}{31}$$

Putting $(r - 1)$ in place of r in (iii), $\dfrac{t_{r-1}}{t_r} = \dfrac{21(r-1)}{\{12-(r-1)\} \cdot 10} = \dfrac{21r-21}{130-10r} \leq 1$

$$\text{or} \quad r \leq 4\frac{27}{31} \qquad ...(iv)$$

From (iii) and (iv) $r = 4$

$$\therefore \quad \text{Greatest term} = t_4 = {}^{11}C_3 \cdot 7^8 \left(5 \cdot \frac{2}{3}\right)^3 = \frac{440}{9} \cdot 7^8 \cdot 5^3$$

Illustration :

Given T_4 *in the expansion of* $\left(2+\frac{3x}{8}\right)^{10}$ *has maximum numerical value find range of x.*

Sol. $T_4 > T_3$ and $T_4 > T_5$

$$\Rightarrow \quad \frac{10-3+1}{3}\left|\frac{3x}{8\times 2}\right| \quad \text{and} \quad \frac{10-4+1}{4}\left|\frac{3x}{16}\right| < 1$$

$$\Rightarrow \quad |x| > 2 \quad \text{and} \quad |x| < \frac{64}{21}.$$

$$\Rightarrow \quad x > 2\ ;\ x < -2 \quad \text{and} \quad \frac{-64}{x} < x < \frac{64}{21}$$

$$\underbrace{\text{Inter sec tion}}_{x\in\left(\frac{-64}{21},-2\right)\cup\left(2,\frac{64}{21}\right)}.$$

$$\underbrace{\qquad\qquad}_{x\in\left(\frac{-64}{21},-2\right)\cup\left(2,\frac{64}{21}\right)}.$$

Illustration :

Find the index n of the binomial $\left(\frac{x}{5}+\frac{2}{5}\right)^n$ *if the* 9^{th} *term of the expansion has numerically the greatest coefficient (n* $\in$ *N) (at x = 1)*

Sol. *If* T_9 *is greatest term then*

$$\frac{T_9}{T_8} > 1 \quad \text{and} \quad \frac{T_{10}}{T_9} < 1$$

$$\left(\frac{n-8+1}{8}\right)\frac{2}{5}\cdot 5 > 1 \quad \text{and} \quad \frac{2}{5}\cdot 5\left(\frac{n-9+1}{9}\right) < 1$$

$2n - 14 > 8$ and $2n - 16 < 9$

$n > 11$ and $11 < n < 12.5$

$\therefore$ $n = 12$. **Ans.**

3.2 BINOMIAL COEFFICIENTS & THEIR PROPERTIES :

Propertices of ^nC_r

(1) $^nC_r = {}^nC_{n-r} \Rightarrow {}^nC_x = {}^nC_y$ has two solution $x = y$ or $x + y = n$.

(2) $^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$

(3) $^nC_r = \frac{n}{r}\ ({}^{n-1}C_{r-1})$

(4) $\frac{{}^nC_r}{{}^nC_{r-1}} = \frac{n-r+1}{r}$.

Note :

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

$$\int x^n\, dx = \frac{x^{n+1}}{n+1} + c$$

$$\int_a^b x^n\, dx = \left[\frac{x^{n+1}}{n+1}\right]_a^b = \frac{b^{n+1} - a^{n+1}}{n+1}$$

In the expansion of $(1+x)^n$; i.e.$(1+x)^n = {}^nC_0 + {}^nC_1x + \ldots\ldots.. + {}^nC_r x^r + \ldots. + {}^nC_nx^n$

The coefficients $^nC_0$, $^nC_1$, ^nC_n of various powers of x, are called binomial coefficients and they are written as

$$C_0, C_1, C_2, \ldots\ldots, C_n$$

Hence

$$(1+x)^n = C_0 + C_1x + C_2x^2 + \ldots.. + C_rx^r + \ldots\ldots + C_nx^n \qquad \ldots(1)$$

Where $C_0 = 1$, $C_1 = n$, $C_2 = \frac{n(n-1)}{2!}$

$$C_r = \frac{n(n-1)\ldots\ldots(n-r+1)}{r!},\ C_n = 1$$

Now, we shall obtain some important expressions involving binomial coefficients-

(a) Sum of Coefficient : putting x = 1 in (1), we get

$$C_0 + C_1 + C_2 + \ldots\ldots + C_n = 2^n \qquad \ldots(2)$$

(b) Sum of coefficients with alternate signs : putting x = –1 in(1)

We get

$$C_0 - C_1 + C_2 - C_3 + \ldots\ldots = 0 \qquad \ldots(3)$$

(c) Sum of coefficients of even and odd terms: from (3), we have

$$C_0 + C_2 + C_4 + \ldots\ldots. = C_1 + C_3 + C_5 + \ldots\ldots.. \qquad \ldots(4)$$

i.e. sum of coefficients of even and odd terms are equal.

from (2) and (4)

$$\Rightarrow \quad C_0 + C_2 + \ldots\ldots.. = C_1 + C_3 + \ldots\ldots. = 2^{n-1}$$

(d) Sum of products of coefficients : Replacing x by 1/x in (1)

We get

$$\left(1+\frac{1}{x}\right)^n = C_0 + \frac{C_1}{x} + \frac{C_2}{x^2} + \ldots. + \frac{C_n}{x^n} + \ldots. \qquad \ldots(5)$$

Multiplying (1) by (5), we get

$$\frac{(1+x)^{2n}}{x^n} = (C_0 + C_1x + C_2x^2 + \ldots.)\ \left(C_0 + \frac{C_1}{x} + \frac{C_2}{x^2} + \ldots.\right)$$

Now, comparing coefficients of x^r on both the sides, we get

$$C_0C_r + C_1C_{r+1} + \ldots\ldots. + C_{n-r}C_n = {}^{2n}C_{n-r}$$

$$= \frac{2n!}{(n+r)!(n-r)!} \qquad \ldots(6)$$

(e) Sum of squares of coefficients :

putting r = 0 in (6), we get

$$C_0^2 + C_1^2 + C_2^2 + \ldots\ldots + C_n^2 = \frac{2n!}{n!n!}$$

(f) putting r = 1 in (6), we get

$$C_0C_1 + C_1C_2 + C_2C_3 + \ldots\ldots. + C_{n-1}C_n = {}^{2n}C_{n-1}$$

$$= \frac{2n!}{(n+1)!(n-1)!} \qquad \ldots(7)$$

(g) putting r = 2 in (6) , we get

$$C_0C_2 + C_1C_3 + C_2C_4 + \ldots. + C_{n-2}C_n = {}^{2n}C_{n-2}$$

$$= \frac{2n!}{(n+2)!(n-2)!} \qquad \ldots(8)$$

Illustration :

If $(1+x)^n = C_0 + C_1x + C_2x^2 + \ldots\ldots + C_nx^n$, *prove that* $C_0^2 - C_1^2 + C_2^2 \ldots\ldots + (-1)^n \cdot (-1)^{n/2} C_n^2 = 0$

or $(-1)^{n/2} \dfrac{n!}{(n/2)!\,(n/2)!}$ *according as n is odd or even.*

Sol. *Since*

$$(1-x)^n = C_0 - C_1x + C_2x^2 - \ldots\ldots + (-1)^n C_nx^n \qquad \ldots(i)$$

and $$(x+1)^n = C_0x^n + C_1x^{n-1} + C_2x^{n-2} + \ldots\ldots + C_n \qquad \ldots(ii)$$

Multiplying (i) and (ii), we get

$$(1-x^2)^n = (C_0 - C_1x + C_2x^2 - \ldots.. + (-1)^n C_nx^n)(C_0x^n + C_1x^{n-1} + C_2x^{n-2} + \ldots\ldots + C_n) \ldots(iii)$$

Now, coefficient of x^n *in R.H.S.*

$$= C_0^2 - C_1^2 + C_2^2 - \ldots\ldots + (-1)^n C_n^2$$

General term in L.H.S. $= T_{r+1} = {}^nC_r(-x^2)^r$

$$= {}^nC_r(-1)^r x^{2r}$$

Putting $2r = n$

$\therefore$ $r = n/2$

$\therefore$ $T_{(n/2)+1} = {}^nC_{n/2}(-1)^{n/2} x^n$

$\therefore$ *Coefficient of* x^n *in L.H.S.* $= {}^nC_{n/2}(-1)^{n/2}$

$$= (-1)^{n/2}\frac{n!}{(n/2)!(n/2)!} = \begin{cases} 0 & \text{if } n \text{ is odd} \\ (-1)^{n/2}\dfrac{n!}{(n/2)!(n/2)!} & \text{if } n \text{ is even} \end{cases}$$

But (iii) is an identity, therefore coefficient of x^n *in R.H.S. = coefficient of* x^n *in L.H.S.*

$$\Rightarrow \quad C_0^2 - C_1^2 + C_2^2 - \ldots\ldots + (-1)C_n^2 = \begin{cases} 0 & \text{if } n \text{ is odd} \\ (-1)^{n/2}\dfrac{n!}{(n/2)!(n/2)!} & \text{if } n \text{ is even} \end{cases}$$

Illustration :

Prove that ${}^{m+n}C_r = {}^mC_r + {}^mC_{r-1}\,{}^nC_1 + {}^mC_{r-2}\,{}^nC_2 + \ldots\ldots + {}^nC_r$ *if* $r < m$, $r < n$ *and m, n, r are positive integers.*

Sol. *Here sum of lower suffices of binomial coefficient in each term is r*

i.e. $r = r - 1 + 1 = r - 2 + 2 = \ldots\ldots = r = r$

since $(1+x)^m = {}^mC_0 + {}^mC_1x + \ldots\ldots + {}^mC_{r-2}x^{r-2} + {}^mC_{r-1}x^{r-1} + {}^mC_rx^r + \ldots\ldots + {}^mC_mx^m \quad \ldots(i)$

and $$(1+x)^n = {}^nC_0 + {}^nC_1x + {}^nC_2x^2 + \ldots\ldots + {}^nC_rx^r + \ldots\ldots + {}^nC_nx^n \qquad \ldots(ii)$$

multiplying (i) and (ii), we get

$$(1+x)^{m+n} = ({}^mC_0 + {}^mC_1x + \dots\dots + {}^mC_{r-2}x^{r-2} + {}^mC_{r-1}x^{r-1} + {}^mC_rx^r + \dots\dots$$
$$+ \dots\dots + {}^mC_mx^m) \times ({}^nC_0 + {}^nC_1x + {}^nC_2x^2 + \dots\dots + {}^nC_rx^r + \dots\dots + {}^nC_nx^n) \qquad \dots(iii)$$

Now coefficient of x^r in R.H.S.

$$= {}^mC_r \cdot {}^nC_0 + {}^mC_{r-1} \cdot {}^nC_1 + {}^mC_{r-2} \cdot {}^nC_2 + \dots\dots + {}^mC_0 \cdot {}^nC_r$$
$$= {}^mC_r + {}^mC_{r-1} \cdot {}^nC_1 + {}^mC_{r-2} \cdot {}^nC_2 + \dots\dots + {}^nC_r$$

Coefficient of x^r in L.HS. $\qquad = {}^{m+n}C_r$

But (iii) is an identity, therefore coefficient of x^r in L.H.S. = coefficient of x^r R.H.S.

$${}^{m+n}C_r = {}^mC_r + {}^mC_{r-1} \cdot {}^nC_1 + {}^mC_{r-2} \cdot {}^nC_2 + \dots\dots + {}^nC_r$$

(h) Use of Differentiation :

$$(1+x)^n = C_0+C_1x + C_2x^2+\dots\dots+ C_rx^r + \dots\dots+ C_nx^n \qquad \dots(1)$$

Differentiating both sides of (1) w.r.t. x, we get

$$n(1+x)^{n-1} = C_1 + 2C_2x + 3C_3\,x^2 +\dots\dots+ nC_n\,x^{n-1}$$

Now putting x = 1 and x = – 1 respectively

$$C_1+ 2C_2 +3C_3+ \dots\dots+ nC_n = n.2^{n-1} \qquad \dots\dots(9)$$

Now putting x = 1 and x = – 1 respectively

$$C_1+ 2C_2 +3C_3+ \dots\dots+ nC_n = n.2^{n-1} \qquad \dots\dots(9)$$

and $\qquad C_1 - 2C_2+ 3C_3 - \dots\dots= 0 \qquad \dots(10)$

(i) Adding (2) and (9)

$$C_0+ 2C_1 + 3C_2 + \dots.+ (n+1)\,C_n = 2^{\,n-1}\,(n+2) \qquad \dots(11)$$

Illustration :

If $(1+x)^n = C_0 + C_1x + C_2x^2 + \dots\dots + C_nx^n$, then prove that

$$C_1 + 2C_2 + 3C_3 + \dots\dots + nC_n = n\cdot 2^{n-1}.$$

Sol. *Given series is*

$$(1+x)^n = C_0 + C_1x + C_2x^2 + C_3x^3 + \dots\dots + C_nx^n$$

then differentiating both sides w.r. to x, we get

$$\Rightarrow \qquad n\,(1+x)^{n-1} = 0 + C_1 + 2C_2x + 3C_3x^2 + \dots\dots + nC_nx^{n-1}$$

Putting x = 1, we get

$$n\,.\,2^{n-1} = C_1 + 2C_2 + 3C_3 + \dots\dots + nC_n$$

or $\qquad C_1 + 2C_2 + 3C_3 + \dots\dots + nC_n = n.2^{n-1}$

Illustration :

If $(1 + x)^n = C_0 + C_1x + C_2x^2 + \ldots\ldots + C_nx^n$, *then prove that*

$$C_0 + 2C_1 + 3C_2 + \ldots\ldots + (n + 1)C_n = (n + 2)2^{n-1}.$$

Sol. *Given series is*

$$(1 + x)^n = C_0 + C_1x + C_2x^2 + \ldots\ldots + C_nx^n$$

Now replacing by x^1 *and multiplying both sides by* x^1, *then*

$$x\,(1 + x)^n = C_0x + C_1x^2 + C_2x^3 + \ldots\ldots + C_nx^{n+1}$$

Now differentiating both sides w.r.t. x, we get

$$x\,.\,n\,(1 + x)^{n-1} + (1 + x)^n\,.\,1 = C_0 + 2C_1x + 3C_2x^2 + \ldots\ldots + (n + 1)C_nx^n$$

Putting $x = 1$, *we get*

$$n\,(2)^{n-1} + 2^n = C_0 + 2C_1 + 3C_2 + \ldots\ldots + (n + 1)C_n$$

or $\quad C_0 + 2C_1 + 3C_2 + \ldots\ldots + (n + 1)C_n = (n + 2)2^{n-1}$

Illustration :

If $(1 + x)^n = C_0 + C_1x + C_2x^2 + \ldots\ldots + C_nx^n$ *then prove that*

$$C_0 + 3C_1 + 5C_2 + \ldots\ldots + (2n + 1)\,C_n = (n + 1)2^n$$

Sol. *The given series is*

$$(1 + x)^n = C_0 + C_1x + C_2x^2 + \ldots\ldots + C_nx^n$$

Now replacing x by x^2

then $\quad (1 + x^2)^n = C_0 + C_1x^2 + C_2x^4 + \ldots\ldots + C_nx^{2n}$

multiplying both sides by x^1 , *we get*

$$x\,(1 + x^2)^n = C_0x + C_1x^3 + C_2x^5 + \ldots\ldots\ C_nx^{2n+1}$$

then differentiating both sides w.r. to x, we get

$$x\,.\,n\,(1 + x^2)^{n-1}\,.\,2x + (1 + x^2)^n\,.1 = C_0 + 3C_1x^2 + 5C_2x^4 + \ldots\ldots + (2n + 1)C_nx^{2n}$$

Putting $x = 1$, *then we get*

$$n.1^{n-1}\,.\,2 + 2^n = C_0 + 3C_1 + 5C_2 + \ldots\ldots\ (2n + 1)C_n$$

or $\quad C_0 + 3C_1 + 5C_2 + \ldots\ldots\ (2n + 1)C_n = (n + 1)\,2^n$

Illustration :

If $(1 + x)^n = C_0 + C_1x + C_2x^2 + \ldots\ldots + C_nx^n$ *then prove that*

$$(1.2)C_2 + (2.3)C_3 + \ldots\ldots\ ((n - 1)\,.\,n)\,C_n = n\,(n - 1)\,2^{n-2}$$

Sol. *The given series is*

$$(1 + x)^n = C_0 + C_1x + C_2x^2 + C_3x^3 + \ldots\ldots + C_nx^n$$

Differentiating both sides w.r. to x, we get

$$n\,(1 + x)^{n-1} = 0 + C_1 + 2C_2x + 3C_3x^2 + \ldots\ldots + nC_nx^{n-1}$$

again differentiating both sides w.r. to x, we get

$$n\,(n-1)\,(1 + x)^{n-2} = 0 + 0 + (1.2)C_2 + (2.3)C_3x + \ldots\ldots + ((n-1).n)C_nx^{n-2}$$

Putting $x = 1$, *then*

$$n\,(n-1)\,(1 + 1)^{n-2} = (1.2)C_2 + (2.3)C_3 + \ldots\ldots + (n-1)n\;C_n$$

or $\quad (1.2)C_2 + (2.3)C_3 + \ldots\ldots + (n-1)n\,.\,C_n = n\,(n-1)\,2^{n-2}\,.$

Illustration :

If $(1 + x)^n = C_0 + C_1x + C_2x^2 + C_3x^3 + \ldots\ldots + C_nx^n$ *then prove that*

$$C_0 + 2C_1 + 3C_2 - 4C_3 + \ldots\ldots + (-1)^n\,(n + 1)\,C_n = 0$$

Sol. *The given series is*

$$(1 + x)^n = C_0 + C_1x + C_2x^2 + C_3x^3 + \ldots\ldots + C_nx^n$$

Multiplying both sides by x, then

$$x\,(1 + x)^n = C_0x + C_1x^2 + C_2x^3 + C_3x^4 + \ldots\ldots + C_nx^{n+1}$$

$$x\,(1 + x)^n = C_0x + C_1x^2 + C_2x^3 + C_3x^4 + \ldots\ldots + C_nx^{n+1}$$

Differentiating both sides w.r. to x, then we get

$$x.n\,(1 + x)^{n-1} + (1 + x)^n\,.\,1 = C_0 + 2C_1x + 3C_2x^2 + 4C_3x^3 + \ldots\ldots + (n-1)C_nx^n$$

Putting $x = -1$, *then we get*

$$0 = C_0 - 2C_1 + 3C_2 - 4C_3 + \ldots\ldots + (-1)^n\,(n + 1)\,C_n$$

or $\quad C_0 - 2C_1 + 3C_2 - 4C_3 + \ldots\ldots\ldots\ldots + (-1)^n\,(n + 1)C_n = 0$

Illustration :

If $(1 + x)^n = C_0 + C_1x + C_2x^2 + C_3x^3 + \ldots\ldots + C_nx^n$ *then prove that*

$$C_1 - 2C_2 + 3C_3 - \ldots\ldots + (-1)^{n-1}\,n\,C_n = 0$$

Sol. $(1 + x)^n = C_0 + C_1x + C_2x^2 + C_3x^3 + \ldots\ldots + C_nx^n$

Differentiating both sides w.r. to x, we get

$$n\,(1 + x)^{n-1} = 0 + C_1 + 2C_2x + 3C_3x^2 + \ldots\ldots + nC_nx^{n-1}$$

Putting $x = -1$, *then we get*

$$0 = C_1 - 2C_2 + 3C_3 - \ldots\ldots + (-1)^{n-1}\,nC_n$$

or $\quad C_1 - 2C_2 + 3C_3 - \ldots\ldots + (-1)^{n-1}\,n\,C_n = 0$

Illustration :

If $(1 + x)^n = C_0 + C_1x + C_2x^2 + C_3x^3 + \ldots\ldots + C_nx^n$, *then prove that*

$$C_0 - 3C_1 + 5C_2 - \ldots\ldots + (-1)^n (2n + 1) C_n = 0$$

Sol. *The given series is*

$$(1 + x)^n = C_0 + C_1x + C_2x^2 + C_3x^3 + \ldots\ldots + C_nx^n \text{ replacing } x \text{ by } x^2 \text{ then}$$

$$(1 + x^2)^n = C_0 + C_1x^2 + C_2x^4 + \ldots\ldots + C_nx^{2n}$$

Multiplying both sides by x, then

$$x\,(1 + x^2)n^2 = C_0x + C_1x^3 + C_2x^5 + \ldots\ldots + C_nx^{2n+1}$$

Differentiating both sides w.r. to x, we get

$$x.n\,(1 + x^2)^{n-1}\,2n + (1 + x^2)^n.1 = C_0 + 3C_1x^2 + 5C_2x^4 + \ldots\ldots + (2n + 1)C_nx^{2n}$$

Putting x = i in both sides, we get

$$0 + 0 = C_0 - 3C_1 + 5C_2 - \ldots\ldots + (2n + 1)\,(-1)^n\,C_n$$

or $$C_0 - 3C_1 + 5C_2 - \ldots\ldots + (-1)^n (2n + 1) C_n = 0$$

Illustration :

If $(1 + x)^n = C_0 + C_1x + C_2x^2 + C_3x^3 + \ldots\ldots + C_nx^n$ *then prove that*

$$C_1^2 + 2C_2^2 + 3C_3^2 + \ldots\ldots nC_n^2 = \frac{(2n-1)!}{((n-1)!)^2}$$

$$C_1^2 + 2C_2^2 + 3C_3^2 + \ldots\ldots nC_n^2 = \frac{[illegible]}{((n-1)!)^2}$$

Sol. *Given* $$(1 + x)^n = C_0 + C_1x + C_2x^2 + C_2x^3 + \ldots\ldots + C_nx^n$$

Differentiating both sides w.r.t. to x, we get

$$n\,(1 + x)^{n-1} = 0 + C_1 + 2C_2x + 3C_3x^2 + \ldots\ldots + nC_nx^{n-1}$$

$$\Rightarrow \quad n\,(1 + x)^{n-1} = C_1 + 2C_2x + 3C_3x^2 + \ldots\ldots + nC_nx^{n-1} \quad \ldots(i)$$

and $$(x + 1)^n = C_0x^n + C_1\,x^{n-1} + C_2x^{n-2} + C_3x^{n-3} + \ldots\ldots + C_n \quad \ldots(ii)$$

Multiplying (i) and (ii), we get

$$n\,(1 + x)^{2n-1} = (C_1 + 2C_2x + 3C_3x^2 + \ldots\ldots + nC_nx^{n-1})$$

$$\times\,(C_0x^n + C_1x^{n-1} + C_2x^{n-2} + C_3x^{n-3} + \ldots\ldots + C_n) \quad \ldots(iii)$$

Now, coefficient of x^{n-1} *on R.H.S.*

$$C_1^2 + 2C_2^2 + 3C_3^2 + \ldots\ldots + nC_n^2$$

and coefficient of x^{n-1} *on L.H.S.* $= n.\ {}^{2n-1}C_{n-1}$

$$= n\cdot\frac{(2n-1)!}{(n-1)!n!} = \frac{(2n-1)!}{(n-1)!(n-1)!} = \frac{(2n-1)!}{\{(n-1)!\}^2}$$

but (iii) is an identity, therefore the coefficient of x^{n-1} *in R.H.S. = coefficient of* x^{n-1} *in L.H.S.*

$$\Rightarrow \quad C_1^2 + 2C_2^2 + 3C_3^2 + \ldots\ldots + nC_n^2 = \frac{(2n-1)!}{\{(n-1)!\}}$$

(j) Use of Integration :

$(1+x)^n = C_0+C_1x + C_2x^2+.....+ C_rx^r + ...+ C_nx^n$ ...(1)

Integrating (1) w.r.t. x between the limits 0 to 1, we get,

$$\left[\frac{(1+x)^{n+1}}{n+1}\right]_0^1 = \left[C_0x + C_1\frac{x^2}{2} + C_2\frac{x^3}{3} + \ldots\ldots + \frac{C_nx^{n+1}}{n+1}\right]$$

$$\Rightarrow \quad C_0 + \frac{C_1}{2} + \frac{C_2}{3} + \ldots + \frac{C_n}{n+1} = \frac{2^{n+1}-1}{n+1} \quad \ldots(12)$$

Integrating (1) w.r.t. x between the limits –1 to 0, we get

$$\left[\frac{(1+x)^{n+1}}{n+1}\right]_{-1}^0 = \left[C_0x + C_1\frac{x^2}{2} + C_2\frac{x^3}{3} + \ldots\ldots + \frac{C_nx^{n+1}}{n+1}\right]_{-1}^0$$

$$\Rightarrow \quad C_0 - \frac{C_1}{2} + \frac{C_2}{3} - \frac{C_3}{4} + \ldots + \frac{(-1)^n.C_n}{n+1} = \frac{1}{(n+1)} \quad \ldots(13)$$

Illustration :

If $C_r = {}^nC_r$, *then prove that* $C_0 + \frac{C_1}{2} + \frac{C_2}{3} + \ldots\ldots + \frac{C_n}{n+1} = \frac{2^{n+1}-1}{n+1}$.

Sol. *Consider the expansion*

$$(1+x)^n = C_0 + C_1x + C_2x^2 + \ldots\ldots + C_nx^n \quad \ldots(i)$$

Integrating both sides of (i) within limits 0 to 1, we get

$$\int_0^1 (1+x)^n\,dx = \int_0^1 (C_0 + C_1x + C_2x^2 + \ldots\ldots + C_nx^n)dx$$

$$\Rightarrow \quad \left[\frac{(1+x)^{n+1}}{n+1}\right]_0^1 = \left[C_0x + \frac{C_1x^2}{2} + \frac{C_2x^3}{3} + \ldots\ldots + \frac{C_nx^{n+1}}{n+1}\right]_0^1$$

$$\Rightarrow \quad \frac{2^{n+1}-1}{n+1} = C_0 + \frac{C_1}{2} + \frac{C_2}{3} + \ldots\ldots + \frac{C_n}{n+1}$$

Hence $C_0 + \frac{C_1}{2} + \frac{C_2}{3} + \ldots\ldots + \frac{C_n}{n+1} = \frac{2^{n+1}-1}{n+1}$

Illustration :

Prove that $C_0 - \frac{C_1}{2} + \frac{C_2}{3} - \ldots\ldots + (-1)^n \frac{C_n}{n+1} = \frac{1}{n+1}$

Sol. *Consider the expansion* $(1 + x)^n = C_0 + C_1x + C_2x^2 + \ldots\ldots + C_nx^n$
Integrating both sides of (i) within limits –1to 0, we get

$$\int_{-1}^{0} (1 + x)^n\, dx = \int_{-1}^{0} (C_0 + C_1x + C_2x^2 + \ldots\ldots + C_nx^n)\, dx$$

$$\Rightarrow \quad \left[\frac{(1+x)^n}{n+1}\right]_{-1}^{0} = \left[C_0x + \frac{C_1x^2}{2} + \frac{C_2x^3}{3} + \ldots\ldots + \frac{C_nx^{n+1}}{n+1}\right]_{-1}^{0}$$

$$\Rightarrow \quad \frac{1-0}{n+1} = 0 - \left(-C_0 + \frac{C_1}{2} - \frac{C_2}{3} + \ldots\ldots(-1)^{n+1}\frac{C_n}{n+1}\right)$$

$$\Rightarrow \quad \frac{1}{n+1} = C_0 - \frac{C_1}{2} + \frac{C_2}{3} - \ldots\ldots(-1)^n\frac{C_n}{n+1}$$

Hence $C_0 - \frac{C_1}{2} + \frac{C_2}{3} - \ldots\ldots + (-1)^n\frac{C_n}{n+1} = \frac{1}{n+1}$

Illustration :

Prove that $\frac{C_0}{1} + \frac{C_2}{3} + \frac{C_4}{5} + \ldots\ldots = \frac{2^n}{n+1}$

Sol. *Consider the expansion*

$$(1 + x)^n = C_0 + C_1x + C_2x^2 + C_3x^3 + C_4x^4 + \ldots\ldots + C_nx^n \;\ldots(i)$$

Integrating both sides of (i) within limits –1 to 1, we get

$$\int_{-1}^{1} (1 + x)^n\, dx = \int_{-1}^{1} (C_0 + C_1x + C_2x^2 + C_3x^3 + C_4x^4 + \ldots\ldots + C_nx^n)\, dx$$

$$= \int_{-1}^{1} (C_0 + C_2x^2 + C_4x^4 + \ldots\ldots)\, dx + \int_{-1}^{1} (C_1x + C_3x^3 + \ldots\ldots)\, dx$$

$$= 2\int_{-1}^{1} (C_0 + C_2x^2 + C_4x^4 + \ldots\ldots)dx + 0 \quad \textit{(By prop. of difinite integral)}$$

(since second integral contains odd function)

$$\Rightarrow \quad \left[\frac{(1+x)^{n+1}}{n+1}\right]_{-1}^{1} = 2\left(C_0x + \frac{C_2x^3}{3} + \frac{C_4x^5}{5} + \ldots\ldots\right)\Big]_{0}^{1}$$

$$\Rightarrow \quad \frac{2^{n+1}}{n+1} = 2\left(C_0 + \frac{C_2}{3} + \frac{C_4}{5} + \ldots\ldots\right)$$

Hence $C_0 + \frac{C_2}{3} + \frac{C_4}{5} + \ldots\ldots = \frac{2^n}{n+1}$

Illustration :

If $(1 + x)^n = C_0 + C_1x + C_2x^2 + \ldots\ldots + C_nx^n$, *prove that*

$$3C_0 + 3^2\,\frac{C_1}{2} + \frac{3^3C_2}{3} + \frac{3^4C_3}{4} + \ldots\ldots + \frac{3^{n+1}C_n}{n+1} = \frac{4^{n+1}-1}{n+1}$$

Sol. *Consider the expansion*

$$(1 + x)^n = C_0 + C_1x + C_2x^2 + C_3x^3 + \ldots\ldots + C_nx^n$$

Integrating both sides of (i) within limits 0 to 3, we get :

$$\int_0^3 (1 + x)^n\,dx = \int_0^3 (C_0 + C_1x + C_2x^2 + C_3x^3 + \ldots\ldots + C_nx^n)\,dx$$

$$\Rightarrow \quad \left.\frac{(1+x)^{n+1}}{n+1}\right]_0^3 = \left.C_0x + \frac{C_1x^2}{2} + \frac{C_2x^3}{3} + \frac{C_3x^4}{4} + \ldots\ldots + \frac{C_nx^{n+1}}{n+1}\right]_0^3$$

$$\Rightarrow \quad \frac{4^{n+1}-1}{n+1} = 3C_0 + \frac{3^2C_1}{2} + \frac{3^3C_2}{3} + \frac{3^4C_3}{4} + \ldots\ldots\frac{3^{n+1}C_n}{n+1}$$

Hence $3C_0 + \dfrac{3^2C_1}{2} + \dfrac{3^3C_2}{3} + \dfrac{3^4C_3}{4} + \ldots\ldots + \dfrac{3^{n+1}C_n}{n+1} = \dfrac{4^{n+1}-1}{n+1}$

Illustration :

Prove that ${}^{15}C_4 + 2\,{}^{15}C_5 + {}^{15}C_6 = {}^{17}C_6$

Sol. ${}^{15}C_4 + 2\,{}^{15}C_5 + {}^{15}C_6 = ({}^{15}C_4 + {}^{15}C_5) + ({}^{15}C_5 + {}^{15}C_6) = {}^{16}C_5 + {}^{16}C_6 = {}^{17}C_6$ ***Ans.***

Illustration :

$$\left(\frac{{}^nC_0 + {}^nC_1}{{}^nC_0}\right)\left(\frac{{}^nC_1 + {}^nC_2}{{}^nC_1}\right)\ldots\ldots\ldots\ldots\left(\frac{{}^nC_{n-1} + {}^nC_n}{{}^nC_{n-1}}\right) = \frac{(15)^{14}}{14\,!}$$ *find value of n .*

Sol. *Given product series may be written as* $\displaystyle\prod_{r=1}^{n}\left(\frac{{}^nC_{r-1} + {}^nC_r}{{}^nC_{r-1}}\right)$

$$= \prod_{r=1}^{n}\frac{{}^{n+1}C_r}{{}^nC_{r-1}} = \prod_{r=1}^{n}\frac{n+1}{r} = \frac{(n+1)}{1}\times\frac{(n+1)}{2}\times\frac{(n+1)}{3}\ldots\ldots\ldots\frac{(n+1)}{n} = \frac{(n+1)^n}{n\,!} \Rightarrow n = 14. \ \textbf{Ans.}$$

Illustration :

Find the sum of $S = {}^nC_0 + 2\,({}^nC_1) + 3\,({}^nC_2) + \ldots\ldots\ldots\ldots + (n+1)\,({}^nC_n)$.

Sol. *Given sum may be written as*

$$S = \sum_{r=1}^{n}(r+1)\left({}^nC_r\right) = \sum_{r=0}^{n} r\cdot\left({}^nC_r\right) + \sum_{r=0}^{n}\left({}^nC_r\right) = \sum_{r=0}^{n} n\cdot\left({}^{n-1}C_{r-1}\right) + 2^n$$

$$= n\sum_{r=0}^{n}{}^{n-1}C_{r-1} + 2^n = n\cdot 2^{n-1} + 2^n. \ \textbf{Ans.}$$

Alternate Method :

Consider $x\,(1+x)^n = {}^nC_0\,x + {}^nC_1\,x^2 \ldots\ldots + {}^nC_n\,x^{n+1}$

Differtentate w.r.t. x $(1+x)^n + nx\,(1+x)^{n-1} = {}^nC_0 + 2\,{}^nC_1\,x + \ldots\ldots + (n+1)\,{}^nC_n\,x^n$

Put x = 1 to get ${}^nC_0 + 2\cdot({}^nC_1) + 3\,({}^nC_2) + \ldots\ldots\ldots\ldots + (n+1)\,({}^nC_n) = n\cdot 2^{n-1} + 2^n$

Ans.

Illustration :

Find value of sum $\frac{{}^nC_0}{2} + \frac{{}^nC_1}{3} + \frac{{}^nC_2}{4}$ *up to n term*

Find value of sum $\dfrac{{}^nC_0}{2} + \dfrac{{}^nC_1}{3} + \dfrac{{}^nC_2}{4} \ldots\ldots$ *up to n term*

Sol. $S = \dfrac{{}^nC_0}{2} + \dfrac{{}^nC_1}{3} + \dfrac{{}^nC_2}{4} \ldots\ldots$

$$= \sum_{r=0}^{n}\frac{{}^nC_r}{r+2}$$

Consider $x\,(1+x)^n = {}^nC_0\,x + {}^nC_1\,x^2 + {}^nC_2\,x^3 + \ldots\ldots + {}^nC_n\,x^{n+1}$

Intergrate both side w.r.t x

$$\int_0^1 x(1+x)^n\,dx = \int_0^1\left({}^nC_0x + {}^nC_1x^2 + \ldots\ldots + {}^nC_nx^{n+1}\right)dx$$

or $$\left[\frac{x\cdot(1+x)^{n+1}}{n+1}\right]_0^1 - \int_0^1\frac{(1+x)^{n+1}}{n+1}\,dx = \left[{}^nC_0\frac{x^2}{2} + {}^nC_1\frac{x^3}{3}\ldots\ldots\right]_0^1$$

or $$\frac{2^{n+1}}{n+1} - \left(\frac{2^{n+2}}{(n+1)(n+2)} - \frac{1}{(n+1)(n+2)}\right) = \frac{{}^nC_0}{2} + \frac{{}^nC_1}{3} + \ldots\ldots + \frac{{}^nC_n}{n+2}$$

$$\Rightarrow \quad S = \frac{2^{n+1}}{n+1} - \frac{2^{n+2}}{(n+1)(n+2)} + \frac{1}{(n+1)(n+2)}$$

Illustration :

Find sum of series $S = {}^nC_1 + 2 \cdot {}^nC_2 + 3 \cdot {}^nC_3 + \ldots\ldots + n\,{}^nC_n$

Sol. $S = \sum_{r=1}^{n} r \cdot {}^nC_r = \sum_{r=1}^{n} r \cdot \frac{n}{r}\,{}^{n-1}C_{r-1} = n\sum_{r=1}^{n} {}^{n-1}C_{r-1}$

$$= n \cdot 2^{n-1}$$

Alternate Method :

$S = 0 \cdot {}^nC_0 + 1 \cdot {}^nC_1 + 2 \cdot {}^nC_2 + \ldots\ldots\ n \cdot {}^nC_n$

$S = n \cdot {}^nC_n + (n-1)^n\, C_{n-1} + \ldots\ldots + 0 \cdot {}^nC_0$

$\Rightarrow \quad 2S = n \cdot {}^nC_0 + n \cdot {}^nC_1 + \ldots\ldots + n \cdot {}^nC_n$

or $\quad S = \frac{n}{2}\,({}^nC_0 + {}^nC_1 + \ldots\ldots + {}^nC_n)$

$\Rightarrow \quad S = \frac{n}{2}\,2^n$

An Important Result :

An Important Result :

For sums involving product of two binomial coefficients use

$${}^nC_0\,{}^mC_k + {}^nC_1\,{}^mC_{k-1} + {}^nC_2\,{}^mC_{k-2} \ldots\ldots\ldots\ldots + {}^nC_k\,{}^mC_0 = {}^{m+n}C_k$$

Proof: Consider two series

$(1 + x)^n = {}^nC_0 + {}^nC_1x + {}^nC_2\,x^2 + \ldots\ldots. + {}^nC_k\,x^k + \ldots\ldots. + {}^nC_n\,x^n$ ....(1)

$(1 + x)^m = {}^mC_mx^m + \ldots\ldots\ldots + {}^mC_k\,x^k + {}^mC_{k-1}\,x^{k-1} + \ldots\ldots\ldots + {}^mC_0$ ....(2)

Here second series is written in reverse order.

Multiplying (1) and (2) and equate coefficients of x^k on both sides to get

$${}^{m+n}C_k = {}^nC_0\,{}^mC_k + {}^nC_1 \cdot {}^mC_{k-1} + \ldots\ldots\ldots + {}^nC_k\,{}^mC_0.$$

Illustration :

Find the sum $S = {}^nC_0{}^nC_1 + {}^nC_1{}^nC_2 + \ldots\ldots\ldots + {}^nC_{n-1}{}^nC_n.$

Sol. $S = {}^nC_0\,{}^nC_{n-1} + {}^nC_1\,{}^nC_{n-2} + \ldots\ldots\ldots\ldots + {}^nC_{n-1}{}^nC_0 \qquad [{}^nC_r = {}^nC_{n-r}]$

If sum of lower suffices is constant then

$\Rightarrow \quad S = {}^{n+n}C_{n-1} = {}^{2n}C_{n-1} = {}^{2n}C_{n+1}.$ ***Ans.***

Illustration :

Find the sum $S = 1 \cdot ({}^nC_0)^2 + 2 \cdot ({}^nC_1)^2 + 3 \cdot ({}^nC_2)^2 + \dots\dots\dots + (n+1)\,({}^nC_n)^2$

Sol. *Given sum may be written as*

$$S = \sum_{r=1}^{n} r \cdot \left({}^nC_r\right)^2 \quad or \quad S = \sum_{r=1}^{n} \left(r \cdot {}^nC_r\right) {}^nC_r \quad or \quad S = \sum_{r=1}^{n} n \cdot {}^{n-1}C_{r-1} \cdot {}^nC_r$$

or $$S = n \sum_{r=1}^{n} {}^{n-1}C_{r-1} \cdot {}^nC_r = n \sum_{r=1}^{n} {}^{n-1}C_{r-1}\ {}^nC_{n-r} \quad \Rightarrow \quad S = n \cdot {}^{2n-1}C_{n-1}\,.\ \textbf{Ans.}$$

Illustration :

If $(1+x)^n = C_0 + C_1x + \dots\dots + C_nx^n$, *then the value of* $\sum_{r=0}^{n}\sum_{s=0}^{n}(C_r + C_s)$ *is equal to*

(A) $(n+1)\,2^{n+1}$ (B) $(n-1)2^{n+1}$ (C) $(n+1)\,2^n$ (D) none of these

Sol. *We have,* $$\sum_{r=0}^{n}\sum_{s=0}^{n}(C_r + C_s)$$

$$= \sum_{r=0}^{n}\sum_{s=0}^{n} C_r + \sum_{r=0}^{n}\sum_{s=0}^{n} C_s = \sum_{s=0}^{n}\left(\sum_{r=0}^{n} C_r\right) + \sum_{r=0}^{n}\left(\sum_{s=0}^{n} C_s\right) = \sum_{s=0}^{n} 2^n + \sum_{r=0}^{n} 2^n$$

$$= \sum_{r=0}^{n}\sum_{s=0}^{n} C_r + \sum_{r=0}^{n}\sum_{s=0}^{n} C_s = \sum_{s=0}^{n}\left(\sum_{r=0}^{n} C_r\right) + \sum_{r=0}^{n}\left(\sum_{s=0}^{n} C_s\right) = \sum_{s=0}^{n} 2^n + \sum_{r=0}^{n} 2^n$$

$$= (n+1)\,2^n + (n+1)\,2^n = (n+1)\,2^{n+1}$$

Hence (A) is correct answer.

Illustration :

If $(1+x)^n = C_0 + C_1x + \dots\dots + C_nx^n$, *then the value of* $\sum\limits_{0 \le r}\sum\limits_{< s \le n} C_rC_s$ *is equal to*

(A) $\frac{1}{2}[2^{2n} - {}^{2n}C_n]$ (B) $\frac{1}{4}[2^{2n} - {}^{2n}C_n]$ (C) $\frac{1}{2}[2^{2n} + {}^{2n}C_n]$ (D) $\frac{1}{2}[2^n - {}^{2n}C_n]$

Sol. *We have*

$$\sum_{r=0}^{n}\sum_{s=0}^{n} C_rC_s = \left(\sum_{r=0}^{n} C_r^2\right) + 2\sum_{0 \le r}\sum_{< s \le n} C_rC_s$$

$$\Rightarrow \quad 2^{2n} = {}^{2n}C_n + 2\sum_{0 \le r}\sum_{< s \le n} C_rC_s$$

$$\Rightarrow \quad \sum_{0 \le r}\sum_{< s \le n} C_rC_s = \frac{1}{2}[\,2^{2n} - {}^{2n}C_n\,]$$

Hence (A) is correct answer.

3.3 AN IMPORTANT CONCEPT :

Finding nature of integral part of expression.

$$N = \left(a+\sqrt{b}\right)^n \qquad (n \in N)$$

Step-1: Consider $N' = \left(a-\sqrt{b}\right)^n$ or $\left(\sqrt{b}-a\right)^n$ according as $a > \sqrt{b}$ or $\sqrt{b} > a$.

Step-2: Use N + N' or N – N' such that result is integer.

Step-3: Use fact N = I + f

'I' stands for [N] and 'f' for {N}.

Illustration :

For $n \in N$ prove that integral part of $N = \left(3+\sqrt{7}\right)^n$ is an odd integer.

Sol. *Consider $N' = \left(3-\sqrt{7}\right)^n$*

$$N = {}^nC_0\, 3^n + {}^nC_1\, 3^{n-1}\sqrt{7} + {}^nC_2\, 3^{n-2}\left(\sqrt{7}\right)^2 + \ldots\ldots\ldots - {}^nC_n\left(\sqrt{7}\right)^n.$$

$$N' = {}^nC_0\, 3^n - {}^nC_1\, 3^{n-1}\sqrt{7} + {}^nC_2\, 3^{n-2}\left(\sqrt{7}\right)^2 + \ldots\ldots\ldots\ldots - {}^nC_n\left(\sqrt{7}\right)^n.$$

Using $N + N' = 2\,[{}^nC_0\, 3^n + {}^nC_2\, 3^{n-2}\left(\sqrt{7}\right)^2 + \ldots\ldots\ldots]$

$\Rightarrow \quad N + N' = 2k \quad (k \in I) \quad \Rightarrow \quad I + f + N' = 2k \qquad \{N = I + f\}$

$0 < N' < 1 \Rightarrow 0 < f + N' < 2$

But $f + N'$ *it self an integer* $f + N' = 1 \Rightarrow I + 1 = 2k$ *or* $I = 2k - 1$.

Hence proved.

Illustration :

Show that integral part of $P = \left(3\sqrt{3}+5\right)^{2n+1}$ *(n* $\in$ *N) is an even number.*

Sol. Consider $P' = \left(3\sqrt{3}-5\right)^{2n+1}$ here $0 < P' < 1$

Using $P - P' = 2\left[{}^{2n+1}C_1\left(3\sqrt{3}\right)^{2n}5' + {}^{2n+1}C_3\left(3\sqrt{3}\right)^{2n-2}(5)^3 + \ldots\ldots\right]$

$\Rightarrow \quad I + f - P' = 2k \; (k \in N) \qquad \{P = I + f\}$

$-1 < f - P' < 1$ but $f - P'$ is an integer $\Rightarrow f - P' = 0 \Rightarrow I = 2k.$

Illustration :

Let $N = \left(7+4\sqrt{3}\right)^n = I + f$ *(n* $\in$ *N), then find the value of (1 – f) N.*

Sol. Consider $N' = \left(7-4\sqrt{3}\right)^n$

Using $N + N' = 2\,[{}^nC_0\,7^n + {}^nC_2\,7^{n-2}\left(4\sqrt{3}\right)^2 + \ldots\ldots]$

or $\quad I + f + N' = 2k \qquad (k \in I)$

Illustration :

Let $N = \left(7+4\sqrt{3}\right)^n = I + f$ *(n* $\in$ *N), then find the value of (1 – f) N.*

Sol. Consider $N' = \left(7-4\sqrt{3}\right)^n$

Using $N + N' = 2\,[{}^nC_0\,7^n + {}^nC_2\,7^{n-2}\left(4\sqrt{3}\right)^2 + \ldots\ldots]$

or $\quad I + f + N' = 2k \qquad (k \in I)$

or $\quad f + N' = 2k - I$

but $\quad 0 < f + N' < 2 \Rightarrow f + N' = 1$ or $f = 1 - N'.$

$(1 - f)N = N'N = \left(7-4\sqrt{3}\right)^n\left(7+4\sqrt{3}\right)^n = (49-48)^n = 1.$

Solved Examples

Single correct question

Q.1 Value of middle term of expansion $\left(\frac{2}{3}x^2 - \frac{3}{2x}\right)^{20}$ for $x = p$ is $32 \cdot {}^{20}C_{10}$ value of 'p' is

(A) 2 (B) $\frac{1}{2}$ (C) $\sqrt{2}$ (D) $\frac{1}{4}$

Sol. Here $n = 20$ so no of terms will be 21 then middle term $t_m = t_{10+1} = {}^{20}C_{10}x^{10}$

Q.2 If $b_1, b_2, \ldots\ldots b_n$ are in G.P. with common ratio '2' then $b_1\,{}^nC_1 + b_2\,{}^nC_2 + \ldots\ldots + b_n \cdot {}^nC_n =$

(A) $\frac{b_1}{2}(3^n)$ (B) $b_1\,(3^n)$ (C) $b_1\,(3^n - 1)$ (D) $\frac{b_1}{2}(3^n - 1)$

Sol. $b_1\,{}^nC_1 + b_2\,{}^nC_2 + \ldots\ldots + b_n\,{}^nC_n = b_1\,({}^nC_1 + 2({}^nC_2) + 2^2\,({}^nC_3) + \ldots\ldots + 2^{n-1}\,{}^nC_n)$

$$= \frac{b_1}{2}[(1+2)^n - 1] = \frac{b_1}{2}(3^n - 1)$$

Q.3 If $(1 + ax)^n = 1 + 8x + 24x^2 \ldots\ldots$ then $\frac{a-n}{a+n}$ is equal to

(A) 3 (B) -3 (C) $-\frac{1}{3}$ (D) $\frac{1}{3}$

Sol. $1 + ax \cdot {}^nC_1 + {}^nC_2\,(ax)^2 + \ldots\ldots = 1 + 8x + 24x^2 + \ldots\ldots\ldots$

By comparison

$\Rightarrow \quad {}^nC_1\,a = 8 \quad \ldots\ldots.(1) \quad$ and $\quad {}^nC_2\,a^2 = 24 \quad \ldots\ldots.(2)$

By equation (1) and (2), we get

$n = 4$ and $a = 2$.

Q.4 In the expansion $\left(x^2 + \frac{2}{x}\right)^n$ $(n \in N)$ has 13^{th} term independent of x, then sum of even divisors of n is equal to –

(A) 39 (B) 63 (C) 13 (D) 26

Sol. $t_{13} = {}^nC_{12} \cdot 2^{12}\, x^{2n-36}$ $\Rightarrow$ $2n - 36 = 0$ or $n = 18$

$n = 18 = 2^1 \times 3^2$

sum of divisors (even) $= (2^1)(3^0 + 3^1 + 3^2)$

$= 2 \times 13 = 26$

Q.5 Cofficient of x^7 in the expansion of $(1 - x^4)(1 + x)^9$ is –

(A) 27 (B) –24 (C) 48 (D) –48

Sol. $(1 - x^4)(1 + x)^9 = (1 + x)^9 - x^4(1 + x)^9$

cofficient of $x^7 = {}^9C_7 - {}^9C_3 = -48$

Q.6 Cofficient of x^{100} in $1 + (1 + x) + (1 + x)^2 + (1 + x)^3 \ldots\ldots (1 + x)^n$ $(n > 100)$ is ${}^{201}C_{101}$, then n =

(A) 100 (B) 200 (C) 101 (D) None of these

Sol. Cofficient of $x^{100} = {}^{100}C_{100} + {}^{101}C_{100} + \ldots\ldots + {}^nC_{100}$

$= {}^{n+1}C_{101} = {}^{201}C_{101}$ $\Rightarrow$ $n = 200$

Multiple correct type question

Q.7 Cofficients of three consecutive terms in the expansion of $(1 + x)^n$ are in the ratio 1 : 7 : 42, value of 'n'

Multiple correct type question

Q.7 Cofficients of three consecutive terms in the expansion of $(1 + x)^n$ are in the ratio 1 : 7 : 42, value of 'n' is always less than or equal to –

(A) 55 (B) 54 (C) 56 (D) 51

Sol. ${}^nC_{r-1} : {}^nC_r : {}^nC_{r+1} = 1 : 7 : 42$ $\Rightarrow$ $n = 55$ and $r = 7$

Q.8 $\alpha = \left\{\dfrac{3^{200}}{8}\right\}$ where {x} factional of x then possible of middle term in $(2 + 5x)^{72\alpha}$ is –

(A) 5^{th} (B) 7^{th} (C) 8^{th} (D) 6^{th}

Sol. $3^{200} = 9^{100} = (1 + 8)^{100} = 1 + [{}^{100}C_1\, 8 + {}^{100}C_2\, 8^2 \ldots\ldots]$

$$\Rightarrow \quad \left\{\frac{3^{200}}{8}\right\} = \frac{1}{8}$$

$(2 + 5x)^{72\alpha} = (2 + 5x)^9$

Q.9 Third term in the expansion of $\left(x + x^{\log_{10} x}\right)^5$ is 10^6 then possible value of x are –

(A) 1 (B) 10 (C) $10^{-5/2}$ (D) 10^6

Sol. $T_3 = {}^5C_2\, x^3 \left(x^{\log_{10} x}\right)^2 \;= 10 \cdot x^{3+2\log_{10} x} = 10^6$

$\Rightarrow \quad x^{3+2\log_{10} x} = 10^5$ taking long on both sides given

$(3 + 2\log_{10} x)\log_{10} x = 5 \quad \Rightarrow \quad \log_{10} x = 1 \quad \text{or} \quad -\frac{5}{2}$

Q.10 If $(1 + x + x^2 + x^3) = C_0 + C_1x + C_2x^2 + \ldots\ldots + C_{3n}\, x^{3n}$ then which of following are correct –

(A) $C_0 + C_1 + C_2 + \ldots\ldots + C_{3n} = 2^{2n}$ (B) $C_0 + C_2 + C_4 + \ldots\ldots = C_1 + C_2 + C_5$

(C) $C_0 = C_{3n}, C_1 = C_{3n-1}, C_2 = C_{3n-2}$ (D) None of these

Sol. Put x = 1 to get $4^n = C_0 + C_1 + C_2 + \ldots\ldots + C_{3n} \ldots$

Put x = –1 to get $\Rightarrow \quad C_0 + C_2 + C_4 \ldots\ldots = C_1 + C_3 + C_5 \ldots\ldots$

Paragraph type

Paragraph type

Paragraph for question nos. 11 to 13

Let P be a product given by $P = (x + \alpha_1)(x + \alpha_2) \ldots\ldots (x + \alpha_n)$ and $S_1 = \alpha_1 + \alpha_2 + \ldots\ldots + \alpha_n = \sum_{i=1}^{n} \alpha_i$

$S_2 = \sum\sum_{i<j} \alpha_i\alpha_j \qquad S_3 = \sum\sum\sum_{i<j<k} \alpha_i\alpha_j\alpha_k$ and so on then $P = x^n + S_1\, x^{n-1} + S_2\, x^{n-2} \ldots\ldots + S_n$

Q.11 Cofficient of x^8 in $(x + 2)^2 (x + 3)^3 (x + 4)^4$ is –

(A) 26 (B) 27 (C) 28 (D) 29

Q.12 Cofficient of x^{203} in $(x – 1)(x^2 – 2)(x^3 – 3) \ldots\ldots (x^{20} – 20)$ is –

(A) 11 (B) 12 (C) 13 (D) 15

Q.13 Cofficient of x^{98} in $(x – 1)(x – 2) \ldots\ldots (x – 100)$ is

(A) $1^2 + 2^2 + 3^2 + \ldots\ldots + 100^2$

(B) $(1 + 2 + 3 + \ldots\ldots + 100)^2 – (1^2 + 2^2 + \ldots\ldots + 100^2)$

(C) $\frac{1}{2}[(1 + 2 + \ldots\ldots + 100)^2 – (1^2 + 2^2 + \ldots\ldots + 100^2)]$

(D) None of these

Sol. (i) $(x^4 + 4 + 4x)\,(x^3 + 3c_1\,x^2 + \ldots\ldots)\,(x^4 + {}^4C_1\,x^3 + \ldots\ldots)$ expand suitably

(ii) $(x - 1)\,(x^2 - 2)\,(x^3 - 3)\ \ldots\ldots\ (x^{20} - 20)$

$$= x^{210}\left[\left(1-\frac{1}{x}\right)\left(1-\frac{2}{x^2}\right)\ldots\ldots\left(1-\frac{20}{x^{20}}\right)\right]$$

(iii) Cofficient x^{98} in S_2 in prodct $(x - 1)\,(x - 2)\ \ldots\ldots\ (x - 100)$

$$=\frac{1}{2}\,[(1 + 2 + \ldots\ldots + 100)^2 - (1^2 + 2^2 + \ldots\ldots + 100^2)]$$

Match the Column

Q.14

	Column-I		Column-II
(A)	Cofficient of x^7 and x^8 are equal in the expansion $\left(3+\frac{x}{2}\right)^n$ value of n is	(P)	9
(B)	${}^nC_2 = 990$ then n is divisible by	(Q)	45
(B)	${}^nC_2 = 990$ then n is divisible by	(Q)	45
(C)	Cofficient of x^2 and x^3 are equal in expansion of $\left(3+\frac{9x}{7}\right)^m$ then m is grater than or equal to	(R)	7
		(S)	55
(D)	${}^pC_6 = {}^{p-1}C_5 + 1$ then p is less than	(T)	15

Sol. (A) ${}^nC_7 \cdot \frac{3^{n-7}}{2^7} = {}^nC_8 \cdot \frac{3^{n-8}}{2^8} \Rightarrow \quad n = 55$

(B) ${}^nC_2 = 990 \quad \Rightarrow \quad n = 45$

(C) ${}^mC_2\ \frac{3^{m+2}}{7^2} = {}^mC_3\,\frac{3^{m+3}}{7^3} \quad \Rightarrow \quad m = 9$

(D) ${}^pC_6 = {}^{p-1}C_5 + 1 \quad$ or $\quad {}^{p-1}C_6 + {}^{p-1}C_5 = {}^{p-1}C_5 + 1$

$\Rightarrow\ {}^{p-1}C_6 = 1 \quad$ or $\quad p = 7$

Reasoning type question

Q.15 **Statement-1:** ${}^{20}C_0\ {}^{15}C_r + {}^{20}C_1\ {}^{15}C_{r-1} + \ldots\ldots + {}^{20}C_r\ {}^{15}C_0$ has maximum value ${}^{35}C_{18}$.

Statement-2: ${}^{n}C_0\ {}^{m}C_s + {}^{n}C_1\ {}^{m}C_{s-1} + {}^{n}C_2\ {}^{m}C_{s-2}\ \ldots\ldots + {}^{n}C_s\ C_0 = {}^{m+n}C_s$

(A) Statement-1 is true, statement-2 is true and statement-2 is correct explanation for statement-1.

(B) Statement-1 is true, statement-2 is true and statement-2 is NOT the correct explanation for statement-1.

(C) Statement-1 is true, statement-2 is false.

(D) Statement-1 is false, statement-2 is true.

Sol. S_2 is true as $\sum\sum {}^{n}C_{r_1}\ {}^{m}C_{r_2}$ where $r_1 + r_2 = k$ (constant) is equal to ${}^{m+n}C_k$. Sum given in S_1 is equal to ${}^{35}C_r$.

Q.16 **Statement-1:** If ${}^{100}C_r$, ${}^{100}C_{r+1}$, ${}^{100}C_{r+2}$ and ${}^{100}C_{r+3}$ are in AP then $r = 49$.

Statement-2: Four consecutive binomial cofficient can never be in AP.

(A) Statement-1 is true, statement-2 is true and statement-2 is correct explanation for statement-1.

(B) Statement-1 is true, statement-2 is true and statement-2 is NOT the correct explanation for statement-1.

(C) Statement-1 is true, statement-2 is false.

(C) Statement-1 is true, statement-2 is false.

(D) Statement-1 is false, statement-2 is true.

Sol. $2 \cdot {}^{100}C_{r+1} = {}^{100}C_r + {}^{100}C_{r+2}$

$$\Rightarrow \qquad 2 = \frac{r+1}{100-r} + \frac{99-r}{r+2}$$

which has no solution

Appendix - I

THE PRINCIPLE OF MATHEMATICAL INDUCTION :

Mathematical induction is a technique for proving any statement any theorem, or a formula that is asserted about every natureal number,.

By "every", or "all" natural numbers, we mean

any one that we might possibly name.

For example,

$$1 + 2 + 3 + \ldots\ldots + n = \frac{1}{2n}(n+1)$$

This asserts that the sum of consecutive numbers from 1 to n is given by the formula on the right. We want to prove that this will be true for $n = 1$, $n = 2$, $n = 3$, and so on. Now we can test the formula for any given number, say $n = 3$.

$$1 + 2 + 3 = \frac{1}{2} \cdot 3 \cdot 4 = 6$$

Which is true. It is also true for $n = 4$.

Which is true. It is also true for $n = 4$.

$$1 + 2 + 3 + 4 = \frac{1}{2} \cdot 4 \cdot 5 = 10.$$

But how are we to prove this rule for every value of n ?

The method of proof is the following. It is called the principle of mathematical induction.

AXIOM OF INDUCTION :

(1) when a statement is true for a natural number $n = k$,
then it will also be true for its successor, $n = k + 1$,
and

(2) the statement is true for $n = 1$.
then the statement will be true for every natural number n.

To prove a statement by induction, we must prove parts 1 and 2 above. For, when the statement is true for n= 1, then according to 1, it will also be true for 2. But that implies it will be true for 3 which implies it will be true for 4. And so on. It will be true for any natural number that we migth name.

Appendix - II

BINOMIAL THEOREM FOR ANY INDEX :

When n is a negative integer or a fraction then the expansion of a binomial is possible only when

(i) Its first term is 1, and

(ii) Its second term is numerically less than 1.

Thus when $n \notin N$ and $|x| < 1$, then it states

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots\ldots + \frac{n(n-1)(n-r+1)}{r!}x^r + \ldots\infty\infty$$

1.1 General Term :

$$T_{r+1} = \frac{n(n-1)(n-2)\ldots(n-r+1)}{r!} \cdot x^r$$

Note :

(i) In this expansion the coefficient of different terms can not be expressed as $^nC_0, ^nC_1, ^nC_2 \ldots$ because **n** is not a positive integer.

(ii) In this case there are infinite terms in the expansion.

(ii) In this case there are infinite terms in the expansion.

1.2 Some Important Expansions :

If $|x| < 1$ and $n \in Q$ but $n \notin N$, then

(a) $(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \ldots + \frac{n(n-1)\ldots(n-r+1)}{r!}x^r + \ldots\ldots$

(b) $(1-x)^n = 1 - nx + \frac{n(n-1)}{2!}x^2 - \frac{n(n-1)(n-2)}{3!}x^3 + \ldots + \frac{n(n-1)\ldots(n-r+1)}{r!}(-x)^r + \ldots$

(c) $(1-x)^{-n} = 1 + nx + \frac{n(n+1)}{2!}x^2 + \frac{n(n+1)(n+2)}{3!}x^3 + \ldots\ldots + \frac{n(n+1)\ldots(n+r-1)}{r!}x^r + \ldots$

(d) $(1+x)^{-n} = 1 - nx + \frac{n(n+1)}{2!}x^2 - \frac{n(n+1)(n+2)}{3!}x^3 + \ldots + \frac{n(n+1)\ldots(n+r-1)}{r!}(-x)^r + \ldots$

By putting n = 1, 2, 3 in the above results (c) and (d), we get the following results-

(e) $(1-x)^{-1} = 1 + x + x^2 + x^3 + \ldots + x^r + \ldots$

General term $T_{r+1} = x^r$

(f) $(1+x)^{-1} = 1 - x + x^2 - x^3 + \ldots(-x)^r + \ldots$

General term $T_{r+1} = (-x)^r$

(g) $(1-x)^{-2} = 1+2x+3x^2+4x^3+.....+(r+1)x^r+.....$

General term $T_{r+1} = (r+1)\,x^r$

(h) $(1+x)^{-2} = 1-2x+3x^2-4x^3+.....+(r+1)\,(-x)^r+.....$

General term $T_{r+1} = (r+1)\,(-x)^r$.

(i) $(1-x)^{-3} = 1+3x+6x^2+10\,x^3+......+\dfrac{(r+1)(r+2)}{2!}x^r+.....$

General term $= \dfrac{(r+1)(r+2)}{2!}x^r$.

(j) $(1+x)^{-3} = 1-3x+6x^2-10x^3+.....+\dfrac{(r+1)(r+2)}{2!}(-x)^r+.....$

General term $= \dfrac{(r+1)(r+2)}{2!}\,(-x)^r$.

Illustration :

If $|x| < 2/3$ then the fourth term in the expansion of $\left(1+\dfrac{3}{2}x\right)^{1/2}$ is–

(A) $\dfrac{27}{128}x^3$ (B) $-\dfrac{27}{128}x^3$ (C) $\dfrac{81}{256}x^3$ (D) $-\dfrac{81}{256}x^3$

Sol. $T_4 = \dfrac{1/2(1/2-1)(1/2-2)}{3!}\cdot\left(\dfrac{3x}{2}\right)^3 = \dfrac{27}{128}x^3$ **Ans.[A]**

(A) $\dfrac{27}{128}x^3$ (B) $-\dfrac{27}{128}x^3$ (C) $\dfrac{81}{256}x^3$ (D) $-\dfrac{81}{256}x^3$

Sol. $T_4 = \dfrac{1/2(1/2-1)(1/2-2)}{3!}\cdot\left(\dfrac{3x}{2}\right)^3 = \dfrac{27}{128}x^3$ **Ans.[A]**

Illustration :

The term independent of x in the expansion of $\left(\dfrac{1-x}{1+x}\right)^2$ is–

(A) 4 (B) 3 (C) 2 (D) 1

Sol. $(1-x)^2\,(1+x)^{-2}$

$\Rightarrow$ $(1-2x+x^2)\,(1-2x+x^2+....)$

$\Rightarrow$ so term independent of $x = 1$. **Ans. [D]**

Illustration :

The coefficient of x^5 in the expansion of $(1-x)^{-6}$ is –

(A) 1260 (B) – 1260 (C) – 252 (D) 252

Sol. x^5 occurs in T_6 of the expansion, so

$$T_6 = T_{5+1} = \frac{6.7.8.9.10}{5!}\,x^5 = 252\,x^5$$

$\therefore$ Coefficient of $x^5 = 252$ **Ans.[D]**

1.3 Applications of binomial theorem :

(I) With the help of binomial theorem, we can find out the value of sq. root, cube root and 4th root etc. of the given number upto any decimal places.

Illustration :

The value of cube root of 1001 upto five decimal places is –

(A) 10.03333 (B) 10.00333 (C) 10.00033 (D) None of these

Sol. $(1001)^{1/3} = (1000+1)^{1/3} = 10\left(1+\frac{1}{1000}\right)^{1/3} = 10 \left\{1+\frac{1}{3}.\frac{1}{1000}+\frac{1/3(1/3-1)}{2!}\frac{1}{1000^2}+.....\right\}$

$= 10\ \{1 + 0.0003333 - 0.00000011 + ...\}$

$= 10.00333$ **Ans.[B]**

(II) To find the sum of Infinite series :

We can compare the given infinite series with the expansion of $(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}$

We can compare the given infinite series with the expansion of $(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}$

$x^2 + ...$ and by finding the value of x and n and putting in $(1 + x)^n$ the sum of series is determined.

Illustration :

The sum of $1 + \frac{1}{4} + \frac{1.3}{4.8} + \frac{1.3.5}{4.8.12} + ...\infty$ is–

(A) $\sqrt{2}$ (B) $\frac{1}{\sqrt{2}}$ (C) $\sqrt{3}$ (D) $2^{3/2}$

Sol. Comparing with $1 + nx + \frac{n(n-1)}{2!}x^2 + ...$

$n\,x = 1/4$ ...(1)

and $\quad \dfrac{n(n-1)x^2}{2!} = 1.3/4.8$

or $\quad \dfrac{nx(nx-x)}{2!} = \dfrac{3}{32} \Rightarrow \dfrac{1}{4}\left(\dfrac{1}{4} - x\right) = \dfrac{3}{16}$ $\quad$ (by (1))

$\Rightarrow \quad \left(\dfrac{1}{4} - x\right) = \dfrac{3}{4} \Rightarrow x = \dfrac{1}{4} - \dfrac{3}{4} = -\dfrac{1}{2}$ $\quad$ ... (2)

putting the value of x in (1)

$n(-1/2) = 1/4 \quad \Rightarrow n = -1/2$

$\therefore$ sum of series $= (1+x)^n$

$= (1-1/2)^{-1/2} = (1/2)^{-1/2} = \sqrt{2}$ **Ans.[A]**

(III) Approximation:

Illustration :

If x is so small so that its square and higher power can be neglected.

If x is so small so that its square and higher power can be neglected.

Find the value of $\dfrac{\left(1+\dfrac{2x}{3}\right)^{-5} + (4+2x)^{1/2}}{(4+x)^{3/2}}$.

Sol. $\dfrac{\left(1+\dfrac{2x}{3}\right)^{-5} + (4+2x)^{1/2}}{(4+x)^{3/2}} = \dfrac{\left(1-\dfrac{10x}{3}\right) + 2\left(1+\dfrac{x}{4}\right)}{8\left(1+\dfrac{3x}{8}\right)} = \dfrac{1}{8}\left(3 - \dfrac{10x}{3} + \dfrac{x}{2}\right)\left(1+\dfrac{3x}{8}\right)^{-1}$

$= \dfrac{3}{8}\left(1 - \dfrac{17x}{18}\right)\left(1 - \dfrac{3x}{8}\right) = \dfrac{3}{8}\left(1 - \dfrac{17x}{18} - \dfrac{3x}{8}\right) = \dfrac{72-95x}{24 \times 8}$

PERMUTATION AND COMBINATION

FUNDAMENTAL PRINCIPAL OF COUNTING :

If an event can occur in m different ways following which another event can occur in n different ways, then total number of simultaneous occurrence of both the events in a **definite order** is $(m \times n)$. This can be extended to any number of events

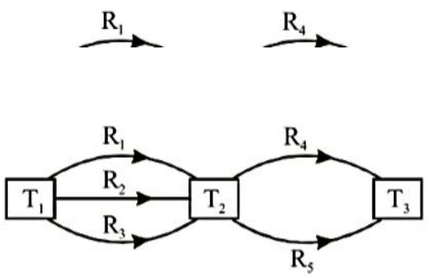
Eg : For an event to be occur in $m_1, m_2, \ldots\ldots, m_n$ ways then number of ways is $m_1 \times m_2 \ldots\ldots m_n$

Important Point :

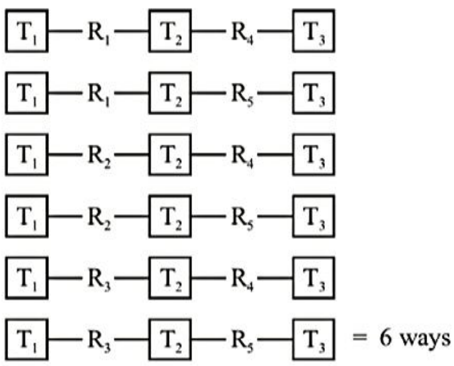
FPC (Fundamental Principal of Counting) is used to count some event without actually counting them.

Let us take help of some model.

Model- I : Find number of ways of in which one can travel from T_1 (town 1) to T_3 (town 3) via T_2 (town 2).



Total ways : –



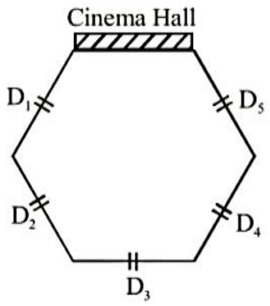
It is easy to proceed by FPC $\quad T_1$ to $T_2 \longrightarrow 3$

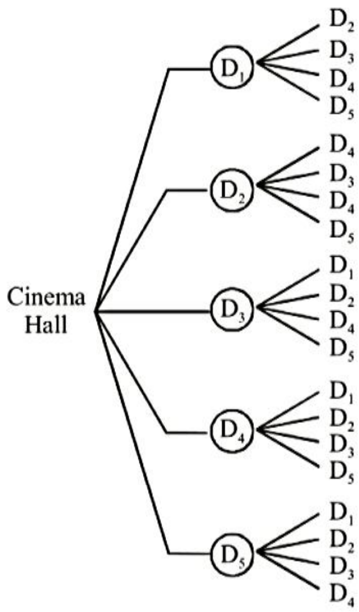
T_2 to $T_3 \longrightarrow 2$

Total ways = $3 \times 2 = 6$

Model- II :

(i) To find the number of ways by which a person can enter and leave cinema hall by a different door.





$4 + 4 + 4 + 4 + 4 = 20$

By F.P.C.
A person can enter in cinema hall by 5 ways & leave by 4 ways $= 5 \times 4 = 20$.

(ii) If he can enter and leave by any door then number of ways $= 5 \times 5 = 25$.

(ii) If he can enter and leave by any door then number of ways $= 5 \times 5 = 25$.

Basic Steps to Remember :

Step-I : Identify the independent events involved in a given problem.
Step-II : Find the number of ways performing/occuring each event
Step-III : Multiply these numbers to get the total number of ways of performing/occuring all the events

Illustration :

How many 3 digit numbers can be formed by the digit 1, 2, 3, 4, 5 without repetition.

Sol. *Hundred's place digit can be selected in 5 ways.*
Ten's place digit can be selected in 4 ways.
Unit's place digit can be selected in 3 ways.
So, $5 \times 4 \times 3 = 60$

Illustration :

In an examination of 10 T/F question, How many sequence of answers are possible.

Sol. *Any question can be answered in two ways , i.e. true or false.*
So total task of anwering tan question can be done in
$2 \times 2 \times 2 \times \ldots\ldots 10 \text{ times} = 2^{10}$ *ways*

Illustration :

10 students complete in a swiming race. In how many ways can they occupy the first 3 positions.

Sol. 1^{st} *place can be occupied in 10 ways*
2^{nd} *place can be occupied in 9 ways*
3^{rd} *place can be occupied in 8 ways.*
So total number of ways $= 10 \times 9 \times 8 = 720$

Illustration :

There are 7 flags of diffrent colour. Find the number of diffrent signals that can be transmitted by the use of 2 flags one above the other.

Sol. 1^{st} *place can be occupied in 7 ways*
2^{nd} *place can be occupied in 6 ways*
So total number of ways $= 7 \cdot 6 = 42$

Illustration :

A letter lock consists of 3 rings each marked with 10 different letters. In how many ways, its is possible to make an unsuccessful attempt to open the lock?

possible to make an unsuccessful attempt to open the lock?

Sol. *2 ring may have same letters at a time. One ring can have any one of 10 different letters in 10 ways.*
Similarly 2^{nd} *and* 3^{rd} *ring can have any one of 10 different letters in 10 ways respectively.*
Total number of attempts $= 10 \times 10 \times 10 = 10^3 = 1000$
But out of these 1000 attempt, only one attempt is successful.
Required number of unsuccessful attempt $= 1000 - 1 \quad = 999$

Illustration :

How many 6 digits odd number greater than 6,00,000 can be formed from the digits 5,6,7, 8, 9, 0 if repetition of digit is allowed ?

Sol. *Numbers greater than 6,00,000 and formed with the digit 5, 6, 7, 8, 9, 0 are of 6 digit but begin with 6, 7, 8 or 9.*
Also, the numbers which end with 5, 7, 9 are odd.
Hence, first place can be filled by 4 ways (out of 6, 7, 8 or 9). Last place can be filled by 3 ways.
Hence, first and last place can be filled by 4×3 *ways.*
Also 2^{nd} *place can be filled by 6 ways.*
3^{rd} *place can be filled by 6 ways*
4^{th} *place can be filled by 6 ways.*
5^{th} *place can be filled by 6 ways*
Hence, all the 6 places can be filled by
$4 \times 3 \times 6 \times 6 \times 6 \times 6 = 15552$ *ways.*

Illustration :

How many integers greater than 5000 can be formed with the digit 7, 6, 5, 4 & 3 using each digit at most once.

Sol. *For 4 digits number ⇒ First position can be filled by 7, 6 or 5 (that is in three ways). Hence*

4 *digit number* $= 3 \times 4 \times 3 \times 2 = 72$

5 *digit number* $= 5 \times 4 \times 3 \times 2 \times 1 = 120$

Total number $= 192$

Illustration :

How many natural number less than 30000 can be made from the digits 0, 1, 2, 3, 4, 5, 6.

Sol. *Let a five digit number be denoted by*

a	b	c	d	e

Each of the places can be filled by either of 0, 1, 2, 3, 4, 5, 6 in 7 ways.

The place marked by "a" can be filled by the digits 0, 1 or 2 (since number is to be less than 30000)

Each of the places can be filled by either of 0, 1, 2, 3, 4, 5, 6 in 7 ways.

The place marked by "a" can be filled by the digits 0, 1 or 2 (since number is to be less than 30000)

Hence number of intergers $= 3 \times 7 \times 7 \times 7 \times 7 = 3 \times 7^4$

In these numbers one case includes "00000" which is not a natural numbers

Hence number of natural numbers $= 3 \times 7^4 - 1$

Illustration :

Consider the word DAUGHTER. How many 4 letter word can be formed from the letter of above word so that each word contain letter G.

Sol. *4 possible positions for G.*

Remaining three by $\Rightarrow 4 \times 7 \times 6 \times 5 = 28 \times 30 = 840.$

Alternative method :

Total number of ways by which 4 letter word can be formd $= 8 \times 7 \times 6 \times 5$

Number of four letter word without G $= 7 \times 6 \times 5 \times 4$

$= 8 \times 7 \times 6 \times 5 - 7 \times 6 \times 5 \times 4.$

Illustration :

How many diffrent words can be formed using all the letters in the word "MIRACLE".

(a) *If vowels may accupy the even position.*

(b) *If vowels may occupy odd position.*

Sol.

(a) *Even position* $\Rightarrow$

1	2	3	4	5	6	7
	×		×		×	

vowels $\rightarrow$ *I, E, A*

consonants $\rightarrow$ *M, R, C, L*

Three vowels at three position $\Rightarrow 3 \times 2 \times 1 = 6$

Four consonants at four position $\Rightarrow 4 \times 3 \times 2 \times 1 = 24$

Total number of ways $= 6 \times 24 = 144$.

(b)

1	2	3	4	5	6	7
×		×		×		×

1^{st} *position can be filled by any one of the four vowel.*

2^{nd} *position can be filled by any one of the three vowel.*

3^{rd} *position can be filled by any one of the two vowel.*

Thus total ways $= (4 \times 3 \times 2) \times (4 \times 3 \times 2 \times 1) = 576$

Illustration :

There are m men and n monkey. Number of ways in which every monkey has a master, if a man can have any number of monkey.

Sol. *Monkey is distributed among in masters, like 1 monkey can go to* $\rightarrow$ *m masters*

Total number of ways $= m \times m \times \ldots\ldots\ldots\ m = m^n$

Illustration :

Number of ways in which m diffrent toys can be distributed in "n" children if every child may receive any number of toys

Sol. *Object of distribution toys*

One toy $\rightarrow$ *n children*

Total number of ways $= \quad n \times n \ldots\ldots\ldots\ldots\ n = n^m$

Illustration :

Find the number of ways in which we can post 5 letters in 10 letter boxes.

Sol. 1^{st} *letter* $\Rightarrow$ *10 boxes*

2^{nd} *letter* $\Rightarrow$ *10 boxes*

3^{rd} *letter* $\Rightarrow$ *10 boxes*

4^{th} *letter* $\Rightarrow$ *10 boxes*

5^{th} *letter* $\Rightarrow$ *10 boxes*

$= 10^5$.

Illustration :

In a car plate number containing only 3 or 4 digits not containing the digit 0. What is the maximum numbers of cars that can be numbered?

Sol. *Here repetition of digits is allowed.*
Also, numbers are formed with the digit1, 2, 3, …… 9

Case-I : *When car plate numbers contain 3 digit number of places to be filled up r = 3.*
Out of the 9 digit first place can be filled by 9 ways.
Similarly, 2^{nd} and 3^{rd} place can be filled in 9 ways respectively.
So, when car plate number contais 3 digit, maximum number of cars = 9^3.

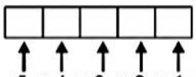
Case-II : *When car plate number contains 4 digit, in this case number of cars to be filled up r = 4.*
1^{st} place can be filled in 9 ways.
2^{nd} place can be filled by 9 ways and so on.
Maximum number of cars that can be numbered.

$$= 9^3 + 9^4 = 7290$$

Lexicography illustration (Lexicography is called science of making words) :

(a) Find total number of 5 letter word that can be formed from letters of word "TOUGH"

(a) Find total number of 5 letter word that can be formed from letters of word "TOUGH".



$5 \times 4 \times 3 \times 2 \times 1 = 120$

(b) Find the rank of "TOUGH" if all the letters of the word are arranged in all possible orders & written out as in a dictionary.

The number of letters in the word "TOUGH" is 5 & all the five letters are different.
Alphabatical order of all the letters is G, H, O, T, U

Number of words begining with	G	=	$4 \times 3 \times 2 \times 1$
Number of words begining with	H	=	$4 \times 3 \times 2 \times 1$
Number of words begining with	O	=	$4 \times 3 \times 2 \times 1$
Number of words begining with	TG	=	$3 \times 2 \times 1$
Number of words begining with	TH	=	$3 \times 2 \times 1$
Number of words begining with	TOG	=	2×1
Number of words begining with	TOH	=	2×1

Next words begining with "TOU" and it is "TOUGH" = 1.
Rank = 24 + 24 + 24 + 6 + 6 + 2 + 2 + 1 = 89

Try Yourself

Find the rank of "SACHIN" if all the letters of the word are arranged in all possible order & written out as in a dicitonary. [Ans. 601]

Practice Problem

Q.1 If the letters of the word "VARUN" are written in all possible ways and then are arranged as in a dictionary, then the rank of the word VARUN is :

(A) 98 (B) 99 (C) 100 (D) 101

Q.2 How many natural numbers are their from 1 to 1000 which have none of their digits repeated.

Q.3 A man has 3 jackets, 10 shirts, and 5 pairs of slacks. If an outfit consists of a jacket, a shirt, and a pair of slacks, how many different outfits can the man make?

Q.4 There are 6 roads between A & B and 4 roads between B & C.

(i) In how many ways can one drive from A to C by way of B?

(ii) In how many ways can one drive from A to C and back to A, passing through B on both trips ?

(iii) In how many ways can one drive the circular trip described in (ii) without using the same road more than once.

Q.5 (i) Find the number of four letter word that can be formed from the letters of the word HISTORY. (each letter to be used at most once)

(ii) How many of them contain only consonants?

(iii) How many of them begin & end in a consonant?

(iv) How many of them begin with a vowel?

(v) How many contain the letters Y?

(iv) How many of them begin with a vowel?

(v) How many contain the letters Y?

(vi) How many begin with T & end in a vowel?

(vii) How many begin with T & also contain S?

(viii) How many contain both vowels?

Q.6 If repetitions are not permitted

(i) How many 3 digit numbers can be formed from the six digits 2, 3, 5, 6, 7 & 9 ?

(ii) How many of these are less than 400 ? (iii) How many are even ?

(iv) How many are odd ? (v) How many are multiples of 5?

Q.7 A letter lock consists of three rings each marked with 10 different letters. Find the number of ways in which it is possible to make an unsuccessful attempts to open the lock.

Q.8 It is required to seat 5 men and 4 women in a row so that the women occupy the even places. How many such arrangements are possible?

Answer key

Q.1 C Q.2 738 Q.3 150 Q.4 (i) 24; (ii) 576; (iii) 360

Q.5 (i) 840 ; (ii) 120 ; (iii) 400; (iv) 240 ; (v) 480; (vi) 40; (vii) 60; (vii) 240

Q.6 (i) 120; (ii) 40 ; (iii) 40; (iv) 80; (v) 20

Q.7 999

Q.8 2880

NOTATION OF FACTORIAL :

Notation of factorial & its Algebra :

The continued product of first n, natural number is called as " n factorial" and denoted by n! or $\lfloor n$

$n! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot \ldots\ldots (n-1)n!$
$4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$
$3! = 1 \cdot 2 \cdot 3 = 6$
$5! = 120; \qquad 61 = 720; \qquad 7! = 5040$

Special Results :

❖ $0! = 1$ i.e. factorial of zero is 1

Proof: $n! = (n-1)! \cdot n$

Putting $n = 1$

$1! = (1-1)!\,1 \quad \Rightarrow \quad 0! = 1$

❖ Factorial of negative number is undefined

$$(n-1)! = \frac{n!}{n} \text{ if } n = 0 \text{ then } (-1)! = \frac{0!}{0} = \frac{1}{0} \text{ Not defined.}$$

Asking:

(i) Find n if $(n+1)! = 12 \times (n-1)!$

(ii) $(n+2)! = 2550\, n!$

Sol.

Sol.

(i) $(n+1)\, n\, (n-1)! = 12 \times (n-1)!$
$n^2 + n - 12 = 0;$
$(n+4)(n-3) = 0 \therefore n = 3$

(ii) $(n+2)(n+1) = 2550;$
$(n+52)(n-49) = 0 \therefore n = 49$

❖ $(2n!) = 2^n \cdot n!\, [1 \cdot 3 \cdot 5 \ldots\ldots (2n-1)]$

Proof:

$(2n)! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \ldots\ldots (2n-1)(2n-2)$

Take 2 common each n even terms

$= 2^n (1 \cdot 3 \cdot 5 \ldots\ldots (2n+1))(1 \cdot 2 \cdot 3 \ldots\ldots n)$
$= 2^n (n!)(1 \cdot 3 \cdot 5 \ldots\ldots (2n+1))$

Illustration :

Using above find exponent of prime 2 in (100 !)

Sol. $(2 \cdot 50)! = 2^{(50)} \cdot 50!\,(1 \cdot 3 \cdot 5 \ldots\ldots 99) \rightarrow 50 \quad 2's$
$(50!) = (2 \cdot 25)! = 2^{(25)} \cdot 25!\,(1 \cdot 3 \cdot 5 \ldots\ldots 49) \rightarrow 25 \quad 2's$
$(24!) = (2 \cdot 12)! = 2^{(12)} \times 12!\,(1 \cdot 3 \cdot 5 \ldots\ldots 23) \rightarrow 12 \quad 2's$
$(12!) = (2 \cdot 6)! = 2^{(6)} \cdot 6!\,(1 \cdot 3 \cdot 5 \ldots\ldots 11) \rightarrow 6 \quad 2's$
$(6!) = (2 \cdot 3)! = 2^{(3)} \cdot 3!\,(1 \cdot 3 \cdot 5 \ldots\ldots 5)$
$= 2^{(3)} \cdot 3!\,(1 \cdot 3 \cdot 5) \rightarrow 3 \quad 2's$
$(3!) = 3 \cdot 2 \cdot 1 = 2^{(1)} \cdot (1 \cdot 3) \rightarrow 1 \quad 2's$
$50 + 25 + 12 + 6 + 3 + 1 = 97$

Alternative Method :

Let p be a prime number and n be a positive integer. Then, the last integer amongst 1, 2, 3, ……, (n – 1), n which is divisible by p is $\left[\frac{n}{p}\right]p$, *where* $\left[\frac{n}{p}\right]$ *denotes the greatest integer less than or equal to* $\frac{n}{p}$.

For example, $\left[\frac{10}{3}\right]=3, \left[\frac{12}{5}\right]=2, \left[\frac{15}{3}\right]=5$ etc

Let $E_p(n)$ *denote the exponent of the prime p in the positive integer n. Then,*

$$E_p(n!) = E_p(1\cdot 2\cdot 3 \ldots\ldots (n-1)n)$$

$$= E_p\left(p\cdot 2p\cdot 3p\ldots\ldots\left[\frac{n}{p}\right]p\right) \qquad \left[\begin{matrix}\because \text{Remaining integers between} \\ \text{1 and n are not divisible by p}\end{matrix}\right]$$

$$= \left[\frac{n}{p}\right] + E_p\left(1\cdot 2\cdot 3\ldots\left[\frac{n}{p}\right]\right)$$

Now, the last integer amongst 1, 2, 3, ……, $\left[\frac{n}{p}\right]$ *which is divisible by p is*

$\left[\frac{n/p}{\ }\right]p = \left[\frac{n}{\ }\right]p$

Now, the last integer amongst 1, 2, 3, ……, $\left[\frac{\ }{p}\right]$ *which is divisible by p is*

$$\left[\frac{n/p}{p}\right]p = \left[\frac{n}{p^2}\right]p$$

$$\therefore \qquad E_p(n!) = \left[\frac{n}{p}\right] + E_p\left(p\cdot 2p\cdot 3p\ldots\ldots\left[\frac{n}{p^2}\right]p\right) \qquad \left[\begin{matrix}\because \text{Remaining integers between} \\ \text{1 and } \left[\frac{n}{p}\right] \text{ are not divisible by p}\end{matrix}\right]$$

$$E_p(n!) = \left[\frac{n}{p}\right] + \left[\frac{n}{p^2}\right] + \left[\frac{n}{p^3}\right] + \ldots\ldots + \left[\frac{n}{p^s}\right]$$

where s is the largest positive integer such that $p^s \le n < p^{s+1}$

Alternative Method :

$$E_2(100!) = \left[\frac{100}{2}\right] + \left[\frac{100}{2^2}\right] + \left[\frac{100}{2^3}\right] + \left[\frac{100}{2^4}\right] + \left[\frac{100}{2^5}\right] + \left[\frac{100}{2^6}\right]$$

$$= 50 + 25 + 12 + 6 + 3 + 1$$

Illustration :

Find number of zeros of the end of (1000)!

Sol. *In any usual factorial of a natural number of 2s are more than number of 5s. Hence number of 10s are same as number of 5s.*

Special Note :

Such kind of counting is possible only on the basis of primes.

Basic Method :

Now we decide number of 5s in (1000)! (1000!)

5, 10, 15 ……… 1000 : 200	$\Rightarrow$	200 number containing at least one 5
25, 50 ……… 1000 : 40	$\Rightarrow$	40 number containing at least two 5
125, 250 ……… 1000 8	$\Rightarrow$	8 number containing at least three 5
625 1	$\Rightarrow$	1 number containing at least four 5

Total = 200 + 40 + 8 + 1 = 249

Objective approach

$$E_5\,(1000!) = \left[\frac{1000}{5}\right] + \left[\frac{1000}{5^2}\right] + \left[\frac{1000}{5^3}\right] + \left[\frac{1000}{5^4}\right]$$

$$= 200 + 40 + 8 + 1 = 249$$

Illustration :

Find number of zeros at the end of 2007!

Find number of zeros at the end of 2007!

Sol. $$E_5\,(2007!) = \left[\frac{2007}{5}\right] + \left[\frac{2007}{5^2}\right] + \left[\frac{2007}{5^3}\right] + \left[\frac{2007}{5^4}\right]$$

$$= 401 + 80 + 16 + 3 = 500$$

Illustration :

Find exponent of 3 in 100!

Sol. *Basic method :*

	Term		
One 3	*3, 6, 9, 12, ……… 99 = 33*	$\Rightarrow$	*33 number containing at least one 3*
Two 3	*9, 18, 27 ……… 99 = 11*	$\Rightarrow$	*11 number containing at least two 3*
Third 3s	*27 + 54 + 81 = 3*	$\Rightarrow$	*3 number containing at least three 3*
4s	*81 = 1*	$\Rightarrow$	*1 number containing at least four 3*

Sum = 33 + 11 + 3 + 1 = 48

Objective approach

$$E_3\,(100!) = \left[\frac{100}{3}\right] + \left[\frac{100}{3^2}\right] + \left[\frac{100}{3^3}\right] + \left[\frac{100}{3^4}\right]$$

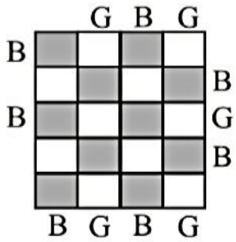
$$= 33 + 11 + 3 + 1 = 48$$

Illustration :

In a class of 20 students having 20 chairs in 5 raw's of 4 each. If the class has 10 boys and 10 girls, in how many ways, can the student's be placed in the chair's such that no boy is sitting in front of, behind, or next to another boy and no girl is sitting in front of, behind or next to another girl.

(A) $^{20}C_{10}\,(10!)^2$ *(B)* $2x^{20}\,(10!)^2$ *(C)* $2x\,(10!)^2$ *(D) not possible*

Sol. *Let us colour the desk of chair's like chessboard pattern. The given arrangement is possible if boys lies on black and girl's on white or girl's on black and boys white can be done* $\underbrace{^2C_1}_{\text{Black or White}}(10!)^2$.



DIVISIBILITY OF NUMBERS :

The following chart shows the conditions of divisibility of numbers by 2,3,4,5,6,8,9,25

Divisible by	Condition	
2	whose last digit is even	(2, 4, 6, 8, 0)
3	sum of whose digits is divisible by 3	
4	whose last two digits number is divisible by 4	
4	whose last two digits number is divisible by 4	
5	whose last digit is either 0 or 5	
6	which is divisible by both 2 and 3	
8	whose last three digits number is divisible by 8	
9	sum of whose digits is divisible by 9	
25	whose last two digits are divisible by 25	

PERMUTATION & COMBINATION :

Permutation :

Permutation means arrangement in a definte order of things which may be alike or different taken some or all at a time. Hence permutation refers to the situation where order of occurrence of the events is important.

Combination :

Combination/selection/collection/committee refers to the situation where order of occurrence of the event is not important. Combination is sellection of one or more things out of n things which may be alike or different taken some or all at a time.

Note :

Things which are alike and which are different. All god made things in general are treated to be different and all man made things are to be spelled weather like or different.

Hence we say that permutation is arrangement of things in definite order.

Example :

(i) Out of A, B, C, D take 3 letters & form number plate of car. [Pemutation]

(ii) Out of four letters A, B, C, D take any 3 letters & form triangle (possible). [Combination]
In 1^{st} case arrangement of letters are there, in 2^{nd} case only selection will form the triangle, arrangment is not required.

Theorm related to application of Permutation and Combination :

Theorm-1 :

Number of permutations of n distinct things taken all at a time symbolised as :

$$^nP_n = P(n, n) = A_n^n = n!$$

Proof :

Let these are n things arranged at n places

$$n \cdot (n-1) \cdot (n-2) \ldots\ldots 3 \cdot 2 \cdot 1 = n!$$

We also say that number of ways in which n distinct objects can be arranged amongst themselves in $^nP_n = n!$
i.e. Find total number of words that of 10 letters that can be formed from all the letters of word GANESHPURI.

$$A = {}^nP_n = 10! = n!$$

$$A = {}^nP_n = 10! = n!$$

Theorm-2 :

Number of permutations of n distinct things taken r at a time

$$0 \le r \le n$$

$$^nP_r = P(n, r) = A_r^n = \frac{n!}{(n-r)!}$$

Things $T_1, T_2 \ldots\ldots T_n$

Places $1, 2, 3, \ldots\ldots r$

Choice $n \cdot (n-1) \cdot (n-2) \ldots\ldots [n-(r-1)]$

$$\text{Total way} = \frac{n(n-1)(n-2)\ldots\ldots(n-r+1)(n-r)!}{(n-r)!} = \frac{n!}{(n-r)!}$$

Hence we can say that

$$^nP_r = \frac{n!}{(n-r)!}$$

Eg : In how many ways can 5 person be made to occupy

(a) Five different chairs

$$^5P_5 = 5! = 120$$

(b) Three different chairs

$$^5P_3 = 5 \times 4 \times 3 = 60$$

Theorm-3 :

Number of combination / selection of n distinct things taken r at a time

$$^nC_r = C(n, r) = \binom{n}{r} = \frac{n!}{(r)!(n-r)!}$$

Proof :

Let 10 different objects are given as A, B, C, D, E, F, G, H, I, J

Let combinations taking 3 at a time = x

Arrangement = (x) × (3!)

$$x \cdot 3! = {}^{10}P_3$$

$$x = \frac{^{10}P_3}{3!} = \frac{10!}{(10-3)3!}$$

Theorm-4 :

Number of combination of n different things taken r at a time when p particular things are always included.

$= {}^{n-p}C_{r-p}$

i.e. Find total number of ways of selecting 11 player out of 15 player when Mahendra Sing Dhoni and Yuvraj Singh are always included $= {}^{15-2}C_{11-2} = {}^{13}C_9$

Theorm-5 :

Number of combination of n different things taken r at a time when p particular thing are always excluded.

$^{n-p}C_r$

Eg : How many different selections of 6 books can be made from 11 different books if two particular

$^{n-p}C_r$

Eg : How many different selections of 6 books can be made from 11 different books if two particular
books are never selected $= {}^{11-2}C_6 = {}^9C_6$

Important Results :

(i) $^nC_0 = 1$

(ii) $^nC_n = 1$

(iii) $^nC_{n-r} = {}^nC_r$ If $^nC_x = {}^nC_y \Rightarrow x + y = n$ or $x = y$

(iv) $^nP_r = {}^nC_r\, r!$

i.e. permutation is defined total number of combination of object then arragement of objects.

(v) $^nC_r + {}^nC_{r+1} = {}^{n+1}C_{r+1}$

Examples on ^nC_r and ^nP_r :

Illustration :

There are n points in a plane, no three of which collinear, find

(a) Number of straight lines *(b) Number of triangles*

(c) Number of digonals in a polygon of n sides.

Sol.

(a) Number of ways by which we can select any two points gives total number of straight lines $= {}^nC_2$.

(b) Number of ways by which we can select any 3 non collinear points, gives total number of triangle $= {}^nC_3$.

(c) In any polygon number of vertices = number of sides

$^nC_2$ = *number of sides + number of diagonals*

$^nC_2 - n$ = *Number of diagonals.*

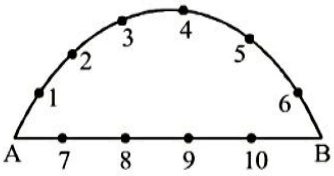
Illustration :

There are 10 points in a plane of which 4 are collinear and rest are non-collinear. Find

(i) *Number of lines* ***(ii)*** *Number of triangles*

Sol.

(i) *Number of lines*



Method-I :

Given 4 points are collinear so total number of ways of selecting any two points = $^{10}C_2$.

If 4 collinear points give only one line (AB). So over counted number of lines formed by collinear

points = $^4C_2 - 1$.

Thus total lines = $^{10}C_2 - {}^4C_2 + 1$

Method-II :

Take any two points from upper arc = $^6C_2$ ways.

Take one point from upper arc and one point from line AB = $^6C_1 \times {}^4C_1$ ways.

Take both the points from line AB then number of lines = 1 (Line AB).

Total number of lines = $^6C_1 \times {}^4C_1 + {}^6C_2 + 1$

Take both the points from line AB then number of lines = 1 (Line AB).

Total number of lines = $^6C_1 \times {}^4C_1 + {}^6C_2 + 1$

(ii) *Method-I :*

Number of ways of taking any three points = $^{10}C_3$

Number of ways of taking any three points from four collinear points = $^4C_3$

Number of triangles formed = $^{10}C_3 - {}^4C_3$

Method-II :

Two points from upper arc and one point from line AB = $^6C_2 \times {}^4C_1$

Two points from line AB and one point from upper arc = $^6C_1 \times {}^4C_2$

All the three points from upper arc = $^6C_3$

Total number of triangle = $^6C_2 \times {}^4C_1 + {}^6C_1 \times {}^4C_2 + {}^6C_3$

Illustration :

The sides AB, BC, CA of a triangle ABC have 3, 4, 5 points respectively on them. Find the number of triangle that can be constructed using these points as vertices.

Sol. *Total ways = $^{12}C_3 - {}^3C_3 - {}^4C_3 - {}^5C_3 = 205$*

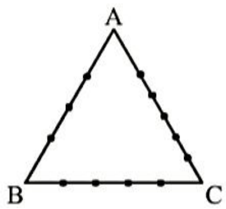


Illustration :

10 diffrent letters of English alphabet are given. Words of 5 letters are formed from these given letters. How many words are formed when at least one letter is repeated.

Sol. $10^5 - 10 \cdot 9 \cdot 7 \cdot 6 = 6976$

Illustration :

Everybody in a room shakes hand with every body else. The total number of handshakes is 66. Find total number of person in the room.

Sol. *Let total number of persons are n. For every two persons possible handshake is 1.*
So, total hand shake = ${}^nC_2 = 66 \Rightarrow n^2 - n - 132 = 0$
$n = 12, \ -11 \Rightarrow n = 12, n \neq -11.$

Illustration :

The number of ways in which a team of 11 players can be selected from 22, players including two of them and excluding four of them

Sol. *As two players are already included so only 9 players to be selected from* $22 - 4 - 2 = 16$
Number of ways = ${}^{16}C_9$

Illustration :

Illustration :

How many differnt words can be formed from the letters of the word GANESHPURI, when the letters E, H, P are never together.

Sol.

	EHP						

$\boxed{\text{EHP}} \longrightarrow$ String

(Consider EHP as a letter or string)

So, we have to arrange one string & 7 letters
Hence total number of ways = $10! - 8! \times 3!$

EHP can be arranged amongst themselves in 3! ways.

Illustration :

How many ways can the seven diffrent colour of a rainbow be arranged so that the blue and green never come together

Sol. *Total number of arrangement without constraint* $= 7!$

	BG					

$\boxed{\text{BG}} \longrightarrow$ String *(Consider BG as a letter or string)*

If BG always come together then number of ways = $(\underbrace{1}_{\text{string}} + \underbrace{5}_{\text{letters}})!\ \underbrace{2!}_{\text{Arrangement of B \& G}}$

Required number of ways = Total possible (without restriction) – (ways when BG together)
$= 7! - 6! \times 2!$

Illustration :

In a morse telegraphy there are 4 arms and each arm is capable of taking 5 distinct position including the position of rest. Find total no. of different signals.

Sol. *One arm is capable of taking 5 positions*
Hence total number of ways = 5^4
Out all above these arise a unique condition in which no signal is transfer. It is R R R R
Hence number of different signal = $5^4 - 1$

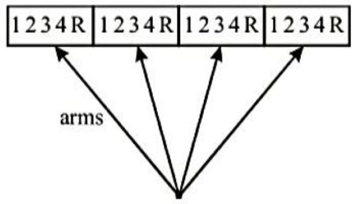
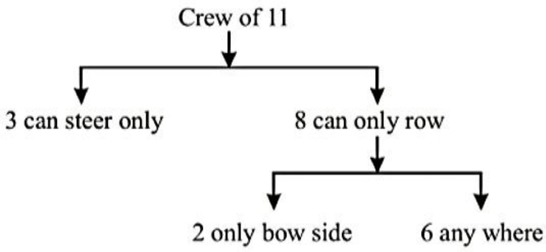
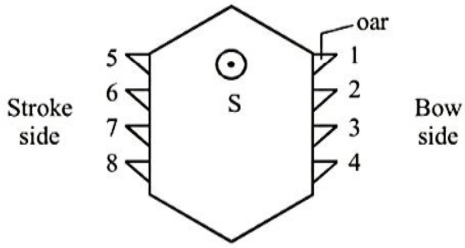


Illustration :

An 8 oared boat to be manned from a crew of 11 of which 3 can only steer. In how many ways the staff can be arranged if 2 of the men can row only on the bow side.





Sol. *For steering position 3 person's can be selected in ${}^3C_1$ ways.*
Out of 4 places of bow sides two are selected for 2 men by ${}^4C_2$ ways and arranged by ${}^4C_2 \times 2!$ ways

Sol. *For steering position 3 person's can be selected in ${}^3C_1$ ways.*
Out of 4 places of bow sides two are selected for 2 men by ${}^4C_2$ ways and arranged by ${}^4C_2 \times 2!$ ways
Remaining six person's can be arranged by 6! ways at six places (As number of seats and number of person's are equal)

Thus required ways = $\underbrace{{}^3C_1}_{\text{Steer}} \times (\underbrace{{}^4C_2\,(2!)}_{\text{Row side}}) \times (\underbrace{6!}_{\text{Any side}})$

Illustration :

4 Boys & 4 Girls are to be seated in a line find number of ways

(i) *They can be seated so that "No two girls are together"*

(ii) *If not all the girls are together*

or

If at least one girl is separated from rest of girls.

(iii) *Boys and girls are alternate*

(iv) *If there are 4 married couples then number of ways in which they can be seated so theat each couple is together.*

Sol.

(i) *Out of four standing boys five gaps are there*

$$| B_1 | B_2 | B_3 | B_4 |$$

Out of five select any four gaps by ${}^5C_4$ ways in which girls are arranged by = ${}^5C_4 \times 4!$ ways.
Also standing boys are arranged by 4! ways
Required ways = $(4!) \times ({}^5C_4 \cdot 4!)$

Note :- Arrangement by this method is called as gap method.

(ii) *Consider* $G_1\,G_2\,G_3\,G_4 \rightarrow$ *as one string.*

If all girls are together then total ways = $\underbrace{(1+\overbrace{4}^{\text{Four boys}})!}_{\text{String}}\ \underbrace{(4!)}_{\text{Arrangement of four girls in string}} = (5! \times 4!)$

Total number of arrangment without any restriction = 8!
Total number of ways by which not all girls are together = $8! - (5! \times 4!)$

(iii) *If boys and girls are alternate then two ways of arranging respective position of boys and girls are shown below*

Number of ways of arranging boys $\Rightarrow 4!$
Number of ways of arranging girls $\Rightarrow 4!$
Required ways = $2 \times 4! \times 4!$

(iv) *Let the couples are as below*

| B G | B G | B G | B G |

(iv) *Let the couples are as below*

| B_1G_1 | B_2G_2 | B_3G_3 | B_4G_4 |

There are four string and each string has husband and wife which can be arranged in 2! ways.
Ways of arranging strings = 4!
Required ways = $(2!)^4 \times 4!$

Illustration :

Let C be the set of 6 consonants (b, c, d, f, g, h) and V be a set of 5 vowels (a, e, i, o, u) and W be the set of seven letter words that can be formed with these 11 letters using both the following rules.
(a) The vowels and consonant in the word must alternate.
(b) No letter can be used more than once in a single word.
If the number of words in the set W are 10 K. Find K.

Sol. *Consonant* $\rightarrow$ *b, c, d, f, g, h (6)*
Vowels $\rightarrow$ *a, e, i, o, u (5)*
W : | × | | × | | × | | × |

Case I : If word begins with consonants
then $({}^6C_4 \times 4!) \times ({}^5C_3 \times 3!) = 360 \times 60 = 21600$
Case II : If word begins with vowels
$({}^5C_4 \times 4!) \times ({}^6C_3 \times 3!) = 120 \times 120 = 14400$
Total $= 36000 \Rightarrow 10K = 36000 \Rightarrow K = 3600$ *Ans.*

Illustration :

Find total number of permutations of n different things taken not more than m at a time when each thing may be repeated any number of times?

Sol. Number of permutation of one thing $= n$
Number of permutation of two things $= n \times n = n^2$
Number of permutation of three things $= n \times n \times n = n^3$
and similarly for m things
So, number of permutation of n things taken not more then m at a time when repetition is allowed

$$= n + n^2 + n^3 \dots\dots + n^m = \frac{n(n^m - 1)}{n-1}$$

Illustration :

In how many ways 10 examination papers be arranged so that the best and the worst papers never come together?

Sol. For the number of arrangements of 10 examination papers without restriction
$n = 10 \qquad r = 10$
total number of arrangement of 10 examination papers with any restriction $= {}^{10}P_{10} = 10!$
Keeping best and worst together, number of different papers = 9.
(i.e. in this case $n = 9,\ r = 9$)
Number of arrangements when best and worst are kept together $= {}^{9}P_{9} \cdot 2!$
Hence, the required number of arrangements of 10 examination papers so that best and worst papers are not together is $10! - {}^{9}P_{9} \cdot 2! = 8 \times 9!$

Hence, the required number of arrangements of 10 examination papers so that best and worst papers are not together is $10! - {}^{9}P_{9} \cdot 2! = 8 \times 9!$

Illustration :

On a new year day, every students of a class sends a card to every other student, the postman delivers 600 cards. How many students are there in class?

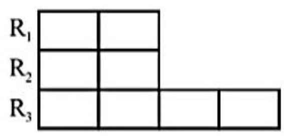
Sol. Total number of students $= n$
Number of pair of students $= {}^{n}C_2$
Two students out of n can be selected in ${}^{n}C_2$ ways.
Here for each pair of students, number of cards sent is 2.
If P sends card to Q, the Q also sends a card to P.
Number of cards sent $= 2 \times {}^{n}C_2$
According to the problem $2 \times {}^{n}C_2 = 600$

$$\Rightarrow \quad 2 \times \frac{n(n-1)}{2!} = 600 \quad n^2 - n - 600 = 0$$

$$\Rightarrow \quad n = 25 \qquad [\because\ n \neq -24]$$

Illustration :

In how many ways the letters of the word 'PERSON' can be placed in the squares of the given figure shown so that no row remain empty?



Sol. There are 6 different letters in the word 'PERSON'
Total required way = Total possible way – (1st row empty + 2nd row empty)
Total possible ways $= {}^{8}P_6$ (arrangement of 6 different letters in 8 boxes)

ways when 1^{st} row empty = 6! (arrangement of 6 different letters in remaining 6 boxes)
way when 2^{nd} row empty = 6! (arrangement of 6 different letters in remaining 6 boxes)
Total required ways = $^8P_6 - (6! + 6!)$
= 18720

Illustration :

Consider all the six digit numbers that can be formed using the digits 1, 2, 3, 4, 5 and 6, each digit being used exactly once. Each of such six digit numbers have the property that for each digit, not more than two digits smaller than that digit appear to the right of that digit. Find the number of such six digit numbers having the desired property.

Sol. *Begin with six open positions.*
To get an arrangements of digits having the desired property, choose a position to place the digit 6.
The 6 can be placed in any of the 3 rightmost positions.
Once the position of the 6 has been chosen, choose a position to place the digit 5.
The 5 can be placed in any of the 3 rightmost positions not occupied by the 6.
Continuing in this way, the digits 6, 5, 4 and 3 can all be placed in one of 3 positions, the 3 rightmost positions which are left open.
Finally, there will be 2 available positions to place the 2, and only one position left for the 1.
Thus, the number of ways to choose an arrangement with the desired property is $3^4 \cdot 2 = 162$.

Practice Problem

Q.1 In how many ways can 5 persons be made to occupy

Practice Problem

Q.1 In how many ways can 5 persons be made to occupy
(i) five different chairs (ii) three different chairs

Q.2 A telegraph has 5 arms and each arm is capable of 4 distinct positions, including the position of rest. What is the total number of signals that canbe made.

Q.3 A gentleman has 6 friends to invite. In how many ways can invitation cards be sent to them if he has three servants.

Q.4 An English school and a Vernacular school are both under one superintendent. Suppose that the superintendentship, the four teachership of English and Vernacular school each, are vacant, if there be altogether 11 candidates for the appointments, 3 of whom apply exclusively for the superintendentship and 2 exclusively for the appointment in the English school, the number of ways in which the different appointments can be disposed of is :
(A) 4320 (B) 268 (C) 1080 (D) 25920

Q.5 If m denotes the number of 5 digit numbers if each successive digits are in their descending order of magnitude and n is the corresponding figure, when the digits are in their ascending order of magnitude then $(m - n)$ has the value
(A) $^{10}C_4$ (B) $^9C_5$ (C) $^{10}C_3$ (D) $^9C_3$

Answer key

Q.1 (i) 120, (ii) 60 Q.2 $4^5 - 1 = 1023$ Q.3 3^6
Q.4 D [Hint :-Similar to boat problem] Q.5 B

FORMATION OF GROUPS :

(I) No. of ways in which $(m+n)$ different things can be divided in two groups, one containing m things and other contains 'n' things is

$$^{m+n}C_n \text{ or } \frac{(m+n)!}{m!\,n!}.$$

Illustration :

Out of four players P_1, P_2, P_3, P_4 form two teams one contain 3 players and other one player.

Sol. *Number of groups* $= \frac{4!}{1!\,3!} = 4$

Note :

$\Rightarrow$ Acutual explanation of above 4 groups

These an 4 answers so obtains

Select - 3	Rejected - 1
P_1, P_2, P_3	P_4
P_2, P_3, P_4	P_1
P_3, P_4, P_1	P_2
P_4, P_1, P_2	P_3

(II) If groups are equal size i.e. $m = n$

P_4, P_1, P_2 | P_3

(II) If groups are equal size i.e. $m = n$

Total number of ways in which 2n different things can be divided into two equal groups $= \frac{^{2n}C_n}{2!}$

$$= \frac{(2n)!}{(n!)(n!)(2!)}$$

We divide by 2! to avoid false counting.

Proof :

Divide P_1, P_2, P_3, P_4 in two groups

Team – A	Team - B
P_1P_2	P_3P_4
P_1P_3	P_2P_4
P_1P_4	P_2P_3
P_2P_3	P_1P_4
P_2P_4	P_1P_3
P_3P_4	P_1P_2

We see that half of the case are repeated.

Thus $\frac{4!}{2!\,2!}$ gives us wrong answer.

Correct answer $= \frac{4!}{(2!)(2!)(2!)}$

Actually counting all such cases we observe that regrouping appear's when equal size group's are required. To avoid false counting we devide by factorial of number of equal size group's.

(III) Total numbers of ways in which $(m+n+p)$ different things can be divided into three unequal groups m, n, p is $\dfrac{(m+n+p)!}{m!\,n!\,p!}$

Illustration :

If we divide 12 different things in three unequal groups (2, 3, 7) then total number of ways are

Sol. 12 digit things $\begin{cases} \nearrow 2 \\ \rightarrow 3 \\ \searrow 4 \end{cases}$ $\qquad \dfrac{12!}{2!\,3!\,7!}$

Explanation :

$$12 \begin{cases} \nearrow 2 \\ \searrow 10 \end{cases} \quad \frac{12!}{2!\,10!} \qquad 12 \begin{cases} \nearrow 2 \\ \searrow 10 \begin{cases} \nearrow 3 \\ \searrow 7 \end{cases} \end{cases} \quad \frac{10!}{3!\,7!}$$

$$\text{Total ways} = {}^{12}C_2 \times {}^{10}C_7 \times {}^{3}C_3 = \frac{12!}{2!\,10!} \times \frac{10!}{3!\,7!} = \frac{12!}{2!\,3!\,7!}$$

If groups are equal (m = n = p) then number of ways

$$= \frac{(3n)!}{(n!)^3\,3!}$$

Important Points :

The number of ways in which 'r' group's of n different object's can be formed in such a way that 'p' groups of n_1 object, q group of n_2 object each is ……

$$\text{Required ways} = \frac{n!}{(n_1!)^p (n_2!)^q (p!)(q!)}$$

$n = (n_1 + n_1 \ldots\ldots\ldots\ldots \text{ p times }) + (n_2 + n_2 \ldots\ldots\ldots\ldots \text{ q times})$

Divide by factorial of number of equal size group.

Illustration :

In how many ways 3 team's of 11 player's each, 4 team's 6 player's each, 2 team's of 15 player's each can be formed out of 87 player's

Sol. $\therefore$ $\quad \text{Required ways} = \dfrac{87!}{(11!)^3 (6!)^4 (15!)^2} \times \dfrac{1}{(3!)(4!)(2!)}$

Illustration :

In how many ways 6 bundles of 12 different toys be made such that 2 bundles are of 3 toys each, 2 bundles are 2 toys each & 2 bundle of 1 toy each

Sol. $\text{Required ways} = \dfrac{(12!)}{(3!)^2 (2!)^2 (1!)^2} \times \dfrac{1}{(2!)(2!)(2!)}$

Illustration :

Total number of ways in which 200 person's can be divided into 100 equal group's.

Sol. Required ways $= \dfrac{200!}{(2!)^{100}(100)!}$

Corollary : Grouping and then arrangement.
If (m + n + p) different thing's can be divided in 3 group's & can be distributed to three person's.

Required ways $= \dfrac{(m+n+p)!}{m!\,n!\,p!} \times 3!$

Illustration :

Find number of ways by which 30 Jawan's can be devided into three group's of 12, 10 & 8 and send to three different boarder's.

Sol. Total ways $= \dfrac{(30!)\times 3!}{(8!)(10!)(12!)}$

In above case if group's are equal size (i.e. group of 10 each)

Send to three boarder's

In above case if group's are equal size (i.e. group of 10 each)

Send to three boarder's

$$= \frac{(30!)\times(3!)}{(10!)^3(3)!}$$

Three equal size groups

Illustration :

In how many ways six diff. books can be distributed between four persons, so that each person gets atleast one book.

Sol. *Two cases possible {1, 1, 1, 3}, {1, 1, 2, 2}*

$$Groups\left[\frac{6!}{(1!)^3\,3!\,3!}+\frac{6!}{(1!)^2\,(2!)^2\,2!\,2!}\right]4!$$

Illustration :

Find number of ways by which five different objects given to three students.

Sol. *Two cases possible {1, 1, 3} {1, 2, 2}*

$$\left[\frac{5!}{(1!)^2\,3!\times 2!}+\frac{5!}{1!\,(2!)^2\times 2!}\right]3!$$

Try yourself :

In how many ways eight different computers can be distributed in 5 institution so that each institute gets atleast one computer.

Illustration :

Number of ways in whcih 8 persons can be seated in three diff. taxies each reaching 3 seats for passengers and duly numbered is

(a) *If internal arrangement of persons inside the taxi is immaterial.*

(b) *If internal arrangement also matters*

Sol.

(a) $8 \begin{cases} \to 2 \\ \to 3 \\ \to 3 \end{cases}$ $\left[\dfrac{8!}{2!\,3!\,3!} \times \dfrac{1}{2!}\right] 3!$

×	×
×	D

(b) *Using grouping* $\left[\left(\dfrac{8\,!}{2!\ \ 3!\ \ 3!} \times \dfrac{1}{2!}\right) 3!\right] 3!\,3!\,3! = 9!$

or arrange 8 people in 9 seat ${}^{9}C_{8} \times 8! = 9!$

Illustration :

During election's 3 districts are to be convassed by 20, 15, 10 people respectively. If 45 volunter's there then number of ways in which they can be sent.

Sol. *Required ways* $= \dfrac{45!}{20!(5!)(10!)}$

Illustration :

In a jeep there are 3 seat in front and three in the back, number of different ways in which six persons of different height can be seated so that every one in front is shorter than the person directly behind him,

Sol. $6 \begin{cases} \to 2 \\ \to 2 \\ \to 2 \end{cases}$ $\dfrac{6! \times 3!}{(2!)^3\, 3!} = \dfrac{6 \times 5 \times 4 \times 3 \times 2 \times 1}{8 \times 6} \times 6 = 15 \times 6 = \mathbf{90\ Ans.}$

×	×	×
×	×	×

Practice Problem

Q.1 In how many ways can a pack of 52 cards be divided
(i) equally in four sets. (ii) equally among four players

Q.2 In how many ways can a pack of 52 cards be
(i) distributed among four players having 10, 12, 14 and 16 cards.
(ii) divided into four sets of 7, 15, 15 and 15 cards.

Q.3 (i) In how many ways can five people be divided into three groups.
(ii) In how many ways can five people be distributed in thre different rooms if no room must be empty.

Q.4 (i) In how many ways can 12 different balls be divided between 2 boys, any one receiving 5 and the other 7 balls?
(ii) In how many ways can these 12 balls be divided into groups of 5, 4 and 3 balls respectively?

Answer key

Q.1 (i) $\frac{52}{(13!)^4}\times\frac{1}{4!}$ (ii) $\left(\frac{52!}{(13!)^4}\times\frac{1}{4!}\right)\times 4!$ Q.2 (i) $\frac{52!}{10!12!14!16!}\times 4!$ (ii) $\frac{52!}{7!(15!)^3}\times\frac{1}{3!}$

Q.3 (i) 25, (ii) 150 Q.4 (i) $\frac{12!}{5!7!}\times 2!$ (ii) $\frac{12!}{5!4!3!}$

Q.3 (i) 25, (ii) 150 Q.4 (i) $\frac{12!}{5!7!}\times 2!$ (ii) $\frac{12!}{5!4!3!}$

PERMUTATION OF ALIKE OBJECTS :

Case – I: taken all at a time
Case – II: taken some at a time

Case-I: ***Permutation of a like objects taken all at a time***

Number of permuation of n things: p of one kind, q of another kind and r are all different, taken all at a time $=\frac{n!}{p!q!r!}$

Proof:
DADDY
Let three D's are different

$D_1 A D_2 D_3 Y$
$D_1 A D_2 D_3 Y$
$D_2 A D_3 D_1 Y$
$D_2 A D_1 D_3 Y$
$D_3 A D_2 D_1 Y$
$D_3 A D_1 D_2 Y$

One D A D D Y is counted as six different, word's

Let x is required ways

$x \times (6) = 6!$

$x = \frac{6!}{3!}$

Illustration :

Find total number of word's formed by using all letters of the word "IITJEE".

Sol. *ways are* $= \dfrac{6!}{2!(2!)}$

Illustration :

Consider word ASSASSINATION, find number of ways of arranging the letters.

(i) Number of words using all.

(ii) If no two vowels are together.

(iii) If all S are seperated.

(iv) Atleast one S is seperated from rest of the S's

(v) vowels are in the same order.

(vi) Relative position of vowels and consonant remain same.

Sol.

(i) ASSASSINATION contains four S, three A, two N and two I.

$13!$

(i) ASSASSINATION contains four S, three A, two N and two I.

Total ways $= \dfrac{13!}{(4!)(3!)(2!)(2!)}$

(ii) We have six vowels as A, A, A, I, I, O and seven consonants as S, S, S, S, N, T, N

$| S | S | S | S | N | T | N |$

Six vowels in 8 gap's

Total ways $= {}^{8}C_{6} \times \dfrac{6!}{(3!)(2!)} \times \dfrac{7!}{(4!)(2!)}$

(iii) $| A | A | I | N | A | T | I | O | N |$

Out of 10 gaps select 4

Total ways $= {}^{10}C_{4} \times \dfrac{9!}{(3!)(2!)(2!)}$

(iv) Total – all four S together

$$\frac{13!}{(4!)(3!)(2!)(2!)} - \frac{10!}{(3!)(2!)(2!)}$$

$\Rightarrow$ Consider $\boxed{\text{SSSS}}$ as one string.

(v) Total ways $= \underbrace{{}^{13}C_6 \times 1}_{\text{arrangement of vowels}} \times \underbrace{\dfrac{7!}{4!\,2!\,1!}}_{\text{arrangement of consonants}}$

(vi)

v			v		v		v		v	v	

Total ways $= \underbrace{\dfrac{6!}{3!\;2!}}_{\text{arrangement of vowels}} \times \underbrace{\dfrac{7!}{4!\;2!}}_{\text{arrangement of consonants}}$

Illustration :

How many words can be formed with the letters of the word 'PATALIPUTRA' without changing the relative positions of vowels and consonents?

Sol. *The word 'PATALIPUTRA' has eleven letters, in which two P's, three A's, two T's, one L, one U, one R, one I, Vowels are AAIUA*

one R, one I, Vowels are AAIUA

These vowels can be arranged themselves in $\dfrac{5!}{3!} = 20$ *ways.*

The consonents are PTLPTR these consonents can be arranged themselves in $\dfrac{6!}{2!\,2!} = 180$ *ways.*

$\therefore$ *Required number of words* $= 20 \times 180 = 3600$ *ways.*

Illustration :

How many words can be formed using all the letters of the word HONOLULU if no two alike letters are together.

Sol. *Let A represent's ways when OO together, B when LL together, C when UU together.*

Required ways = Total ways – [When all three alike letters together + when 2 alike letters together + when one alike letter together]

$= \text{Total ways} - [n(E_3) + n(E_2) + n(E_1)]$ ...(i)

$$\text{Total ways} = \frac{8!}{(2!)(2!)(2!)} = 5040$$

(a) $n(E_3) = A \cap B \cap C$ (Region 7)

i.e.,

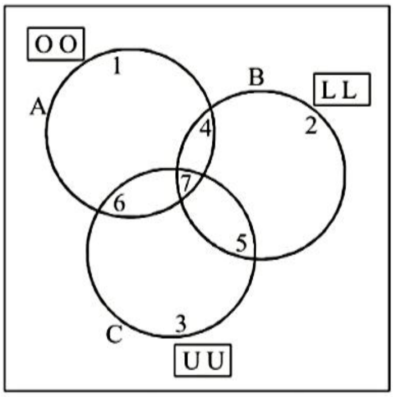
H	N	OO	LL	UU

$n(E_3) = 5! = 120$

(b) $n(E_2) = 3\,[(A \cap B) - (A \cap B \cap C)]$ or [Region 4 + 5 + 6]

$$\underbrace{}_{\to\, n(E_3)}$$

$$= 3\left(\frac{6!}{2!} - 5!\right) = 720$$



i.e., | H | OO | N | LL | UU |

(c) $n(E_1) = 3\,[A - \{(A \cap B) + (A \cap C)\} + (A \cap B \cap C)]$

$$= \left[\frac{7!}{(2!)(2!)} - \left(\frac{6!}{2!} \times 2\right) + 5!\right] = 1980$$

Put in 1^{st}

Required ways = 5040 – [120 + 720 + 1980] = 2220   *Ans.*

Try Yourself :

How many 8 digit numbers can be formed using two 1's, two 2's, two 3's, 4 and one 5. So that no two consecutive digit is identical. [Ans. 2220]

Illustration :

consecutive digit is identical. [Ans. 2220]

Illustration :

Four faces of a tetrahedral dice are marked with 2, 3, 4, 5. The lowest face being considered as the outcome. In how many way a total of 30 can occur in 7 throws.

Sol. *7 throws outcome whose sum is equal to 30 can be obtained in following way.*

Category	Numebr of ways
5,5,5,5,5,2,3	$\frac{7!}{5!} = 42$
5,5,5,5,4,4,2	$\frac{7!}{(4!)(2!)} = 105$
5,5,5,5,4,3,3	$\frac{7!}{(4!)(2!)} = 105$
5,5,5,4,4,4,3	$\frac{7!}{(3!)(3!)} = 140$
5,5,4,4,4,4,4	$\frac{7!}{(2!)(5!)} = 21$

Total ways = 42 + 105 + 105 + 140 + 21 = 413   *Ans.*

Illustration :

Total number of ways of forming 6 letter word from the letters of word "PROPORTION".

Sol. *There are*

$P \to 2$: $R \to 2$, $O \to 3, T \to 1$, $I \to 1$, $N \to 1$

	Category	Process of Selection	Selection	Arrangement
(1)	2 alike, 4 different	(a) 2 alike PP, RR, OOO $\to {}^3C_1$ (b) 4 different from 5 possible option $\to {}^5C_4$	${}^3C_1 \times {}^5C_4 = 15$	$15 \times \frac{6!}{2!}$
(2)	3 alike, 3 different	(a) 3 alike OOO $\to {}^1C_1$ (b) 3 different out of 5 different option $\to {}^5C_3$	${}^1C_1 \times {}^5C_3 = 10$	$10 \times \frac{6!}{3!}$
(3)	2 alike of one kind	(a) 2 alike of one and 2 alike of (b) 3 different out of 5 different option $\to {}^5C_3$	${}^1C_1 \times {}^5C_3 = 10$	$10 \times \frac{6!}{3!}$
(3)	2 alike of one kind 2 alike other kind & 2 different	(a) 2 alike of one and 2 alike of other PP, RR, OOO $\to {}^3C_2$ (b) 2 out of 4 different $\to {}^4C_2$	${}^3C_2 \times {}^4C_2 = 18$	$18 \times \frac{6!}{2!\,2!}$
(4)	3 alike of one kind + 2 a like of another kind + 1 different	(a) OOO, PP, RR $\to {}^1C_1 \times {}^2C_1$ (b) 2 out of 4 different $\to {}^4C_2$	${}^2C_1 \times {}^4C_1 = 8$	$8 \times \frac{6!}{3!\,2!}$
(5)	2 alike, 2 alike + 2 alike	PP, RR, OOO $\to {}^1C_1 \times {}^1C_1 \times {}^1C_1$	${}^1C_1 = 1$	$1 \times \frac{6!}{2!\,2!\,2!}$
(6)	All 6 different	6 possible option P, R, O, T, I, N $\to {}^6C_6$	${}^6C_6 = 1$	$6!$
	Total		Number of selection = 53	Number of words = 11130

Then total number of words formed = 11130

PERMUTATION OF THE OBJECTS TAKEN SOME AT A TIME :

Illustration :

How many 5 lettered words can be formed using the letters of the words "INDEPENDENCE".

Sol. *There are* $E \to 4, N \to 3, D \to 2, P \to 1, I \to 1, C \to 1$

Category	Selection	Arrangement
4 alike & diff.	$1 \times {}^5C_1 = 5$ eg: EEEED	$\frac{5!}{4!} \times 5 = 25$
3 alike & 2 diff.	${}^2C_1 \times {}^5C_2 = 20$ EEEDD	$\frac{5!}{3!} \times 20 = 400$
3 alike & 2 alike of diff. kind	${}^2C_1 \times {}^2C_1 = 4$ EEEDD	$\frac{5!}{3! \times 2!} \times 4 = 40$
2 alike & 3 diff.	${}^3C_1 \times {}^5C_3 = 30$ EENDI	$\frac{5!}{2!} \times 30 = 1800$
2 alike + 2 other alike and 1 diff.	${}^3C_2 \times 4 = 12$ EENND	$\frac{5!}{2! \times 2!} \times 12 = 360$
all five diff.	${}^6C_5 = 6$ EDIPC	$6 \times 5! = 720$

EDIPC

Illustration :

Find the number of words each consisting 5 letters from the letters of the word "M I S S S I S S I P P I"

Sol. $S \to 4, I \to 4, P \to 2, M \to 1$

	Category	***Selection***	***Arrangement***
(1)	*4 alike, 1 different*	${}^2C_1 \times {}^3C_1 = 6$	$6 \times \frac{5!}{4!}$
(2)	*3 alike, 2 alike*	${}^2C_1 \times {}^2C_1 = 4$	$4 \times \frac{6!}{2!3!}$
(3)	*3 alike, 2 different*	${}^2C_1 \times {}^3C_1 = 6$	$6 \times \frac{6!}{3!}$
(4)	*2 alike, 2 alike 1 different*	${}^3C_2 \times {}^2C_1 = 6$	$6 \times \frac{5!}{2!2!}$
(5)	*2 alike, 3 different*	${}^3C_1 \times {}^3C_3 = 3$	$3 \times \frac{5!}{2!}$
	Total	*Number of selection = 25*	*Number of arrangement = 1350*

Then total number of words formed = 1350

Practice Problem

Q.1 Determine the number of permutations of the letters of the word MATHEMATICS.

Q.2 Find the number of numbers greater than 10^6 that can be formed using the digits of the number 2334203 if all the digits of the given number must be used.

Q.3 In how many ways can be letters of the word RESTRICTION be arranged so that the vowels never occur together.

Q.4 How many ways can be formed with the letters of the word RESTRICTIONwithout changing the relative order of the vowels and consonants.

Q.5 In how many ways can the letters AAABBCD be arranged so that
(i) the two B's are together but no two A's are together.
(ii) no two B's and no two A's are together.

Answer key

Q.1 $\frac{11!}{2!\,2!\,2!}$ Q.2 360 Q.3 $\frac{8!}{2!\,2!}\times\frac{4!}{2!}$ Q.4 $\frac{7!}{(2!)^2}\times\frac{4!}{2!}$ Q.5 (i) 24, (ii) 96

CIRCULAR PERMUTATION :

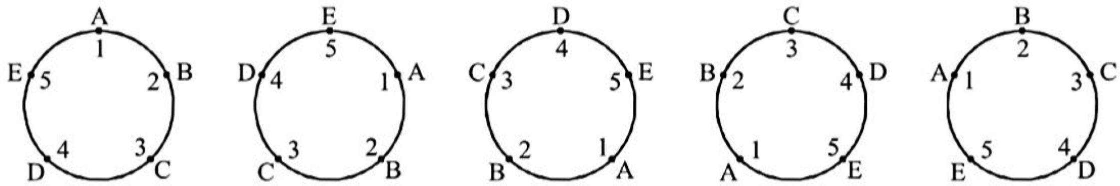
When object are different

Permutation of objects in a row is called as linear permutation. If we arrange the objects along a closed curve it is called as circular permutation.
Thus in, circular permutation, we consider one object fixed and the remaining objects are arranged as in the case of a linear arrangements.

Theorm-I :

The number of circular permutation of **n** distinct objects is **(n – 1) !**

Proof :-Consider 5objects A, B, C, D, E to be arranged around a closed curve is called circular permutation.



All are Same

Let the total number of circular permutation be x. Above circular permutation is equivalent to 5 linear permutations given by ABCDEF, EABCD, DEABC, CDEAB, BCDEA

that is one circular permutation is equivalent to 5x linear permutation given by

$$x \cdot 5 = 5\,!$$

$$x = \frac{5!}{5} = \frac{5.(5-1)!}{5} = (5 - 1)\,!$$

Similarly for n objects nx = n !

$$\boxed{x = \frac{n!}{n} = (n-1)!}$$

(i) n distinct things taken all at a time and arranged along circle in $(n-1)\,!$ ways

(ii) Taken r things out of n distinct things at a time and arranged along circle in ${}^{n}C_{r} \cdot (r-1)\,!$ ways.

Note :-

In the above theorem anti–clockwise and clockwise order of arrangements are considered as distinct permutations.

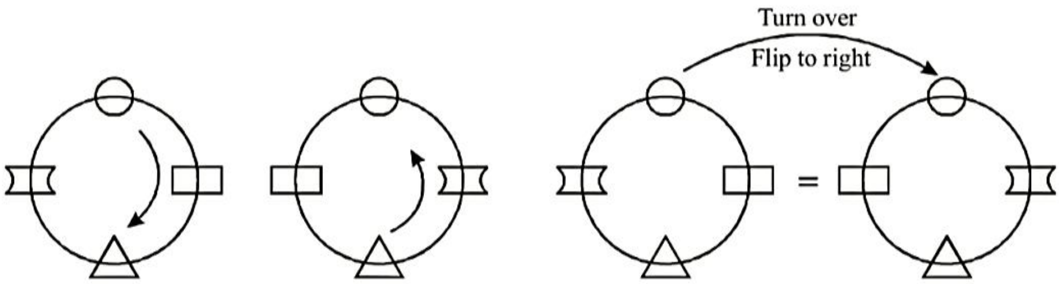
Theorm-II :

If aniclockwise and clockwise are considered to be same total number of circular permutation given by

If aniclockwise and clockwise are considered to be same total number of circular permutation given by

$\frac{(n-1)!}{2}$.

If we arrange flowers or garland beads in a neckless then there is no distinction between clockwise & anticlockwise direction.



Clock wise   Counter clockwise   Both Same

Note:-

(i) If we have n diffrent things taken r at a time in form of a garland or neckless

Required number of arrangements = $\frac{{}^{n}C_{r} \cdot (r-1)!}{2}$.

(ii) The distinction between clockwise and anticlockwise is ignored when a number of people have to be seated around a table so as not to have the same neighours.

Illustration :

Find the number of ways in which 9 people can be seated on a round table so that all shall not have the same neighbours in any 2 arrangements.

Sol. *For same neighbour, clockwise and anticlockwise arrangements are same.*
So total number of ways will be arrangement of 9 people taken clockwise and anticlockwise same and equal to $\frac{(9-1)!}{2} = \frac{8!}{2}$

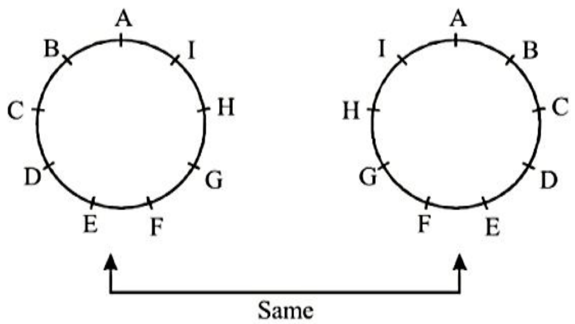


Illustration :

Illustration :

Find the number of ways in which 10 children can sit in a mary go round relative to one another.

Sol. *Here clockwise and anticlockwise arrangements are diffrent.*
Thus required ways = (10 – 1)! = 9!

Illustration :

The 10 students of batch B feel they have some conceptual doubt on "Circular permutation". Mr. Mathew called them in disussion room and asked them to sit down around a circular table which is surrounded by 13 chairs. Mr mathew told that his adjacent seat should not remain empty. The number of ways, in which the students can sit around a round table if Mr. Mathew also sit around a chair.

Sol.

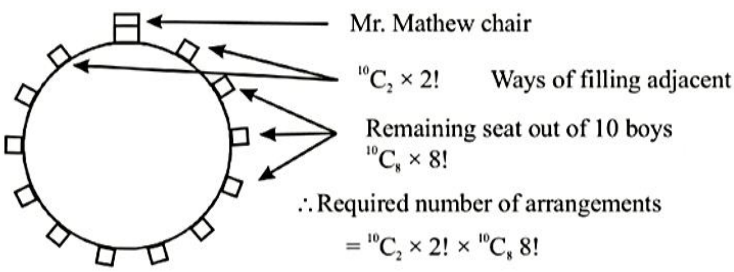
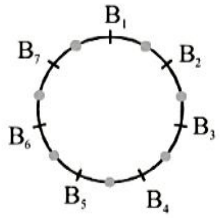


Illustration :

Find number of ways in which 7 American and 7 British people can be seated on a round table so that no two Americans are consecutive.

Sol. *Circular arrangement of 7 British = (7 – 1) !*
There are 7 gap among 7 British.



Out of 7 gap's 7 American can be filled by (7!) ways.
Total ways = (7!) (7 – 1)!

Illustration :

There are n intermediate stations on a railway line from one terminus to another. In how many ways can the train stop at 3 of these intermediate stations if

(a) all the three stations are consecutive
(b) at least two of the stations are consecutive
(c) no two of these stations are consecutive.

Sol.

(a) The number of triples of consecutive stations, viz.

$S_1S_2S_3,\ S_2S_3S_4,\ S_3S_4S_5,\ \ldots\ldots,\ S_{n-2}S_{n-1}S_n$

is (n – 2).

(b) The total number of consecutive pair of stations, viz.

$S_1S_2,\ S_2S_3,\ \ldots\ldots\ S_{n-1}S_n$

is (n – 1).

Each of the above pair can be associated with a third station in (n – 2) ways. Thus, choosing a pair of stations and any third station can be done in (n – 1) (n – 2) ways. The above count also includes the case of three consecutive stations. However, we can see that each such case has counted twice. For example, the pair S_4S_5 combined with S_6 and the pair S_5S_6 combined with S_4 are identical.

Hence, subtracting the excess counting, the number of ways which three stations can be chosen so that at least two of them are consecutive

$$= (n-1)(n-2) - (n-2) = (n-2)^2.$$

(c) Without restriction, the train can stop at any three stations in ${}^nC_3$ ways.
Hence, the number of ways the train can stop so that no two stations are consecutive

$$= {}^nC_3 - (n-2)^2 = \frac{n(n-1)(n-2)}{1\cdot2\cdot3} - (n-2)^2$$

$$= (n-2)\left(\frac{n^2-n-6n+12}{6}\right) = \frac{(n-2)(n-3)(n-4)}{6} = {}^{n-2}C_3$$

Illustration :

n different things are arranged in a circle. In how many ways can three objects be selected if

(a) *all the three objects are consecutive*

(b) *at least two of the objects are consecutive*

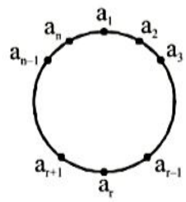
(c) *no two objects are consecutive*

Sol.

(a) *The number of triaples of consecutive objects, viz.*

$a_1a_2a_3,\ a_2a_3a_4,\ \ldots\ldots\ a_na_1a_2$

is n



(b) *The total number of consecutive pair of objects, viz.*

$a_1a_2,\ a_2a_3,\ \ldots\ldots\ a_na_1$

is n

Each of the above pair can be associated with a third objects as

$a_1a_2a_3,\ a_1a_2a_4,\ a_1a_2a_5,\ \ldots\ldots\ a_1a_2a_{n-1}$

in (n – 3) ways.

Note that $a_1a_2a_n$ has not been counted since it will be counted when we write $a_na_1a_2$.
Thus, choosing a pair of objects and any third object can be done in n(n – 3) ways.
Hence, the number of ways of selecting three objects so that at least two of them are consecutive.

Note that $a_1a_2a_n$ has not been counted since it will be counted when we write $a_na_1a_2$.
Thus, choosing a pair of objects and any third object can be done in n(n – 3) ways.
Hence, the number of ways of selecting three objects so that at least two of them are consecutive.

$= n\,(n - 3)$

(c) *Without restriction, three objects can be selected in ${}^nC_3$ ways*

Hence, the number of ways of selecting three objects so that no two of them are consecutive

$= {}^nC_3 - n\,(n - 3)$

Illustration :

Find number of circular permutation of n persons if two specific people are never together.

Sol. *Required ways = Total – when A & B are always together*

$= (n - 1) - (n - 2)! \times 2 = (n - 2)!\,[n - 1 - 2]$

$= (n - 2)!\,(n - 3).$

Illustration :

In how many ways 7 diffrent flowers can be formed into a garland.

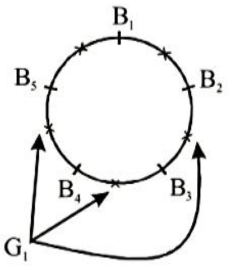
Sol. *Here clockwise and anticlockwise permutations are same*

Hence total ways $= \dfrac{6!}{2}.$

Illustration :

Find number of ways in which 5B and 5G can be seated on a circle alternately if a particular B_1 and G_1 are never adjacent to each other in any arrangement.

Sol.



For B_1, G_1 not to be together. G_1 must select from 3 possible gaps in ${}^3C_1$ ways

$$\underbrace{4!}_{\text{Boy}} \times \underbrace{{}^3C_1}_{G_1} \times 4! = 1728 \quad \textit{Ans.}$$

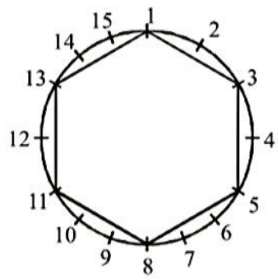
Try Yourself :

Out of 10 flowers of diffrent colours, how many diffrent garlands can be made if each garland condsist of 6 flowers of diffrent colurs. [Ans. ${}^{10}C_6\frac{5!}{2}$]

Illustration :

How many hexagon's can be constructed by joining vertices of a quindecagon (15 side's polygon)

How many hexagon's can be constructed by joining vertices of a quindecagon (15 side's polygon) if none of the sides of Hexagon is also the sides of quindecagon .



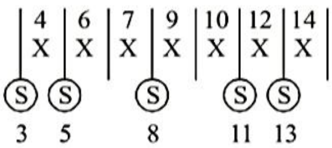
Sol. *Step*

(i) Select the initial vertex say '1' (in ${}^{15}C_1$ ways)

(ii) Now 2 and 15 cannot be selected. From the remaining vertices 3 to 14 (twelve) we have to select 5 more for our hexagon.

(iii) Symbolise the vertices to be taken by Ⓢ Ⓢ Ⓢ Ⓢ Ⓢ and the vertices not be taken (7 in this case) by X X X X X X X.

(iv) Identify the gaps between these crosses (8 in this case) and select



any five out of these gaps in ${}^8C_5$ ways.

(v) Serial number are to be alloted either to Ⓢ or to 'X' whichever comes earlier. In the present problem the vertices corresponding to one selection are 3, 5, 8, 11, 13 and the hexagon as shown

(vi) For each selection we therefore have a haxagon with two non consecutive vertices.

(vii) Number of hexagons = ${}^{15}C_1 \times {}^8C_5$. However this particular hexagon 1, 3, 5, 8, 11, 13 will occur 6 times when we select the initial vertex as 3 or 5 or 8 or 11 or 13. Hence our answer is 6 times more.

(viii) Required number of hexagons $= \dfrac{15 \times {}^8C_5}{6} = \dfrac{15 \times 56}{6}$ $=$ ***140 Ans.***

Alternative Solution :

As in linear let us open this chain to have 1, 2, 3, …… 13, 14, 15 O O O O O O to be selected ;
| X | X | X | X | X | X | X | X | X | not to be selected number of ways = $^{10}C_6$
Required number of ways = $^{10}C_6$ *– number of ways when 1 and 15 are included, since in circular these become consecutive*
Now if 1 and 15 are laready selected , 2 and 14 cannot be taken. Remaining vertices are 3, 4, 5, ……, 11, 12, 13 (11); O O O O (4); | X | X | X | X | X | X | (7)
Cases to be rejected = $^{8}C_4$
Required number of ways = $^{10}C_6 - {}^{8}C_4 \Rightarrow {}^{10}C_4 - {}^{8}C_4 \Rightarrow 210 - 70 = \mathbf{140}$ ***Ans.***

Illustration :

In how many ways triangle can be constructed by joining vertices of a queendecagen if name of the sides of triangle can be sides of quindecagon.

Sol. *By method in above question*

$$= \frac{^{15}C_1 \times {}^{11}C_2}{3} = 275$$

Case-II : Circular permutation which objects are alike.

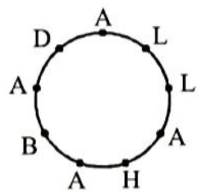
Illustration :

In how many ways letter's word ALLAHABAD can be arranged in a circle.

In how many ways letter's word ALLAHABAD can be arranged in a circle.

Sol. *There are four A's & two L's*

Required ways = $\frac{(9-1)!}{4!\,2!} = \frac{8!}{4!\,2!}$



Practice Problem

Q.1 In how many ways can 5 men and 5 women be seated at a round table if
(i) there is no restriction (ii) all the five women sit together
(iii) no two women sit together (iv) not more than four women sit together

Q.2 In how many ways can 5 men and 3 women be seated at a round table if
(i) no two women sit together
(ii) two particular women must sit together, while the third one must not sit beside those two.

Q.3 Find the number of ways in which n different beads can be arranged to form a necklace.

Answer key

Q.1 (i) 9!, (ii) 5! × 5! (iii) 4! × 5!, (iv) 9! – 5! Q.2 (i) $4! \times {}^{5}P_3$ (ii) 5700

Q.3 $\frac{1}{2}(n-1)!$

TOTAL NUMBER OF COMBINATIONS OR SELECTION OR COLLECTION :

We know that

$$(1 + x)^n = {}^nC_0 + {}^nC_1x + {}^nC_2x^2 + {}^nC_3x^3 + ... + {}^nC_rx^r + ... + {}^nC_n\, y^n.$$

Now replace x by 1

then $\quad 2^n = {}^nC_0 + {}^nC_1 + {}^nC_2 + {}^nC_3 + \ldots\ldots + {}^nC_n$

Case-I : Selection of one or more things out of n things. When all the things are different total number of selections.

One thing can be selected in ${}^nC_1$ ways

Two things can be selected in ${}^nC_2$ ways

Three things can be selected in ${}^nC_3$ ways

⋮

⋮

n things can be selected in ${}^nC_n$ ways

n things can be selected in ${}^nC_n$ ways

Total ways = ${}^nC_1 + {}^nC_2 + {}^nC_3 + \ldots\ldots + {}^nC_n = 2^n - 1$

Illustration :

Sanjeev has 7 friend's. In how many ways can be invite one or more of them to dinner.

Sol. $\quad {}^7C_1 + {}^7C_2 + {}^7C_3 \ldots\ldots {}^7C_7 = 2^7 - 1$

Case-II : The number of selecting zero or more things out of n identical things is n + 1

Proof :

Selecting none thing = 1 way

Selecting 1 thing = 1 ways

Selecting 2 things = 1 way

⋮

⋮

Selecting n things = 1 way

Total number of ways = 1 + 1 + 1 ……(n + 1) times = (n + 1) Ans.

Case-III : Number of ways selecting one or more things out of which p are alike of one kind, q are alike of second kind, r alike of third kind, while s are different is $(p + 1)(q + 1)(r + 1)2^S - 1$

Proof :

Selecting none thing (out of p alike things) = 1 way
Selecting 1 thing (out of p alike things) = 1 ways
Selecting 2 things (out of p alike things) = 1 way
⋮
⋮
Selecting p things (out of p alike things) = 1 way

Total number of ways = 1 + 1 + 1 ……(p + 1) times = (p + 1)   Ans.

Similarly for q alike, total ways = q + 1
Similarly for r alike, total ways = r + 1

For s different things total ways of selection will be 2^S, i.e. any item is selected or not.

So total number of required ways = $(p + 1)(q + 1)(r + 1)2^S - 1$

(1 is subtracted when no item is selected)

Illustration :

Find the number of ways in which one or more letter be selected from the letters "AAAABBCCCDEF"

Sol. Total number of ways = $\underset{A}{(4 + 1)}\,\underset{B}{(2 + 1)}\,\underset{C}{(3 + 1)}\,\underset{DEF}{2^3} - 1 = 479$

Sol. Total number of ways = $\underset{A}{(4 + 1)}\,\underset{B}{(2 + 1)}\,\underset{C}{(3 + 1)}\,\underset{DEF}{2^3} - 1 = 479$

Illustration :

It is given that 4 Apples, 3 Mangoes, 2 Bananas, 2 Oranges, consider the following cases.

Case-I: Fruits of same species are alike and rests are different ,then

(i) Find the number of ways, atleast one fruit is selected.

(ii) Find the number of ways, atleast one fruit of each kind is selected.

Case-II: Fruits of same species are different and rests are also different , then

(i) Find the number of ways, atleast one fruit is selected.

(ii) Find the number of ways, atleast one fruit of each kind is selected.

Sol. *Case-I :*

(i) *Apples can be selected in (4 + 1) ways*
Total number of ways = $(4 + 1)(3 + 1)(2 + 1)(2 + 1) - 1 = 179$

(ii) *Since we need atleast one fruit of each kind, Apples can be selected in 4 ways.*
So total number of ways = $4 \times 3 \times 2 \times 2 = 48$

Case-II :

(i) *Since fruits of same kind are different.*
Apples can be selected in 2^4 *ways*
Total number of ways = $2^4 \times 2^3 \times 2^2 \times 2^2 - 1 = 2047$

(ii) *Since we need atleast one fruit of each kind, Apples can be selected in* $2^4 - 1$ *ways.*
So total number of ways = $(2^4 - 1) \times (2^3 - 1) \times (2^2 - 1) \times (2^2 - 1)$

Total number of combinations in different cases :

(a) The number of combinations of n **different things** taking some or all (or atleast one) at a time $= {}^nC_1 + {}^nC_2 + \ldots.. + {}^nC_n = \mathbf{2^n - 1}$

(b) The number of ways to select some or all out of $(p + q + r)$ things where p are alike of first kind, q are alike of second kind and r are alike of third kind is $= \mathbf{(p+1)(q+1)(r+1) - 1}$

(c) The number of ways to select some or all out of $(p + q + t)$ things where p are alike of first kind, q are alike of second kind and remaining t are different is $= \mathbf{(p+1)(q+1)2^t - 1}$.

PROMBLEMS BASED ON NUMBER THEORY :

Note that every natural number except 1 has atleast 2 divisors. If it has exactly two divisors then it is called a prime. System of prime numbers begins with 2. All primes except 2 are odd. A number having more than 2 divisors is called a composite. 2 is the only even number which is not composite. A pair of natural numbers are said to be relatively prime or coprime if their HCF is one. For two natural numbers to be relatively it is not necessay that one or both should be prime. It is possible that they both are composite but still coprime. eg. 4 and 25. Note that 1 is neither prime nor composite however it is coprime with every other natural number.A pair of primes are said to be twin if their non-negative difference is 2 e.g. 3 & 5 ; 5 & 7 e.t.c.

Number of divisors and their sum :

Number of divisors and their sum :

(a) Every natural number N can always be put in the form $N = p_1^{\alpha_1}, p_2^{\alpha_2} \ldots p_k^{\alpha_k}$ where $p_1, p_2, \ldots p_k$ are distinct primes and $\alpha_1, \alpha_2, \ldots, \alpha_k$ are non-negative integers.

(b) If $N = p_1^{\alpha_1}, p_2^{\alpha_2} \ldots p_k^{\alpha_k}$ then number of divisor of N is equivalent of number of ways of selecting zero or more objects from the groups of identical objects, $(p_1, p_1, \ldots, \alpha_1$ times) $(p_2, p_2, \ldots \alpha_2$ times), $(p_k, p_k, \ldots, \alpha_k$ times) $= (\alpha_1 + 1)(\alpha_2 + 1) \ldots (\alpha_k + 1)$ which includes 1 and N also.

(c) All the divisors excluding 1 and N are called proper divisors.
Also number of divisiors of N can be seen as number of different terms in the expansion of

$$(p_1{}^0 + p_1{}^1 + p_1{}^2 + \ldots + p_1^{\alpha_1}) \times (p_2{}^0 + p_2{}^1 + p_2{}^2 + \ldots + p_2^{\alpha_2}) \times \ldots.. \times (1 + p_k + p_k{}^2 + \ldots + p_k^{\alpha_k})$$

Hence, sum of the divisors of N is

$$(1 + p_1 + p_1{}^2 + \ldots + p_1^{\alpha_1}) \times (1 + p_2 + p_2{}^2 + \ldots + p_2^{\alpha_2}) \ \ldots. \ (1 + p_k + p_k{}^2 + \ldots + p_k^{\alpha_k})$$

$$= \frac{p_1^{\alpha_1+1} - 1}{p_1 - 1} \ \frac{p_2^{\alpha_2+1} - 1}{p_2 - 1} \ \ldots \ \frac{p_k^{\alpha_k+1} - 1}{p_k - 1}$$

(d) Number of ways of putting N as a product of two natural numbers is $\frac{1}{2}(a_1 + 1)(a_2 + 1) \ldots..(a_k + 1)$ if N is not a perfect square.

If N is a perfect square, then this is $\frac{1}{2}[(a_1 + 1)(a_2 + 1) \ldots...(a_k + 1) + 1]$.

(e) If $N = p^a \times q^b \times r^c$ …… p, q are prime number & a, b are natural number.

$$I = \frac{(\text{Total number of divisors})}{2}$$

where I is the number of required ways if N is not a perfect square.

$$I = \frac{(\text{Total number of divisors} + 1)}{2}$$

where I is the number of required ways if N is a perfect square.

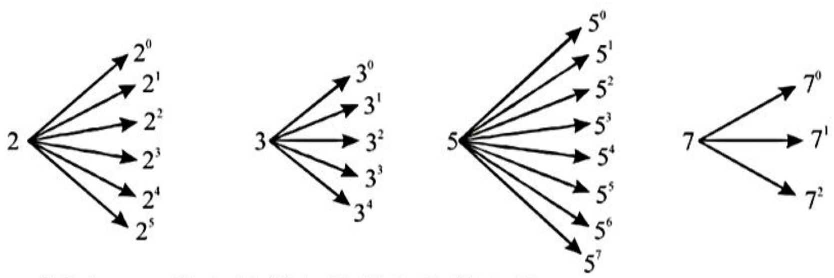
Illustration :

Consider the number $N = 2^5 \times 3^4 \times 5^7 \times 7^2$,
Now answer the following question.

(i) *Total number of divisor,*
(ii) *Number of proper divisor*
(iii) *Number of odd divisor*
(iv) *Number of even divisor.*
(v) *Number of divisors divisible by 5.*
(vi) *Number of divisors divisible by 10.*
(vii) *Number of divisors divisible by 2 but not by 4.*
(viii) *Sum of all divisors*
(ix) *Sum of even divisors*
(vii) *Number of divisors divisible by 2 but not by 4.*
(viii) *Sum of all divisors*
(ix) *Sum of even divisors*
(x) *Sum of odd divisors*
(xi) *Sum of divisor of the form* $(4n + 2),\ n \in N\ (4n + 2) \Rightarrow$ *Even number but not divisible by 4, so exactly one 2.*
(xii) *Number of ways in which N can be resolved as product of two divisor.*

Sol.

(i)



Number of divisor $= (5 + 1)\,(4 + 1)\,(7 + 1)\,(2 + 1)$
$= 6 \times 5 \times 8 \times 3 = 720$

(ii) *Proper divisor will be other than 1 and number itself.*
So proper divisor $= 720 - 2 = 718$.

(iii) *Number of odd divisor can be obtained by choosing* $3^4 \times 5^7 \times 7^2$
Which can be formed by $(4 + 1)\,(7 + 1) \times (2 + 1) = 120$ *ways*

(iv) It can be obtain by selecting atleast one '2' in $2^5 \cdot 3^4 \cdot 5^7 \cdot 7^2$
which can be done in $5 \times (4 + 1) \times (7 + 1) \times (2 + 1) = 600$

(v) Atleast one 5 must be there in $2^5 \times 3^4 \times 5^7 \times 7^2$
$(5 + 1)(4 + 1)(7)(2 + 1) = 630$

(vi) For divisibility by 10 number most be divisible by 2 & 5.
So total divisor must contain atleast one 2 and atleast one 5 in $2^5 \times 3^4 \times 5^7 \times 7^2$
So that ways $= 5 \times (4 + 1) \times (7) \times (2 + 1) = 525$

(vii) Exact one 2 should be there
Total ways $= 1 \times 5 \times 8 \times 3 = 120$

(viii) It can be obtained from
$(2^0 + 2^1 + 2^2 + 2^3 + 2^4 + 2^5)(3^0 + 3^1 + 3^2 + 3^3 + 3^4)$
$(5^0 + 5^1 + 5^2 + 5^3 + 5^4 + \ldots\ldots + 5^7)(7^0 + 7^1 + 7^2)$

$$= (2^6 - 1) \times \frac{(3^5-1)}{2} \times \frac{(5^8-1)}{4} \times \frac{(7^3-1)}{6}$$

(ix) At least one 2 must be there
Sum $= (2^1 + 2^2 + \ldots\ldots + 2^5)(3^0 + 3^1 + \ldots\ldots + 3^4)(5^0 + 5^1 + \ldots\ldots + 5^7)(7^0 + 7^1 + 7^2)$

$$Sum = 2 \cdot (2^5 - 1) \times \frac{(3^5-1)}{2} \times \frac{(5^8-1)}{4} \times \frac{(7^3-1)}{6}$$

(x) No 2 can be selected
Sum $= (3^0 + 3^1 + 3^2 + 3^3 + 3^4)(5^0 + 5^1 + \ldots\ldots + 5^7)(7^0 + 7^1 + 7^2)$

(xi) Sum $= (2)(3^0 + 3^1 + 3^2 + 3^3 + 3^4)(5^0 + 5^1 + \ldots\ldots + 5^7)(7^0 + 7^1 + 7^2)$

$$= 2 \times \frac{(3^5-1)}{2} \times \frac{(5^8-1)}{4} \times \frac{(7^3-1)}{6} = \frac{(3^5-1)(5^8-1)(7^3-1)}{24}$$

(xii) When N is perfect square $N = 2^5 \times 3^4 \times 5^7 \times 7^2$ is not a perfect square

So $I = \frac{\text{(Total number of divisors)}}{2} = \frac{720}{2} = 360$ Ans.

Illustration :

For $N = 75600$, find number of ways by which N can be resolved as product of 2 devisor.

Sol. $N = 2^4 \times 3^3 \times 5^2 \times 7^1$
Total number of divisor $(N_T) = (4 + 1) \times (3 + 1) \times (2 + 1) \times (1 + 1) = 120$

If (N_T) is even then required number of ways $= \frac{N_T}{2} = 60$

Illustration :

Prove that number of ways in which N can be resolved as a product of 2 divisors which are relatively prime = 2^{n-1}. Where n is the number of prime involed in the prime factorisation of

$N = 2^6 \times 3^5 \times 5^4 \times 7^3 \times 11^2 \times 13^1$

Sol. $N = \underset{a}{2^6} \times \underset{b}{3^5} \times \underset{c}{5^4} \times \underset{d}{7^3} \times \underset{e}{11^2} \times \underset{f}{13^1}$ *a, b, c, d, e, f are relatively prime*

$\text{Total ways} = {}^6C_0 + {}^6C_1 + {}^6C_2 + \frac{{}^6C_3}{2}$

${}^6C_0 \longrightarrow 1 \times abcdef$

${}^6C_1 \longrightarrow a \times bcdef,\ b \times acdef,\ c \times abdef,\ d \times abcef,\ e \times abcdf,\ f \times abcde,$

${}^6C_2 \longrightarrow ab\times cdef,\ ae\times bcdf,\ \ldots\ldots$

$\frac{{}^6C_3}{2} \longrightarrow$ *2 for equal size group*

$\text{Total ways} = {}^6C_0 + {}^6C_1 + {}^6C_2 + \frac{{}^6C_3}{2} = 2^{6-1} = 32$

Similarly for n prime factorisation total ways $= 2^{n-1}$

Illustration :

Illustration :

Find the number of permutation of 6 digits from the set {1, 2, 3, 4, 5, 6} where each digit is to be used exactly once, so that the chosen permutation changes from increasing to decreasing or decreasing to increasing at most once ***e.g.*** *the strings like 1 2 3 4 5 6, 6 5 4 3 2 1, 1 2 6 5 4 3 and 6 3 2 1 4 5 are acceptable but strings like 1 3 2 4 5 6 or 6 5 3 2 4 1 are not.*

Sol. *{1, 2, 3, 4, 5, 6}*

Case-1:	up	up	up	up	up	6	${}^5C_0$
Case-2:	up	up	up	up	6	down	${}^5C_1$
Case-3:	up	up	up	6	down	down	${}^5C_2$
Case-4:	up	up	6	down	down	down	${}^5C_3$
Case-5:	up	6	down	down	down	down	${}^5C_4$
Case-6:	6	down	down	down	down	down	${}^5C_5$
Case-7:	down	down	down	down	1	up	${}^5C_1$
Case-8:	down	down	down	1	up	up	${}^5C_2$
Case-9:	down	down	1	up	up	up	${}^5C_3$
Case-10:	down	1	up	up	up	up	${}^5C_4$

Total = 62   ***Ans.***

Aliter: ${}^6C_1 \times 2 + {}^6C_2 \times 2 + {}^6C_3 = 12 + 30 + 20 = 62$ ***Ans.***

Illustration :

Let N be the number of ordered pairs of non empty sets A and B that have the following properties:

(a) $A \cup B = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

(b) $A \cap B = \phi$

(c) *The number of elements of A is not the element of A.*

(d) *The number of elements of B is not an element of B.*

Find N.

Sol. ***Explanation:***

(a) and (b) $\Rightarrow n(A) + n(B) = 10$

(c) and (d) $\Rightarrow$ *if A is the two elements set then B is an eight elements set, therefore 2 is not in A and must be in B and 8 is not in B and must be in A.*

Also note that both set can not have equal number of elements because if A is a 5 element set then B will also be five element set and the elements can not be both in A and B.

No. of elements in A i.e. $n(A)$	$n(B)$	number of ways
1 element set (say) $\{9\}$	$\{1..........\}$ 9 element set	$\underbrace{{}^8C_0}_{\text{group of 0 and 8}}$
2 element set $\{8............\}$	$\{2..........\}$ 8 element set	$\underbrace{{}^8C_1}_{\text{group of 1 and 7}}$
3 element set $\{7............\}$	$\{3..........\}$ 7 element set	$\underbrace{{}^8C_2}_{\text{group of 2 and 6}}$
4 element set $\{6............\}$	$\{4..........\}$ 6 element set	${}^8C_3$
5 element set	Not possible	
6 element set	4 element	${}^8C_5$
7 element set	3 element	${}^8C_6$
8 element set	2 element	${}^8C_7$
9 element set	1 element	${}^8C_8$

$$\text{Total} = ({}^8C_0 + {}^8C_1 + {}^8C_2 + ... + {}^8C_8) - {}^8C_4 = 2^8 - 70 = 256 - 70 = 186 \textbf{ Ans.}$$

Illustration :

Find the number of non-empty subsets S of {1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12} such that, no two consecutive integers belong to S and if, S contains k elements, then S contains no number less than k.

Sol. $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$

single element subset = 12

two element subset (1 cannot be taken – think! why?)

$$\begin{array}{ll} \quad S \quad S & 2 \\ \underbrace{|\times|\times|\times|\times|\times|\times|\times|\times|\times|}_{\text{nine to be taken}} & 9 \end{array}$$

gaps $10 \rightarrow {}^{10}C_2 = 45$

$|||^{ly}$ 3 element subset (1 and 2 cannot be taken)

3, 4, ... 11, 12

$$0\ 0\ 0$$
$$|\times|\times|\times|\times|\times|\times|\times|$$
$${}^8C_3 = 56$$

4 element set (1, 2, 3, rejected)

$$0\ 0\ 0\ 0$$
$$|\times|\times|\times|\times|\times|$$
$${}^6C_4 = 15$$

5 element (1, 2, 3, 4 rejected)

$$0\,0\,0\,0\,0$$
$$|\times|\times|\times|$$
$${}^4C_5 = 0 \text{ Not possible}$$

Total = 12 + 45 + 56 + 15 = 68 + 60 = 128 ***Ans.***

Practice Problem

Q.1 Consider the number $N = 75600\ (2^4.3^3.5^2.7^1)$ Find

(i) Number of divisors (ii) Number of proper divisors

(iii) Number of odd divisors (iv) Number of even divisors

(v) Number of divisors divisible by 5 (vi) Number of divisors divisible by 10

(vii) sum of all the divisors

(iii) Number of odd divisors (iv) Number of even divisors

(v) Number of divisors divisible by 5 (vi) Number of divisors divisible by 10

(vii) sum of all the divisors

Answer key

Q.1 (i) 120, (ii) 118, (iii) 24, (iv) 96, (v) 80, (vi) 64,

(vii) $(2^0 + 2^1 + 2^2 + 2^3 + 2^4)\ (3^0 + 3^1 + 3^2 + 3^3)(5^0 + 5^1 + 5^2)(7^0 + 7^1)$

SUMMATION OF NUMBERS (3 DIFFERENT WAYS) :

(a) Sum of all the numbers greater then 10000 formed by the digits 1,3,5,7,9 if no digit being repeated.

Method - 1 : All possible numbers = 5! = 120

If one occupies the units place then total numbers = 24.

Hence 1 enjoys units place 24 times

				1

$|||^{ly}$ 1 enjoys each place 24 times

Sum due to $1 = 1 \times 24\ (1 + 10 + 10^2 + 10^3 + 10^4)$

$|||^{ly}$ Sum due to the

digit $3 = 3 \times 24\ (1 + 10 + 10^2 + 10^3 + 10^4)$

$\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots$

Required total sum $= 24\ (1 + 10 + 10^2 + 10^3 + 10^4)\ (1 + 3 + 5 + 7 + 9)$

Method –2 : In 1^{st} column there are twenty four 1's , Twenty four 3's & so on and their sum is
$= 24 \times 25 = 600$
Hence add. in vertical column normally we get = 6666600

$$
\begin{array}{r|ccccc}
 & 5^{th} & & & 2^{nd} & 1^{st} \\
 & X & X & X & X & X \\
120 & X & X & X & X & X \\
\text{Number} & \vdots & \vdots & \vdots & \vdots & \vdots \\
 & X & X & X & X & X \\
\hline
 & 666 \quad 6 & 6 & 0 & 0 & = 6666600
\end{array}
$$

Method–3 : Applicable only if the digits used are such that they have the same common difference. (valid even if the digits are repeating)
Writing all the numbers in ascending order of magnitude
S = (13579 + 13597 + ... + 97513 + 97531)
S = (13579 + 99531) + (13597 + 97513) + ...
= (111110) 60 time = 6666600 Ans

$S = \frac{n}{2}(l + L)$ where n = number of numbers, l = smallest, L = Largest

Illustration :

Illustration :

Find sum of all the numbers greater than 10000 formed by the digit 0, 1, 2, 4, 5, no digit being repeated.

Sol. *Using all the given digits we can form a five digit number except when zero is at first place.*
So to find the sum of all the possible five digit number
= (Sum of all possible arrangement) – (Sum of all the arrangements when zero is at first place)

$\Rightarrow$ *5 different digits can be arranged in 5! ways so each digit will appear at every place* $= \frac{5!}{5}$ *times*
i.e. 24 times
Sum of all digits at unit place = 24 (0 + 1 + 2 + 4 + 5)
Sum of all digits at ten's place = 24 (0 + 1 + 2 + 4 + 5)
……………………………………………………………
Sum of all digits at 10000^{th} *place = 24 (0 + 1 + 2 + 4 + 5)*
In this way sum of all possible arrangement $= 24\,(0 + 1 + 2 + 4 + 5)\,[1 + 10 + 10^2 + 10^3 + 10^4]$

when zero is at first place 4 digit number willbe formed.
Each number will appear 6 times at every place.
Sum of all 4 digit number at unit place = 6 (1 + 2 + 4 + 5)
Sum of all 4 digit number at ten's place = 6 (1 + 2 + 4 + 5)
……………………………………………………………
Hence sum of all four digit numbers $= 6\,(1 + 2 + 4 + 5)\,(1 + 10 + 10^2 + 10^3)$
Required sum $= 24\,[0 + 1 + 2 + 3 + 4 + 5]\,[1 + 10 + 10^2 + 10^3 + 10^4]$
$- 6\,(1 + 2 + 4 + 5)\,(1 + 10 + 10^2 + 10^3)$

Illustration :

Find the sum of all the four digit numbers that can be formed with the digits 3, 2, 3, 4.

Sol. *The number of numbers having 2 in units place*

$= \frac{3!}{2!} = 3$ [$\because$ the other three places are to be filled by 3, 3 and 4]

Similarly the number of numbers having 4 in units places

$= \frac{3!}{2!} = 3$ [$\because$ the other three places are to be filled by 3, 3 and 2]

and the number of numbers having 3 in units places

$= 3! = 6$ [$\because$ the other three places are to be filled by 2, 3 and 4]

Thus, sum of the digits occurring in the units place

$= 2 \times 3 + 3 \times 6 + 4 \times 3 = 36$

We can see that the given digits (3, 2, 3, 4) occur at the tens, hundreds and thousands place, the same number of times as they occur at the units place.

Hence, the required sum of the numbers formed

$= 36\,(1 + 10 + 100 + 1000) = 39996$

Illustration :

Find the sum of the five digit numbers that can be formed using the digits 3, 4, 5, 6, 7 not using any digit more than once in any number.

Sol. *If 3 is placed at units place, the remaining 4 places can be filled in 4! = 24 ways*

Sol. *If 3 is placed at units place, the remaining 4 places can be filled in 4! = 24 ways*

Thus, 3 occurs at unit place 24 times.

The other digits similarly, each occurs at the unit places 24 times.

Similarly, each of the digit occurs at the other places tens, hundreds and so on, 24 times.

Hence, the required sum, is

$= 24\,(3 + 4 + 5 + 6 + 7)\,(10^0 + 10^1 + 10^2 + 10^3 + 10^4)$

$= 24 \times 25 \times 11111 = 6666600$

Illustration :

Find the number of permutations of the digits 1, 2, 3, 4 and 5 taken all at a time so that the sum of the digits at the first two places is smaller than the sum of the digits at the last two places.

Sol. *Total = 120;* $a_1\ a_2\ a_3\ a_4\ a_5$

Number of ways in which $a_1 + a_2 > a_4 + a_5$ = *number of ways in which* $a_1 + a_2 < a_4 + a_5$

$\therefore$ *required number* $= \frac{(120) - (a_1 + a_2 = a_4 + a_5)^*}{2}$

(* *denotes the number of ways when* $a_1 + a_2 = a_4 + a_5$)

Now $a_1 + a_2 = a_4 + a_5$ (1, 2, 3, 4, 5)

If $a_3 = 1$, 4 3 1 2 5 8 *ways*

$a_3 = 3$, 1 5 3 2 4 8 *ways*

$a_3 = 5$, 4 1 5 2 3 8 *ways*

If $a_3 = 2 / 4 / 6$ *not possible (think !)*

$\therefore$ *required number* $= \frac{(120) - 24}{2} = 48$ ***Ans.***

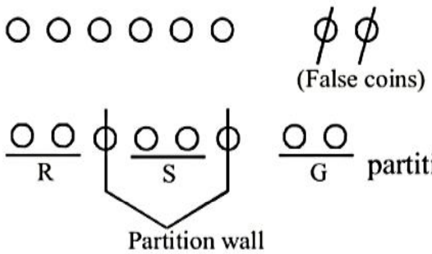
DESTRIBUTION OF ALIKE OBJECTS :

TYPE-1:

Total number of ways in which n identical coins can be distributed among p persons so that each person may get any number of coin is $^{n+p-1}C_{p-1} = \dfrac{(n+p-1)!}{(p-1)!\,(n)!}$

Proof:-

Let 6 identical coins can be distributed among 3 persons R|S|G



Out of eight places if we select any two 3 partitions are formed by $^8C_2$ ways

Illustration :

Find number of ways in which 30 mangos can be distributed among 5 persons.

Sol. $^{30+5-1}C_{5-1} = {}^{34}C_4$

Find number of ways in which 30 mangos can be distributed among 5 persons.

Sol. $^{30+5-1}C_{5-1} = {}^{34}C_4$

TYPE-2:

Total number of ways in which n identical items can be distributed among p persons such that each of them receive at least one item $^{n-1}C_{p-1}$.

Illustration :

Find total number of ways of distributing 7 identical computers to R|S|G. So that each receive atleast one computer

Sol. $^{7-1}C_{3-1} = {}^6C_2$.

Illustration :

Find number of natural solutions of equation $x + y + z = 102$, where $x, y, z \in N$.

Sol. $x + y + z = 102$ consider x, y, z as 3 beggers and 102 as coins, give 1 coin to each equation becomes $X + Y + Z = 99$. $X = x + 1, Y = y + 1, Z = z + 1$.
Now distributed 99 coins without constrants total ways $^{99+2}C_2 = {}^{101}C_2$.

Imp. Point :

Number of different terms in a complete homogeneous expression of degree m in n variables is equivalent to distribution of m identical coins among n beggers.
If expression is $(x_1 + x_2 + x_3 \ldots\ldots + x_n)^m$ number of terms $^{m+n-1}C_{n-1}$.

Illustration :

Find number of different terms in expansion of $(x+y+z)^{10}$.

Sol. $^{10+3-1}C_{3-1} = {}^{12}C_2$.

Illustration :

A man has to buy 25 mangoes in four different variety buying at least 4 of each variety. In how many ways can he plan his purchases; if mangoes of each variety are identical and available in abundance.

Sol. V_1, V_2, V_3, V_4 are considered as beggers give 4 to each

$$\circ \circ \circ \circ \circ \circ \circ \circ \circ \qquad \phi \ \phi \ \phi$$

Now 9 mangoes to 4 verieties without any ristriction given as $^{9+4-1}C_{4-1} = {}^{12}C_3$.

Illustration :

Number of non-negative integral solution of the inequality $x+y+z+t \le 30$.

Sol. $\underset{\text{(False begger)}}{(x+y+z+t) + \underset{\downarrow}{w} = 30}$

Required condition is equivalent to giving 30 coin's to 5 begger's $= {}^{30+5-1}C_{5-1} = {}^{34}C_4$.

Illustration :

Required condition is equivalent to giving 30 coin's to 5 begger's = $C_{5-1} = C_4$.

Illustration :

Number of ways in which 16 identical toys are to be distributed among 3 children such that each child does not receive less than 3 toys will be

(A) 96 (B) 16 (C) 36 (D) 46

Sol. Let x_1, x_2, x_3 be the number of toys received by the three children

Then, $x_1, x_2, x_3, \ge 3$ and $x_1, x_2, x_3 = 16$

Let $u_1 = x_1 - 3$, $u_2 = x_2 - 3$ and $u_3 = x_3 - 3$

Then, $u_1, u_2, u_3 \ge 0$ and $u_1 + u_2 + u_3 = 7$

Here, $n = 7$ and $r = 3$

$\therefore$ Number of ways $= {}^{n+r-1}C_{r-1} = {}^9C_2 = 36$

Illustration :

Number of non-negative integral solutions of $x_1 + x_2 + x_3 + 4x_4 = 20$ will be

(A) 496 (B) 516 (C) 536 (D) 546

Sol. Here, clearly $0 \le x_4 \le 5$, $x_1, x_2, x_3, \ge 0$ and $x_1 + x_2 + x_3 = 20 - 4x_4$

$\Rightarrow$ $r = 3$ and $n = 20 - 4x_1$

If $x_4 = 0$ number of ways $^{20+3-1}C_{3-1} = {}^{22}C_2$

If $x_4 = 1$ number of ways $^{16+3-1}C_{3-1} = {}^{18}C_2$ Similarly, if $x_4 = 2, 3, 4, 5$, number of ways $= {}^{14}C_2, {}^{10}C_2, {}^6C_2, {}^2C_2$ respectively

$\therefore$ Total number of ways

$= {}^{22}C_2 + {}^{18}C_2 + {}^{14}C_2 + {}^{10}C_2 + {}^6C_2 + {}^2C_2 = 536$

Illustration :

Number of ways 5 identical balls can be distributed into 3 different boxes so that no box remains empty will be

(A) 1 (B) 3 (C) 6 (D) 15

Sol. The required number of ways = ${}^{5-1}C_{3-1}$

$$= {}^{4}C_{2} = \frac{4.3}{1.2} = 6$$

Try yourself :

A shelf contain 6 separate compartment's. Find number of ways in which 12 identical marbles can be placed in the compartment so that no compartment is empty. [Ans. ${}^{11}C_5$]

MAXIMISE ${}^{n}C_{r}$:

$${}^{n}C_{r} \text{ is maximum at } \begin{cases} r = \dfrac{n}{2} & \text{if n is even} \\ r = \dfrac{n-1}{2} \text{ or } \dfrac{n+1}{2} & \text{if n is odd} \end{cases}$$

e.g. ${}^{15}C_r$ is maximum when r = 7 or 8, ${}^{12}C_r$ is maximum when r = 6

GRID PROBLEM :

Complete cartesian plane is partitioned by drawing line || to x and y-axis equidistant apart like the lines

GRID PROBLEM :

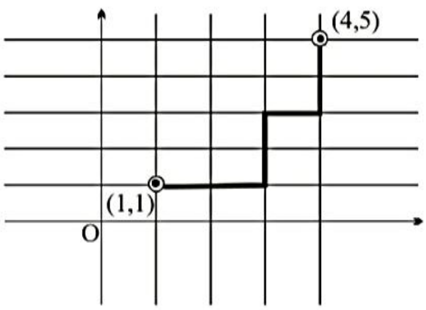
Complete cartesian plane is partitioned by drawing line || to x and y-axis equidistant apart like the lines
on a chess board. Then the

Illustration :

Number of ways in which an ant can reach from (1, 1) to (4, 5) via shortest path.

Sol. Whatever may be the mode of travel of the ant; it has to traverse 3H (Horizontal) and 4V (Vertical) paths.

Hence required number of ways = $\frac{7!}{4!\,3!} = {}^{7}C_3$



Note : If there are n vertical and m horizontal lines then there will be (n – 1) horizontal and (m – 1) vertical paths

DERANGEMENT :

If n things are arranged in a row, the number of a ways they can be deranged so that r things occupy wrong places while (n – r) things occupy their original places, is

$$= {}^{n}C_{n-r}\, D_r$$

where $D_r = r!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \ldots\ldots + (-1)^r \frac{1}{r!}\right)$

If n things are arranged in a row, the number of ways they can be deranged so that none of them occupies its original place, is

$$= {}^{n}C_0\, D_n = n!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \ldots\ldots + (-1)^n \frac{1}{n!}\right)$$

Aliter :

D_n = ways in which n things are arranged so that all n things occupy wrong places
= arranged without restriction
– ways in which 1 thing is in correct position while (n – 1) things are deranged
– ways in which 2 things are in correct position while (n – 2) things are deranged
....
....
– ways in which all n things are in correct position and there is no derangement

$$= n! - {}^{n}C_1\, D_{n-1} - {}^{n}C_2 D_{n-2} - \ldots\ldots\ldots\ldots\ldots - {}^{n}C_n D_0$$

$$= n! - \sum^{n} {}^{n}C_r D_{n-r}$$

$$= n! - \sum_{r=1}^{n} {}^{n}C_r D_{n-r}$$

Thus, we have

$$D_0 = 0! = 1$$

$$D_1 = 1! - {}^{1}C_1 D_0 = 0$$

$$D_2 = 2! - {}^{2}C_1 D_1 - {}^{2}C_2 D_0 = 1$$

$$D_3 = 3! - {}^{3}C_1 D_2 - {}^{3}C_2 D_1 - {}^{3}C_3 D_0 = 2$$

$$D_4 = 4! - {}^{4}C_1 D_3 - {}^{4}C_2 D_2 - {}^{4}C_3 D_1 - {}^{4}C_4 D_0 = 9$$

$$D_5 = 5! - {}^{5}C_1 D_4 - {}^{5}C_2 D_3 - {}^{5}C_3 D_2 - {}^{5}C_4 D_1 - {}^{5}C_5 D_0 = 44$$

and so on.

Illustration :

A person writes letters to five friends and addresses the corresponding envelopes. In how many ways can the letters be placed in the envelops so that

(a) all letters are in the wrong envelopes.

(b) at least three of them are in the wrong envelopes.

Sol.

(a) Required number of ways

$$= 5!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!}\right) = \frac{5!}{2!} - \frac{5!}{3!} + \frac{5!}{4!} - \frac{5!}{5!}$$

(b) *Required number of ways*

$$= n!\sum_{n=1}^{n} {}^{n}C_{r}D_{n-r} \qquad \textit{where } n = 5$$

$$= {}^{5}C_{2}D_{3} + {}^{5}C_{1}D_{4} + {}^{5}C_{0}D_{5}$$

$$= 10 \times 3!\left(1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}\right) + 5 \times 4!\left(1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}\right) + 1 \times 5!\left(1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}-\frac{1}{5!}\right)$$

$$= 10\,(3-1) + 5\,(12-4+1) + (60-20+5-1) = 20 + 45 + 44 = 109$$

Practice Problem

Q.1 The number of ways to put 4 letters in 4 addressed envelops so that all in wrong envelopes.

Q.2 The number of ways to put 5 letters in 5 addressed envelops so that all are in wrong envelopes.

Answer key

Q.1 9 Q.2 44

Q.1 9 Q.2 44

SOME IMPORTANT RESULTS ABOUT POINTS :

If there are n points in a plane of which m (< n) are collinear, then

(a) Total number of different straight lines obtained by joining these n points is

$$\mathbf{{}^{n}C_{2} - {}^{m}C_{2} + 1}$$

(b) Total number of different triangles formed by joining these n points is

$$\mathbf{{}^{n}C_{3} - {}^{m}C_{3}}$$

(c) Number of diagonals in polygon of n sides is

$$\mathbf{{}^{n}C_{2} - n} \qquad \text{i.e. } \frac{n(n-3)}{2}$$

(d) If m parallel lines in a plane are intersected by a family of other n parallel lines. Then total number of parallelograms so formed is

$$\mathbf{{}^{m}C_{2} \times {}^{n}C_{2}} \quad \text{i.e. } \frac{mn(m-1)\,(n-1)}{4}$$

(e) Number of triangles formed by joining vertices of convex polygon of n sides is $\mathbf{{}^{n}C_{3}}$ of which

(i) Number of triangles having exactly two sides common to the polygon = n

(ii) Number of triangles having exactly one side common to the polygon = n(n – 4)

(iii) Number of triangles having no side common to the polygon = $\frac{n(n-4)(n-5)}{6}$

Solved Examples

Single correct question

Q.1 In a bag there is a minimum of six old indian coins of every denominations (i.e. Athanni, Chavanni, Duanni, Ekanni). Number of ways in which one can take 6 coins from the bag is

(A) 120 (B) 90 (C) 84 (D) 60

Sol. Let Athanni $\to$ A Duanni $\to$ C

Chavanni $\to$ B Ekanni $\to$ D

$A + B + C + D = 6$ where $A \geq 0,\ B \geq 0, C \geq 0, D \geq 0$

Total number of arrangement $= {}^{n+r-1}C_{r-1} = {}^{9}C_{3} = 84$

Q.2 3 Indian and 3 American men and their wives are to be seated round to circular table. Let m denotes the number of ways when the indian couples are together and n denotes the number of ways when all the six couples are together. If $m = kn$ then k equals.

(A) 36 (B) 42 (C) 45 (D) 48

Sol. Indian couples can be seated together and rest 6 person are siting in a round table in $(6 + 3 - 1)! \times 2^3$

$= 8! \times 2^3$

where 2^3 are arrangement among themselves

$m = 8! \times 2^3$

Similarly 6 couples in a round table can be seated in

$n = (6 - 1)! \times 2^6$ ways $= 5! \times 2^6$

$$m = kn \quad \Rightarrow \quad \frac{m}{n} = k \quad \Rightarrow \quad \frac{8! \times 2^3}{5! \times 2^6} = k \quad \Rightarrow \quad k = 42$$

Similarly 6 couples in a round table can be seated in

$n = (6 - 1)! \times 2^6$ ways $= 5! \times 2^6$

$$m = kn \quad \Rightarrow \quad \frac{m}{n} = k \quad \Rightarrow \quad \frac{8! \times 2^3}{5! \times 2^6} = k \quad \Rightarrow \quad k = 42$$

Q.3 Golden temple express going from Amritsar to mumbai stops at 5 intermediate stations. 10 passengers enter the train during the journey with 10 different tickets of 'k' classes. If number of different sets of tickets they have is ${}^{45}C_{35}$ then k equals.

(A) 1 (B) 2 (C) 3 (D) 4

Sol. Let $I_1\ I_2\ I_3\ I_4\ I_5$ are the intermediate stations.

A $I_1\ I_2\ I_3\ I_4\ I_5$ M

Total number of tickets of one class are $= {}^{6}C_{2}$ (Selecting two station as origin & destination station)

Total tickets for k classes $= {}^{6}C_{2} \times k$

$= 15k$

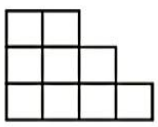
10 different tickets for 10 person can be choosen in ${}^{15k}C_{10}$

${}^{15k}C_{10} = {}^{45}C_{35} = {}^{45}C_{10}$

$15k = 45 \quad \Rightarrow \quad k = 3$

Q.4 The number of ways in which 5 X's can be placed in the squares of the figure so that no horizontal row remains empty is

(A) 97 (B) 98 (C) 100 (D) 126



Sol. Total number of x's are 5 so two horizontal rows can not be empty at a time that one horizontal row could be empty.

Total number of required ways $= {}^{9}C_{5}$ (total possible)

$-$ [top row empty + middle row empty + bottom row empty]

$= {}^{9}C_{5} - [{}^{7}C_{5} + {}^{6}C_{5} + {}^{5}C_{5}] = 98$ Ans.

Q.5 Six boys and six girls sit along a line alternativelyin x ways and along a circle (again alternatively) in y ways then :

(A) x = y (B) y = 12 x (C) x = 10 y (D) x = 12 y

Sol. Lencar arrangement of 6 boy = 6! ways

$| B_1 | B_2 | B_3 | B_4 | B_5 | B_6$

Arrangement girls can be done in $6! \times 2$ ways

$x = 6! \times 6! \times 2$

For circular arrangement boys can be arranged in 5!

6 places can be filled by 6 girls in 6! ways

total ways $y = 5! \times 6!$

$x = 12\,y$ Ans.

Q.6 A forecast is to be made of the result of five cricket matches, each of which can be a win or a draw or a loss for indian team.

Let p = number of forecast with exactly 1 error

q = number of forecast with exactly 3 errors and

r = number of forecast with all five errors, then incorrect statement is :

(A) 8p = 5r (B) 2q = 5r (C) 8p = q (D) 2 (p + q) > q

Sol. Selection of one wrong forecast = ${}^5C_1$

Wrong forecast of 1 match can be done in 2 ways

$p = {}^5C_1 \times 2 = 10$

$q = {}^5C_3 \times 2 \times 2 \times 2 = 80$

$r = {}^5C \times 2 \times 2 \times 2 \times 2 \times 2 = 32$

Hence (A) Ans.

Hence (A) Ans.

Q.7 The number of ten digit numbers that contain only 2 and 3 as its digit, but no any pairwise 3's joins together, is

(A) 145 (B) 143 (C) 129 (D) None

Sol.

One	3'	and	Nine	2's	$\rightarrow$	$\|2\|2\|2\|2\|2\|2\|2\|2\|2\| \longrightarrow {}^{10}C_1$
Two	3's	and	8	2's	$\rightarrow$	${}^9C_2$
Three	3's	and	7	2's	$\rightarrow$	${}^8C_3$
Four	3's	and	6	2's	$\rightarrow$	${}^7C_4$
Five	3's	and	5	2's	$\rightarrow$	${}^6C_5$
Six	3's	and	6	2's	$\rightarrow$	Not possible as repetition of 3's will come

Total number of ways = ${}^{10}C_1 + {}^9C_2 + {}^8C_3 + {}^7C_4 + {}^6C_5$

= 143 Ans.

Q.8 If the sum of all even positive dirisors of 100000 can be expressed in the form $k\,(5^2 + 5 + 1)\,(5^3 + 1)$ then the value of k is

(A) 31 (B) 62 (C) 64 (D) 93

Sol. $100000 = 2^5 \times 5^5$

Sum of all even divisor = $(2 + 2^2 + 2^3 + 2^4 + 2^5)\,(5^0 + 5^1 + \ldots\ldots + 5^5)$

$$= 2\cdot\frac{2^5-1}{2-1}\cdot\frac{5^6-1}{5-1} = 2\cdot\ \frac{31}{4}(5^3-1)\,(5^3+1)$$

$$= 62\,(5^2+5+1)\,(5^3+1) \quad \Rightarrow \quad k = 62$$

Q.9 Number of ways in which 15 indistinguishable oranges can be distributed in 3 different boxes so that every box R as atmost 8 oranges, are

(A) 52 (B) 108 (C) 76 (D) 28

Sol. Requried ways = (Total possible ways without restriction) – (ways when any box can $\geq$ 9 oranges)

Total possible ways are

$$x + y + z = 15 \quad \Rightarrow \quad {}^{15+3-1}C_{3+1} = {}^{17}C_2$$

Ways when any box can have 9 orarges

$$x + y + z = 15$$

either one of x, y, z can have more then 9 orages.

$$x + y + z = 15 - 9 = 6 \quad \text{with} \quad x \geq 0, \quad y \geq 0, \quad z \geq 0$$

Number of ways are ${}^3C_1 \times {}^{6+3-1}C_{3-1} = {}^3C_1 \times {}^8C_2$

Required ways are $= {}^{17}C_2 - {}^3C_1 \times {}^8C_2$
$= 52$ Ans.

Paragraph type

Paragraph for question nos. 10 to 12

Consider a polygon of sides 'n' which satisfies the equation $3 \cdot {}^np_4 = {}^{n-1}p_5$.

Q.10 Rajdhani express travelling from Delhi to Mumbai has n station enroute. Number of ways in which a train can be stopped at 3 stations if no two of the stopping station are consecutive, is

(A) 20 (B) 35 (C) 56 (D) 84

Q.11 Number of quadrilaterals that can be made using the vertices of the polygon of sides 'n' if exactly two adjacent sides of the quadrilateral are common to the sides of n - gon is

(A) 50 (B) 60 (C) 70 (D) None

Q.11 Number of quadrilaterals that can be made using the vertices of the polygon of sides 'n' if exactly two adjacent sides of the quadrilateral are common to the sides of n - gon is

(A) 50 (B) 60 (C) 70 (D) None

Q.12 Number of quadrilterals that can be formed using the vertices of a polygon of sides 'n' if exactly 1 side of the quadrilateral is common with the side of the n- gon is

(A) 150 (B) 100 (C) 96 (D) None

Sol. $3 \cdot \dfrac{n!}{(n-4)!} = \dfrac{(n-1)!}{(n-6)!} \qquad 3n = (n-4)(n-5)$

$\Rightarrow \quad 3n = n^2 - 9n + 20 \quad \Rightarrow \quad n^2 - 12n + 20 = 0$

$n = 10, \quad n = 2$ not possible

so $n = 10$

(i) Delhi $I_1 \; I_2 \; I_3 \; I_4 \ldots\ldots I_{10}$ Mumbai

3 intermediate station such that no two are consecutive

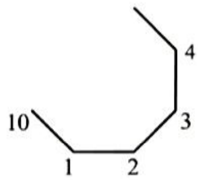
Train is stopping at 3 station, for 7 remaining station there are 8 gaps.

Filling these 3 station out of 8 gaps can be done in ${}^8C_3 = 56$ ways

(ii) $n = 10$

Contecutive two sides can be formed by (1, 2, 3) point so 4, 10 can't be Seleted but one point out of remain 5 points can be selected in ${}^5C_1$ ways

So total quadrilaterals $= 10 \times {}^5C_1 = 50$



(iii) One side common means any two consecutive vertices can be selected in 10 ways. Now (1, 2) is selected 3, 10 can't be selected rest two non consecutive vertices can be selected from ${}^5C_2$ ways

Total number of quadrilateral $= 10 \cdot {}^5C_2 = 100$ ways.

Paragraph for question nos. 13 to 15

Consider the word "w" = C O M M I S I O N E R containing 12 letter of which five vowels and 7 consonants

Q.13 Number of 5 lettered word each comprising of 2 vowels and 3 consonants is

(A) 5120 (B) 6720 (C) 4960 (D) None

Q.14 Number of ways in which the letters of word 'w' can be arranged if alike letters are together but separated from the other alike letters is

(A) 2880 (B) 1120 (C) $\frac{12!-8!}{16}$ (D) None

Q.15 Number of ways is which the letters of the word 'w' can be arranged without changing the order of alike letters is

(A) $\frac{12!}{(2!)^4}$ (B) ${}^{12}C_8$ (C) ${}^{12}P_8$ (D) ${}^{12}P_4$

Sol. Vowels are O O I I E

Consonants are M M S S R C N

(i)	2 vowel &		3 consonant
(a)	vowels are alike	(a')	2 alike , 1 different
	${}^2C_1 = 2$		${}^2C_1 \times {}^4C_1 = 8$
(b)	vowels are different	(b')	All 3 different
	${}^3C_2 = 3$		${}^5C_3 = 10$

(i)	2 vowel &		3 consonant
(a)	vowels are alike	(a')	2 alike , 1 different
	${}^2C_1 = 2$		${}^2C_1 \times {}^4C_1 = 8$
(b)	vowels are different	(b')	All 3 different
	${}^3C_2 = 3$		${}^5C_3 = 10$

Total arrangement

$$aa' \times \frac{5!}{2! \times 2!} \times ab' \times \frac{5!}{2!} + ba' + \frac{5!}{2!} + bb' \times 5!$$

$$2 \times 8 \times \frac{5!}{4} + 2 \times 10 \times \frac{5!}{2!} + 3 \times 8 \times \frac{5!}{2!} + 3 \times 10 \times 5!$$

$$= 480 + 1200 + 1440 + 3600 = 6720 \text{ ways}$$

(ii) Alike word are N M, O O, I I, S S are together both seperated from other alike i.e. M M is seperated from OO.

so C N E R are arranged in 4! ways

5 placed are filled & arranged by ${}^5C_4 \times 4!$

total ways are ${}^5C_4 \times 4! \times 4! = 2880$

(iii) Total possible ways = 12! (considering all defferent)

ways of arranging 8 distinct letters are 8!

total required ways = $\frac{12!}{8!}$

Match the Column

Q.16 **Column-I** **Column-II**

(A) Number of all six digit natural numbers such that sum of their digits is 10 and each of the digit 0, 1, 2, 3 occurs atleast once in them is — (P) 350

(B) A question paper consists of 2 parts A and B. Part A has 4 questions with 1 alternative each and Part-B has 3 question without any alternative. Number of ways in which one can select the question when atleast one question must be attempted from each part, is — (Q) 405, (R) 490

(C) Number of ordered pairs of positive integers (a, b) such that the least common multiple of 'a' and 'b' is $2^2 \cdot 5^4 \cdot 11^4$ — (S) 560

Sol.(A) 6 digit number containg 0, 1, 2, 3 atleast once are

S.No.	Fixed digit	Required digit	Total ways
(i)	0, 1, 2, 3	(4, 0)	$4 \times \frac{5!}{2!} = 240$
(ii)	0, 1, 2, 3	(3, 1)	$\frac{6!}{2! \times 2!} - \frac{5!}{2! \times 2!} = 150$
(iii)	0, 1, 2, 3	(2, 2)	$\frac{6!}{3!} - \frac{5!}{3!} = 100$

(i)	0, 1, 2, 3	(4, 0)	$4 \times \frac{5!}{2!} = 240$
(ii)	0, 1, 2, 3	(3, 1)	$\frac{6!}{2! \times 2!} - \frac{5!}{2! \times 2!} = 150$
(iii)	0, 1, 2, 3	(2, 2)	$\frac{6!}{3!} - \frac{5!}{3!} = 100$

Total required ways = 240 + 150 + 100 = 490

(B) 3 option for question of paper A (i.e. 1^{st} alternative 2^{nd} alternative no solution)

2 option for question of paper B

Total ways = $(3^4 - 1)(2^3 - 1)$

$= (80)(7) = 560$

(C) $a = 2^r \cdot 5^s \cdot 11^t$ $\quad b = 2^{r'} \cdot 5^{s'} \cdot 11^{t'}$

If $r = 0, 1, 2$ then $r' = 2$

If $r = 2$ then $r' = 0, 1, 2$

Total ways = $3 \times 2 - 1$ (for the repetition of (2, 2))

If $s = 0, 1, 2, 3, 4$ then $s^1 = 4$

If $s = 4$ then $s' = 0, 1, 2, 3, 4$

$= 4 \times 2 - 1$ (for repetition (4, 4))

= 9 ways

Similar for t & t' number of ways = 9 ways

total ways $5 \times 9 \times 9 = 405$ Ans.

Q.17 **Column-I** **Column-II**

	Column-I		Column-II
(A)	If the number of ways in which n different toys can be distributed in n children if exactly one child doesn't get any toy is 1200 then n equals	(P)	3
(B)	There are 2n white and 2n red counters. Counters are all alike exept for the colour. If the number of ways in which they can be arranged in a line so that they are symmetric w.r.t. a central mark is 70 then n equals to	(Q)	4
		(R)	5
(C)	Total number of divisors of the number N = 360 which are of the form 4n + 2, n ≥ 0 is	(S)	6

Sol.(A) Child can be rejected in $^nC_1$ ways

Now dividing n toys in (n – 1) boys.

$$= \frac{n!(n-1)!}{2!(n-2)!}$$

$$\text{Total ways} = {}^nC_1 \times \frac{n! \times (n-1)!}{2! \times (n-2)!} = n! \times {}^nC_2 = 1200$$

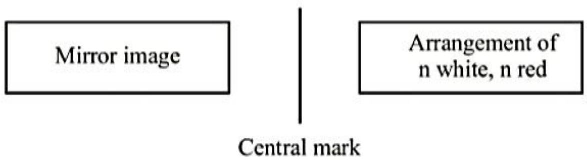
$n = 5$

(B) 2n red counters, 2n white counters

$$\text{Total ways} = {}^nC_1 \times \frac{n! \times (n-1)!}{2! \times (n-2)!} = n! \times {}^nC_2 = 1200$$

$n = 5$

(B) 2n red counters, 2n white counters



n white and n red can be arranged in $\frac{2n!}{n!\,n!}$ ways (Rest n white and n red are the mirror image)

$$\frac{2n!}{n!\,n!} = 70 \quad \Rightarrow \quad n = 4$$

(C) $360 = 2^3 \cdot 3^2 \cdot 5$

$4n + 2 \Rightarrow$ even number which is not divisible by 4 hence exacthy one '2' must be taken.

number of divisor $(1) \times (3) \times (2) = 6$

Q.18

	Column-I	Column-II	
(A)	Number of five digit numbers of the form $d_1\, d_2\, d_3\, d_4\, d_5$ where d_{is} $i = 1, 2, 3, 4, 5$ are digit and satisfying $d_1 < d_2 \le d_3 < d_4 \le d_5$, is	(P)	${}^{10}C_5$
(B)	Number of five digit numbers of the form $d_1\, d_2\, d_3\, d_4\, d_5$ where $d_{is} = 1, 2, 3, 4, 5$ are digits satisfying $d_1 > d_2 \ge d_3 > d_4 > d_5$, is	(Q)	${}^{11}C_4$
(C)	Boby fischer and Boris Spassky play a unique game series in a chess tournament. They decide to play on till one of them won 5 matches. If each match end only in win or loss. Number of ways in which series can be won by either of them, is	(R)	${}^{11}C_6$
(D)	A badminton team has to be selected comprising of 5 students out of 10 students for inter shcool tournament. Number of ways this can be done if a perticular player is to be always included or always excluded from the team, is	(S)	$2 \cdot {}^{9}C_5$

Sol.(A) $d_1 < d_2 < d_3 < d_4 < d_5 \Rightarrow$ Number of numbers ${}^{9}C_5$ (0 not included)

If $d_1 < d_2 = d_3 < d_4 < d_5 \Rightarrow$ Number of numbers ${}^{9}C_4$ (0 not included and two same digit)

If $d_1 < d_2 < d_3 < d_4 = d_5 \Rightarrow$ Number of numbers ${}^{9}C_4$ (0 not included and two same digit)

If $d_1 < d_2 = d_3 < d_4 = d_5 \Rightarrow$ Number of number ${}^{9}C_3$ (0 not included & two pair of same digit)

Total $= {}^{9}C_5 + {}^{9}C_4 + {}^{9}C_4 + {}^{9}C_3 = {}^{10}C_5 + {}^{10}C_4 = {}^{11}C_6$

(B) $d_1 > d_2 \ge d_3 > d_4 > d_5$ ${}^{10}C_5$ (If all different)

${}^{10}C_4$ (If $d_2 = d_3$)

digit)

Total $= {}^{9}C_5 + {}^{9}C_4 + {}^{9}C_4 + {}^{9}C_3 = {}^{10}C_5 + {}^{10}C_4 = {}^{11}C_6$

(B) $d_1 > d_2 \ge d_3 > d_4 > d_5$ ${}^{10}C_5$ (If all different)

${}^{10}C_4$ (If $d_2 = d_3$)

$= {}^{10}C_5 + {}^{10}C_4 = {}^{11}C_5 = {}^{11}C_6$ Ans.

(C) Total number of ways in which the tournament can be won by either player $= {}^{10}C_5 = 2 \times {}^{9}C_5$

(D) Always excluded $= {}^{9}C_5$

Always included $= {}^{9}C_4$

Total ways $= {}^{9}C_5 + {}^{9}C_4 = {}^{10}C_5 = 2 \cdot {}^{9}C_5$

Reasoning type question

Q.19 Consider the 10 digits 0, 1, 2, 3, ... 9

Statement-1: Number of four digit even numbers that can be formed if each digit is to used only once in the number is 2268.

because

Statement-2: Total 4 digit numbers that can be formed if eah digit is used only once is 4536.

(A) Statement-1 is true, statement-2 is true and statement-2 is correct explanation for statement-1.

(B) Statement-1 is true, statement-2 is true and statement-2 is NOT the correct explanation for statement-1.

(C) Statement-1 is true, statement-2 is false.

(D) Statement-1 is false, statement-2 is true.

Sol. Four digit even number

Case-I : If zero as least digit

$9 \times 8 \times 7\ \boxed{0} = 9 \times 8 \times 7$

Case-II : Zero is not last digit

$8 \times 8 \times 7\ \boxed{\uparrow} = 8 \times 8 \times 7$

4 ways (2, 4, 6, 8)

Four digit even numbers = $17 \times 56 = 952$
Total four digit numbers

$$\underset{9}{\square} \times \underset{9}{\square} \times \underset{8}{\square} \times \underset{7}{\square} = 4536$$

Q.20 **Statement-1:** Number of different terms in the expansive of $(a + b + c + d)^{12}$ is $^{15}C_3$
because
Statement-2: Number of ways in which n distinguishable objects can be distributed in p persons if each reciving none or one or more is $^{n+p-1}C_n$.
(A) Statement-1 is true, statement-2 is true and statement-2 is correct explanation for statement-1.
(B) Statement-1 is true, statement-2 is true and statement-2 is NOT the correct explanation for statement-1.
(C) Statement-1 is true, statement-2 is false.
(D) Statement-1 is false, statement-2 is true.

Sol. Statement-1 :
Power of a $\rightarrow x_1$
Power of b $\rightarrow x_2$ $\quad x_1 + x_2 + x_3 + x_4 = 12$
Power of c $\rightarrow x_3$ $\quad$ total ways = $^{15}C_3$
Power of d $\rightarrow x_4$
Statement-2 :
n different object can be distributed to person in p^n ways.

Q.21 With usual notation
Statement-1: $c(n, r) \cdot p(r, r) = p(n, r)$ where $r, n \in N$
because
Statement-2: Every permutation of n distinct objects taken r at a time can be uniquely determine by

n different object can be distributed to person in p^n ways.

Q.21 With usual notation
Statement-1: $c(n, r) \cdot p(r, r) = p(n, r)$ where $r, n \in N$
because
Statement-2: Every permutation of n distinct objects taken r at a time can be uniquely determine by
first choosing r object out of these n object and then arranging thdese r objects.
(A) Statement-1 is true, statement-2 is true and statement-2 is correct explanation for statement-1.
(B) Statement-1 is true, statement-2 is true and statement-2 is NOT the correct explanation for statement-1.
(C) Statement-1 is true, statement-2 is false.
(D) Statement-1 is false, statement-2 is true.

Sol. $c(n, r) = {}^nc_r$

$$p(r, r) = {}^rp_r = \frac{r!}{(r-r)!} = r!$$

$$c(n, r) \cdot p(r, r) = p(n, r)$$

Q.22 **Statement-1:** Number of rectangles on chessboard (which may be overlapped also) is $^8C_2 \times {}^8C_2$
because
Statement-2: To form a rectangles we have to select any two of the horizontal lines and any two from the vertical lines.
(A) Statement-1 is true, statement-2 is true and statement-2 is correct explanation for statement-1.
(B) Statement-1 is true, statement-2 is true and statement-2 is NOT the correct explanation for statement-1.
(C) Statement-1 is true, statement-2 is false.
(D) Statement-1 is false, statement-2 is true.

Sol. Rectangle can be formed by selecting any two horizontal line and any two vertical lines.
Chess board has 9 horizontal and 9 vertical lines so total rectangles from a chessboard
$= {}^9C_2 \times {}^9C_2$

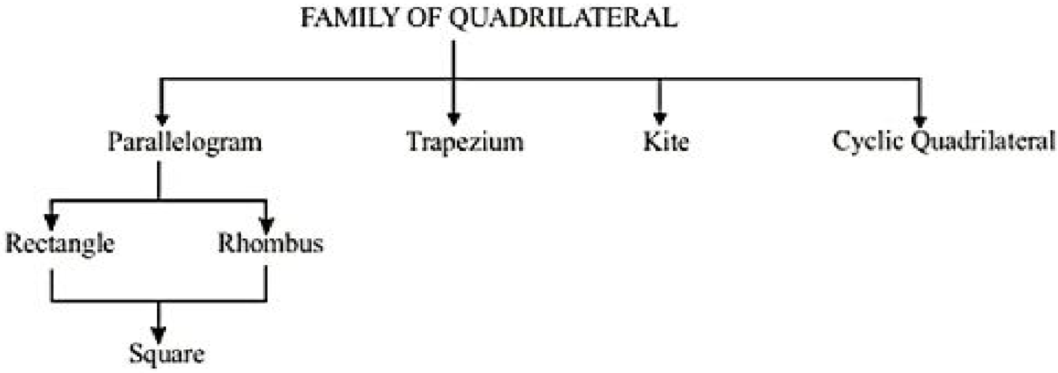
STRAIGHT LINE

1. INTRODUCTION :

We are familiar with 'Geometry' which is simply the study of the properties of figures and curves. The geometry usually studied upto the high school is known as 'Euclidean 'Geometry' as it is based upon the axioms laid by famous Greek mathematician Euclid in his first systematic treatise on geometry about 300 B.C. During this period and upto the seventeenth century geometric reasoning alone was employed in the study of geometry. This study of geometry is named as 'synthetic geometry'. There were problems whose solution were not available in the synthetic geometry. It was until about 17th century, A.D. that the geometry was linked with algebra, which is employed in solution of problems in synthetic geometry. By this mean the methods of algebra are applied in the study of geometry which is referred now as 'analytic geometry'. A systematic study of geometry by the use of algebra was first carried out by celebrated French philosopher and mathematician Rene' Descartes' (1596 – 1650), in his book 'La Geometrie', which was published in 1637. The book 'La Geometric is mainly concerned with the algebraic solutions of geometric problems and geometric interpretation of algebraic equations.

In order to relate algebra with geometry Descartes established a relationship between the basic geometric concept of 'point' with ordered pairs of real numbers. This association is named after Rene' Descartes' as Cartesian coordinate system which will be studied in this chapter.

2. TYPES OF QUADRILATERAL (EUCLIDIAN FIGURES) :



2.1 PARALLELOGRAM :

Definition :-

If opposite sides of quadrilateral are parallel and equal, then quadrilateral is called parallelorgam.

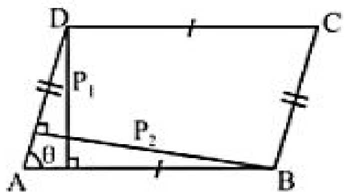
Four ways to prove that a quadrilateral is parallogram.

(i) Opposite sides are parallel i.e. AB || DC and AD || BC

(ii) Opposite sides are equal i.e. AB = DC and AD = BC

(iii) One pair of opposite sides are equal and parallel.

(iv) Diagonals bisect each other



Note: Area of parallelogram $= \dfrac{P_1P_2}{\sin\theta}$

Proof : Area = Base × Height (where P_1 and P_2 are distances between pair of parallel sides and θ is angle between two adjacent sides)

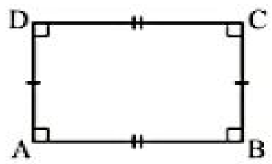
$$= AB \times P_1$$

$$= \frac{P_1P_2}{\sin\theta} \qquad \left\{\because \sin\theta = \frac{P_2}{AB}\right\}$$

(i) Rectangle :

Definition :

If all angles of parallelogram are equal then it is called rectangle.



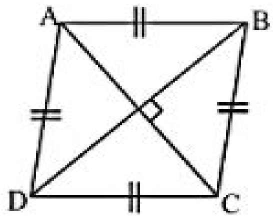
(i) Diagonal are equal and bisect each other.

(ii) Each diagonal divides the rectangle into two triangles of equal area.

(ii) Rhombus :

Definition :

If all sides of a parallelogram are equal then it is called Rhombus



(i) Diagonals are perpendicular

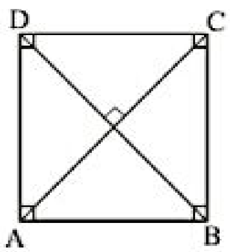
(ii) Area $= \dfrac{1}{2}\, d_1d_2$ where d_1 and d_2 are diagonal.

Note: If distance between pair of parallel sides are equal then it is a rhombus.

(iii) Square :

Definition :

If all the sides and all the interior angles of a parallelogram are equal then it is called a square.



(i) All sides are equal

$AB = BC = CD = DA$

(ii) Diagonals are equal and bisect each other at 90°.

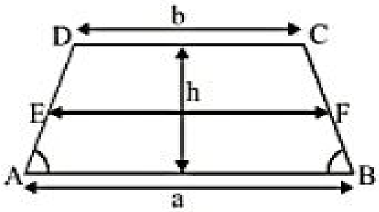
(iii) Area $= \dfrac{d^2}{2}$ (d = diagonal)

Note: Every square is a rectangle but not the converse.

2.2 TRAPEZIUM :

Definition :-

Trapezium is a quadrilateral which has exactly one pair of opposite sides parallel.



$\angle DAB$ & $\angle CBA$ = Base angle

(i) Area of trapezium = $\frac{1}{2}$ (sum of parallel sides) $\times$ distance between sides

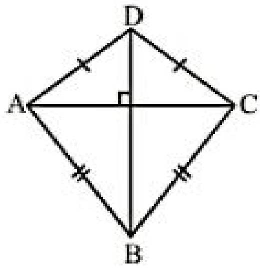
(iii) Median (EF) = $\frac{1}{2}(a + b)$

(iv) For equilateral /isosceles trapezium, non parallel sides are equal i.e. AD = BC

2.3 KITE :

Definition :

It is a quadrilateral in which two pairs of adjacent sides are equal.



(i) AD = DC and AB = BC

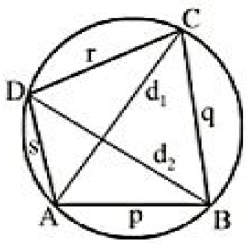
(ii) Diagonals are perpendicular but not bisect

(iii) Only one diagonal divide the figure into two congruent triangles.

2.4 CYCLIC QUADRILATERAL :

Definition :

If all vertices of quadrilateral lies on the circumference of a circle, then it is called cyclic quadrilateral.

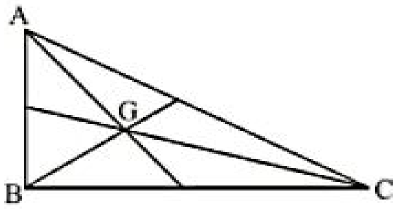


(i) Opposite angles are supplementary.

(ii) Sum of product of opposite sides are equal to product of diagonals

$pr + qs = d_1d_2$

3. FIVE IMPORTANT POINTS WITH RESPECT TO A TRIANGLE :



3.1 CENTROID (G) :

Definition :

Centroid is a point of concurrency of medians.

$\Rightarrow$ Centroid always lies inside the triangle.

$\Rightarrow$ Centroid divides the median in the ratio 2 : 1, reckoning from the vertex.

$\Rightarrow$ The area of ΔGBC, ΔGCA, ΔGAB are equal (G is centroid)

$\Rightarrow$ Each median divides the triangle into the two triangles of equal areas.

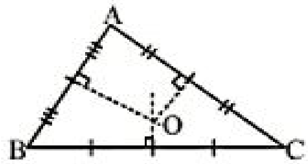
3.2 CIRCUMCENTRE (O) :

Definition :

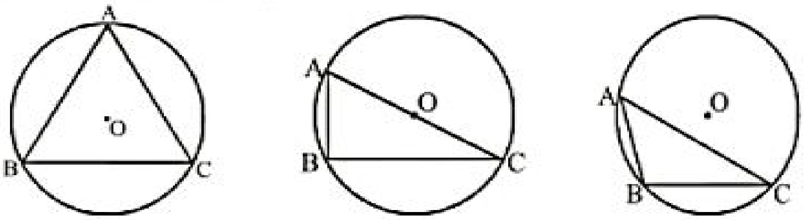
Circumcentre is a point of concurrency of perpendicular bisector of sides of triangle.

or

Circumcentre is a centre of circle circumscribing the triangle.



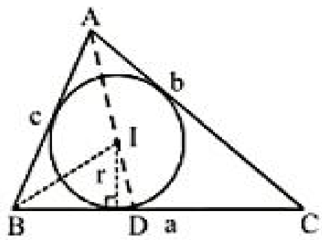
⇒ In case of acute angle triangle circumcentre lies inside the triangle
For right angle triangle it lies on mid-point of hypotenuse and in case of obtuse angle triangle it lies outside the triangle.



3.3 INCENTRE (I) :

Definition :

Incentre is a point of concurrency of internal angle bisector of triangle.



⇒ Incentre always lies inside the triangle.

⇒ Internal angle bisector AD divides the base BC in the ratio of the sides containing the angle i.e. BD : DC = c : b

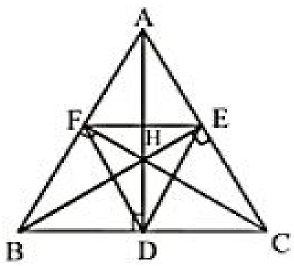
⇒ Incentre I divides AD in the ratio AB : BD

⇒ AI : ID = (b + c) : a

3.4 ORTHOCENTRE (H) :

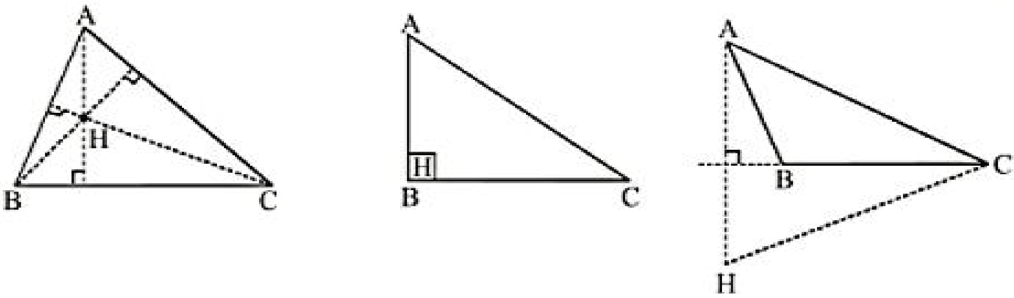
Definition :

Orthocentre is a point of concurrency of altitudes of triangle.



H- Orthocentre

⇒ In case of acute angle triangle orthocentre lies inside the triangle. For right angle triangle orthocentre lies at the vertex where it is right angled and in case of obtuse triangle orthocentre lies outside the triangle.



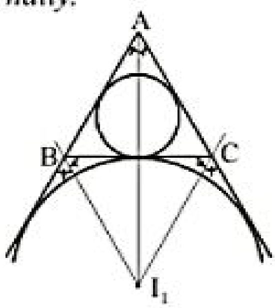
3.5 EXCENTRES (I_1, I_2, I_3) :

Definition :

Excentre is a point of concurrency of two external angle bisectors and one interior angle bisector.

or

Centre of a circle (excircle) which touches all the sides of the triangle externally.



⇒ There are three excentres with respect to a given triangle.

I_1 : Centre of the excircle opposite to vertex A (as shown in figure)

I_2 : Centre of the excircle opposite to vertex B

I_3 : Centre of the excircle opposite to vertex C

Imp. Points :

☞ For isosceles triangle centroid, circumcentre, orthocentre and incentre are collinear.

☞ For a triangle Orthocentre (O), Centroid (G), Circumcentre (C) are collinear and centroid divides orthocentre and circumcentre in the ratio 2 : 1 internally.

☞ For equilateral Δ, centroid, circumcentre, orthocentre and incentre coincide.

Illustration :

Find the distance between the orthocentre and circumcentre of a triangle whose vertices are

$P(3, 0), Q(0, 0)$ *and* $R\left(\frac{3}{2}, \frac{-3\sqrt{3}}{2}\right)$

Sol. ∴ $\text{side } PQ = \sqrt{(3-0)^2+(0-0)^2} = 3$

$$QR = \sqrt{\left(\frac{3}{2}-0\right)^2+\left(\frac{-3\sqrt{3}}{2}-0\right)^2} = 3$$

$$PR = \sqrt{\left(3-\frac{3}{2}\right)^2+\left(0+\frac{3\sqrt{3}}{2}\right)^2} = 3$$

Hence $PQ = QR = PR$

Hence, the triangle is equilateral.

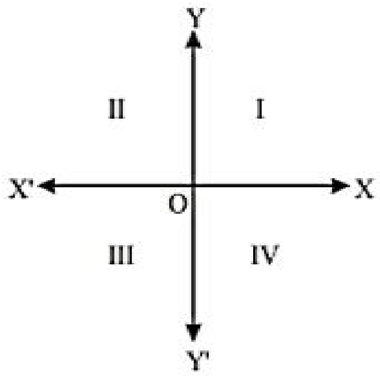
Now, since in an equilateral triangle orthocentre and circumcentre coincides therefore distance between them is zero.

4. CARTESIAN COORDINATE SYSTEM :

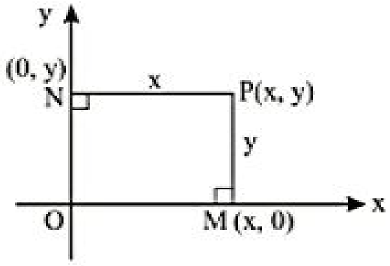
4.1 QUADRANT :

Two mutually perpendicular lines meeting at 'O' (origin) are called axes. Horizontal line X' OX is known as x-axis and vertical line Y'OY is called y-axis. These two perpendicular lines divide the plane into four quadrants, viz., as follows their names are given in anti-clockwise sense.

XOY = First quadrant, X'OY = Second quadrant
X'OY' = Third quadrant, Y'OX = fourth quadrant



4.2 COORDINATES OF A POINT :



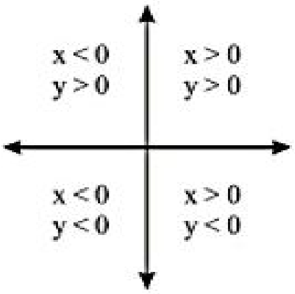
Co-ordinates of a point are given by ordered pair (x, y) whose first entry (x) denotes the x-coordinate or abscissa of the point and second entry (y) denotes the y-coordinate or ordinate of the point.
For x-coordinate (Abscissa) of the point, $|x|$ is the perpendicular distance of the point from y-axis.
For y-coordinate (ordinate) of the point, $|y|$ is the perpendicular distance of the point from x-axis.

Note :

(i) y-coordinate of any point lying on the x-axis is always zero.

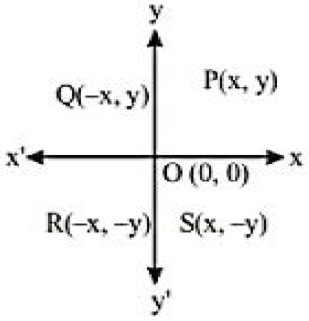
(ii) x-coordinate of any point lying on the y-axis is always zero.

In 1st quadrant $x > 0, y > 0$
IInd quadrant $x < 0, y > 0$
IIIrd quadrant $x < 0, y < 0$ and
IVth quadrant $x > 0, y < 0$



Quadrant wise sign of abscissa (x) and ordinate (y)

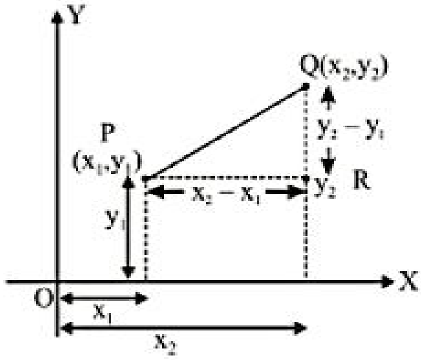
Let x and y are positve real numbers.



5. DISTANCE BETWEEN TWO POINTS :

Let $P(x_1, y_1)$ and $Q(x_2\ y_2)$ be two given points in the xy plane then distance between them is given by

$$|PQ| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$



Proof : For triangle PQR

$$PQ^2 = PR^2 + RQ^2$$

$$= (x_2 - x_1)^2 + (y_2 - y_1)^2$$

$$|PQ| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Note : distance of (x_1, y_1) from origin $= \sqrt{x_1^2 + y_1^2}$

Illustration :

Match the Column

	Column-I		Column-II
(A)	*The triangle with vertices A (7, 10), B (4, 5) and C (10, 15) is*	(P)	*Equilateral*
(B)	*The triangle with vertices P (2, 7), Q(4, –1) and R (–2, 6) is*	(Q)	*Isosceles*
(C)	*The triangle with verties L(3, 1), M (5, 6) and N(9, 16) is*	(R)	*Right angled*
(D)	*The triangle with vertices R (a, a), S(–a, –a) and T $\left(\sqrt{3}\,a, -\sqrt{3}\,a\right)$ is*	(S)	*Collinear*

Sol. (A) *In a ΔABC*

$\therefore$

$$AB = \sqrt{(4-7)^2 + (5-10)^2} = \sqrt{34}$$

$$BC = \sqrt{(10-4)^2 + (15-5)^2} = \sqrt{136}$$

$$AC = \sqrt{(10-7)^2 + (15-10)^2} = \sqrt{34}$$

Hence $AB = AC$

Hence ΔABC is isosceles triangle.

(B) $\therefore$ $PQ = \sqrt{(2-4)^2+(7+1)^2} = \sqrt{68}$

$QR = \sqrt{(4+2)^2+(-1-6)^2} = \sqrt{85}$

$PR = \sqrt{(-2-2)^2+(6-7)^2} = \sqrt{17}$

Hence $PQ^2 + PR^2 = QR^2$

Hence ΔABC is right angled.

(C) $\therefore$ $LM = \sqrt{(5-3)^2+(6-1)^2} = \sqrt{29}$

$MN = \sqrt{(9-5)^2+(16-6)^2} = \sqrt{116} = 2\sqrt{29}$

$LN = \sqrt{(9-3)^2+(16-1)^2} = \sqrt{261} = 3\sqrt{29}$

Hence $LM + MN = LN$

Hence points L, M, N are collinear.

(D) $\therefore$ side $RS = \sqrt{(a+a)^2+(a+a)^2} = 2\sqrt{2}\,a$

$ST = \sqrt{(\sqrt{3}a+a)^2+(-\sqrt{3}a+a)^2} = 2\sqrt{2}\,a$

$RT = \sqrt{(\sqrt{3}a-a)^2+(-\sqrt{3}a-a)^2} = 2\sqrt{2}\,a$

Hence $RS = ST = RT$

Hence ΔRST is equilateral.

Illustration :

If A(0, –1), B(6, 7), C(–2, 3) and D(λ, 3) forms a rectangle then find the value of λ.

Sol. $AB = CD$ and $AC = BD$.

$AB = \sqrt{6^2+8^2} = 10$

$CD = \sqrt{(\lambda+2)^2} = |\lambda+2|$

$\lambda + 2 = 10$ or $\lambda + 2 = -10$

$\lambda = 8$ or $\lambda = -12$ ... (1)

Now $AC = BD$

$4 + 16 = (6-\lambda)^2 + 4^2$

$4 = 36 + \lambda^2 - 12\lambda$

$\Rightarrow$ $\lambda^2 - 12\lambda + 32 = 0$

$\lambda = 4, 8$

Hence from (1) and (2)

$\lambda = 8$

Illustration :

Prove that the four points A(0, 0), B(2, 2), C(2 ($\sqrt{2}$ + 1), 2) and D($2\sqrt{2}$, 0) form a Rhombus but not a rectangle.

Sol. *Sides are*

$AB = 2\sqrt{2}, \quad BC = 2\sqrt{2}, \quad CD = 2\sqrt{2} \quad DA = 2\sqrt{2}$

Diagonals

$$AC = \sqrt{2^2(\sqrt{2}+1)^2+4}$$

$$BD = \sqrt{2^2(\sqrt{2}-1)^2+4}$$

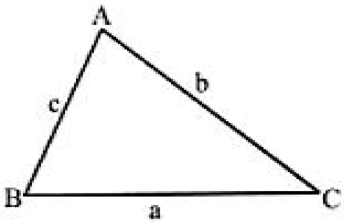
Since $AC \neq BD$

and all sides are equal hence given points from a Rhombus but not a rectangle.

Illustration :

The vertices of a triangle are A(0, 0), B(2, 3) and C(0, 4). Find sinA.

Sol.



$$a = BC = \sqrt{4+1} = \sqrt{5}$$

$$b = AC = 4$$

$$c = AB = \sqrt{4+9} = \sqrt{13}$$

$$\cos A = \frac{b^2+c^2-a^2}{2bc} = \frac{16+13-5}{2.4.\sqrt{13}} = \frac{3}{\sqrt{13}}$$

$$\sin^2 A = 1 - \cos^2 A = 1 - \frac{9}{13} = \frac{4}{13}$$

$$\therefore \quad \sin A = \frac{2}{\sqrt{13}}$$

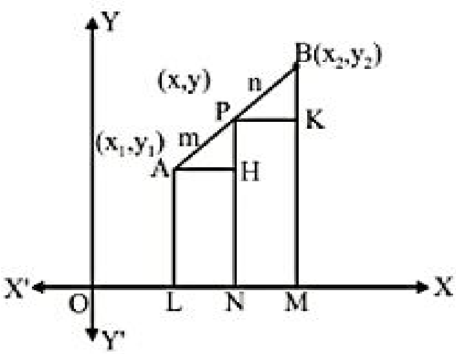
6. SECTION FORMULA :

6.1 FORMULA FOR INTERNAL DIVISION :

Coordinates of the point that divides the line segment joining the points (x_1, y_1) and (x_2, y_2) internally in the ratio m : n are given by

$$x = \frac{mx_2+nx_1}{m+n}, \ y = \frac{my_2+ny_1}{m+n}$$

Proof :



From the figure,

Clearly, ΔAHP and ΔPKB are similar.

$$\Rightarrow \quad \frac{AP}{BP} = \frac{AH}{PK} = \frac{PH}{BK}$$

$$\Rightarrow \quad \frac{m}{n} = \frac{x-x_1}{x_2-x} = \frac{y-y_1}{y_2-y}$$

Now, $\frac{m}{n} = \frac{x - x_1}{x_2 - x}$

$\Rightarrow \quad mx_2 - mx = nx - nx_1$

$\Rightarrow \quad x = \frac{mx_2 + nx_1}{m + n}$ and $\frac{m}{n} = \frac{y - y_1}{y_2 - y}$

$\Rightarrow \quad my_2 - my = ny - ny_1$

$\Rightarrow \quad y = \frac{my_2 + ny_1}{m + n}$

Thus, the coordinates of P are $\left(\frac{mx_2 + nx_1}{m + n}, \frac{my_2 + ny_1}{m + n}\right)$

6.2 FORMULA FOR EXTERNAL DIVISION :

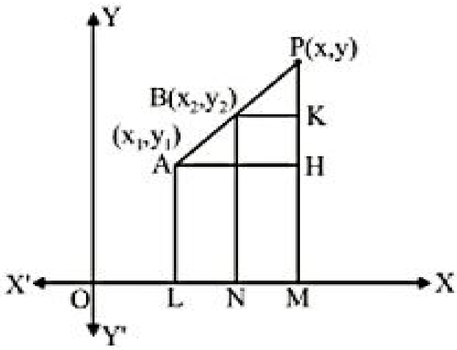
Coordinates of the point that divides the line segment joining the points (x_1, y_1) and (x_2, y_2) externally in the ratio m : n are given by

$$x = \frac{mx_2 - nx_1}{m - n}, \qquad y = \frac{my_2 - ny_1}{m - n}$$

Proof :

From the figure,

Clearly, trianlges PAH and PBK are similar. Therefore,



$$\frac{AP}{PB} = \frac{AH}{BK} = \frac{PH}{PK}$$

$\Rightarrow \quad \frac{m}{n} = \frac{x - x_1}{x - x_2} = \frac{y - y_1}{y - y_2}$

$\Rightarrow \quad mx - mx_2 = nx - nx_1$

$\Rightarrow \quad x = \frac{mx_2 - nx_1}{m - n}$

and $\quad \frac{m}{n} = \frac{y - y_1}{y - y_2}$

$\Rightarrow \quad my - my_2 = ny - ny_1$

$\Rightarrow \quad y = \frac{my_2 - ny_1}{m - n}$

Thus, the coordinates of P are $\left(\frac{mx_2 - nx_1}{m - n}, \frac{my_2 - ny_1}{m - n}\right)$.

7. HARMONIC CONJUGATES :

If two points P and Q divides the line A B internally and externally in the same ratio m : n , then P and Q are said to be harmonic conjugate of each other with respect to A and B.

$A(x_1, y_1)$ P $B(x_2, y_2)$ Q

$$\text{i.e.} \quad \frac{AP}{PB} = \frac{AQ}{BQ} = \lambda \qquad .....(1)$$

Also, AP, AB and AQ are in H.P. ie. $\frac{2}{AB} = \frac{1}{AP} + \frac{1}{AQ}$

Proof : from (1), $\frac{AP}{AB - AP} = \frac{AQ}{AQ - AB}$

$$\frac{AB - AP}{AP} = \frac{AQ - AB}{AQ}$$

$$\frac{AB}{AP} - 1 = 1 - \frac{AB}{AQ}$$

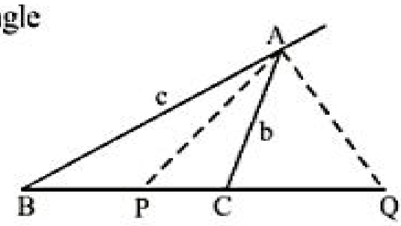
$$2 = \frac{AB}{AQ} + \frac{AB}{AP}$$

$$\frac{2}{AB} = \frac{1}{AP} + \frac{1}{AQ}$$

Examples:

(i) Internal & exeternal angle bisector of an angle of a triangle divide the opposite base harmonically.

$$\frac{BP}{PC} = \frac{BQ}{CQ} = \frac{c}{b}$$



(ii) External and internal common tangents divide the line joining the centres of the two circles externally and internally in the ratio of the their radii.

$$\frac{C_1P}{PC_2} = \frac{C_1Q}{C_2Q} = \frac{r_1}{r_2}$$

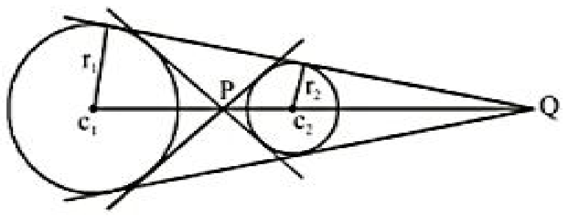


Illustration :

If mid points of the sides of a triangle are (1, 1), (2, 4) and (3, 5). Find the co-ordinates of vertices.

Sol. $h_1 + h_2 = 2, \qquad h_1 + h_3 = 6, \qquad h_3 + h_2 = 4$

$\Rightarrow \quad h_1 + h_2 + h_3 = 6$

$h_1 = 2, h_2 = 0, h_3 = 4$

$k_1 + k_2 = 2, \qquad k_2 + k_3 = 8, \qquad k_3 + k_1 = 10$

$k_1 + k_2 + k_3 = 10$

$k_3 = 8, k_2 = 0, k_1 = 2$

Points are A(2, 2), B(0, 0), C(4, 8).

Illustration :

Find the harmonic conjugate of point R(2, 4) with respect to the points P(2, 2) and Q(2, 5).

Sol. *Let R divides the PQ in ratio k : 1.*

$$\frac{5k+2}{k+1} = 4$$

$$5k + 2 = 4k + 4$$

$$k = 2$$

harmonic conjugate is $\left(\frac{2\times 2-1\times 2}{2-1}, \frac{2\times 5-1\times 2}{2-1}\right) = (2, 8)$

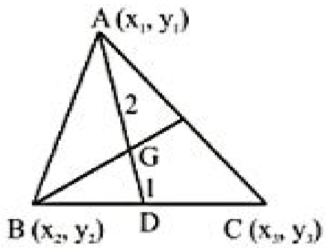
8. COORDINATES OF SPECIAL POINTS WITH RESPECT TO A TRIANGLE :

If co-ordinates of vertices of a triangle are given as $A \equiv (x_1, y_1)$, $B \equiv (x_2, y_2)$ and $C \equiv (x_3, y_3)$ then the co-ordinates of its Centroid, Incentre, Excentre, Circumcentre and Orthocentre are as follows :

8.1 CENTROID :

As it is known that centroid is the point of corcurrency of medians.

From the figure D is the mid point of BC



co-ordinates of D are $\qquad D \equiv \left(\frac{x_2+x_3}{2}, \frac{y_2+y_3}{2}\right)$

We know that centroid G divides AD in the ratio 2 : 1

$\therefore$ co-ordinate of G are

x co-ordinate of G is $= \left(\frac{1(x_1)+2\left(\frac{x_2+x_3}{2}\right)}{1+2}\right) = \frac{x_1+x_2+x_3}{3}$

Similarly $\qquad$ y-co-ordinate of G is $= \frac{y_1+y_2+y_3}{3}$

8.2 INCENTRE :

We know that point of concurrency of the internal bisector of the angles of a triangle is the incentre of the triangle.

Angle bisector AL divides the base BC internally in the ratio c : b

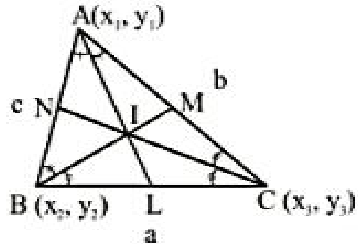
$\therefore$ co-ordinates of L are

$$=\left(\frac{bx_2+cx_3}{b+c},\frac{by_2+cy_3}{b+c}\right)$$

Now, I divides AL internally in the ratio b + c : a

$$\therefore \quad \text{x-co-ordinates} = \frac{ax_1+(b+c)\left(\frac{bx_2+cx_3}{b+c}\right)}{a+b+c}$$

$$=\frac{ax_1+bx_2+cx_3}{a+b+c}$$

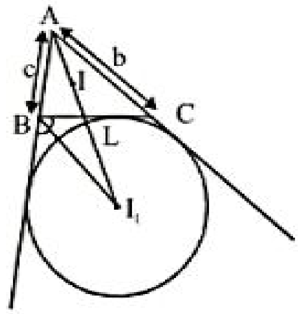


Hence, $\quad I \equiv \left(\frac{ax_1+bx_2+cx_3}{a+b+c},\frac{ay_1+by_2+cy_3}{a+b+c}\right)$

8.3 EX-CENTRES :

Let $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$ be the vertices of the triangle ABC, and let a, b, c be the length of the sides BC, CA, AB, respectively. The circle which touches the side BC and the other two sides AB and AC produced is called the escribed angle B and C meet at a point I_1 which is the centre of the escribed circle opposite to the angle A.

$$\frac{BL}{LC}=\frac{c}{b}, \text{ also } \frac{AI_1}{I_1L}=-\frac{(b+c)}{a}$$



The coordinates of I_1 are given by

$$\left(\frac{-ax_1+bx_2+cx_3}{-a+b+c},\frac{-ay_1+by_2+cy_3}{-a+b+c}\right)$$

Similarly, coordinates of I_2 and I_3 (centres of escribed circles opposite to the angles B and C, respectively) are given by

$$I_2=\left(\frac{ax_1-bx_2+cx_3}{a-b+c},\frac{ay_1-by_2+cy_3}{a-b+c}\right)$$

$$I_3=\left(\frac{ax_1+bx_2-cx_3}{a+b-c},\frac{ay_1+by_2-cy_3}{a+b-c}\right)$$

8.4 ORTHOCENTRE :

Important ratios which are useful to determine the coordinates of orthocentre are :

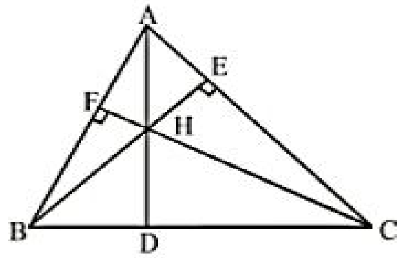
BD : DC = c cos B : b cos C

and AH : HD = 2 R cos A : 2R cos B cos C

where R is the circumradius of the triangle.

$\therefore$ Coordinates of orthocentre are

$$\left(\frac{x_1 \tan A + x_2 \tan B + x_3 \tan C}{\tan A + \tan B + \tan C}, \frac{y_1 \tan A + y_2 \tan B + y_3 \tan C}{\tan A + \tan B + \tan C}\right)$$



8.5 CIRCUMCENTRE :

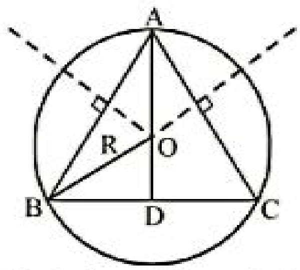
Important ratios which are useful to determine the coordinates of circumcentre are

BD : DC = sin2C : sin2B

and AO : OD = sin2B + sin 2C : sin2A

$\therefore$ Coordinates of circumcentre are

$$\left(\frac{x_1 \sin 2A + x_2 \sin 2B + x_3 \sin 2C}{\sin 2A + \sin 2B + \sin 2C}, \frac{y_1 \sin 2A + y_2 \sin 2B + y_3 \sin 2C}{\sin 2A + \sin 2B + \sin 2C}\right)$$



R = Radius of circumcircle

• Students are suggested to remeber the coordinates of circumcentre and orthocentre as it is, proof is not important.

Illustration :

If α, β and γ are the roots of equation $x^3 - 12x^2 + 44x - 48 = 0$. Find the centroid of the Δ whose co-ordinates are $A\left(\alpha, \frac{1}{\alpha}\right)$, $B\left(\beta, \frac{1}{\beta}\right)$ and $C\left(\gamma, \frac{1}{\gamma}\right)$.

Sol. *Centroid* $= \left(\frac{\alpha+\beta+\gamma}{3}, \frac{\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}}{3}\right)$

$$\frac{\alpha+\beta+\gamma}{3} = \frac{12}{3} = 4$$

$$\frac{1}{3}\left(\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}\right) = \frac{1}{3}\left(\frac{\alpha\beta+\beta\gamma+\gamma\alpha}{\alpha\beta\gamma}\right) = \frac{1}{3}\left(\frac{44}{48}\right) = \frac{11}{36}$$

Centroid is $\left(4, \frac{11}{36}\right)$

Illustration :

Find the co-ordinates of circumcentre of the triangle whose vertices are (8, 6), (8, −2) and (2, −2).

Sol. Let $A(8, 6)$ and $B(8, -2)$ and $C(2, -2)$

$P(h, k)$ be the circumcentre

$PA = PB = PC$

$\Rightarrow \quad PA^2 = PB^2$

$(h-8)^2 + (k-6)^2 = (h-8)^2 + (k+2)^2$

$16k = 32 \quad \Rightarrow \quad k = 2$

$PB^2 = PC^2$

$(h-8)^2 + (k+2)^2 = (h-2)^2 + (k+2)^2$

$12h = 60 \quad \Rightarrow \quad h = 5$

Hence the co-ordinate of the circumcentre is (5, 2).

Illustration :

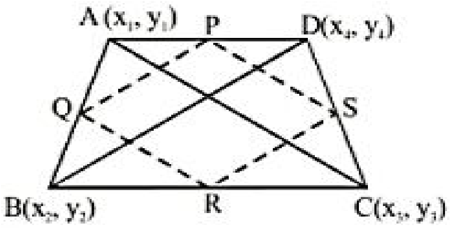
Prove the following results, analytically.

(i) Line joining the middle points of a quadrilateral forms a parallelogram.

(ii) Median to the hypotenuse in a right angled triangle is half as long as the hypotensue.

Sol. (i) P, Q, R & S are the mid point of sides AD, AB, CB & DC respectively.

Mid point of AD $= P\left(\frac{x_1+x_4}{2}, \frac{y_1+y_4}{2}\right)$



Similarly $Q\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$, $R\left(\frac{x_2+x_3}{2}, \frac{y_2+y_3}{2}\right)$,

$S\left(\frac{x_3+x_4}{2}, \frac{y_3+y_4}{2}\right)$

Mid point of $PR = \left(\frac{x_1+x_2+x_3+x_4}{2}, \frac{y_1+y_2+y_3+y_4}{2}\right)$

Mid point of $QS = \left(\frac{x_1+x_2+x_3+x_4}{2}, \frac{y_1+y_2+y_3+y_4}{2}\right)$

Since PR & QS bisects each other therefore PQRS is a parallelogram

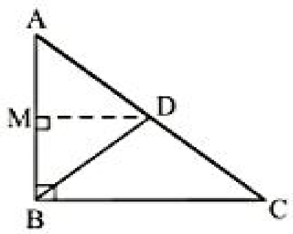
hence PQRS form a parallelogram

(ii) BD is a median to AC

where $\angle B = 90^o$

Let M is the mid point of AB

hence $DM \,||\, BC$



In ΔADM & ΔBDM

$AM = BM$

$DM = DM$ (common)

$\angle AMD = \angle BMD = 90^o$

$\therefore \quad \Delta ADM$ & ΔBDM are congruent

hence $AD = BD = \frac{1}{2}AC$

Practice Problem

Single correct question

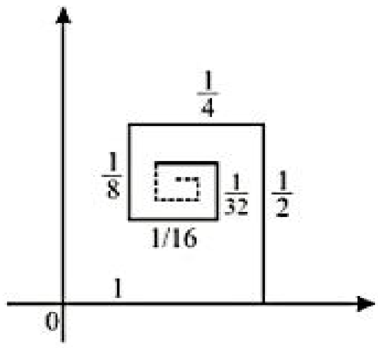
Q.1 The distance between the points $\left(t_1^3, \sqrt{2}\, t_1\right)$ and $\left(t_2^3, -\sqrt{2}\, t_2\right)$, if t_1 and t_2 are the roots of $x^2 - 2\sqrt{2}\,x + 1 = 0$ is

(A) 3 (B) $6\sqrt{6}$ (C) $5\sqrt{2}$ (D) 10

Q.2 Let ΔABC have vertices A (3, 4), B (4, 6) and C (–1, 6). The distance between orthocentre of ΔABC and the origin is :

(A) 0 (B) 3 (C) 8 (D) 5

Q.3 A particle begins at the origin and moves successively in the following manner as shown, 1 unit to the right, $\frac{1}{2}$ unit up, $\frac{1}{4}$ unit to the left, $\frac{1}{8}$ unit down, $\frac{1}{16}$ unit to the right, $\frac{1}{32}$ unit up etc. The length of each move is half the length of previous move and movement continues indefinitely. The co-ordinates of the point to which the movement converges is :



(A) $\left(\frac{4}{5}, \frac{2}{5}\right)$ (B) $\left(\frac{2}{5}, \frac{4}{5}\right)$ (C) $\left(\frac{4}{3}, \frac{2}{3}\right)$ (D) $\left(\frac{2}{3}, \frac{4}{3}\right)$

Q.4. Harmonic conjugate of the point C(5, 1) with respect to the point A(2, 10) and B(6, –2) is

(A) (8, –8) (B) (–1, 3) (C) (6, –4) (D) (2, 5)

Q.5 If in a triangle $A \equiv (1, 10)$, circumcentre $\equiv \left(-\frac{1}{3}, \frac{2}{3}\right)$ and orthocentre $\equiv \left(\frac{11}{3}, \frac{4}{3}\right)$ then the co-ordinate of mid-point of side opposite to A is

(A) (1, 6) (B) (1, 5) (C) (1, –3) (D) $\left(1, -\frac{11}{3}\right)$

Multiple correct type question

Q.6 Coordinate of the vertices of ΔABC are (12, 8), (–2, 6) and (6, 0) then the correct statement (s) are :

(A) triangle is right but not isosceles

(B) triangle is right as well as isosceles

(C) triangle is obutse

(D) The product of abscissae of the centroid, orthocentre and circumcentre is 160

Paragraph type

Paragraph for question nos. 7 to 8

Consider the equation $x^3 - 3x^2 + 6x - 1 = 0$, let α, β, γ are the roots of the equation then

Q.7 The centroid of the triangle the coordinate of whose vertices are $\left(\alpha^2, \frac{1}{\alpha^2}\right), \left(\beta^2, \frac{1}{\beta^2}\right)$ and $\left(\gamma^2, \frac{1}{\gamma^2}\right)$ is

(A) (0, 3) (B) (–1, 10) (C) (2, 5) (D) (–3, 8)

Q.8 The ratio in which x-axis divides the join of centroid of ΔABC and the point (3, –4) is

(A) 1 : 3 externally (B) 2 : 5 externally

(C) 2 : 1 internally (D) 5 : 2 internally

Match the Column

Q.9

	Column-I		Column-II
(A)	The point (2, 0), (3, 3), (0, 2) and (–5, –5) taken in order are the vertices of	(P)	Rectangle
(B)	The points (2, –2), (8, 4), (5, 7) and (–1, 1) taken in order are the vertices of	(Q)	Trapezium
(C)	The point (–3, 4), (–1, 0), (1, 0) and (3, 4) taken in order are vertices of	(R)	Parallelogram
(D)	The points (3, –5), (–5, –4), (7, 10) and (15, 9) taken in order are the vertices of	(S)	Cyclic qudrilateral
		(T)	kite

Q.10 Prove the following result analytically

(i) Diagonals of an isosceles trapezium are equal

(ii) $l_1^2 + l_2^2 + l_3^2 = \frac{3}{4}(a^2 + b^2 + c^2)$ (where l_1, l_2, l_3 are the lengths of median of ΔABC)

(iii) Medians of the equal sides of an isosceles triangle are equal and converse.

Answer key

Q.1	B	Q.2	D	Q.3	A	Q.4.	A	Q.5	D
Q.6	B,D	Q.7	B	Q.8	D				

Q.9 (A) $\to$ T; (B) $\to$ P, R, S; (C) $\to$ Q, S; (D) $\to$ Q

9. DETERMINANT :

9.1 INTRODUCTION :

The quantity $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$ is called a second order determinant.

It consists of two rows and two columns, and it stands for the quantity $a_1b_2 - a_2b_1$. Thus

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1$$

The symbol $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ is called the determinant of order 3. Its value can be found as

$$D = a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} \qquad \text{or}$$

$$= a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - a_2\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + a_3\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}$$

In the way we can expand a determinant in 6 ways using elements of $R_1, R_2, R_3, C_1, C_2, C_3$.

9.2 MINORS AND COFACTORS OF A DETERMINANT :

Minors :

Minors (M_{ij}) of an element (a_{ij})is defined as the value of the determinant obtained by deleting i^{th} row and j^{th} column in which that element lies. e.g. in the determinant

$$D = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \quad \text{minor of } a_{12} \text{ denoted as } M_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} \quad \text{and so on}$$

Cofactor :

It has no separate identity and is related to the minor as

$C_{ij} = (-1)^{i+j} M_{ij}$ where 'i' denotes the row and 'j' denotes the column.

Hence the value of a determinant of order three in terms of 'Minor' and 'Cofactor' can be written as

$$D = a_{11} M_{11} - a_{12} M_{12} + a_{13} M_{13} \text{ or}$$
$$= a_{11} C_{11} + a_{12} C_{12} + a_{13} C_{13}$$

Illustration :

Expand $\begin{vmatrix} 2 & 3 & 0 \\ -2 & 1 & 2 \\ 6 & 5 & -1 \end{vmatrix}$.

Sol. $D = 2\begin{vmatrix} 1 & 2 \\ 5 & -1 \end{vmatrix} - 3\begin{vmatrix} -2 & 2 \\ 6 & -1 \end{vmatrix} + 0\begin{vmatrix} -2 & 1 \\ 6 & 5 \end{vmatrix}$

$= 2(-1 - 10) - 3(2 - 12) + 0 = -22 + 30 = 8$

9.3 PROPERTIES OF DETERMINANTS :

P-1: The value of a determinant remains unaltered, if the rows & columns are inter changed. e.g. if

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = D'$$

D & D' are transpose of each other. If $D' = -D$ then it is **Skew symmetric** determinant but $D' = D \Rightarrow 2D = 0 \Rightarrow D = 0 \Rightarrow$ Skew symmetric determinant of third order has the value zero. The value of a skew symmetric determinant of odd order is zero.

P-2: If any two rows (or columns) of a determinant be interchanged, the value of determinant is changed in sign only. e.g.

Let $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ & $D' = \begin{vmatrix} a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix}$ Then $D' = -D$.

P-3: If a determinant has any two rows (or columns) identical, then its value is zero.

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0.$$

P-4: If all the elements of any row (or column) be multiplied by the same number, then the determinant is multiplied by that number.

If $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix}$ and $D' = \begin{vmatrix} Ka_1 & Kb_1 & Kc_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ Then D'= KD

P-5: If each element of any row (or column) can be expressed as a sum of two terms then the determinant can be expressed as the sum of two determinants. We can say

$$\begin{vmatrix} a_1+x & b_1+y & c_1+z \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} x & y & z \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

P-6: The value of a determinant is not altered by adding to the elements of any row (or column) the same multiples of the corresponding elements of any other row (or column).

e.g. Let $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ and $D' = \begin{vmatrix} a_1+ma_2 & b_1+mb_2 & c_1+mc_2 \\ a_2 & b_2 & c_2 \\ a_3+na_1 & b_3+nb_1 & c_3+nc_1 \end{vmatrix}$. Then D'=D.

Note that while applying this property **ATLEAST ONE ROW (OR COLUMN)** must remain unchanged.

P-7: If by putting $x = a$ the value of a determinant vanishes then $(x - a)$ is a factor of the determinant.

Note : Factorisation in respect the following determinants are very useful and should be remembered.

(i) $$\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} = (x-y)(y-z)(z-x)$$

(ii) $$\begin{vmatrix} 1 & x & x^3 \\ 1 & y & y^3 \\ 1 & z & z^3 \end{vmatrix} = (x-y)(y-z)(z-x)(x+y+z)$$

(iii) $$\begin{vmatrix} x & x^2 & yz \\ y & y^2 & zx \\ z & z^2 & xy \end{vmatrix} = (x-y)(y-z)(z-x)(xy+yz+zx)$$

(iv) $$\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = -(a^3+b^3+c^3-3abc) < 0 \quad \text{where a, b, c are different and positive}$$

Proof : (i) $$D = \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}$$

$$D = 1\begin{vmatrix} y & y^2 \\ z & z^2 \end{vmatrix} - x\begin{vmatrix} 1 & y^2 \\ 1 & z^2 \end{vmatrix} + x^2\begin{vmatrix} 1 & y \\ 1 & z \end{vmatrix}$$

$$= (yz^2 - y^2z) - x(z^2 - y^2) + x^2(z-y)$$
$$= yz(z-y) - x(z^2-y^2) + x^2(z-y)$$
$$= (z-y)(yz - x(z+y) + x^2)$$
$$= (x-y)(y-z)(z-x)$$

Proof : (ii) $$D = \begin{vmatrix} 1 & x & x^3 \\ 1 & y & y^3 \\ 1 & z & z^3 \end{vmatrix}$$

Applying $R_1 \to R_1 - R_2$ and $R_2 \to R_2 - R_3$, we get

$$D = \begin{vmatrix} 0 & x-y & x^3-y^3 \\ 0 & y-z & y^3-z^3 \\ 1 & z & z^3 \end{vmatrix}$$

$$= (x-y)(y-z)\begin{vmatrix} 0 & 1 & x^2+xy+y^2 \\ 0 & 1 & y^2+yz+z^2 \\ 1 & z & z^3 \end{vmatrix}$$

$$= (x-y)(y-z)\begin{vmatrix} 1 & x^2+xy+y^2 \\ 1 & y^2+yz+z^2 \end{vmatrix}$$

$$= (x-y)\,(y-z)\,(y^2+yz+z^2-x^2-xy-y^2)$$
$$= (x-y)\,(y-z)\,(z-x)\,(x+y+z)$$

Proof : (iii) $D = \begin{vmatrix} x & x^2 & yz \\ y & y^2 & zx \\ z & z^2 & xy \end{vmatrix}$

Applying $R_1 \to R_1 - R_2$ & $R_2 \to R_2 - R_3$

$$\begin{vmatrix} x-y & x^2-y^2 & z(y-z) \\ y-z & y^2-z^2 & x(z-y) \\ z & z^2 & xy \end{vmatrix} = (x-y)\,(y-z)\begin{vmatrix} 1 & x+y & -z \\ 1 & y+z & -x \\ z & z^2 & xy \end{vmatrix}$$

Applying $R_1 \to R_1 - R_2$

$$= (x-y)\,(y-z)\begin{vmatrix} 0 & x-z & x-z \\ 1 & y+z & -x \\ z & z^2 & xy \end{vmatrix}$$

$$= (x-y)\,(y-z)\,(x-z)\begin{vmatrix} 0 & 1 & 1 \\ 1 & y+z & -x \\ z & z^2 & xy \end{vmatrix}$$

Applying $C_2 \to C_2 - C_3$

$$= (x-y)\,(y-z)\,(x-z)\begin{vmatrix} 0 & 0 & 1 \\ 1 & x+y+z & -x \\ z & z^2-xy & xy \end{vmatrix}$$

$$= (x-y)\,(y-z)\,(x-z)[z^2-xy-z\,(x+y+z)]$$
$$= (x-y)\,(y-z)\,(z-x)(xy+yz+zx)$$

Proof : (iv) $D = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$

Applying $R_1 \to R_1 + R_2 + R_3$

$$= \begin{vmatrix} a+b+c & b+c+a & c+a+b \\ b & c & a \\ c & a & b \end{vmatrix}$$

$$= (a+b+c)\begin{vmatrix} 1 & 1 & 1 \\ b & c & a \\ c & a & b \end{vmatrix}$$

Applying $C_1 \to C_1 - C_2$ & $C_2 \to C_2 - C_3$

$$= (a+b+c)\begin{vmatrix} 0 & 0 & 1 \\ b-c & c-a & a \\ c-a & a-b & b \end{vmatrix}$$

$$= (a+b+c)\begin{vmatrix} b-c & c-a \\ c-a & a-b \end{vmatrix} = (a+b+c)\,[(b-c)(a-b)-(c-a)^2]$$

$$= -(a+b+c)(a^2+b^2+c^2-ab-bc-ca) = -(a^3+b^3+c^3-3abc)$$

Illustration :

If a, b,c are the roots of the equation $x^3 - 3x^2 + 2x - 1 = 0$ then find the value of

$$\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix}.$$

Sol. $D = \begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix}$

Taking a, b & c common from row 1, row 2 and row 3 respectively.

$$= abc\begin{vmatrix} \frac{1}{a}+1 & \frac{1}{a} & \frac{1}{a} \\ \frac{1}{b} & 1+\frac{1}{b} & \frac{1}{b} \\ \frac{1}{c} & \frac{1}{c} & 1+\frac{1}{c} \end{vmatrix}$$

Applying $R_1 \to R_1 + R_2 + R_3$

$$= abc\begin{vmatrix} 1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c} & 1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c} & 1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \\ \frac{1}{b} & 1+\frac{1}{b} & \frac{1}{b} \\ \frac{1}{c} & \frac{1}{c} & 1+\frac{1}{c} \end{vmatrix}$$

$$= abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\begin{vmatrix} 1 & 1 & 1 \\ \frac{1}{b} & 1+\frac{1}{b} & \frac{1}{b} \\ \frac{1}{c} & \frac{1}{c} & 1+\frac{1}{c} \end{vmatrix}$$

Applying $C_2 \to C_2 - C_1$, $C_3 \to C_3 - C_1$

$$= abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\begin{vmatrix} 1 & 0 & 0 \\ \frac{1}{b} & 1 & 0 \\ \frac{1}{c} & 0 & 1 \end{vmatrix}$$

$$= abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \; = abc\left(\frac{abc+ab+bc+ca}{abc}\right)$$

$$= abc + ab + bc + ca = 1 + 2 = 3$$

Illustration :

If $a \neq p,\ b \neq q,\ c \neq r$ *and* $\begin{vmatrix} p & b & c \\ a & q & c \\ a & b & r \end{vmatrix} = 0$*, then find the value of* $\frac{p}{p-r}+\frac{q}{q-b}+\frac{r}{r-c}$.

Sol. $\begin{vmatrix} p & b & c \\ a & q & c \\ a & b & r \end{vmatrix} = 0$

Applying $R_1 \to R_1 - R_2$ *and* $R_2 \to R_2 - R_3$

$$\begin{vmatrix} p-a & b-q & 0 \\ 0 & q-b & c-r \\ a & b & r \end{vmatrix} = 0$$

$(p-a)\,[(q-b)r - b\,(c-r)] + a[(b-q)\,(c-r)] = 0$

$(p-a)\,(q-b)r - (p-a)\,(c-r)b + a(b-q)\,(c-r) = 0$

Dividing the equation by $(p-a)\,(q-b)\,(r-c)$

$$\frac{r}{r-c}+\frac{b}{q-b}+\frac{a}{p-a}=0$$

$$\frac{r}{r-c}+\frac{b-q+q}{q-b}+\frac{a-p+p}{p-a}=0$$

$$\frac{r}{r-c}+\frac{q}{q-b}+\frac{p}{p-a}=2$$

Practice Problem

Single correct question

Q.1 The value of $\begin{vmatrix} 1 & 1+ac & 1+bc \\ 1 & 1+ad & 1+bd \\ 1 & 1+ae & 1+be \end{vmatrix}$ equals

(A) 0 (B) 1 (C) 3 (D) a + b + c

Q.2 If $\begin{vmatrix} x+y & y & z \\ y+z & x & y \\ z+x & z & x \end{vmatrix} = k(x+y+z)(x-z)^2$ then k equals

(A) $x^2y^2z^2$ (B) xyz (C) 1 (D) 0

Q.3 The value of $\begin{vmatrix} \cos(\alpha+\beta) & -\sin(\alpha+\beta) & 1 \\ \sin\alpha & \cos\alpha & \sin\beta \\ -\cos\alpha & \sin\alpha & \cos\beta \end{vmatrix}$ is dependent on

(A) α (B) β (C) both α and β (D) Neither α nor β

Multiple correct type question

Q.4 Let α, β, γ are the roots of the equation $\begin{vmatrix} 1 & -1 & x \\ x & 0 & 2 \\ -3 & x & -5 \end{vmatrix} = 0$ then, which of the following are correct?

(A) $(\alpha+\beta)^{-1} + (\beta+\gamma)^{-1} + (\gamma+\alpha)^{-1} = \dfrac{-7}{6}$ (B) $\alpha^3 + \beta^3 + \gamma^3 = 3\alpha\beta\gamma$

(C) $\alpha^2 + \beta^2 + \gamma^2 = 8$ (D) $\Sigma\alpha - \Sigma\alpha\beta = 4$

Q.5 The determinant $\begin{vmatrix} a & b & a\alpha+b \\ b & c & b\alpha+c \\ a\alpha+b & b\alpha+c & 0 \end{vmatrix} = 0$ if

(A) a, b, c are in A.P. (B) a, b, c are in G.P.

(C) α is a roots of $ax^2 + bx + c = 0$ (D) $(x - a)$ is a factor of $ax^2 + 2bx + c = 0$

Answer key

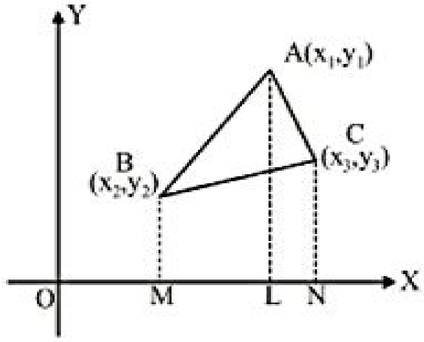
Q.1	A	Q.2	C	Q.3	D	Q.4	AB	Q.5	BD

10. AREA OF A TRIANGLE :

The area of a triangle, whose vertices are (x_1, y_1), (x_2, y_2) and (x_3, y_3) is

$$\frac{1}{2}\left| x_1(y_2-y_3)+x_2(y_3-y_1)+x_3(y_1-y_2)\right| = \frac{1}{2}\left\|\begin{matrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1\end{matrix}\right\|$$

Proof :



Let ABC be a triangle whose vertices are $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$. Draw AL, BM and CN as perpendicualr from A, B and C on the x-axis. Clearly, ABML, ALNC and BMNC all are trapziums. We have

Area of ΔABC = Area of trapezium ABML + Area of trapedezium ALNC – Area of trapezium BMNC

$\Rightarrow$ Area of $\Delta ABC = \frac{1}{2}(BM+AL)(ML)+\frac{1}{2}(AL+CN)(LN)-\frac{1}{2}(BM+CN)(MN)$

$$=\frac{1}{2}(y_2+y_1)(x_1-x_2)+\frac{1}{2}(y_1+y_3)(x_3-x_1)-\frac{1}{2}(y_2+y_3)(x_3-x_2)$$

$$=\frac{1}{2}[x_1(y_2-y_3)+x_2(y_3-y_1)+x_3(y_1-y_2)]$$

$$=\frac{1}{2}\left\|\begin{matrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1\end{matrix}\right\| \qquad \text{(Remember)}$$

Illustration :

If the area of the triangle formed by the points (1, 2), (2, 3) and (x, 4) is 40 sq. units then find x.

Sol. *Let A (1, 2), B (2, 3) & C(x, 4) be three points.*

$$Area\ (D) = \frac{1}{2}\begin{vmatrix} 1 & 2 & 1 \\ 2 & 3 & 1 \\ x & 4 & 1\end{vmatrix} = 40$$

$$=\left|1(3-4)-2(2-4)+x(2-3)\right| = 80$$

$$=\left|-1+4-x\right| = 80$$

$$=\left|3-x\right| = 80$$

$3-x=80$ *or* $3-x=-80$

$x=-77$ *or* $x=83$

Illustration :

If A(1, 1), B(3, 4), C(5, −2) and D(4, −7) in order are the vertices of a quadrilateral. Find its area.

Sol. *To find the area of quadrilateral ABCD, we can calculate area of ΔABC & ΔADC*

$$\text{Area of } \Delta ABC = \frac{1}{2}\begin{vmatrix} 1 & 1 & 1 \\ 3 & 4 & 1 \\ 5 & -2 & 1 \end{vmatrix} = \frac{18}{2}$$

$$\text{Area of } \Delta ADC = \frac{1}{2}\begin{vmatrix} 1 & 1 & 1 \\ 4 & -7 & 1 \\ 5 & -2 & 1 \end{vmatrix} = \frac{23}{2}$$

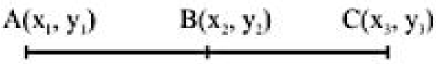
Hence Area of quadrilateral ABCD = ΔABC + ΔADC $= \frac{18}{2} + \frac{23}{2} = \frac{41}{2}$

10.1 COLLINEARITY OF THREE POINTS :

Different conditions for three points $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ to be collinear are as follows

(i) AB + BC = AC, AC – AB = BC

(ii)



Slope of AB = Slope of BC

$$\Rightarrow \quad \frac{y_2 - y_1}{x_2 - x_1} = \frac{y_3 - y_2}{x_3 - x_2}$$

(iii) If the area of triangle ABC be zero then the three points will be collinear.

$$\Rightarrow \quad \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$$

Illustration :

Find the value of k for which points (2, 3), (3, 5) and (5, k) arecollinear.

Sol. *If the points are collinear then*

Area (Δ) = 0

$$\frac{1}{2}\begin{vmatrix} 2 & 3 & 1 \\ 3 & 5 & 1 \\ 5 & k & 1 \end{vmatrix} = 0$$

$$|5(3 - k) - k\,(2 - 3) + 1\,(10 - 9)| = 0$$

$$|-10 + k + 1| = 0$$

$$|k - 9| = 0$$

$$k = 9$$

Illustration :

Show that points (b, c + a), (c, a + b) and (a, b + c) are always collinear where a, b, c ∈ R

Sol. *Area* $(\Delta) = \frac{1}{2}\begin{vmatrix} a & b+c & 1 \\ b & c+a & 1 \\ c & a+b & 1 \end{vmatrix}$

$$\alpha\Delta = \begin{vmatrix} a & b+c & 1 \\ b & c+a & 1 \\ c & a+b & 1 \end{vmatrix}$$

Applying $c_1 \to c_1 + c_2$

$$= \begin{vmatrix} a+b+c & b+c & 1 \\ b+c+a & c+a & 1 \\ c+a+b & a+b & 1 \end{vmatrix} = (a+b+c)\begin{vmatrix} 1 & b+c & 1 \\ 1 & c+a & 1 \\ 1 & a+b & 1 \end{vmatrix} = 0$$

Since $\Delta = 0$*, hence points are collinear.*

Practice Problem

Single correct question

Q.1 The area of quadrilateral whose vertices are A(1, 1), B(3, 4), C(5, –2) and D(4, –7) is :

(A) 12 sq. unit (B) $\frac{3\sqrt{3}}{2}$ sq. unit (C) 25 sq. unit (D) $\frac{41}{2}$ sq. unit

Q.2 If the points $(\sin\theta, \cos\theta)$, $(-2\sqrt{2}, 2)$ and $(-\sqrt{2}, 1)$ are collinear, then the number of values of $\theta \in [0, 2\pi]$ is :

(A) 0 (B) 1 (C) 2 (D) Infinite

Q.3 Let Δ_1 denotes the area of the triangle formed by the vertices $(a^3m_1^3, am_1)$, $(a^3m_2^3, am_2)$, $(a^3m_3^3, am_3)$ and Δ_2 denotes the area of the triangle formed by the vertices $(2am_1m_2, a^2(m_1^2+m_2^2))$, $(2am_2m_3, a^2(m_2^2+m_3^2))$ and $(2am_3m_1, a^2(m_3^2+m_1^2))$. Then $\frac{\Delta_1}{\Delta_2}$ $(a > 0)$ equals

(A) $\frac{a}{2}$ (B) 2a (C) $\frac{a^3}{8}$ (D) $8a^3$

Multiple correct type question

Q.4 If area of Hexagon whose vertices taken in order are (0, 0), (1, 1), (1, 3), (–1, 4), (–3, 2) and $\left(-2, \frac{1}{2}\right)$ can be written in the form $\frac{a}{b}$ (where $a, b \in N$ and a and b are in their lowest form) then

(A) a and b are co-prime (B) a + b is an even number
(C) a + b is an odd prime (D) $a^2 + b^2 = 1000$

Q.5 If the area of triangle formed by the points (5, 2), (0, 3) and (a, 4) is 8 square units. Then
(A) sum of all possible value(s) of 'a' is equal to 5
(B) sum of all possible value(s) of 'a' is equal to –10
(C) product of all possible value(s) of 'a' is equal to 125
(D) product of all possible value(s) of 'a' is equal to –210

Answer key

Q.1	D	Q.2	C	Q.3	A	Q.4	AC	Q.5	BD

11. LOCUS AND EQUATION TO A LOCUS :

11.1 LOCUS :

The curve described by a point which moves under given condition or conditions is called its locus. For example, suppose C is a point in the plane of the paper and P is a variable point in the plane of the paper such that its distance from C is always equal to a (say). It is clear that all the positions of the moving point P lie on the circumference of a circle whose centre is C and whose radius is a. The circumference of this circle is, therefore, the "Locus" of point P when it moves under the condition that its distance from the point C is always equal to constant a.

Let A and B be two fixed points in the plane of the paper, and P be a variable point in the plane of the paper which moves in such a way that its distance from A and B is always same. Thus, the "locus" of P is the perpendicular bisector of AB when it moves under the condition that its distance from A and B is always equal.

11.2 EQUATION TO LOCUS OF A POINT :

The equation to the locus of a point is the relation which is satisfied by the coordinates of every point on the locus of the point.

Note : Steps to find locus of a point.

Step I : Assume the coordinates of the point say (h, k) whose locus is to be determined.
Step II: Write the given condition in mathematical form involving h, k.
Step III : Eliminate the variable (s), if any.
Step IV : Replace h by x and k by y in the result obtained in step III.
The equation so obtained is the locus of the point which moves under some condition(s).

Illustration :

Find the locus of point P if P is equidistant from points

(i) $A(3, 4)$ & $B(5, -2)$

(ii) $A(a + b, a - b)$ & $B(a - b, a + b)$

Sol.

(i) If point $P(h, k)$ is equidistant from $A(3, 4)$ & $B(5, -2)$ then

$PA = PB$

$\Rightarrow \quad \sqrt{(h-3)^2+(k-4)^2} = \sqrt{(h-5)^2+(k+2)^2}$

$\Rightarrow \quad h^2 - 6h + 9 + k^2 - 8k + 16 = h^2 - 10h + 25 + k^2 + 4k + 4$

$\Rightarrow \quad 4h - 12k = 4$

$\Rightarrow \quad h - 3k = 1$

hence locus of P is $\quad x - 3y = 1$

(ii) $PA = PB$

$\Rightarrow \quad [h - (a + b)]^2 + [k - (a - b)]^2 = [h - (a - b)]^2 + [k - (a + b)]^2$

$\Rightarrow \quad -2h(a + b) - 2k(a - b) = -2h(a - b) - 2k(a + b)$

$\Rightarrow \quad 2h(2b) + 2k(2b) = 0$

$\Rightarrow \quad h + k = 0$

hence locus of P is $x + y = 0$

Illustration :

Find the equation to the locus of a point which moves so that

(i) Its distance from the point $(a, 0)$ is always four times its distance from the axis of y.

(ii) Sum of the squares of its distances from the axes is equal to 3.

(iii) Its distance from x-axis is 3 times of its distance from y-axis.

Sol.

(i) Let the point be $P(h, k)$

Distance of P from axis of $y = |h|$

Distance of P from $(a, 0) = \sqrt{(h-a)^2+k^2}$

$\Rightarrow \quad \sqrt{(h-a)^2+k^2} = 4|h|$

$\Rightarrow \quad (h - a)^2 + k^2 = 16h^2$

$\Rightarrow \quad h^2 - 2ah + a^2 + k^2 = 16h^2$

$\Rightarrow \quad 15h^2 - k^2 + 2ah = a^2$

hence locus of P is

$15x^2 - y^2 + 2ax = a^2$

(ii) *Let the point be P(h, k)*

Distance of P from y-axis $= |h|$

Distance of P from x-axis $= |k|$

$$h^2 + k^2 = 3$$

hence Locus of P is

$$x^2 + y^2 = 3$$

(iii) *Let the point be P(h, k)*

Distance from x-axis $= |k|$

Distance from y-axis $= |h|$

$$|k| = 3|h|$$

$3h - k = 0$ *or* $3h + k = 0$

$3x - y = 0$ *or* $3x + y = 0$

Illustration :

A(0, 1) and B(0, – 1) are 2 points if a variable point P moves such that sum of its distance from A and B is 4. Then the locus of P is the equation of the form of $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. *Find the value of* $(a^2 + b^2)$.

Sol. *Let the point P is (h, k)*

Given that PA + PB = 4 where A (0, 1) and B(0, –1)

$$\Rightarrow \quad \sqrt{(h-0)^2 + (k-1)^2} + \sqrt{(h-0)^2 + (k+1)^2} = 4$$

$$\Rightarrow \quad \sqrt{h^2 + k^2 - 2k + 1} = 4 - \sqrt{h^2 + k^2 + 2k + 1}$$

squaring both sides, we get

$$\Rightarrow \quad h^2 + (k-1)^2 = 16 + h^2 + (k+1)^2 - 8\sqrt{h^2 + (k+1)^2}$$

$$\Rightarrow \quad 8\sqrt{h^2 + (k+1)^2} = 16 + 4k$$

squaring again, we get

$$\Rightarrow \quad 4h^2 + 3k^2 = 12$$

$$\Rightarrow \quad \frac{h^2}{3} + \frac{k^2}{4} = 1$$

hence locus of P is $\frac{x^2}{3} + \frac{y^2}{4} = 1$

$$\Rightarrow \quad a^2 + b^2 = 3 + 4$$

$$= 7$$

Practice Problem

Single correct question

Q.1 The equation of locus of all points equidistant from the point $(-1, 2)$ and the origin is :

(A) $x - 3y - 7 = 0$ (B) $2x - 4y + 5 = 0$ (C) $4x + y - 3 = 0$ (D) $x + 2y + 1 = 0$

Q.2 Let A $(1, -3)$ and B$(-2, 5)$ be vertices of ΔABC. If the third vertex C of ΔABC move on the line $3x + y = 1$, then locus of centroid is :

(A) $x - y = 0$ (B) $x - y = 3$ (C) $x + y = 1$ (D) $3x + y = 0$

Q.3 If A $(\cos\alpha, \sin\alpha)$, B$(\sin\alpha, -\cos\alpha)$, C$(1, 2)$ are the vertices of a ΔABC, then as α varies, the locus of its centroid is

(A) $x^2 + y^2 - 2x - 4y + 1 = 0$ (B) $3(x^2 + y^2) - 2x - 4y + 1 = 0$

(C) $x^2 + y^2 - 2x - 4y + 3 = 0$ (D) None of these

Multiple correct type question

Q.4 A stick of length 10 units rests against the floor and a wall of a room. If the stick begins to slide on the floor then

(A) Locus of middle point is $x^2 + y^2 = 25$

(B) Locus of middle point is $x^2 + y^2 = 100$

(C) Locus of centroid of triangle formed by axes and stick is $x^2 + y^2 = 25/9$

(D) Locus of centroid of triangle formed by axes and stick is $x^2 + y^2 = 100/9$

Q.5 If the equation of locus of a point which moves so that its distance from the point $(ak, 0)$ is k $(k > 0, \neq 1)$ times the distance from the point $\left(\frac{a}{k}, 0\right)$ then

(A) Locus of the point depends on k (B) Locus of the point is independent of k

(C) Locus is $x^2 + y^2 = a^2$ (D) Locus is $(1 - k^2)x^2 + y^2 = k^2$

Answer key

Q.1	B	Q.2	D	Q.3	B	Q.4	AD	Q.5	BC

12. STRAIGHT LINE

12.1 DEFINITION :

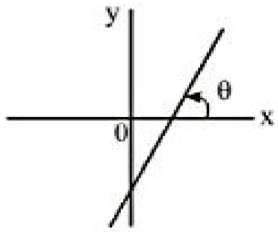
It is a locus of a point P (h, k) which moves in such a way that point P(h, k) is collinear with the two given points.

'or'

A straight line is a curve such that every point on the line segment joining any two points on it lies on it.

12.2 INCLINATION OF A LINE (θ) :

If a straight line intersects the x-axis, the inclination of the line is defined as the measure of the smallest non-negative angle which the line makes with the positive direction of the x-axis.

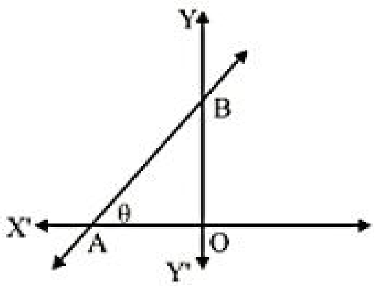


12.3 SLOPE (GRADIENT) OF A LINE :

If the inclination of a line (i.e. non vertical line) is θ and $\theta \neq \frac{\pi}{2}$, then the slope of a line is defined to be $\tan\theta$.

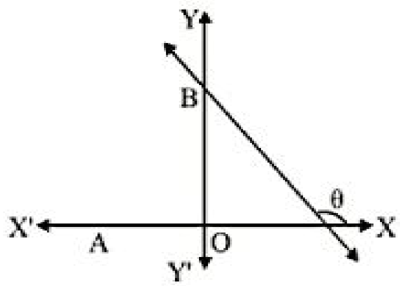
Imp.Point :-

(i) (a)



Slope = $\tan\theta$ = positive
$(0^\circ < \theta < 90^\circ)$

(b)



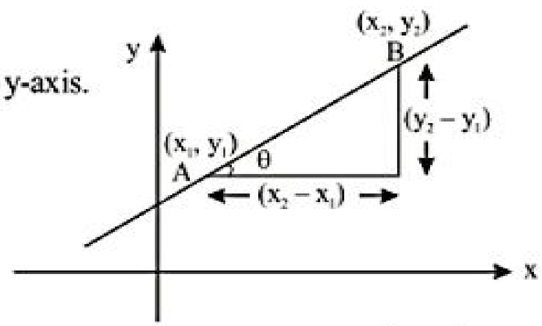
Slope = $\tan\theta$ = negative
$(90^\circ < \theta < 180^\circ)$

(ii) $0^\circ \leq \theta < 180^\circ$ $(\theta \neq 90^\circ)$

(iii) If $\theta = 0$ then line is parallel to x-axis
If $\theta = 90^\circ$ then line is perpendicular to x-axis or parallel to y-axis.

(iv) If $A(x_1, y_1)$ & $B(x_2, y_2)$, $x_1 \neq x_2$ are points on a straight line then the slope m of the line is given by

$$m = \frac{(y_2 - y_1)}{(x_2 - x_1)}$$



slope = $\tan\theta = m = \frac{(y_2 - y_1)}{(x_2 - x_1)}$

Example : Find the slopes if

(i) $\theta = \frac{\pi}{8}$ (ii) $\theta = \frac{3\pi}{8}$ (iii) $\theta = \frac{5\pi}{12}$

Ans. (i) $(\sqrt{2} - 1)$ (ii) $(\sqrt{2} + 1)$ (iii) $(2 + \sqrt{3})$

12.4 INTERCEPT :

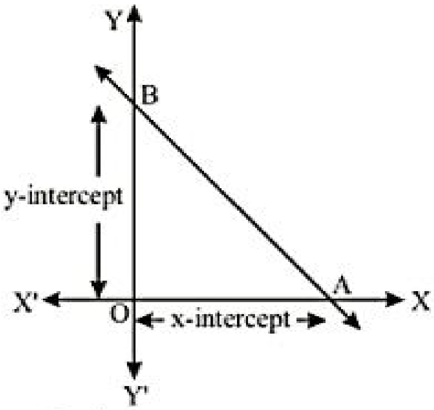
Definition :

The abscissa of the point where a line cuts the x-axis is called its x-intercept and ordinate of the point where it cuts the y-axis is called its y-intercept. If a line is parallel to x-axis its x-intercept is infinite, and if parallel to y-axis then y-intercept is not defined.

Intercepts of a line on the Axes :

If a straight line cuts x-axis at A and the y-axis at B then OA and OB are known as the intercepts of the line on x-axis and y-axis respectively.

The intercepts are positive or negative according as the line meets with positive or negative directions of the coordinate axes.



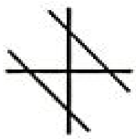
In figure OA = x-intercept, OB = y-intercept

OA is positive or negative according as A lies on OX or OX' respectively.

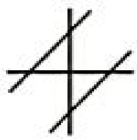
Similarly OB is positive or negative according as B lies on OY or OY' respectively.

Note:

(i) Line having equal interecept then m = –1.



(ii) Line having intercept equal in magnitude but opposite in sign then m = +1



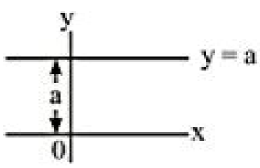
(iii) Line equally inclined with coordinate axes then m = ±1

(iv) Line cutting of equal non zero distance from origin then m = ±1.

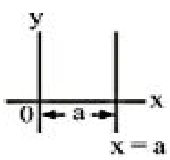
(v) $0.x + 0.y + c = 0$ $(c \neq 0)$ represents a straight line with x and y intercept both infinity

$\Rightarrow$ Straight line approaches at infinity.

(vi) Equations of lines parallel to x-axis are of form y = a.



(vii) Equations of lines parallel to y-axis are of form x = a.



13. DIFFERENT FORMS OF A LINE :

13.1 POINT-SLOPE FORM OF A LINE :

The equation of a line which passes through the point (x_1, y_1) and has the slope 'm' is

$$y - y_1 = m(x - x_1)$$

Proof :

Let $Q(x_1, y_1)$ be the point through which the line passes and let $P(x, y)$ be any point on the line. Then, the slope of the line is

$$\frac{y - y_1}{x - x_1}$$

But m is the slope of the line. Therefore,

$$m = \frac{y - y_1}{x - x_1} \quad \Rightarrow \quad y - y_1 = m(x - x_1)$$

Thus, $y - y_1 = m(x - x_1)$ is the required equation of the line.

Illustration :

A line passes through the point (1, 0). Find the equation of line if

(a) *It is inclined at an angle of $\frac{3\pi}{8}$ with positive x-axis.*
(b) *It passes through (2, 1).*
(c) *It passes through the point of intersection of lines $y = x$ and $y = 2x + 1$.*
(d) *It has equal non zero intercepts.*
(e) *It has intercepts equal in magnitude but opposite in sign.*
(f) *It cuts an intercept of 4 units on x-axis.*
(g) *It cuts an intercept of –3 units on y-axis.*
(h) *It cuts equal non zero distances on co-ordinate axes from origin.*
(i) *It is equally inclined with co-ordinate axes.*
(j) *It has an angle of 30° with positive y-axis.*

Sol. *Since equation passes through (1, 0)*

(a) $\theta = \frac{3\pi}{8}$

$m = \tan\theta = \sqrt{2} + 1$

$y - 0 = m(x - 1)$

$y = (\sqrt{2} + 1)(x - 1)$

(b) *If it passes through (2, 1) & (1, 0)*

$$slope = \frac{1 - 0}{2 - 1} = 1$$

Equation of line $\quad y - 0 = 1(x - 1)$

$$y = x - 1$$

(c) *Point of intersection of* $y = x$ *and* $y = 2x + 1$ *is* $x = -1, y = -1$
Line passing through $(-1, -1)$ & $(1, 0)$

$$\text{slope} \quad (m) = \frac{0-(-1)}{1-(-1)} = \frac{1}{2}$$

$$\text{Equation of line } (y - 0) = \frac{1}{2}(x - 1)$$

$$2y = x - 1$$

(d) *If it has equal non zero intercepts*
then slope $(m) = -1$
Equation of line $(y - 0) = -1\ (x - 1)$
$x + y = 1$

(e) *If it has intercepts equal in magnitude but opposite in sign then,* $m = 1$
$y - 0 = 1\ (x - 1)$
$y = x - 1$

(f) *It cuts an intercept of 4 units on x-axis then it passes through* $(4, 0)$
Slope of line through $(1, 0)$ *and* $(4, 0)$ *is*

$$m = \frac{0-0}{4-1} = 0$$

Equation of line $y - 0 = 0(x - 1)$
$y = 0$

(g) *If cuts an intercept of* -3 *on y-axis, then it passes through* $(0, -3)$ & $(1, 0)$

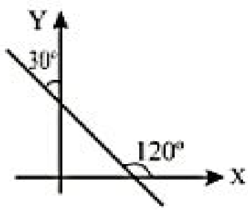
$$\text{slope} \quad (m) = \frac{0-(-3)}{1-0} = 3$$

Equation of line $(y - 0) = 3(x - 1) \Rightarrow 3x - y = 3$

(h) *If it cuts equal non zero distances then slope* $(m) = \pm 1$
Equation of lines are
$(y - 0) = 1\ (x - 1)$ *or* $(y - 0) = -1\ (x - 1)$
$x - y = 1$ *or* $x + y = 1$

(i) *If it is equally inclined with co-roordinate axes then* $m = \pm 1$
Equation of lines are
$x - y = 1$ *or* $x + y = 1$

(j) *If angle with y-axis is* 30^o *then angle with positive x-axis* $= 120^o$
slope $(m) = \tan 120^o = -\sqrt{3}$
Equation of line is



$$y - 0 = -\sqrt{3}\ (x - 1)$$

$$\sqrt{3}\,x + y = \sqrt{3}$$

13.2 TWO-POINT FORM OF A LINE :

The equation of a line passing through two points (x_1, y_1) and (x_2, y_2) is

$$y - y_1 = \left(\frac{y_2 - y_1}{x_2 - x_1}\right)(x - x_1)$$

Proof :

Let m be the slope of the line passing through (x_1, y_1) and (x_2, y_2) then

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

So, the equation of the line is

$$y - y_1 = m(x - x_1) \text{ (Using point – slope form)}$$

Substituting the value of m, we obtain

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

This is the required equation of the line in two-point form.

Illustration :

Find the equation of a line passing through (2, 3) and (4, 5).

Sol. *Equation of line is given by*

$$y - y_1 = \left(\frac{y_2 - y_1}{x_2 - x_1}\right)(x - x_1)$$

$$y - 3 = \left(\frac{5 - 3}{4 - 2}\right)(x - 2)$$

$$y = x + 1$$

Illustration :

Find number of straight lines passing through (2, 4) & forming a triangle of 16 sq. cm with the co-ordinate axes.

Sol. *Let the co-ordinates on x-axis be (h, 0)*

Equation of line becomes

$$y - 0 = \frac{4}{2 - h}(x - h)$$

$$y\text{-intercept} = \frac{4h}{h-2}$$

$$\text{area of triangle} = \frac{1}{2}\left|h \times \frac{4h}{h-2}\right| = 16$$

$$\left|\frac{h^2}{h-2}\right| = 8$$

$h^2 = 8h - 16$ or $h^2 = -8h + 16$

$h^2 - 8h + 16 = 0$ or $h^2 + 8h - 16 = 0$

$(h - 4)^2 = 0$

Three values of h are possible hence three equations are possible.

13.3 SLOPE INTERCEPT FORM OF A LINE :

The equation of a line with slope m that makes an intercept c on y-axis is

$$y = mx + c$$

Proof :

Since y intercept = 'c'

Hence it passes through (0, c).

Equation of line with slope m and passing through (0, c) is given by

$$y - c = m(x - 0)$$

$$y = mx + c$$

Illustration :

Find the equation to the straight line cutting off an intercept 3 from the negative direction of the axis of y and inclined at 120° to the positive direction of x-axis.

Sol. Slope of line (m) = $\tan 120^\circ = -\sqrt{3}$

y intercept (c) = -3

Equation of line is

$$y = mx + c$$

$$y = -\sqrt{3}\,x + (-3)$$

$$y + \sqrt{3}\,x + 3 = 0$$

13.4 INTERCEPT FORM OF A LINE :

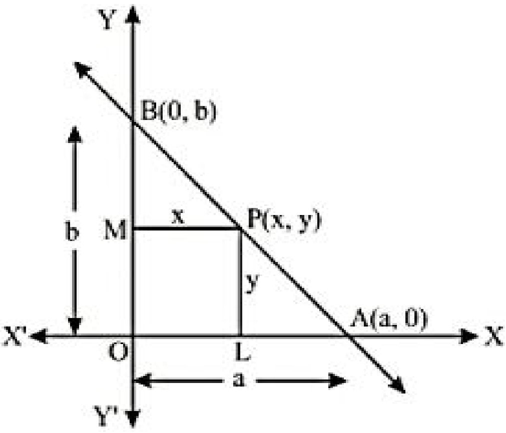
The equation of a line which cut-off intercepts a and b, respectively from the x and y-axes is

$$\frac{x}{a}+\frac{y}{b}=1$$

Proof :

Line cut-off intercepts a and b from the x and y-axes respectively,

Equation of line passes through the points (a, 0) and (0, b) is



$$y-0=\frac{-b}{a}(x-a)$$

$\Rightarrow$ $bx+ay=ab$

$\Rightarrow$ $\frac{x}{a}+\frac{y}{b}=1$

This is the equation of the line in the intercept form.

Illustration :

Find the equation to the straight line passing through the point (3, –4) and cutting off intercepts, equal but of opposite signs from the two axes.

Sol. *Let the intercepts cut off from the two axes are a & –a, then equation of straight line is given by*

$$\frac{x}{a}+\frac{y}{-a}=1$$

$$x-y=a$$

Since it passes through (3, –4)

hence $3-(-4)=a$

$$a=7$$

Required equation is

$$x-y=7$$

13.5 NORMAL FORM OR PERPENDICULAR FORM OF A LINE :

The equation of the straight line upon which the length of the perpendicular from the origin is p and this perpendicular makes an angle α with positive direction of x-axis is

$$x \cos \alpha + y \sin \alpha = p$$

Proof :

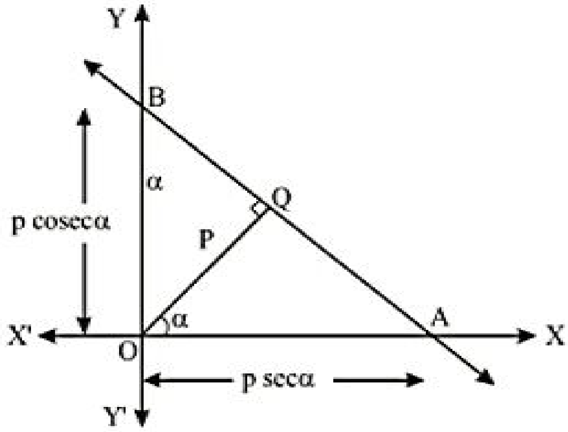
Let the line AB be such that the length of the perpendicular OQ from the origin O to the line be p and $\angle XOQ = \alpha$.

From the diagram, using the intercept form, we get

Equation of line AB is

$$\frac{x}{p \sec \alpha} + \frac{y}{p \operatorname{cosec} \alpha} = 1$$

or $\quad x \cos \alpha + y \sin \alpha = p$



(i) $\quad 0 < \alpha < \frac{\pi}{2}$ $\quad$ (ii) $\frac{\pi}{2} < \alpha < \pi$ $\quad$ (iii) $\pi < \alpha < 3\pi$ $\quad$ (iv) $\frac{3\pi}{2} < \alpha < 2\pi$

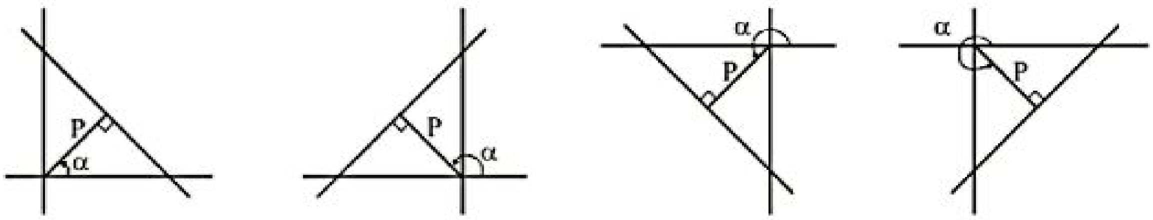


Illustration :

Find equation of a straight line on which length of perpendicular from the origin is four units and the line makes an anlge of 120° with the positive direction of x-axis.

Sol. $\alpha = 30^o$

$P = 4$

Equation line is given by

$$x \cos \alpha + y \sin \alpha = P$$

$$x \cos 30^o + y \sin 30^o = 4$$

$$x \cdot \frac{\sqrt{3}}{2} + \frac{y}{2} = 4$$

$$\sqrt{3}x + y = 8$$

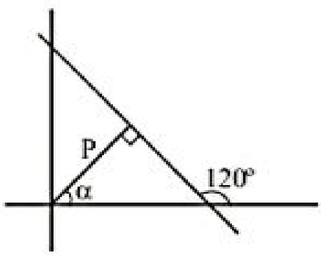


Illustration :

Match the Column

	Column-I		Column-II
(A)	Equation of line which cuts off an intercepts of 4 units on the x-axis & pass through (2, –3)	(P)	$5x - 3y = 0$
(B)	Equation of line which cuts off equal non-zero intercepts on co-ordinate axes and pass through (2, 5)	(Q)	$x + y = 1$
(C)	Equation of line passing through (0, 0) & (3, 5)	(R)	$3x - 2y = 12$
(D)	Equation of line making an angle 135° with positive x-axis and pass through (1, 0)	(S)	$x + y = 7$
(E)	Equation of line passing through (1, 0) and equally inclined with co-ordinate axes	(T)	$x - y = 1$

Sol. (A) *Line cutting off intercept 4 on the x-axis, then line passes through (4, 0)*
Equation of line passing through (4, 0) & (2, –3)

$$y + 3 = \frac{3}{2}(x - 2)$$
$$2y + 6 = 3x - 6$$
$$3x - 2y = 12$$

(B) *Line cutting off equal intercepts = a*
Let the line be

$$\frac{x}{a} + \frac{y}{a} = 1$$
$$x + y = a$$

it passes through (2, 5)
$2 + 5 = a \quad \Rightarrow \quad a = 7$
Equation of line $\quad x + y = 7$

(C) *Equation of line passing through (0, 0) & (3, 5)*

$$y - 0 = \frac{5}{3}(x - 0)$$
$$3y = 5x$$
$$5x - 3y = 0$$

(D) *Slope of line = tan 135° = –1*
Equation of line through (1, 0)

$$y - 0 = -1\ (x - 1)$$
$$y + x = 1$$

(E) *If line is equally inclined then slope (m) = ± 1*
Equation of lines are $y - 0 = \pm 1\ (x - 1)$

$$x + y = 1 \quad \text{or} \quad x - y = 1$$

Practice Problem

Single correct question

Q.1 The gradient of the line joining the points on the curve $y = x^2 - 2x + 3$ whose abscissa are -1 and 2 is

(A) 1 (B) -3 (C) -1 (D) $\frac{1}{2}$

Q.2 If the equation $x \cos\theta + y \sin\theta = p$ is the normal form of the line $\sqrt{3}x + y + 2 = 0$ then values of θ and p are

(A) $\frac{7\pi}{6}, 1$ (B) $\frac{5\pi}{3}, \frac{3}{2}$ (C) $\frac{2\pi}{3}, 1$ (D) $\frac{11\pi}{6}, 4$

Q.3 A variable straight line passes through a fixed point (a, b) intersecting the co-ordinates axes at A and B. If 'O' is the origin then the locus of the centroid of the triangle OAB is

(A) $bx + ay - 3xy = 0$ (B) $bx + ay - 2xy = 0$
(C) $ax + by - 3xy = 0$ (D) none

Q.4 The graph of function, $y = \cos x \cos(x + 2) - \cos^2(x + 1)$ is

(A) a straight line passing through $(0, -\sin^2 1)$ with slope 2
(B) a straight line passing through (0, 0)
(C) a parabola with vertex $(1, -\sin^2 1)$
(D) a straight line passing through the point $\left(\frac{\pi}{2}, -\sin^2 1\right)$ and parallel to the x-axis.

Multiple correct type question

Q.5 The equations of lines which cut off intercepts on the axes whose sum and product are 1 and -6 respectively are

(A) $5x + 3y + 2 = 0$ (B) $2x - 3y - 6 = 0$
(C) $3x - 4y + 1 = 0$ (D) $3x - 2y + 6 = 0$

Q.6 The equation to the straight lines each of which passes through the point (3, 2) and intersects the x-axis and y-axis in A, B respectively such that $OA - OB = 2$ are

(A) $x - y = 1$ (B) $8x - 5y - 2 = 0$
(C) $3x - y - 5 = 0$ (D) $2x + 3y = 12$

Answer key

Q.1 C	Q.2 A	Q.3 A	Q.4 D	Q.5 BD	Q.6 AD

14. ANGLE BETWEEN TWO STRAIGHT LINES WHEN THEIR EQUATIONS ARE GIVEN :

Let the equation of lines are

$$y_1 = m_1x + c_1 \quad \text{and} \quad y_2 = m_2x + c_2$$

then angle ϕ between lines L_1 & L_2 is given by

$$\boxed{\tan\phi = \left|\frac{m_1 - m_2}{1 + m_1m_2}\right|}$$

Proof :

By figure

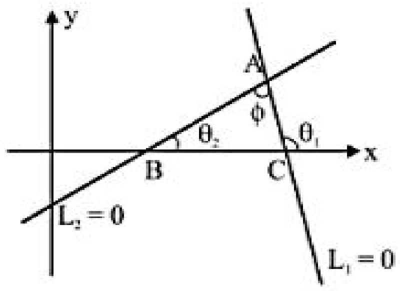
$$\text{slope of line } L_1 = \tan\theta_1 = m_1$$
$$\text{slope of line } L_2 = \tan\theta_2 = m_2$$

In triangle ABC

$$\theta_1 = \phi + \theta_2$$

or $\quad \phi = \theta_1 - \theta_2$

or $\quad \tan\phi = \dfrac{\tan\theta_2 - \tan\theta_1}{1 + \tan\theta_2 \cdot \tan\theta_1}$



$$\tan\phi = \frac{m_1 - m_2}{1 + m_1m_2}$$

& $\quad$ other angle of line $L_2 = 180° - \phi$

$\therefore \quad \tan(180° - \phi) = -\tan\phi = \dfrac{m_1 - m_2}{1 + m_1m_2}$

$\therefore \quad \tan\phi = \left|\dfrac{m_1 - m_2}{1 + m_1m_2}\right|$

Note :

If angle of any one line is 90° then slope of line is not define so to find angle between them.

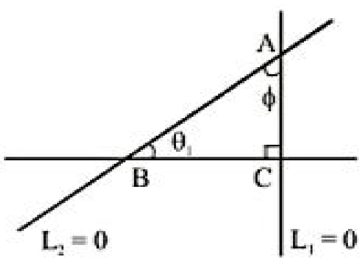
By figure, in triangle ABC

$$\theta_1 + \phi = 90°$$

or $\quad \phi = 90° - \theta_1$

or $\quad \tan\phi = \cot\theta_1$

or $\quad \tan\phi = \dfrac{1}{\tan\theta_1} = \dfrac{1}{m_1}$



and other angle of line is $180° - \phi$ or is equal to $-\dfrac{1}{m_1}$.

Illustration :

Find the equation of a line passing through (1, 2) and making an angle of 45° with the line $2x + 3y = 10$.

Sol. Slope of $2x + 3y = 10$ is $\frac{-2}{3}$

Let the slope of the required line is m, then

$$\tan 45^o = \left|\frac{m - \left(-\frac{2}{3}\right)}{1 + m\left(-\frac{2}{3}\right)}\right|$$

$$1 = \left|\frac{3m+2}{3-2m}\right|$$

$3m + 2 = \pm(3 - 2m)$

$\Rightarrow$ $3m + 2 = 3 - 2m$ or $3m + 2 = -(3 - 2m)$

$m = \frac{1}{5}$ or $m = -5$

Equation of newly formed lines

$y - 2 = -5(x - 1)$ or $y - 2 = \frac{1}{5}(x - 1)$

$y + 5x = 7$ or $5y - x = 9$

Illustration :

Find the tangent of angle between pair of straight lines $x - y + 5 = 0$ and $x + 2y = 0$.

Sol. Slope of $x - y + 5 = 0$ is 1

Slope of $x + 2y = 0$ is $\frac{-1}{2}$

$m_1 = 1$, $m_2 = \frac{-1}{2}$

Angle between lines

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right| = \left|\frac{1 + \frac{1}{2}}{1 + \left(\frac{-1}{2}\right)}\right| = 3$$

$\Rightarrow$ $\tan\theta = 3$

14.1 CONDITION FOR THE LINES TO BE PARALLEL :

If the lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ are parallel, then

$$m_1 = m_2$$

$$\Rightarrow \quad -\frac{a_1}{b_1} = -\frac{a_2}{b_2}$$

$$\Rightarrow \quad \frac{a_1}{a_2} = \frac{b_1}{b_2}$$

14.2 CONDITION FOR THE LINES TO BE PERPENDICULAR :

If the lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ are perpendicular, then

$$m_1\, m_2 = -1 \quad \Rightarrow \quad \left(-\frac{a_1}{b_1}\right)\times\left(-\frac{a_2}{b_2}\right) = -1$$

$$\Rightarrow \quad a_1a_2 + b_1b_2 = 0$$

If follows from the above discussion that the lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ are

(i) Coincident, if $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

(ii) Parallel if $\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$

(iii) Interesecting, if $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$

(iv) Perpendicular, if $a_1a_2 + b_1b_2 = 0$

14.3 EQUATION OF A LINE PARALLEL TO A GIVEN LINE :

The equation of a line parallel to a given line $ax + by + c = 0$ is

$$ax + by + \lambda = 0$$

where λ is a constant and value of λ can be determined using another given condition.

Illustration :

Find the equation of straight line parallel to $3x + 2y + 4 = 0$ and passing through $(1, 1)$.

Sol. *Let the line | | to $3x + 2y + 4 = 0$ be*

$$3x + 2y + c = 0$$

Since it passes through $(1, 1)$

$$3(1) + 2(1) + c = 0$$

$$c = -5$$

Equation of line $3x + 2y = 5$

14.4 EQUATION OF A LINE PEREPENDICULAR TO A GIVEN LINE :

The equation of a line perpendicular to a given line $ax + by + c = 0$ is

$$bx - ay + \lambda = 0$$

where λ is a constant and value of λ can be determined using another given condition.

Illustration :

Find the equation of line perpendicular to $2x - y = 7$ and passing through point of intersections of line $3x + 4y = 8$ and y-axis.

Sol. *Point of intersection of $3x + 4y = 8$ & $x = 0$ is*

$$x = 0,\ y = 2$$

Let the equation of line $\perp$ to $2x - y = 7$ be

$$x + 2y = c$$

Passes through (0, 2)

$\Rightarrow \quad 0 + 4 = c$

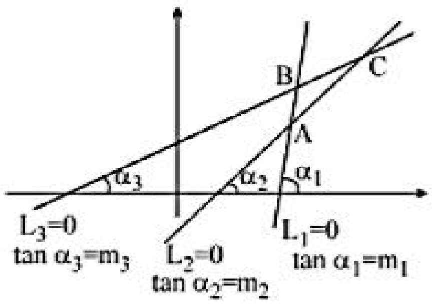
$\Rightarrow \quad c = 4$

hence equation is $x + 2y = 4$

14.5 THE TANGENTS OF THE INTERIOR ANGLES OF A TRIANGLE FORMED BY 3 GIVEN LINES :

Arrange the lines L_1, L_2 and L_3 in their descending order of slopes (as $m_1 > m_2 > m_3$) then tangents of interior angles of ΔABC can be written directly as

$$\tan A = \frac{m_1 - m_2}{1 + m_1 m_2}, \qquad \tan B = \frac{m_2 - m_3}{1 + m_2 m_3}, \tan C = \frac{m_3 - m_1}{1 + m_3 m_1}$$



Explanation : $A = \alpha_1 - \alpha_2$ (from the figure)

$$\tan A = \tan(\alpha_1 - \alpha_2) = \frac{\tan\alpha_1 - \tan\alpha_2}{1 + \tan\alpha_1 \tan\alpha_2} = \frac{m_1 - m_2}{1 + m_1 m_2}$$

$$B = \alpha_2 - \alpha_3$$

$$\tan B = \tan(\alpha_2 - \alpha_3) = \frac{\tan\alpha_2 - \tan\alpha_3}{1 + \tan\alpha_2 \tan\alpha_3} = \frac{m_2 - m_3}{1 + m_2 m_3}$$

$$\pi - C = \alpha_1 - \alpha_3 \qquad \therefore\ C = \pi + \alpha_3 - \alpha_1$$

$$\tan C = \tan(\alpha_3 - \alpha_1) = \frac{\tan\alpha_3 - \tan\alpha_1}{1 + \tan\alpha_3 \tan\alpha_1} = \frac{m_3 - m_1}{1 + m_3 m_1}.$$

Illustration :

If a ΔABC is formed by the lines $2x + y - 3 = 0$; $x - y + 5 = 0$ and $3x - y + 1 = 0$, then obtain a cubic equation whose roots are the tangent of the interior angles of the triangle.

Sol. *Given lines are*

$$2x + y - 3 = 0 \qquad \ldots(i)$$
$$x - y + 5 = 0 \qquad \ldots(ii)$$
$$3x - y + 1 = 0 \qquad \ldots(iii)$$

Slope of line (i) = -2

Slope of line (ii) = 1

Slope of line (iii) = 3

Arranging the lines in descending order of their slopes

$\Rightarrow \quad m_1 = 3, \qquad m_2 = 1, \qquad m_3 = -2$

$$\Rightarrow \quad \tan A = \frac{m_1 - m_2}{1 + m_1 + m_2} = \frac{3-1}{1+(3)(1)} = \frac{1}{2}$$

$$\tan B = \frac{m_2 - m_3}{1 + m_2 m_3} = \frac{1-(-2)}{1+1(-2)} = -3$$

$$\tan C = \frac{m_3 - m_1}{1 + m_3 m_1} = \frac{(-2)-(3)}{1+(-2)3} = 1$$

Roots of cubic are -3, $\frac{1}{2}$ and 1.

$\Rightarrow \quad$ *Equation of cubic is* $(x + 3)\left(x - \frac{1}{2}\right)(x - 1) = 0$

$\Rightarrow \quad 2x^3 + 3x^2 - 8x + 3 = 0$

15. LENGTH OF THE PERPENDICULAR :

The length of the perpendicular from a point (x_1, y_1) to a line $ax + by + c = 0$ is

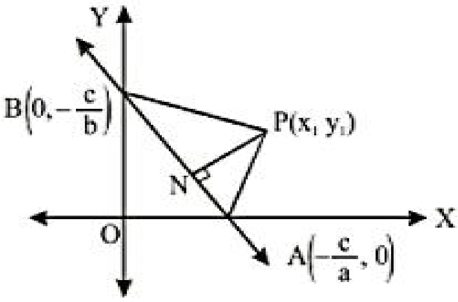
$$\left|\frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}}\right|$$

Proof : The line $ax + by + c = 0$ meets x-axis at $A\left(\frac{-c}{a}, 0\right)$ and y-axis at $B\left(0, -\frac{c}{b}\right)$.

Let $P(x_1, y_1)$ be the point. Draw $PN \perp AB$.
Now, area of ΔPAB

$$= \frac{1}{2}\left|x_1\left(0+\frac{c}{b}\right) - \frac{c}{a}\left(-\frac{c}{b} - y_1\right) + 0(y_1 - 0)\right|$$

$$= \frac{1}{2}\left|\frac{cx_1}{b} + \frac{cy_1}{a} + \frac{c^2}{ab}\right| = \left|(ax_1 + by_1 + c)\frac{c}{2ab}\right| \quad \ldots(i)$$



Also, area of ΔPAB

$$= \frac{1}{2} AB \times PN$$

$$= \frac{1}{2}\sqrt{\frac{c^2}{a^2} + \frac{c^2}{b^2}} \times PN$$

$$= \frac{c}{2ab}\sqrt{a^2 + b^2} \times PN \quad \ldots(ii)$$

From equation (i) and (ii), we get

$$\left|(ax_1 + by_1 + c)\frac{c}{2ab}\right| = \frac{c}{2ab}\sqrt{a^2 + b^2} \times PN$$

$$\Rightarrow \quad PN = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$$

Illustration :

Find the point on y-axis whose perpendicular distance from the line $4x - 3y - 12 = 0$ is 3

Sol. *Let the point on y-axis be $P(0, k)$*
Distance of $P(0, k)$ from $4x - 3y - 12 = 0$ is

$$\left|\frac{4(0) - 3(k) - 12}{\sqrt{4^2 + (-3)^2}}\right| = 3$$

$$|3k + 12| = 15$$
$$|k + 4| = 5$$
$$k = 1, -9$$

hence the points are $(0, 1)$ & $(0, -9)$

Illustration :

Three lines $x + 2y + 3 = 0$, $x + 2y - 7 = 0$ and $2x - y - 4 = 0$ form three sides of two squares find the equations to the fourth sides of squares.

Sol. Distance between the lines $x + 2y + 3 = 0$ & $x + 2y - 7 = 0$ is

$$d = \left|\frac{c_1 - c_2}{\sqrt{a^2 + b^2}}\right|$$

$$= \left|\frac{3-(-7)}{\sqrt{1+4}}\right| = \frac{10}{\sqrt{5}}$$

The fourth side is parallel to $2x - y - 4 = 0$
Let the fourth side be $2x - y + k = 0$

Distance between two sides $2x - y - 4 = 0$ and $2x - y + k = 0$ should be $\frac{10}{\sqrt{5}}$

$$\left|\frac{k+4}{\sqrt{4+1}}\right| = \frac{10}{\sqrt{5}}$$

$$|k + 4| = 10$$

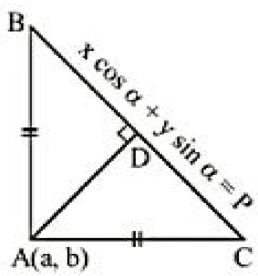
$k = 6$ or $k = -14$

hence the 4^{th} sides of squares are $2x - y + 6 = 0$ or $2x - y - 14 = 0$

Illustration :

Two mutually perpendicular lines are drawn through the point (a, b) and enclose an isosceles triangle together with the line $x \cos\alpha + y \sin\alpha = P$. Find the area of triangle.

Sol. ΔABC is right angled at A.
$AB = AC$
AD perpendicular BC



Length of perpendicular from A (a, b) $= \left|\frac{a\cos\alpha + b\sin\alpha - P}{\sqrt{\cos^2\alpha + \sin^2\alpha}}\right|$

$= |a\cos\alpha + b\sin\alpha - P|$

Since ΔABC is isosceles $\therefore$ $AD = BD = DC$

$BC = 2\,(AD)$

$= 2\,|(a\cos\alpha + b\sin\alpha - P)|$

Area of $\Delta ABC = \frac{1}{2} \times |(a\cos\alpha + b\sin\alpha - P)| \cdot 2|(a\cos\alpha + b\sin\alpha - P)|$

$= (a\cos\alpha + b\sin\alpha - P)^2$

Illustration :

Find the condition so that lines $a_1x + b_1y + c_1 = 0$, $a_2x + b_2y + c_2 = 0$, $a_1x + b_1y + c_2 = 0$ and $a_2x + b_2y + c_1 = 0$, form a rhombus.

Sol. *If the given lines form a rhombus then perpendicular distances between opposite sides are equal*

Distance between $a_1x + b_1y + c_1 = 0$ & $a_1x + b_1y + c_2 = 0$

$$D_1 = \left|\frac{c_1 - c_2}{\sqrt{a_1^2 + b_1^2}}\right|$$

Distance between $a_2x + b_2y + c_1 = 0$ & $a_2x + b_2y + c_2 = 0$

$$D_2 = \left|\frac{c_1 - c_2}{\sqrt{a_2^2 + b_2^2}}\right|$$

$D_1 = D_2$

$$\left|\frac{c_1 - c_2}{\sqrt{a_1^2 + b_1^2}}\right| = \left|\frac{c_1 - c_2}{\sqrt{a_2^2 + b_2^2}}\right|$$

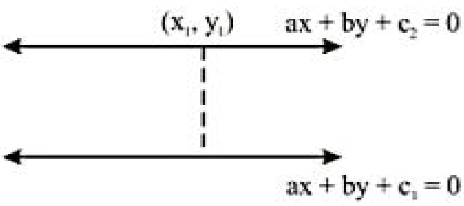
$$\Rightarrow \quad a_1^2 + b_1^2 = a_2^2 + b_2^2$$

15.1 DISTANCE BETWEEN TWO PARALLEL LINES :

Let (x_1, y_1) be any point on the line $ax + by + c_2 = 0$

Distance of point (x_1, y_1) from the line $ax + by + c_1 = 0$ is

$$p = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$$



Now point (x_1, y_1) lies on $ax + by + c_2 = 0$ then

$$ax_1 + by_1 + c_2 = 0$$

$$\Rightarrow \quad ax_1 + by_1 = -c_2$$

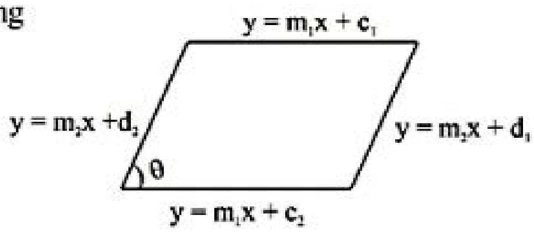
$$\Rightarrow \quad p = \frac{|c_1 - c_2|}{\sqrt{a^2 + b^2}}$$

15.2 AREA OF THE PARALLELOGRAM :

Area of the | |gm whose 4 sides are as shown in the fig. using

$A = p_1 p_2 \operatorname{cosec}\theta$ is given by

$$\left|\frac{(c_1 - c_2)(d_1 - d_2)}{m_1 - m_2}\right| = \left(\frac{p_1 p_2}{\sin\theta}\right)$$



Practice Problem

Single correct question

Q.1 The product of the perpendiculars drawn from the two points $\left(\pm\sqrt{a^2-b^2}, 0\right)$ upon the straight line $\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1$ is

(A) a^2 (B) b^2 (C) $a^2 + b^2$ (D) $2ab$

Q.2 Le the co-ordinates of the two points A and B be (1, 2) and (7, 5) respectively. The line AB is rotated through 45° in anti clockwise direction about the point of trisection of AB which is nearer to B. The equation of the line in new position is :

(A) $2x - y - 6 = 0$ (B) $x - y - 1 = 0$ (C) $3x - y - 11 = 0$ (D) None of these

Q.3 The line L_1 given by $\frac{x}{5} + \frac{y}{b} = 1$ passes through the point M (13, 32). The line L_2 is parallel to L_1 and has the equation $\frac{x}{c} + \frac{y}{3} = 1$. Then the distance between L_1 and L_2 is

(A) $\sqrt{17}$ (B) $\frac{17}{\sqrt{15}}$ (C) $\frac{23}{\sqrt{15}}$ (D) $\frac{23}{\sqrt{15}}$

Multiple correct type question

Q.4 The equations of the straight lines which pass through the origin and are inclined at 75° to the straight line $x + y + \sqrt{3}(y - x) = a$ are

(A) $x = 0$ (B) $y = 0$ (C) $\sqrt{3}x + y = 0$ (D) $x + \sqrt{3}y = 0$

Q.5 The equation(s) of the straight lines drawn through the point (0, 1) on which the perpendiculars let fall from the point (2, 2) are each of length unity, is(are)

(A) $x - y = 1$ (B) $x - 1 = 0$ (C) $y - 1 = 0$ (D) $4x - 3y + 3 = 0$

Answer key

Q.1 B Q.2 C Q.3 C Q.4 AC Q.5 CD

16. PARAMETRIC FORM OF A LINE :

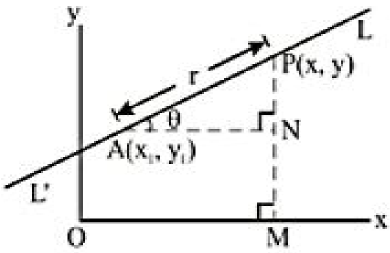
The equation of straight line passing through a given point A (x_1, y_1) and making an angle θ from positive direction of x-axis in anticlockwise sence is –

$$\frac{x-x_1}{\cos\theta}=\frac{y-y_1}{\sin\theta}=\pm r$$

where, r is the distance of any point on the line from the given point $A(x_1, y_1)$.

Explanation :

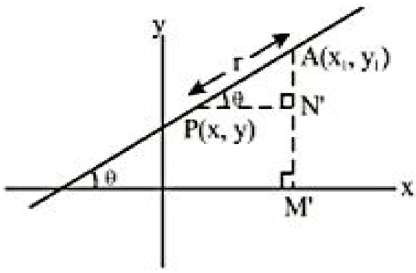
Let P(x, y) be taken on the line above the given point (x_1, y_1) then from the ΔPAN.



$$x-x_1=r\cos\theta$$
$$y-y_1=r\sin\theta$$

$$\frac{x-x_1}{\cos\theta}=\frac{y-y_1}{\sin\theta}=r \qquad \text{... (1)}$$

Again, if point is taken on the line below the given point $A(x_1, y_1)$ then from the ΔAPN'



$$x_1-x=r\cos\theta$$
$$y_1-y=r\sin\theta$$

$$\therefore \qquad \frac{x-x_1}{\cos\theta}=\frac{y-y_1}{\sin\theta}=-r \qquad \text{... (2)}$$

Combining (1) & (2)

$$\frac{x-x_1}{\cos\theta}=\frac{y-y_1}{\sin\theta}=\pm r$$

Here, $x=x_1\pm r\cos\theta$, $y=y_1\pm r\sin\theta$ are the co-ordinates of the points situated on the line at a distance r from the given point $A(x_1, y_1)$.

Illustration :

A straight line is drawn through the point P(2, 3) and is inclined at an angle of 30° with the x-axis. Find the coordinates of two points on it at a distance 4 from P.

Sol. *Here $(x_1, y_1) = (2, 3)$, $\theta = 30°$, the equation of the line is*

$$\frac{x-2}{\cos 30^\circ}=\frac{y-3}{\sin 30^\circ}$$

$$\Rightarrow \qquad \frac{x-2}{\frac{\sqrt{3}}{2}}=\frac{y-3}{\frac{1}{2}}$$

$$\Rightarrow \qquad x-2=\sqrt{3}\,(y-3)$$

$$\Rightarrow \qquad x-\sqrt{3}\,y=2-3\sqrt{3}$$

Points on the line at a distance 4 from P(2, 3) are

$$(x_1\pm r\cos\theta,\ y_1\pm r\sin\theta)$$

$$\Rightarrow \qquad (2\pm 4\cos 30^\circ,\ 3\pm 4\sin 30^\circ)$$

$$\Rightarrow \qquad (2\pm 2\sqrt{3},\ 3\pm 2) \Rightarrow (2+2\sqrt{3},\ 5) \text{ or } (2-2\sqrt{3},\ 1)$$

Illustration :

Find the equation of the line passing through the point A(2, 3) and making an angle of 45° with the x-axis. Also determine the length of intercept on it between A and the line $x + y + 1 = 0$.

Sol. The equation of a line passing through A and making an angle of 45° with the x-axis is

$$\frac{x-2}{\cos 45^\circ} = \frac{y-3}{\sin 45^\circ}$$

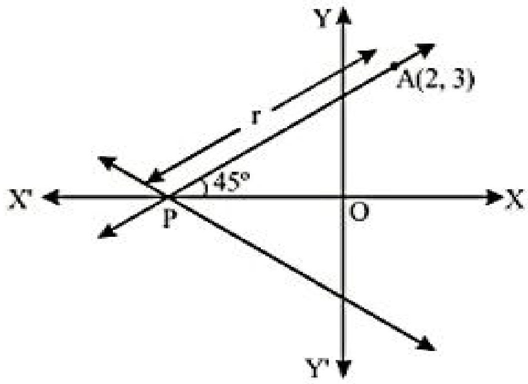
$$\frac{x-2}{\frac{1}{\sqrt{2}}} = \frac{y-3}{\frac{1}{\sqrt{2}}}$$

$$\Rightarrow \quad x - y + 1 = 0$$

Suppose this line meets the line $x + y + 1 = 0$ at P such that $AP = r$.

Then, the coordinates of P are given by

$$\frac{x-2}{\cos 45^\circ} = \frac{y-3}{\sin 45^\circ} = r$$



$$\Rightarrow \quad x = 2 + r\cos 45^\circ,\ y = 3 + r\sin 45^\circ$$

$$\Rightarrow \quad x = 2 + \frac{r}{\sqrt{2}},\ y = 3 + \frac{r}{\sqrt{2}}$$

Thus, the coordinates of P are $\left(2+\frac{r}{\sqrt{2}}, 3+\frac{r}{\sqrt{2}}\right)$

Since P lies on $x + y + 1 = 0$. Therefore,

$$2+\frac{r}{\sqrt{2}}+3+\frac{r}{\sqrt{2}}+1=0$$

$$\Rightarrow \quad \sqrt{2}r = -6 \quad \Rightarrow \quad r = -3\sqrt{2}$$

Therefore, length $AP = |r| = 3\sqrt{2}$

Illustration :

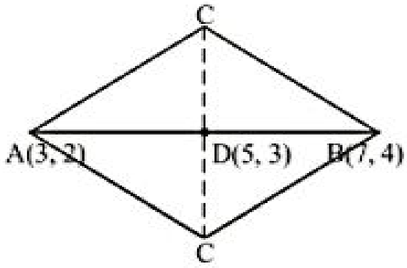
A (3, 2) and B(7, 4) are two vertices of a triangle. Find the third vertex C so that ABC is an equilateral triangle.

Sol. Let the point be C(h, k). If the triangle is equilateral then length of altitude is $\frac{\sqrt{3}}{2}$ times the length of side.

Length of side (AB) = $\sqrt{4^2+2^2} = \sqrt{20}$

Length of altitude = $\sqrt{15}$

coordinates of point D = (5, 3)



slope of side (AB) $= \frac{1}{2}$

slope of altitude $= -2$

$\tan\theta = -2$

$\sin\theta = \frac{2}{\sqrt{5}}, \cos\theta = \frac{-1}{\sqrt{5}}$

Equation of line

$$\frac{x-h}{\cos\theta} = \frac{y-k}{\sin\theta} = \pm\sqrt{15}$$

$$\frac{x-5}{\frac{-1}{\sqrt{5}}} = \frac{y-3}{\frac{2}{\sqrt{5}}} = \pm\sqrt{15}$$

The points are

$$\left(5-\sqrt{3},\ 3+2\sqrt{3}\right) \quad \text{and} \quad \left(5+\sqrt{3},\ 3-2\sqrt{3}\right)$$

Illustration :

A line through A(–5, –4) meets the line $x + 3y + 2 = 0$, $2x + y + 4 = 0$ *and* $x - y - 5 = 0$ *at B, C and D. If* $(15/AB)^2 + (10/AC)^2 = (6/AD)^2$ *then find the equation of the line.*

Sol. *Let the inclination of line be* θ *then equation of line through point A (–5, –4) is*

$$\frac{x+5}{\cos\theta} = \frac{y+4}{\sin\theta} = r_i \qquad \text{where } i = 1, 2, 3$$

$r_1 = AB, \qquad r_2 = AC, \quad \text{and } r_3 = AD$

$\therefore$ *point B* $(r_1\cos\theta - 5, r_1\sin\theta - 4)$

Points B satisfying the line $x + 3y + 2 = 0$

$\therefore$ $(r_1\cos\theta - 5) + 3(r_1\sin\theta - 4) + 2 = 0$

or $r_1(\cos\theta + 3\sin\theta) - 5 - 12 + 2 = 0$ *or* $\cos\theta + 3\sin\theta = \frac{15}{r_1} = \frac{15}{AB}$

Similarly

$$2\cos\theta + \sin\theta = \frac{10}{AC}$$

and $\cos\theta - \sin\theta = \frac{6}{AD}$

Using the given relation

$$\left(\frac{15}{AB}\right)^2 + \left(\frac{10}{AC}\right)^2 = \left(\frac{6}{AD}\right)^2$$

$\Rightarrow \quad (\cos\theta + 3\sin\theta)^2 + (2\cos\theta + \sin\theta)^2 = (\cos\theta - \sin\theta)^2$

$\Rightarrow \quad (1 + 3\tan\theta)^2 + (2 + \tan\theta)^2 = (1 - \tan\theta)^2$

$\Rightarrow \quad 9\tan^2\theta + 12\tan\theta + 4 = 0$

$\Rightarrow \quad (3\tan\theta + 2)^2 = 0$

$\Rightarrow \quad \tan\theta = \dfrac{-2}{3}$

Equation of line is $y + 4 = \dfrac{-2}{3}(x + 5)$

$\Rightarrow \quad 3y + 2x + 22 = 0$

17. POSITION OF A POINT W.R.T. A LINE :

If the points $P(x_1, y_1)$ and $Q(x_2, y_2)$ lies on the same side of the line $Ax + Bx + C = 0$ then the expressions $Ax_1 + By_1 + C$ and $Ax_2 + By_2 + C$ will be of the same sign and if $P(x_1, y_1)$ and $Q(x_2, y_2)$ are on the opposite side of the line then the expressions $Ax_1 + By_1 + C$ and $Ax_2 + By_2 + C$ will be of opposite sign.

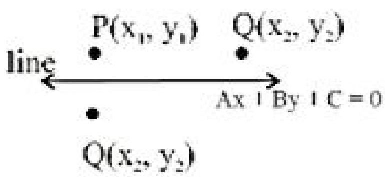


Illustration :

Are the points (3, 4) and (2, –6) on the same or opposite sides of the line $3x - 4y = 8$?

Sol. *Let* $L = 3x - 4y - 8$. *Then the value of L at (3, 4) is* $L_1 = -15$ *and the value of L at (2, –6) is* $L_2 = 22$. *Since* L_1 *and* L_2 *are of opposite signs, therefore the two points are on the opposite sides of the given line.*

Illustration :

If the point (a, a^2) *lies between the lines* $x + y - 2 = 0$ *and* $4x + 4y - 3 = 0$ *then find the range of values of a.*

Sol. *If* (a, a^2) *lies between the lines* $x + y - 2 = 0$ *and* $4x + 4y - 3 = 0$ *then sign of* $a + a^2 - 2$ *and* $4a + 4a^2 - 3$ *should be opposite hence*

$\quad (a + a^2 - 2)(4a + 4a^2 - 3) < 0$

$\Rightarrow \quad (a - 1)(a + 2)(2a + 3)(2a - 1) < 0$

$\Rightarrow \quad a \in \left(-2, \dfrac{-3}{2}\right) \cup \left(\dfrac{1}{2}, 1\right)$

Practice Problem

Single correct question

Q.1 If points $(\sin\theta, \cos\theta)$ and $(3, 2)$ lies on the same side of the line $x + y = 1$, then θ lies between

(A) $\left(0, \frac{\pi}{4}\right)$ (B) $\left(\frac{\pi}{4}, \frac{\pi}{2}\right)$ (C) $\left(\frac{\pi}{2}, \frac{3\pi}{4}\right)$ (D) $\left(\frac{3\pi}{4}, \pi\right)$

Q.2 A triangle ABC is formed by the lines $2x - 3y - 6 = 0$; $3x - y + 3 = 0$ and $3x + 4y - 12 = 0$. If the points $P(\alpha, 0)$ and $Q(0, \beta)$ always lie on or inside the ΔABC, then :

(A) $\alpha \in [-1, 2]$ and $\beta \in [-2, 3]$ (B) $\alpha \in [-1, 3]$ and $\beta \in [-2, 4]$

(C) $\alpha \in [-2, 4]$ and $\beta \in [-3, 4]$ (D) $\alpha \in [-1, 3]$ and $\beta \in [-2, 3]$

Q.3 A line $4x + y = 1$ passes through the point A $(2, -7)$ meets the line BC whose equation is $3x - 4y + 1 = 0$ at the point B. The equation to the line AC so that AB = AC, is

(A) $52x + 89y - 519 = 0$ (B) $52x + 89y + 519 = 0$

(C) $89x + 52y + 519 = 0$ (D) $89x + 52y - 519 = 0$

Q.4 If the straight line through the point P(3, 4) makes an angle $\frac{\pi}{6}$ with the x-axis and meets the line $12x + 5y + 10 = 0$ at Q, then the length PQ is

(A) $\frac{132}{5\sqrt{3}+12}$ (B) $\frac{132}{5\sqrt{3}-12}$ (C) $\frac{132}{12\sqrt{3}+5}$ (D) $\frac{132}{12\sqrt{3}-5}$

Multiple correct type question

Q.5 If the slope of a line passing through the point A (3, 2) be 3/4, then the points on the line which are 5 units away from A, are

(A) (5, 5) (B) (7, 5) (C) (–1, –1) (D) (3, 4)

Q.6 The direction in which a straight line must be drawn through the point (1, 2), so that its point of intersection with the line $x + y = 4$ may be at a distance $\frac{1}{3}\sqrt{6}$ from this point.

(A) 15° (B) 30° (C) 60° (D) 75°

Answer key

Q.1 A	Q.2 D	Q.3 B	Q.4 C	Q.5 BC	Q.6 AD

18. CONCURRENCY OF THREE LINES :

Three lines are said to be concurrent if they pass through a common point, i.e. they meet at a point. Thus, if three line are concurrent the point of intersection of two lines lies on the third line. Let the three concurrent lines be

$$a_1x + b_1y + c_1 = 0 \qquad \ldots(i)$$
$$a_2x + b_2y + c_2 = 0 \qquad \ldots(ii)$$
$$a_3x + b_3y + c_3 = 0 \qquad \ldots(iii)$$

Then the point of intersection of equation (i) and (ii) must lie on the third.

The coordinates of the point of intersection of equation (i) and (ii) are

$$\left(\frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1}, \frac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1}\right)$$

This point lies on the line (iii). Therefore, we get

$$\Rightarrow \quad a_3\left(\frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1}\right) + b_3\left(\frac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1}\right) + c_3 = 0$$

$$\Rightarrow \quad a_3 (b_1c_2 - b_2c_1) + b_3 (c_1a_2 - c_2a_1) + c_3 (a_1b_2 - a_2b_1) = 0$$

$$\Rightarrow \quad \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$$

This is the required condition of concurrency of three lines.

Alternative Method :

Three lines $L_1 = a_1x + b_1y + c_1 = 0$; $L_2 = a_2x + b_2y + c_2 = 0$; $L_3 = a_3x + b_3y + c_3 = 0$ are concurrent iff there exist constants $\lambda_1, \lambda_2, \lambda_3$ not all zero at the same time so that $\lambda_1L_1 + \lambda_2L_2 + \lambda_3L_3 = 0$, i.e., $\lambda_1 (a_1x + b_1y + c_1) + \lambda_2 (a_2x + b_2y + c_2) + \lambda_3 (a_3x + b_3y + c_3) = 0$.

Illustration :

If lines $(\cos^2 A)x + (\cos A)\cdot y + 1 = 0$, $(\cos^2 B)x + (\cos B)y + 1 = 0$ and $(\cos^2 C)x + (\cos C)y + 1 = 0$ are concurrent, where A, B, C are angle of a triangle then prove that the triangle must be isosceles.

Sol. *If the given three lines are concurrent then*

$$\begin{vmatrix} \cos^2 A & \cos A & 1 \\ \cos^2 B & \cos B & 1 \\ \cos^2 C & \cos C & 1 \end{vmatrix} = 0$$

$R_1 \to R_1 - R_2,\ R_2 \to R_2 - R_3$

$$\begin{vmatrix} \cos^2 A - \cos^2 B & \cos A - \cos B & 0 \\ \cos^2 B - \cos^2 C & \cos B - \cos C & 0 \\ \cos^2 C & \cos C & 1 \end{vmatrix} = 0$$

$= (\cos A - \cos B)\,(\cos B - \cos C)\,(\cos C - \cos A) = 0$

hence $\cos A = \cos B$ *or* $\cos B = \cos C$ *or* $\cos C = \cos A$

we can say that triangle is isosceles.

Illustration :

If the lines $p_1x + q_1y + 1 = 0$, $p_2x + q_2y + 1 = 0$ *and* $p_3x + q_3y + 1 = 0$ *are concurrent then prove that the points* (p_1, q_1), (p_2, q_2) *and* (p_3, q_3) *are collinear.*

Sol. *If the given lines are concurrent then*

$$\begin{vmatrix} p_1 & q_1 & 1 \\ p_2 & q_2 & 1 \\ p_3 & q_3 & 1 \end{vmatrix} = 0$$

Above determinant also shows twice the area of triangle formed by points (p_1, q_1), (p_2, q_2) *and* (p_3, q_3)

Since area of triangle formed by these points is zero, hence the given points are collinear.

Illustration :

Let λ & $\alpha \in R$. *If lines* $\left.\begin{array}{l} \lambda x + (\sin\alpha)y + \cos\alpha = 0 \\ x + (\cos\alpha)y + \sin\alpha = 0 \\ -x + (\sin\alpha)y - (\cos\alpha) = 0 \end{array}\right\}$; *are concurrent, then find the set of values of* λ.

Sol.

$$\begin{vmatrix} \lambda & \sin\alpha & \cos\alpha \\ 1 & \cos\alpha & \sin\alpha \\ -1 & \sin\alpha & -\cos\alpha \end{vmatrix} = 0$$

Applying $R_1 \to R_1 - R_3$

$$\begin{vmatrix} \lambda + 1 & 0 & 2\cos\alpha \\ 1 & \cos\alpha & \sin\alpha \\ -1 & \sin\alpha & -\cos\alpha \end{vmatrix} = 0$$

$\Rightarrow\quad (\lambda + 1)\,[-\cos^2\alpha - \sin^2\alpha] + 2\cos\alpha\,[\sin\alpha + \cos\alpha] = 0$

$\Rightarrow\quad -(\lambda + 1) + 2\cos\alpha\,[\sin\alpha + \cos\alpha]$

$\Rightarrow\quad \lambda + 1 = 2\cos\alpha\,[\sin\alpha + \cos\alpha]$

$\Rightarrow\quad \lambda = \sin 2\alpha + \cos 2\alpha$

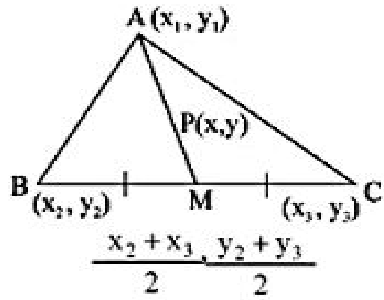
$\Rightarrow\quad \lambda \in \left[-\sqrt{2}, \sqrt{2}\right]$

19. EQUATION OF STRAIGHT LINE IN DETERMINANT FORM :

(i) Line passing through two points (x_1,y_1) and (x_2, y_2) is $\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$

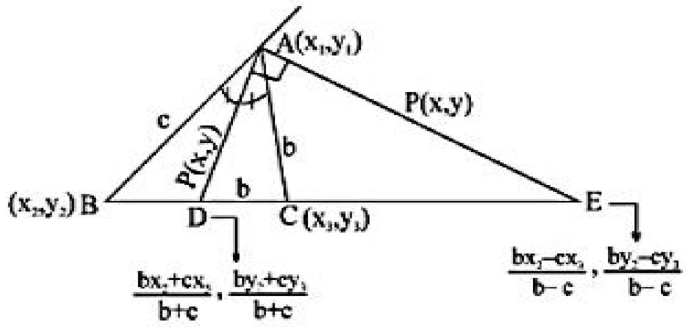
(ii) Equation of the median through $A(x_1, y_1)$ is

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ \frac{x_2+x_3}{2} & \frac{y_2+y_3}{2} & 1 \end{vmatrix} = 0 \text{ or } \begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} + \begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$$



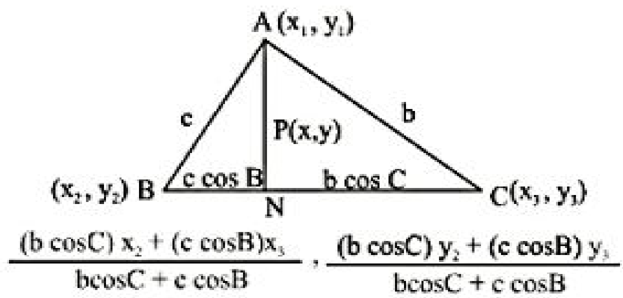
(iii) Equation of internal and external angle bisectors of A are

$$b\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} \pm c\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$$



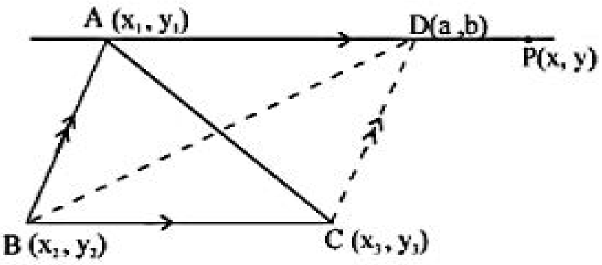
(iv) Equation of the altitude through 'A' is

$$b\cos C\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} + c\cos B\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$$



(v) Equation of the line through A and parallel to the base BC is

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ a & b & 1 \end{vmatrix} = 0$$ where (a, b) are assumed to be co-ordinates of D.



Now , equating the middle point of BD and AC

$a + x_2 = x_1 + x_3 \Rightarrow a = x_1 - x_2 + x_3$

$b + y_2 = y_1 + y_3 \Rightarrow b = y_1 - y_2 + y_3$

Hence the equation of the line is

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_1 + x_3 - x_2 & y_1 + y_3 - y_2 & 1 \end{vmatrix} = 0$$

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_3 - x_2 & y_3 - y_2 & 1-1 \end{vmatrix} = 0 \Rightarrow \begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} - \begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$$

Illustration :

Find the equation of median through A if three points of triangle are given by A(2, 1), B(3, 6), C(1, 0)

Sol. *Mid-point of BC = (2, 3)*

Equation of line passing through (2, 1) and (2, 3)

$$\begin{vmatrix} x & y & 1 \\ 2 & 1 & 1 \\ 2 & 3 & 1 \end{vmatrix} = 0$$

$x(1-3) - y(2-2) + (6-2) = 0$

$-2x + 4 = 0$

$x = 2$

Practice Problem

Single correct question

Q.1 Let (x_1, y_1); (x_2, y_2) and (x_3, y_3) are the vertices of a triangle ABC respectively. D is a point on BC such that BC = 3BD. The equation of the line through A and D, is

(A) $\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} + 2\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$

(B) $3\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} + 2\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$

(C) $\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} + 3\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$

(D) $2\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} + \begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$

Q.2 Let (x_r, y_r) r = 1, 2, 3 are the coordinates of the vertices of a triangle ABC. If D is the point on BC dividing it in the ratio of 1 : 2 reckoning from the vertex B, then the equation of the line AE in the similar from where E is the harmonic conjugate of D w.r.t. the points B and C.

(A) $\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} - 2\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$

(B) $2\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} - \begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$

(C) $2\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} - 3\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$

(D) $3\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} - 2\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$

Q.3 If the line $x + 2ay + a = 0$, $x + 3by + b = 0$ & $x + 4cy + c = 0$ are concurrent then
(A) a, b, c are in A.P. (B) a, b, c are in G.P. (C) a, b, c are in H.P. (D) None of these

Multiple correct type question

Q.4 If $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$ are the vertices of a triangle, then the equation

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} + \begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0 \text{ represents}$$

(A) the median through A
(B) the altitude through A
(C) the perpendicular bisector of BC
(D) the line joining the centroid with a vertex A

Q.5 The line $x + y - 1 = 0$, $(m - 1)x + (m^2 - 7)y - 5 = 0$ and $(m - 2)x + (2m - 5)y = 0$ are
(A) concurrent for three values of m
(B) concurrent for one value of m
(C) concurrent for no value of m
(D) are parallel for m = 3

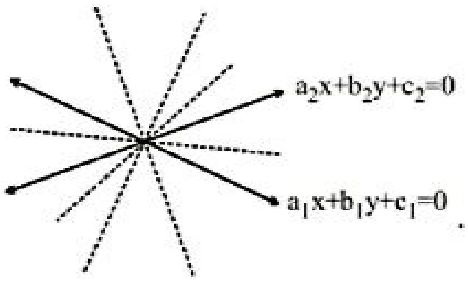
Answer key

Q.1	D	Q.2	B	Q.3	C	Q.4	AD	Q.5	CD

20. FAMILY OF STRAIGHT LINES :

Let $L_1 \equiv a_1x + b_1y + c_1 = 0$ and $L_2 \equiv a_2x + b_2y + c_2 = 0$

Then, the general equation of any straight line passing through the point of intersection of lines L_1 and L_2 is given by $L_1 + \lambda L_2 = 0$, where $\lambda \in R$



These lines form a family of straight line

Also this general equation satisfies point of intersection of L_1 and L_2 for any value of λ.

Conversely, if a variable line is expressed in the form of $L_1 + \lambda L_2 = 0$ $(\lambda \in R)$ then it always passes through fixed point which is the point of intersection of $L_1 = 0$ and $L_2 = 0$.

Illustration :

Two lines are given by $L_1 : 3x - 4y + 6 = 0$ & $L_2 : x + y + 2 = 0$. Find the equation of line passing through the intersection by $L_1 = 0$ & $L_2 = 0$, if

(i) It passes through (1, 2)

(ii) It is parallel to $y = 2x + 3$.

(iii) It has intercepts equal in magnitude but opposite in sign.

(iv) It is parallel to x-axis

(v) It is at a distance of $\sqrt{2}$ units from origin.

Sol. Let equation of line through the intersection of L_1 & L_2 is

$$L_1 + \lambda L_2 = 0$$

$$3x - 4y + 6 + \lambda (x + y + 2) = 0$$

$$x (3 + \lambda) + y (\lambda - 4) + 6 + 2\lambda = 0$$

(i) If it passes through (1, 2)

$$3 + \lambda + 2\lambda - 8 + 6 + 2\lambda = 0$$

$$5\lambda = -1 \quad \Rightarrow \quad \lambda = -1/5$$

Equation of line becomes

$$x\left(3 - \frac{1}{5}\right) + y\left(\frac{-1}{5} - 4\right) + 6 + 2\left(\frac{-1}{5}\right) = 0$$

$$14x - 21y + 28 = 0 \Rightarrow 2x - 3y + 4 = 0$$

(ii) If it is parallel to $y = 2x + 3$

$$-\left(\frac{3+\lambda}{\lambda-4}\right) = 2$$

$$-3 - \lambda = 2\lambda - 8$$

$$3\lambda = 5$$

$$\lambda = 5/3$$

Equation becomes $x\left(3 + \frac{5}{3}\right) + y\left(\frac{5}{3} - 4\right) + 6 + 2\left(\frac{5}{3}\right) = 0$

$$14x - 7y + 28 = 0 \Rightarrow 2x - y + 4 = 0$$

(iii) It has intercepts equal in magnitude but opposite in sign, then

slope = 1

$$\frac{-(3+\lambda)}{\lambda - 4} = 1$$

$$-3 - \lambda = \lambda - 4$$

$$2\lambda = 1$$

$$\lambda = 1/2$$

$$x - y + 2 = 0$$

(iv) *Its is parallel to x-axis*

slope = 0

$$\frac{-(3+\lambda)}{\lambda-4}=0$$

Equation becomes

$$x(3-3)+y(-3-4)+6+2(-3)=0$$

$$y=0$$

(v) *It is at a distance of* $\sqrt{2}$ *units from origin equation of line* $(3+\lambda)x+y(\lambda-4)+6+2\lambda=0$

Perpendicular distance from origin

$$\left|\frac{0(3+\lambda)+0(\lambda-4)+6+2\lambda}{\sqrt{(\lambda+3)^2+(\lambda-4)^2}}\right|=\sqrt{2}$$

$$\left|\frac{(2\lambda+6)}{\sqrt{(\lambda+3)^2+(\lambda-4)^2}}\right|=\sqrt{2}$$

$$(2\lambda+6)^2=2[2\lambda^2-2\lambda+25]$$

$$4\lambda^2+24\lambda+36=4\lambda^2-4\lambda+50$$

$$28\lambda=14$$

$$\lambda=1/2$$

$\therefore$ *Equation of the line is* $x-y+2=0$

Illustration :

It the family of straight lines $x(a+2b)+y(a+3b)=a+b$ *passes through a fixed point for all values of a and b. Find the point.*

Sol. $x(a+2b)+y(a+3b)=a+b$

$\Rightarrow\quad a(x+y-1)+b(2x+3y-1)=0$

This equation will always be satisfied for $x+y-1=0$ & $2x+3y-1=0$ *solving these equation we get*

$$x=2,\ y=-1$$

Illustration :

A variable line $ax+by+c=0$ *passes through a fixed point if a, b, c are in arithmetic progression. Find the fixed pont.*

Sol. *If a, b, c are in A.P. then*

$$2b=a+c$$

$$ax+by+c=0$$

$$ax+\left(\frac{a+c}{2}\right)y+c=0$$

$$a\left(x+\frac{y}{2}\right)+c\left(\frac{y}{2}+1\right)=0$$

Solving $\quad x+\frac{y}{2}=0\quad$ & $\quad\frac{y}{2}+1=0,\quad$ *we get*

$$y=-2,\qquad x=1$$

Illustration :

If $a^2 + 9b^2 = 6ab + 4c^2$ and $ax + by + c = 0$ is a straight line that passes through one or the other of the two fixed points. Find the points.

Sol. *Given*

$$a^2 + 9b^2 = 6ab + 4c^2$$
$$a^2 - 6ab + 9b^2 - 4c^2 = 0$$
$$(a - 3b)^2 - (2c)^2 = 0$$
$$(a - 3b - 2c) \quad or \quad (a - 3b + 2c) = 0$$

Case-I

$$a = 3b + 3c$$
$$ax + by + c = 0$$
$$(3b + 2c)x + by + c = 0$$
$$b\,(3x + y) + c\,(2x + 1) = 0$$
$$3x + y = 0, \qquad 2x + 1 = 0$$
$$x = \frac{-1}{2}, \qquad y = \frac{3}{2}$$

Case II

$$If\ a = 3b - 2c$$
$$ax + by + c = 0$$
$$(3b - 2c)x + by + c = 0$$
$$b\,(3x + y) + c\,(-2x + 1) = 0$$
$$3x + y = 0, \qquad -2x + 1 = 0$$
$$x = \frac{1}{2}, \qquad y = \frac{-3}{2}$$

Hence it passes through one of the fixed points

$$\left(-\frac{1}{2}, \frac{3}{2}\right) \quad or \quad \left(\frac{1}{2}, \frac{-3}{2}\right)$$

Illustration :

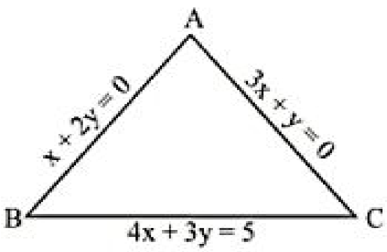
The equations of the sides of a triangle are $x + 2y = 0$, $4x + 3y = 5$ and $3x + y = 0$. Find the co-ordinate of the orthocentre of the triangle without finding the vertices of triangle.

Sol. *Equation of line passing through A can be given by*

$x + 2y + \lambda\,(3x + y) = 0$

$x\,(1 + 3\lambda) + y\,(2 + \lambda) = 0$

Altitude through A is $\perp$ to BC ($4x + 3y - 5 = 0$)



Slope of altitude through A is $= \dfrac{-1}{-\frac{4}{3}} = \dfrac{3}{4}$

$$\frac{-(3\lambda + 1)}{\lambda + 2} = \frac{3}{4}$$
$$-12\lambda - 4 = 3\lambda + 6$$
$$15\lambda = -10$$
$$\lambda = -\frac{2}{3}$$

Equation of altitude A is $x(-1) + y\left(\frac{4}{3}\right) = 0$

$$-3x + 4y = 0 \qquad \dots (1)$$

Equation of line passing through B is

$$x + 2y + k(4x + 3y - 5) = 0$$
$$x(1 + 4k) + y(2 + 3k) - 5k = 0$$

line perpendicular to $3x + y = 0$*, has slope* $= \frac{1}{3}$

$$-\left(\frac{4k+1}{3k+2}\right) = \frac{1}{3}$$
$$-(12k + 3) = 3k + 2$$
$$-15k = 5$$
$$k = -\frac{1}{3}$$

Equation of altitude through B

$$x\left(1 - \frac{4}{3}\right) + y(2 - 1) - 5\left(-\frac{1}{3}\right) = 0$$
$$-x + 3y + 5 = 0$$
$$x - 3y - 5 = 0 \qquad \dots (2)$$

solving (1) and (2) we get orthocentre as $(-4, -3)$

Illustration :

Find equation of the diagonals of the parallelogram formed by the lines

$$2x - y + 7 = 0,\ 2x - y - 5 = 0,\ 3x + 2y - 5 = 0 \ \text{ and } \ 3x + 2y + 4 = 0$$

Sol. *Equation of diagonals can be formed by*

$$\ell_1\ell_2 - \ell_3\ell_4 = 0 \quad \text{or} \quad \ell_1\ell_4 - \ell_2\ell_3 = 0$$
$$\ell_1 : 2x - y + 7 = 0, \qquad \ell_2 : 3x + 2y - 5 = 0$$
$$\ell_3 : 2x - y - 5 = 0, \qquad \ell_4 : 3x + 2y + 4 = 0$$
$$(2x - y + 7)(3x + 2y - 5) - (2x - y - 5)(3x + 2y + 4) = 0$$
$$\Rightarrow \quad -5(2x - y) + 7(3x + 2y) - 35 - 4(2x - y) + 5(3x + 2y) + 20 = 0$$
$$\Rightarrow \quad -10x + 5y + 21x + 14y - 35 - 8x + 4y + 15x + 10 + 20 = 0$$
$$18x + 33y - 15 = 0$$
$$6x + 11y - 5 = 0$$

Other diagonal

$$(2x - y + 7)(3x + 2y + 4) - (2x - y - 5)(3x + 2y - 5) = 0$$
$$4(2x - y) + 7(3x + 2y) + 28 + (2x - y)5 + 5(3x + 2y) - 25$$
$$18x + 5y + 1 = 0$$

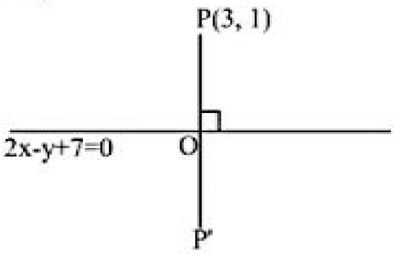
Optics Based Problem :

Illustration :

Find the image of (3, 1) across the line $y = 2x + 7$.

Sol. *Let the point P(3, 1) has image P′ across the line $2x - y + 7 = 0$*

Now PP′ is perpendicular to $2x - y + 7 = 0$



$$\text{Slope of } PP' = -\frac{1}{2}$$

Equation of PP′ is

$$y - 1 = -\frac{1}{2}(x - 3)$$

$$2y - 2 = -x + 3$$

$$x + 2y = 5$$

Point of intersection of lines $2x - y + 7 = 0$ and $x + 2y = 5$ is $O\left(\frac{-9}{5}, \frac{17}{5}\right)$

O is the mid point of PP′

Let P′ (h, k)

$$\frac{h+3}{2} = \frac{-9}{5}, \quad \frac{k+1}{2} = \frac{17}{5}$$

$$h = \frac{-33}{5}, \ k = \frac{29}{5} \Rightarrow \text{Image}\left(-\frac{33}{5}, \frac{29}{5}\right)$$

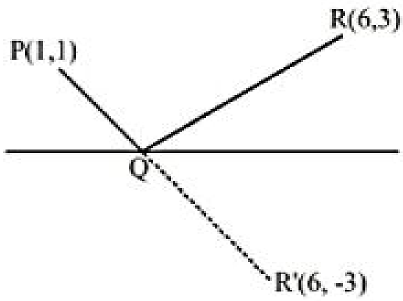
Illustration :

A ray starts from point (1, 1) and is reflected by x-axis and then it passes through the point (6, 3). Find the equation of

(i) Incident ray   *(ii) Reflected ray*

Sol. *If the ray started from (1, 1) and after reflection it passes through (6, 3) then incident ray was supposed to pass from the image of (6, 3) across x-axis.*

Image of R(6, 3) in x-axis is R′(6, –3)



$$\text{Equation of } PQ \Rightarrow y - 1 = \left(\frac{-3-1}{6-1}\right)(x - 1)$$

$$5y - 5 = -4x + 4$$

$$4x + 5y - 9 = 0$$

to find equaiton of reflected ray we have to find Q.

Q is point of intersection of ray with x-axis $Q\left(\frac{9}{4}, 0\right)$

Equation of QR.

$$y - 3 = \left(\frac{3-0}{6-9/4}\right)(x - 6)$$

$$y - 3 = \frac{4}{5}(x - 6)$$

$$5y - 15 = 4x - 24$$

$$4x - 5y - 9 = 0$$

Practice Problem

Single correct question

Q.1 The equation of the line through the point of intersection of the lines $5x - 3y = 1$ and $2x + 3y = 23$ and perpendicular to the line $5x - 3y - 1 = 0$ is

(A) $33x - 47y + 21 = 0$ (B) $3x + 22y + 14 = 0$

(C) $63x + 105y - 781 = 0$ (D) $21x - 84y + 1023 = 0$

Q.2 The line $(k + 1)^2 x + ky - 2k^2 - 2 = 0$ passes through a point regardless of the value of k. Which of the following is the line with slope 2 passing through the point?

(A) $y = 2x - 8$ (B) $y = 2x - 5$ (C) $y = 2x - 4$ (D) $y = 2x + 8$

Q.3 Family of lines represented by the equation $(\cos\theta + \sin\theta)x + (\cos\theta - \sin\theta) - 3(3\cos\theta + \sin\theta) = 0$ passes through a fixed point M for all real values of θ. The reflection of M in the line $x - y = 0$, is

(A) (6,3) (B) (3, 6) (C) (–6, 3) (D) (3, –6)

Multiple correct type question

Q.4 If (–2, 6) is the reflection of the point (4, 2) with respect to line $L = 0$, If L can be written in the form $ax + by + c = 0$ (where a, b and c in their lowest form and $a, b, c \in N$) then

(A) $a + b = c$ (B) $a = b + c$ (C) $ab + bc + ac > 0$ (D) $a^2 + b^2 + c^2 > (a+b+c)^2$

Match the Column

Q.5 Set of family of lines are described in column-I and their mathematical equation are given in column-II. Match the entry of column-I with suitable entry of column-II. (*m* and *a* are parameters)

	Column-I		Column-II
(A)	having gradient 3	(P)	$mx - y + 3 - 2m = 0$
(B)	having y intercept three times the x-intercept	(Q)	$mx - y + 3m = 0$
(C)	having x intercept (–3)	(R)	$3x + y = 3a$
(D)	concurrent at (2, 3)	(S)	$3x - y + a = 0$

Answer key

Q.1 C Q.2 A Q.3 B Q.4 BD

Q.5 (A) $\rightarrow$ S; (B) $\rightarrow$ R; (C) $\rightarrow$ Q; (D) $\rightarrow$ P

21. SHIFTING OF ORIGIN :

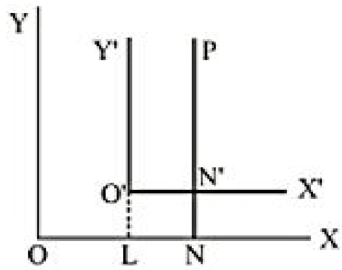
Let OX and OY be the original axes and let the new axes, parallel to the original, be

O'X' and O'Y'

Let the coordinates of the new origin O', referred to the original axes be h and k, so that, if O'L be perpendicular to OX, we have

$$OL = h \text{ and } LO' = k$$

Let P be any point in the plane of the paper, and let its coordinates, referred to the original axes, be x and y and referred to the new axes let them x' and y'.



Draw PN perpendicular to OX to meet O'X' in N'.

then, $ON = x$, $NP = y$, $O'N' = x'$ and $N'P = y'$

we therefore, have

$$x = ON = OL + O'N' = h + x'$$

and $$y = NP = LO' + N'P = k + y'$$

The origin is therefore, transferred to the point (h, k) when we substitute for the coordinates x and y the quantities.

$$x' + h \text{ and } y' + k$$

The above article is true whether the axes be oblique or rectangular.

Illustration :

Find the new coordinates of point (3, –4) if the origin is shifted to (1, 2) by translation.

Sol. *Since origin is shifted to $x = 1$ and $y = 2$*

Hence $x - 1 = X$ and $y - 2 = Y$

Point (3, –4) is shifted to

$$X = 3 - 1, \qquad Y = -4 - 2$$

$$X = 2,\ Y = -6$$

Hence new coordinates are (2, –6)

Illustration :

Find the newly tranformed equation of the straight line $2x - 3y + 5 = 0$ if origin is shifted to (3, –1).

Sol. *If origin is shifted (3, –1)*

$$x - 3 = X,\ y - (-1) = Y$$

$$x = 3 + X,\ y = Y - 1$$

New equation is $2(X + 3) - 3(Y - 1) + 5 = 0$

$$2X - 3Y + 14 = 0$$

Illustration :

Find the point at which the origin be shifted so that the equation $x^2 + y^2 - 5x + 2y - 5 = 0$ has no first degree terms.

Sol. $x^2 + y^2 - 5x + 2y - 5 = 0$

$$\Rightarrow \quad \left(x^2 - 5x + \frac{25}{4}\right) - \frac{25}{4} + (y^2 + 2y + 1) - 1 - 5 = 0$$

$$\Rightarrow \quad \left(x - \frac{5}{2}\right)^2 + (y + 1)^2 - \frac{49}{4} = 0$$

$$\Rightarrow \quad X^2 + Y^2 = \frac{49}{4}$$

If new equation to be formed has no first degree terms then shift the origin to

$$x = \frac{5}{2}, y = -1$$

$$= \left(\frac{5}{2}, -1\right).$$

22. ROTATION OF AXES :

22.1 ROTATION OF AXES WITHOUT CHANGING THE ORIGIN :

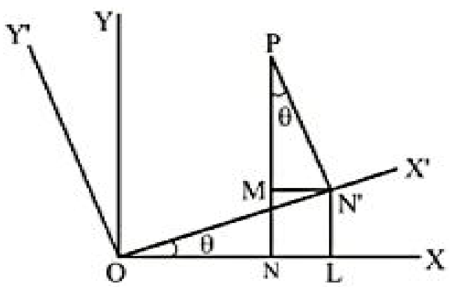
Let OX and OY be the original system of axes and OX' and OY' the new system, and let the angle, XOX', through which the axes are turned be called θ.
Take any point P in the plane of the paper.

Draw PN and PN' perpendicular to OX and OX', and also N'L and N'M perpendicular to OX and PN. If the coordinates of P, referred to the original axes, be x and y, and referred to the new axes, be x' and y', we have

$$ON = x, NP = y, ON' = x' \text{ and } N'P = y'$$

The angle,

$$MPN' = 90^\circ - \angle MN'P = \angle MN'O = \angle XOX' = \theta$$



We then have

$$x = ON = OL - MN' = ON' \cos\theta - N'P \sin\theta$$
$$= x' \cos\theta - y' \sin\theta$$

and

$$y = NP = LN' + MP = ON' \sin\theta + N'P \cos\theta$$
$$= x' \sin\theta + y' \cos\theta$$

If therefore, in any equation we wish to turn the axes, being rectangular, through an angle θ we must substitute

$$x = x'\cos\theta - y'\sin\theta \quad \text{and} \quad y = x'\sin\theta + y'\cos\theta$$

	x	y
x'	$\cos\theta$	$\sin\theta$
y'	$-\sin\theta$	$\cos\theta$

23. ANGLE BISECTOR :

23.1 EQUATION OF BISECTORS OF THE ANGLES BETWEEN TWO LINES :

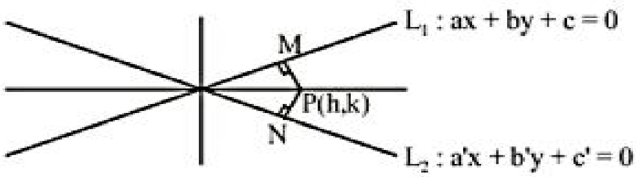
(i) Equations of the bisectors of angles between the lines $ax + by + c = 0$ & $a'x + b'y + c' = 0$ $(ab' \neq a'b)$ are : $\dfrac{ax + by + c}{\sqrt{a^2 + b^2}} = \pm \dfrac{a'x + b'y + c'}{\sqrt{a'^2 + b'^2}}$

Explanation : As we know that angle bisector between two lines is the locus of the point which moves in a plane such that its perpendicular distance from both the lines are equal.

Let P(h, k) be a moving point

PM = PN

$$\left|\frac{ah + bk + c}{\sqrt{a^2 + b^2}}\right| = \left|\frac{a'h + b'k + c'}{\sqrt{a'^2 + b'^2}}\right|$$



∴ Locus of P(h, k) is $\dfrac{ax + by + c}{\sqrt{a^2 + b^2}} = \pm \dfrac{a'x + b'y + c'}{\sqrt{a'^2 + b'^2}}$.

(ii) **To discriminate between the acute angle bisector & the obtuse angle bisector**

If θ be the angle between one of the lines & one of the bisectors, find tan θ.

If $|\tan\theta| < 1$, then $2\theta < 90°$ so that this bisector is the acute angle bisector.

If $|\tan\theta| > 1$, then we get the bisector to be the obtuse angle bisector.

(iii) To discriminate between the bisector of the angle containing the origin & that of the angle not containing the origin. Rewrite the equations, $ax + by + c = 0$ & $a'x + b'y + c' = 0$ such that the constant terms c, c' are positive. Then ;

$\dfrac{ax + by + c}{\sqrt{a^2 + b^2}} = + \dfrac{a'x + b'y + c'}{\sqrt{a'^2 + b'^2}}$ gives the equation of the bisector of the angle containing the origin

& $\dfrac{ax + by + c}{\sqrt{a^2 + b^2}} = - \dfrac{a'x + b'y + c'}{\sqrt{a'^2 + b'^2}}$ gives the equation of the bisector of the angle not containing the origin.

(iv) To discriminate between acute angle bisector & obtuse angle bisector proceed as follows Write $ax+by+c=0$ & $a'x+b'y+c'=0$ such that constant terms are positive.

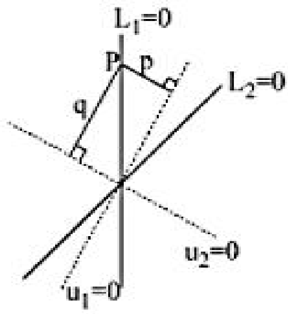
If $aa'+bb'<0$, then the angle between the lines that contains the origin is acute and the equation of the bisector of this acute angle is $\frac{ax+by+c}{\sqrt{a^2+b^2}}=+\frac{a'x+b'y+c'}{\sqrt{a'^2+b'^2}}$ therefore $\frac{ax+by+c}{\sqrt{a^2+b^2}}=-\frac{a'x+b'y+c'}{\sqrt{a'^2+b'^2}}$ is the equation of other bisector.

If, however, $aa'+bb'>0$, then the angle between the lines that contains the origin is obtuse & the equation of the bisector of this obtuse angle is:

$\frac{ax+by+c}{\sqrt{a^2+b^2}}=+\frac{a'x+b'y+c'}{\sqrt{a'^2+b'^2}}$; therefore $\frac{ax+by+c}{\sqrt{a^2+b^2}}=-\frac{a'x+b'y+c'}{\sqrt{a'^2+b'^2}}$

is the equation of other bisector.

(v) Another way of identifying an acute and obtuse angle bisector is as follows :
Let $L_1=0$ & $L_2=0$ are the given lines & $u_1=0$ and $u_2=0$ are the bisectors between $L_1=0$ & $L_2=0$. Take a point P on any one of the lines $L_1=0$ or $L_2=0$ and drop perpendicular on $u_1=0$ & $u_2=0$ as shown. If,



$|p|<|q| \Rightarrow u_1$ is the acute angle bisector.

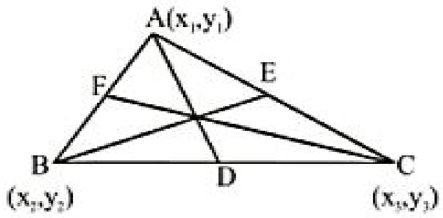
$|p|>|q| \Rightarrow u_1$ is the obtuse angle bisector.

$|p|=|q| \Rightarrow$ the lines L_1 & L_2 are perpendicular.

Note : Equation of straight lines passing through $P(x_1, y_1)$ & equally inclined with the lines $a_1x+b_1y+c_1=0$ & $a_2x+b_2y+c_2=0$ are those which are parallel to the bisectors between these two lines and passing through the point P.

(vi) **Bisector in case of triangle**

CASE-I : When vertices of triangle are known.



Method-1 :

1. Find the length of sides of triangle AB(c), BC(a) and CA(b) by distance formula.
2. Find incentre I of triangle where $I=\left(\frac{ax_1+bx_2+cx_3}{a+b+c}, \frac{ay_1+by_2+cy_3}{a+b+c}\right)$
3. With the help of incentre we can find all angle bisectors of triangle ABC.

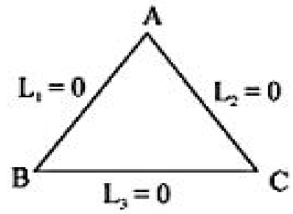
Method-2 :

1. Find the point D which divides the side BC in ratio c : b. and also find the points E and F similarly.
2. Find the equations of AD, BE and CF.

CASE-II : When the equation of sides are given.

Method-1 :

1. Compute tanA , tanB , tanC and arrange the lines in descending order of their slopes.
2. With the help of angles we can find all acute/obtuse angle bisectors.



Method-2 :

1. Plot the lines approximately and compute bisectors containing or not containing the origin.

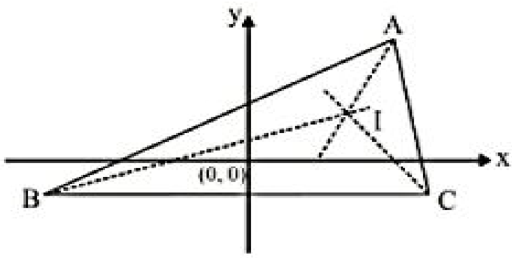


Illustration :

Find the angle bisectors between the lines $4x + 3y - 7 = 0$ and $24x + 7y - 31 = 0$. Also identify the acute angle bisector and bisector contianing origin.

Sol. *If the given equation are*

$$4x + 3y - 7 = 0 \text{ and } 24x + 7y - 31 = 0$$

These can be written as

$$-4x - 3y + 7 = 0 \text{ and } -24x - 7y + 31 = 0$$

Equation of bisector is given by

$$\left(\frac{-4x-3y+7}{\sqrt{4^2+3^2}}\right) = \pm\left(\frac{-24x-7y+31}{\sqrt{24^2+7^2}}\right)$$

$$5(-4x - 3y + 7) = \pm(-24x - 7y + 31)$$

Equation of bisector containing origin is

$$5(-4x - 3y + 7) = (-24x - 7y + 31)$$

$$x - 2y + 1 = 0$$

Equation of bisector not containing origin is

$$5(-4x - 3y + 7) = (-24x - 7y + 31)$$

$$2x + y - 3 = 0$$

Acute Angle Bisector

$$aa' + bb' = (-4)(-24) + (-3)(-7) > 0$$

Since $aa' + bb' > 0$ hence the angle bisector that contains origin is obtuse angle bisector i.e., $x - 2y + 1 = 0$ and equation not containing origin is acute angle bisector i.e., $2x + y - 3 = 0$.

Illustration :

Find the equation of bisectors between the lines $x + \sqrt{3}y = 6 + 2\sqrt{3}$ *and* $x - \sqrt{3}y = 6 - 2\sqrt{3}$.

Sol. *Slope of line* $x + \sqrt{3}y = 6 + 2\sqrt{3}$ *is* $m_1 = -\frac{1}{\sqrt{3}}$.

Slope of line $x - \sqrt{3}y = 6 - 2\sqrt{3}$ *is* $m_2 = \frac{1}{\sqrt{3}}$.

Now $m_1 + m_2 = 0$

Since sum of slopes is zero therefore x = h, and y = k are equations of angle bisector where (h, k) is point of intersection. Hence bisectors are x = 6 and y = 2.

Practice Problem

Single correct question

Q.1 The equation of the bisector of the acute angle between the lines $3x - 4y + 7 = 0$ and $12x + 5y - 2 = 0$ is

(A) $21x + 77y - 101 = 0$ (B) $11x - 3y + 9 = 0$
(C) $21x + 77y + 101 = 0$ (D) $11x - 3y - 9 = 0$

Q.2 The equation of the bisector of the lines $3x - 4y + 1 = 0$ and $5x + 12y - 11 = 0$ which do not contain origin is

(A) $7x - 56y + 34 = 0$ (B) $32x + 4y - 21 = 0$
(C) $3x + 7y + 11 = 0$ (D) $16x - 3y + 7 = 0$

Q.3 When the origin is shifted to (1, 1), the equation $xy - x - y + 1 = 0$ becomes

(A) $X^2 - Y^2 = 0$ (B) $XY = 0$
(C) $X^2 - Y^2 - 2X + 2Y = 0$ (D) $X^2 + Y^2 - 5X + 2Y - 5 = 0$

Multiple correct type question

Q.4 The bisectors of angle between the st. lines, $y - b = \frac{2m}{1-m^2}(x - a)$ and $y - b = \frac{2m'}{1-m'^2}(x - a)$ are

(A) $(y - b)(m + m') + (x - a)(1 - mm') = 0$
(B) $(y - b)(m + m') - (x - a)(1 - mm') = 0$
(C) $(y - b)(1 - mm') + (x - a)(m + m') = 0$
(D) $(y - b)(1 - mm') - (x - a)(m + m') = 0$

Q.5 Let (p, q) is the point to which the origin should be shifted so that the equation $y^2 - 6y - 4x + 13 = 0$ is transformed to the form $Y^2 + AX = 0$ then

(A) $p + q = 4$ (B) $p^2 + q^2 = 25$
(C) $p^2 + q^2 = 10$ (D) p and q are twin prime

Answer key

Q.1	A	Q.2	A	Q.3	B	Q.4	AD	Q.5	ACD

24. PAIR OF STRAIGHT LINES :

24.1 A PAIR OF STRAIGHT LINES THROUGH ORIGIN :

(i) A homogeneous equation of degree two of the type $ax^2 + 2hxy + by^2 = 0$ always represents a pair of straight lines passing through the origin & if:

(a) $h^2 > ab \Rightarrow$ lines are real & distinct.

(b) $h^2 = ab \Rightarrow$ lines are coincident.

(c) $h^2 < ab \Rightarrow$ lines are imaginary with real point of intersection i.e. (0, 0).

Note: A homogeneous equation of degree n represents n straight lines through the origin.

(ii) If $y = m_1x$ & $y = m_2x$ be the two equations represented by $ax^2 + 2hxy + by^2 = 0$, then; $m_1 + m_2 = -\frac{2h}{b}$ & $m_1 m_2 = \frac{a}{b}$.

Explanation :

$b(y - m_1x)(y - m_2x) = ax^2 + 2hxy + by^2$

Comparing co-efficients of x^2 and xy on both sides

$$-b(m_1 + m_2) = 2h \Rightarrow m_1 + m_2 = \frac{-2h}{b}$$

$$b\, m_1 m_2 = a \Rightarrow m_1 m_2 = \frac{a}{b}$$

24.2 ANGLE BETWEEN STRAIGHT LINES REPRESENTED BY THE EQUATION $ax^2 + 2hxy + by^2 = 0$:

Let the equation $ax^2 + 2hxy + by^2 = 0$ represents two lines which are $y - m_1x = 0$ and $y - m_2x = 0$

$\therefore$ Angle between these lines is

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1m_2}\right| = \left|\frac{\sqrt{(m_1 + m_2)^2 - 4m_1m_2}}{1 + m_1m_2}\right| = \left|\frac{\sqrt{\left(\frac{-2h}{b}\right)^2 - 4\frac{a}{b}}}{1 + \frac{a}{b}}\right| = \left|\frac{2\sqrt{h^2 - ab}}{a + b}\right|$$

$\therefore$ acute angle θ between the pair of straight lines represented by the equation,

$ax^2 + 2hxy + by^2 = 0$ is $\tan\theta = \left|\frac{2\sqrt{h^2 - ab}}{a + b}\right|$.

The condition that these lines are :

(a) At right angles to each other if $a + b = 0$. i.e. co–efficient of x^2 + co–efficient of $y^2 = 0$.

(b) Coincident if $h^2 = ab$.

(c) Equally inclined to the axis of x if $h = 0$. i.e. coeff. of $xy = 0$.

24.3 BISECTORS OF ANGLE BETWEEN THE LINES REPRESENTED BY $ax^2 + 2hxy + by^2 = 0$:

Let the given equation represent the straight lines

$$y - m_1x = 0 \qquad \ldots(i)$$

$$y - m_2x = 0 \qquad \ldots(ii)$$

where $m_1 + m_2 = -2h/b$ and $m_1m_2 = a/b$ $\qquad \ldots(iii)$

The equation to the bisectors of the angles between the straight lines in equation (i) and (ii) are

$$\frac{y - m_1x}{\sqrt{1+m_1^2}} = \pm \frac{y - m_2x}{\sqrt{1+m_2^2}}$$

Therefore, the combined equation of the bisectors is

$$\left\{\frac{y - m_1x}{\sqrt{1+m_1^2}} - \frac{y - m_2x}{\sqrt{1+m_2^2}}\right\}\left\{\frac{y - m_1x}{\sqrt{1+m_1^2}} + \frac{y - m_2x}{\sqrt{1+m_2^2}}\right\} = 0$$

or
$$\frac{(y - m_1x)^2}{1+m_1^2} - \frac{(y - m_2x)^2}{1+m_2^2} = 0$$

hence, by equation (iii), we get

$$\frac{-2h}{b}(x^2 - y^2) + 2\left(\frac{a}{b} - 1\right)xy = 0$$

or
$$\frac{x^2 - y^2}{a - b} = \frac{xy}{h}$$

Important point

The product of the perpendiculars, dropped from (x_1, y_1) to the pair of lines represented by the equation, $ax^2 + 2hxy + by^2 = 0$ is $\dfrac{ax_1^2 + 2hx_1y_1 + by_1^2}{\sqrt{(a-b)^2 + 4h^2}}$.

24.4 GENERAL EQUATION OF SECOND DEGREE REPRESENTING A PAIR OF STRAIGHT LINES :

(i) $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a pair of straight lines if:

$$abc + 2fgh - af^2 - bg^2 - ch^2 = 0, \quad \text{i.e. if} \quad \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0.$$

(ii) The angle θ between the two lines representing by a general equation is the same as that between the two lines represented by its homogeneous part only.

Illustration :

Prove that the $x^2 - 4xy + y^2$ and $x + y = 1$ enclose an equilateral triangle. Find also its area.

Sol. Angle between the two lines $x^2 - 4xy + y^2 = 0$ is

$$\theta = \tan^{-1}\left|\frac{2\sqrt{h^2 - ab}}{a+b}\right|$$

$$= \tan^{-1}\left|\frac{2\sqrt{4-1}}{1+1}\right|$$

$$= \tan^{-1}\sqrt{3} = 60°$$

Again $x^2 - 4xy + y^2 = 0$

$\Rightarrow \quad x^2 - 4xy + 4y^2 - 3y^2 = 0$

$\Rightarrow \quad (x - 2y)^2 - (\sqrt{3}\,y)^2 = 0$

$\Rightarrow \quad (x - 2y + \sqrt{3}\,y)\,(x - 2y - \sqrt{3}\,y) = 0$

Equation of lines represented by $x^2 - 4xy + y^2 = 0$

are $\quad x - (2 - \sqrt{3})y = 0 \quad$ ... (i)

$\quad x - (2 + \sqrt{3})y = 0 \quad$ ... (ii)

Now angle between the line $x + y = 1$ and line (i) is

$$\theta = \tan^{-1}\left|\frac{m_1 + m_2}{1 + m_1 m_2}\right|$$

$$= \tan^{-1}\left|\frac{(-1) - \left(\frac{1}{2-\sqrt{3}}\right)}{1 + (-1)\left(\frac{1}{2-\sqrt{3}}\right)}\right|$$

$$= \tan^{-1}\left|\frac{-1-(2+\sqrt{3})}{1-(2+\sqrt{3})}\right|$$

$$= \tan^{-1}\left|\frac{-3-\sqrt{3}}{-1-\sqrt{3}}\right| = \tan^{-1}|-\sqrt{3}| = \tan^{-1}\sqrt{3} = 60°$$

Similarly, angle between the line $x + y = 1$ and line (ii) is $60°$.

$\therefore$ Angle between the lines taking any two of the given three lines be $60°$.
thus, the triangle formed by these lines is an equilateral triangle.

Again, point of intersection of lines $x^2 - 4xy + y^2 = 0$ is origin. So, distance of the line

$x + y - 1 = 0$ from the origin is $\quad h = \left|\frac{(0)+(0)-1}{\sqrt{1^2+1^2}}\right| = \frac{1}{\sqrt{2}}$

Let a is the side of the equilateral triangle
Hence, a sin 60° = h.

$$\Rightarrow \quad a = \frac{h}{\sin 60°} = \frac{\left(\frac{1}{\sqrt{2}}\right)}{\left(\frac{\sqrt{3}}{2}\right)} = \sqrt{\frac{2}{3}}$$

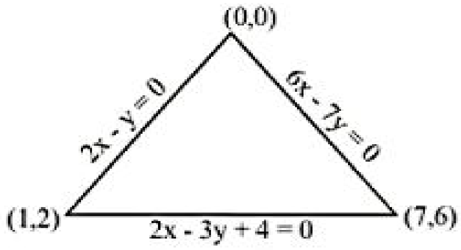
$$\text{Area of triangle} = \frac{\sqrt{3}}{4}a^2 = \frac{\sqrt{3}}{4}\left(\frac{2}{3}\right) = \frac{\sqrt{3}}{6} \text{ sq. units.}$$

Illustration :

Find the centroid of the triangle the equation of whose sides $12x^2 - 20xy + 7y^2 = 0$ *and* $2x - 3y + 4 = 0$.

Sol.

$12x^2 - 20xy + 7y^2 = 0$
$12x^2 - 14xy - 6xy + 7y^2 = 0$
$2x(6x - 7y) - y(6x - 6y) = 0$
$\Rightarrow (2x - y)(6x - 7y) = 0$



So three lines are $2x - y = 0$ .... (i)
$6x - 7y = 0$ .... (ii)
$2x - 3y + 4 = 0$ .... (iii)

Point of intersection of lines (i) and (ii) is (0, 0).
Point of intersection of lines (i) and (iii) is (1, 2).
and Point of intersection of lines (ii) and (iii) is (7, 6).

$$\text{Centroid is } \left(\frac{0+1+7}{3}, \frac{0+2+6}{3}\right) = \left(\frac{8}{3}, \frac{8}{3}\right).$$

Illustration :

Find the distance between the parallel lines $4x^2 + 4xy + y^2 - 6x - 3y - 4 = 0$

Sol. *Given lines is* $4x^2 + 4xy + y^2 - 6x - 3y - 4 = 0$

Again, $4x^2 + 4xy + y^2 = 0$
$(2x + y)^2 = 0$

$\Rightarrow \quad (2x + y + A)(2x + y + B) = 4x^2 + 4xy + y^2 - 6x - 3y - 4$

$\Rightarrow \quad 4x^2 + 4xy + y^2 + (2A + 2B)x + (A + B)y + AB = 4x^2 + 4xy + y^2 - 6x - 3y - 4$

Comparing both sides, we get

$A + B = -3$ ... (i)
$AB = -4$ ... (ii)

By solving (i) and (ii) we get

$A = -4, B = 1$

$\Rightarrow$ *Two parallel lines are* $2x - y - 4 = 0$ *and* $2x + y + 1 = 0$

$$\text{distance} = \left|\frac{-4-1}{\sqrt{2^2+1^2}}\right| = \sqrt{5}.$$

25. HOMOGENISATION :

The joint equation of a pair of straight lines joining origin to the points of intersection of the line given by

$$lx + my + n = 0 \qquad \text{......(i)} \qquad \&$$

the 2nd degree curve : $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ ......(ii)

is $ax^2 + 2hxy + by^2 + 2gx\left(\frac{lx+my}{-n}\right) + 2fy\left(\frac{lx+my}{-n}\right) + c\left(\frac{lx+my}{-n}\right)^2 = 0$ ...... (iii)

(iii) is obtained by homogenizing (ii) with the help of (i), by writing (i) in the form : $\left(\frac{lx+my}{-n}\right) = 1.$

Important concept :

Any second degree curve through the four point of intersection of $f(x\,y) = 0$ & $xy = 0$ is given by $f(x\,y) + \lambda\, xy = 0$ where $f(xy) = 0$ is also a second degree curve.

Illustration :

Find the equation of the line pair joining origin and the point of intersections of the line $2x - y = 3$ and the curve $x^2 - y^2 - xy + 3x - 6y + 18 = 0$. Also find the angle between these two lines.

Sol. *Given*

curve : $x^2 + y^2 - xy + 3x - 6y + 18 = 0$

and line : $2x - y = 3$

covert the curve in homogeneous equation with the help of line

$$\Rightarrow x^2 - y^2 - xy + 3x\left(\frac{2x-y}{3}\right) - 6y\left(\frac{2x-y}{3}\right) + 18\left(\frac{2x-y}{3}\right)^2 = 0$$

$$\Rightarrow x^2 - y^2 - xy + x(2x - y) - 2y(2x - y) + 2(2x - y)^2 = 0$$

$$\Rightarrow x^2 - y^2 - xy + 2x^2 - xy - 4xy + 2y^2 + 8x^2 + 2y^2 - 8xy = 0$$

$$\Rightarrow 11x^2 + 3y^2 - 14xy = 0$$

Angle between them is

$$\tan\theta = \frac{2\sqrt{h^2 - ab}}{a+b} \quad \text{where } a = 11,\ b = 3 \text{ and } h = -7$$

$$= \frac{2\sqrt{49-33}}{14}$$

$$\text{or } \tan\theta = \frac{4}{7} \Rightarrow \quad \theta = \tan^{-1}\left(\frac{4}{7}\right)$$

Illustration :

Find the value of 'm' if the lines joining the origin to the points common to $x^2 + y^2 + x - 2y - m = 0$ *&* $x + y = 1$ *are at right angles.*

Sol. Given : $x^2 + y^2 + x - 2y - m = 0$

and a line : $x + y = 1$

Homogenize the curve with the help of line

$\Rightarrow \quad x^2 + y^2 + x(x + y) - 2y(x + y) - m(x + y)^2 = 0$

$\Rightarrow \quad x^2 + y^2 + x^2 + xy - 2xy - 2y^2 - mx^2 - my^2 - 2mxy = 0$

$\Rightarrow \quad x^2(2 - m) - y^2(1 + m) - xy(1 + 2m) = 0$

If lines are at right angle then

$$(2 - m) - (1 + m) = 0 \Rightarrow m = \frac{1}{2}$$

Illustration :

A line L passing through the point (2, 1) intersects the curve $4x^2 + y^2 - x + 4y - 2 = 0$ *at the points A, B. If the lines joining origin and the points A, B are such that the coordinate axis are the bisectors between them then find the equation of line L.*

Sol. Given :

Curve : $4x^2 + y^2 - x + 4y - 2 = 0$ and point (2, 1).

Let the slope of a line is m, then the equation of the line passing through a point (2, 1) is

$(y - 1) = m(x - 2)$

$\Rightarrow \quad y - 1 = mx - 2m \Rightarrow y - mx + 2m - 1 = 0$

Make the curve homogeneous with the help of line then

$$\Rightarrow \quad 4x^2 + y^2 - x\left(\frac{y - mx}{1 - 2m}\right) + 4y\left(\frac{y - mx}{1 - 2m}\right) - 2\left(\frac{y - mx}{1 - 2m}\right)^2 = 0$$

$\Rightarrow \quad 4x^2(1 - 2m)^2 + y^2(1 - 2m)^2 - x(y - mx)(1 - 2m) + 4y(y - mx)(1 - 2m) - 2(y - mx)^2 = 0$

$\therefore$ If the co-ordinate axis are the bisectors between them the coefficient of xy should be zero.

$(2m - 1) + 4m(2m - 1) + 4m = 0$

$8m^2 + 2m - 1 = 0$

$8m^2 + 4m - 2m - 1 = 0$

$4m(2m + 1) - 1(2m + 1) = 0$

$(4m - 1)(2m + 1) = 0$

$$m = \frac{1}{4} \quad or \quad -\frac{1}{2}$$

$\therefore$ lines are

$$(y - 1) = \frac{1}{4}(x - 2) \qquad or \qquad (y - 1) = -\frac{1}{2}(x - 2)$$

$$x - 4y + 2 = 0 \qquad or \qquad x + 2y - 4 = 0$$

Practice Problem

Single correct question

Q.1 The combined equation of straight lines which pass through (1, 2) and perpendicular to the line pair $3x^2 - 8xy + 5y^2 = 0$ is

(A) $5x^2 - 8xy + 3y^2 + 13x - 15y + 7 = 0$ (B) $5x^2 + 8xy + 3y^2 - 26x - 20y + 33 = 0$

(C) $3x^2 + 4xy + 3y^2 - 15x + 25y + 12 = 0$ (D) $3x^2 + 4xy - 3y^2 - 46x + 8y + 33 = 0$

Q.2 If the pair of lines $ax^2 + 2(a + b)xy + by^2 = 0$ lie along diameters of a circle and divide the circle into four sectors such that the area of one of the sectors is thrice the area of another sector, then

(A) $3a^2 + 2ab + 3b^2 = 0$ (B) $3a^2 + 10ab + 3b^2 = 0$

(C) $3a^2 - 2ab + 3b^2 = 0$ (D) $3a^2 - 10ab + 3b^2 = 0$

Q.3 If $x^2 - 2pxy - y^2 = 0$ and $x^2 - 2qxy - y^2 = 0$ bisect angles between each other, then

(A) $p + q = 1$ (B) $pq = 1$ (C) $pq + 1 = 0$ (D) $p^2 + pq + q^2 = 0$

Q.4 If the straight lines joining the origin and the points of intersection of the curve $5x^2 + 12xy - 6y^2 + 4x - 2y + 3 = 0$ and $x + ky - 1 = 0$ are equally inclined to the co-ordinate axes then the value of k :

(A) is equal to 1 (B) is equal to –1

(C) is equal to 2 (D) does not exist in the set of real numbers

Multiple correct type question

Q.5 The lines L_1 and L_2 denoted by $3x^2 + 10xy + 8y^2 + 14x + 22y + 15 = 0$ intersect at the point P and have gradients m_1 and m_2 respectively. The acute angle between them is θ. Which of the following relations hold good?

(A) $m_1 + m_2 = 5/4$

(B) $m_1 m_2 = 3/5$

(C) acute angle between L_1 and L_2 is $\sin^{-1}\left(\frac{2}{5\sqrt{5}}\right)$

(D) sum of the abscissa and ordinate of the point P is –1.

Q.6 If angle between the lines joining origin and the point of intersections of the line $x - y = 2$ and the curve $x^2 - 4xy + 2y^2 - 2x + y + k = 0$ is 45° then

(A) Sum of all possible value (s) of k is equal to 3

(B) Sum of all possible value (s) of k is equal to –2

(C) Product of all possible value (s) of k is equal to –16

(D) Product of all possible value (s) of k is equal to 12

Answer key

Q.1	B	Q.2	A	Q.3	C
Q.4	B	Q.5	BCD	Q.6	BC

SOLVED EXAMPLES

Q.1 If A (1, 2) and B (3, 8) be two given points, find a point P such that |PA| = |PB| and ΔPAB = 10.

Sol. Let $P = (\alpha, \beta)$, then $|PA| = |PB| \Rightarrow PA^2 = PB^2$

$\Rightarrow (\alpha - 1)^2 + (\beta - 1)^2 = (\alpha - 3)^2 + (\beta - 8)^2$

$\Rightarrow \alpha^2 - 2\alpha + 1 + \beta^2 + 4 - 4\beta = \alpha^2 + 9 - 6\alpha + \beta^2 + 64 - 16\beta$

$\Rightarrow 4\alpha + 12\beta = 68 \quad \Rightarrow \quad \alpha + 3\beta = 17 \qquad \ldots(i)$

Also, area of ΔPAB = 10

$$\Rightarrow \frac{1}{2}|2\alpha - \beta + 8 - 6 + 3\beta - 8\alpha| = 10$$

$\Rightarrow -6\alpha + 2\beta + 2 = \pm 20$

$\Rightarrow -3\alpha + \beta + 1 = \pm 10$

$\Rightarrow -3\alpha + \beta = -1 \pm 10$

$\Rightarrow -3\alpha + \beta = -11 \qquad \ldots(ii)$

or $-3\alpha + \beta = 9 \qquad \ldots(iii)$

Solving (i) and (ii), we obtain $\alpha = 5, \beta = 4$, solving (i) and (iii), we obtain $\alpha = -1, \beta = 6$

Hence, the point P is either (5, 4) or (–1, 6)

Q.2 Factorize $\begin{vmatrix} (b+c)^2 & a^2 & bc \\ (c+a)^2 & b^2 & ca \\ (a+b)^2 & c^2 & ab \end{vmatrix}$.

Sol. Applying $C_1 \to C_1 + C_2 - 2C_3$

$$D = \begin{vmatrix} a^2+b^2+c^2 & a^2 & bc \\ a^2+b^2+c^2 & b^2 & ca \\ a^2+b^2+c^2 & c^2 & ab \end{vmatrix} = (a^2+b^2+c^2)\begin{vmatrix} 1 & a^2 & bc \\ 1 & b^2 & ca \\ 1 & c^2 & ab \end{vmatrix}$$

Applying $R_1 \to R_1 - R_2, R_2 \to R_2 - R_3$

$$= (a^2+b^2+c^2)\begin{vmatrix} 0 & a^2-b^2 & c(b-a) \\ 0 & b^2-c^2 & a(c-b) \\ 1 & c^2 & ab \end{vmatrix}$$

$$= (a^2+b^2+c^2)(a-b)(b-c)\begin{vmatrix} 0 & a+b & -c \\ 0 & b+c & -a \\ 1 & c^2 & ab \end{vmatrix}$$

$$= (a^2+b^2+c^2)(a-b)(b-c)\begin{vmatrix} a+b & -c \\ b+c & -a \end{vmatrix}$$

$C_1 \to C_1 - C_2$

$$= (a^2+b^2+c^2)(a-b)(b-c)\begin{vmatrix} a+b+c & -c \\ b+c+a & -a \end{vmatrix}$$

$$= (a^2+b^2+c^2)(a-b)(b-c)(a+b+c)\begin{vmatrix} 1 & -c \\ 1 & -a \end{vmatrix}$$

$$= (a^2+b^2+c^2)(a+b+c)(a-b)(b-c)(c-a)$$

Q.3 If $\left(\frac{a^3}{a-1}, \frac{a^2-3}{a-1}\right), \left(\frac{b^3}{b-1}, \frac{b^2-3}{b-1}\right)$ and $\left(\frac{c^3}{c-1}, \frac{c^2-3}{c-1}\right)$ are collinear and $\alpha\, abc + \beta\,(a + b + c) = \gamma\,(ab + bc + ca)$, $\alpha, \beta, \gamma \in N$, then find the least value of $\alpha + \beta + \gamma$.

Sol. Let the equation of line on which these three points lie be

$$lx + my + n = 0$$

and the point $\left(\frac{t^3}{t-1}, \frac{t^2-3}{t-1}\right)$ lie on the line where t = a, b, c

$$l\left(\frac{t^3}{t-1}\right) + m\left(\frac{t^2-3}{t-1}\right) + n = 0$$

$$l\,t^3 + m\,(t^2 - 3) + n\,(t - 1) = 0$$

$$t^3\, l + t^2\, m + t\, n - 3m - n = 0$$

If a, b, c are the roots of given equation then

$$a + b + c = \frac{-m}{l} \qquad ...(i)$$

$$ab + bc + ca = \frac{n}{l} \qquad ...(ii)$$

$$abc = \frac{3m}{l} + \frac{n}{l} \qquad ...(iii)$$

using (i), (ii) & (iii) we get

$$abc = -3\,(a + b + c) + ab + bc + ca$$

$$abc + 3\,(a + b + c) = ab + bc + ca$$

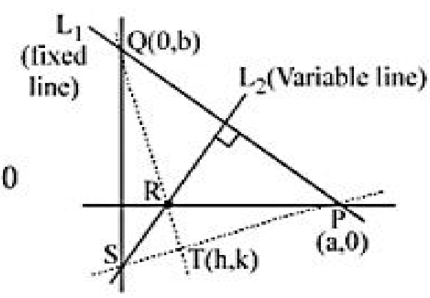
$\Rightarrow$ $\alpha + \beta + \gamma = 1 + 3 + 1 = 5.$

Q.4 A fixed line PQ : $\frac{x}{a} + \frac{y}{b} = 1$; cut the x and y axis at P & Q. and a variable line perpendicular to it cut the x-axis at R and y axis at S. Find the locus of the point of intersection of QR and PS.

Sol. Line PQ cut the x-axis and y-axis at P & Q respectively.

So, P = (a, 0) and Q = (0, b)

$$\text{Slope of PQ} = \frac{-b}{a}$$



Slope of line perpendicular to PQ = $\frac{a}{b}$

Equation of line perpendicular to PQ is $ax - by + \lambda = 0$

It intersect x-axis at R and y-axis at S

Hence, $R = \left(\frac{-\lambda}{a}, 0\right)$, $\qquad S = \left(0, \frac{\lambda}{b}\right)$

$$\text{Slope of QR} = \frac{b-0}{0-\left(\frac{-\lambda}{a}\right)} = \frac{ab}{\lambda}$$

$$\text{Slope of PS} = \frac{\frac{\lambda}{b}-0}{0-a} = \frac{-\lambda}{ab}$$

$\Rightarrow$ $QR \perp PS$

Let the point of intersection of QR and PS is T(h, k)

$\Rightarrow$ $PT \perp QT$

$$\Rightarrow \quad \left(\frac{k-0}{h-a}\right)\left(\frac{k-b}{h-0}\right) = -1$$

$\Rightarrow$ $h(h-a) + k(k-b) = 0$

$\Rightarrow$ $h^2 + k^2 - ah - bk = 0$

So, Locus of T is $x^2 + y^2 - ax - by = 0$

Q.5 A line perpendicular to the line segment joining the points (1, 0) and (2, 3) divides it in the ratio 1 : n. Find the equation of the line, $(n \neq -1)$

Sol. The given points are A (1, 0) and B (2, 3). The required line divides the segment [AB] in the ratio 1 : n $(n \neq -1)$.

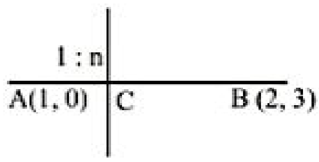
$\therefore$ Dividing point C is $\left(\frac{1\times 2 + n\times 1}{1+n}, \frac{1\times 3 + n \times 0}{n+1}\right) = \left(\frac{2+n}{1+n}, \frac{3}{n+1}\right)$

$\because$ Slope of line $AB = \frac{3-0}{2-1} = 3$

$\therefore$ Slope of the required line $(\perp AB) = -1/3$

As the line passes through C, therefore, its equation is $y - \frac{3}{n+1} = -\frac{1}{3}\left(x - \frac{n+2}{n+1}\right)$

Point slope form



or $\quad \frac{1}{3}x + y - \frac{3}{n+1} - \frac{n+2}{3(n+1)} = 0$

or $\quad \frac{1}{3}x + y - \left(\frac{9+n+2}{3(n+1)}\right) = 0$

or $\quad (n+1)x + 3(n+1)y - (n+11) = 0$

Q.6 The owner of a milk store finds that, he can sell 980 litres of milk each week at Rs 14/litre and 1220 liters of milk each week at Rs 16/litre. Assuming a linear relationship between selling price and demand, how many liters could he sell weekly at Rs 17/litre?

Sol. Let the milk store owner sell L litre of milk at Rs p per litre and let the linear relation ship between L and p be

$$L = mp + b \quad \ldots(i)$$

when $p = 14, L = 980$

$\therefore \quad 980 = 14m + b \quad \ldots(ii)$

when $p = 16, L = 1220$

$\therefore \quad 1220 = 16m + b \quad \ldots(iii)$

Substituting (ii) from (iii), we get $240 = 2m$

$$\Rightarrow \quad m = \frac{240}{2} = 120$$

Substituting this value of m in (ii), we get

$$980 = 14 \times 120 + b \quad \Rightarrow \quad 980 - 1680 = b \quad \Rightarrow \quad b = -700$$

Substituting $m = 120$ and $b = -700$ in (i), we obtain $L = 120p - 700$.

when $p = 17$, then $L = 120 \times 17 - 700 = 2040 - 700 = 1340$

Hence, the man can sell 1340 litres of milk at Rs. 17 per litre.

Q.7 If p and q are the lengths of perpendiculars from the origin to the lines $x \cos\theta - y \sin\theta = 3 \cos 2\theta$ and $x \sec\theta + y \operatorname{cosec}\theta = 3$, respectively, then find the value of $p^2 + 4q^2$.

Sol. Here, $p = \dfrac{|0\cos\theta - 0\sin\theta - 3\cos 2\theta|}{\sqrt{(\cos\theta)^2 + (-\sin\theta)^2}}$

or $\quad p = \dfrac{|3\cos 2\theta|}{1} \quad$ or $\quad p = |3 \cos 2\theta| \quad \ldots(i)$

and $\quad q = \dfrac{|0\sec\theta + 0\operatorname{cosec}\theta - 3|}{\sqrt{\sec^2\theta + cosec^2\theta}}$

$$= \frac{|3|}{\sqrt{\dfrac{1}{\cos^2\theta} + \dfrac{1}{\sin^2\theta}}} = |3\cos\theta\sin\theta| = \frac{|3\sin 2\theta|}{2} \qquad (\because \sin 2\theta = 2\sin\theta\cos\theta)$$

$\Rightarrow \quad 2q = |3 \sin 2\theta| \quad \ldots(ii)$

Squaring (i) and (ii) and adding

or $\quad p^2 + 4q^2 = (3\cos 2\theta)^2 + (3\sin 2\theta)^2$

or $\quad p^2 + 4q^2 = 9(\cos^2 2\theta + \sin^2 2\theta)$

or $\quad p^2 + 4q^2 = 9.$

Q.8 A variable line through origin meets two fixed lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ at P and Q. On it is taken a point R. If $\frac{2}{OR} = \frac{1}{OP} + \frac{1}{OQ}$ then prove that locus of R is also a st. line.

Sol. $a_1x + b_1y + c_1 = 0, \quad a_2x + b_2y + c_2 = 0$

Let the point R be (h, k)

Equation of line through origin with inclination θ is

$$\frac{x-0}{\cos\theta} = \frac{y-0}{\sin\theta} = r_i \qquad \text{where } i = 1, 2, 3$$

$$r_1 = OP, \qquad r_2 = OQ \quad \text{and} \quad r_3 = OR$$

P lies on $a_1x + b_1y + c_1 = 0$

$$a_1r_1\cos\theta + b_1r_1\sin\theta + c_1 = 0$$

$$r_1 = \frac{-c_1}{a_1\cos\theta + b_1\sin\theta}$$

Similarly $\quad r_2 = \frac{-c_2}{a_2\cos\theta + b_2\sin\theta}$

$$R = (r_3\cos\theta, r_3\sin\theta)$$

$$h = r_3\cos\theta, \quad k = r_3\sin\theta$$

If $\quad \frac{2}{OR} = \frac{1}{OP} + \frac{1}{OQ}$

$$\frac{2}{r_3} = \frac{1}{r_1} + \frac{1}{r_2}$$

$$\frac{2}{r_3} = -\left(\frac{a_1\cos\theta + b_1\sin\theta}{c_1} + \frac{a_2\cos\theta + b_2\sin\theta}{c_2}\right)$$

$$2 = -\left[\frac{a_1r_3\cos\theta + b_1r_3\sin\theta}{c_1}\right] - \left[\frac{a_2r_3\cos\theta + b_2r_3\sin\theta}{c_2}\right]$$

$$2 = -\left[\frac{a_1h + b_1k}{c_1}\right] - \left[\frac{a_2h + b_2k}{c_2}\right]$$

$$-2 = h\left(\frac{a_1}{c_1} + \frac{a_2}{c_2}\right) + k\left(\frac{b_1}{c_1} + \frac{b_2}{c_2}\right)$$

$Ax + By + 2 = 0 \qquad$ which is the equation of line.

Q.9 If $bc \neq ad$ and the lines $\left.\begin{array}{l}(\sin 3\theta)x + ay + b = 0 \\ (\cos 2\theta)x + cy + d = 0 \\ 2x + (a+2c)y + (b+2d) = 0\end{array}\right\}$ are concurrent then find the number of values of θ in $[0, 2\pi]$.

Sol. If the lines are concurrent then

$$\begin{vmatrix} \sin 3\theta & a & b \\ \cos 2\theta & c & d \\ 2 & (a+2c) & (b+2d) \end{vmatrix} = 0$$

Applying $R_3 \to R_3 - R_1 - 2R_2$

$$\begin{vmatrix} \sin 3\theta & a & b \\ \cos 2\theta & c & d \\ 2 - \sin 3\theta - 2\cos 2\theta & 0 & 0 \end{vmatrix} = 0$$

$(2 - \sin\theta - 2\cos 2\theta)(ad - bc) = 0$

since $bc \neq ad$, hence

$2 - \sin 3\theta - 2\cos 2\theta = 0$

$2 - (3\sin\theta - 4\sin^3\theta) - 2(1 - 2\sin^2\theta) = 0$

$-3\sin\theta + 4\sin^3\theta + 4\sin^2\theta = 0$

$\sin\theta\,[4\sin^2\theta + 4\sin\theta - 3] = 0$

$\sin\theta\,(2\sin\theta - 1)(2\sin\theta + 3) = 0$

$\sin\theta = 0,\ \sin\theta = \dfrac{1}{2},\ \sin\theta = \dfrac{-3}{2}$ (rejected)

$\Rightarrow \quad \theta = 0, \pi, 2\pi \quad \theta = \dfrac{\pi}{6}, \dfrac{5\pi}{6}$

$\therefore$ number of values of θ in $[0, 2\pi]$ is 5.

Q.10 Let ABC be a given isosceles triangle with AB = AC. Sides AB and AC are extended up to E and F, respectively, such that $BE \times CF = AB^2$. Prove that the line EF always passes through a fixed point.

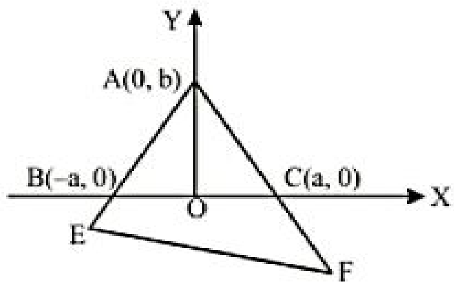
Sol. Let ABC be the triangle having vertices $(-a, 0)$, $(0, b)$ and $(a, 0)$. Now,

$BE \times CF = AB^2 \quad \Rightarrow \quad \dfrac{BE}{AB} = \dfrac{AB}{CF} = \lambda$ (let)

$\Rightarrow \quad \dfrac{BE}{AB} = \dfrac{AB}{CF} = \lambda$

Hence, the coordinates of E and F are $(-a(\lambda + 1), -\lambda b)$ and $(a(1 + 1/\lambda) - b/\lambda)$. Equation of line EF is

$$y + \lambda b = \frac{-\lambda b + \dfrac{b}{\lambda}}{-a(\lambda+1) - \dfrac{a(\lambda+1)}{\lambda}}\,[x + a(\lambda + 1)]$$



or $$y + \lambda b = \frac{\frac{b}{\lambda}(1-\lambda^2)}{-\frac{a(\lambda+1)}{\lambda}[1+\lambda]}\,[x + a(\lambda + 1)]$$

or $$y + \lambda b = \frac{b(\lambda - 1)}{a(\lambda + 1)}\,[x + a(\lambda + 1)]$$

or $a(\lambda + 1)y + ab\,\lambda\,(\lambda + 1) = b\,(\lambda - 1)\,x + ab\,(\lambda^2 - 1)$

or $(bx + ay + ab) - \lambda\,(bx - ay - ab) = 0$

which is the equation of a family of lines passing through the point of intersection of the lines $bx + ay + ab = 0$ and $bx - ay - ab = 0$, the point of intersection being $(0, -b)$. Hence, the line EF passes through a fixed point.

Q.11 The straight line $2x + 3y + 1 = 0$ bisects the angle between two straight lines, one of which is $3x + 2y + 4 = 0$. Find the equation of the other straight line.

Sol. The given line is $3x + 2y + 4 = 0$ ...(i)

and bisector is $2x + 3y + 1 = 0$ ...(ii)

Since the other line passes through the intersection of (i) and (ii), its equation is

$$2x + 3y + 1 + k\,(3x + 2y + 4) = 0. \qquad \text{...(iii)}$$

Since, P lies on one bisector

$\therefore$ length of perpendicular from P on (i) = length of perpendicular from P on (iii)

$$\Rightarrow \quad \left|\frac{3.1 + 2.(-1) + 4}{\sqrt{9+4}}\right| = \frac{|(2+3k).1 + (3+2k)(-1) + (1+4k)|}{\sqrt{(2+3k)^2 + (3+2k)^2}}$$

$$\Rightarrow \quad \frac{5}{\sqrt{15}} = \frac{5|k|}{\sqrt{13k^2 + 24k + 13}}$$

$$\Rightarrow \quad 13k^2 + 24k + 13 = 13k^2$$

$$\Rightarrow \quad 24k + 13 = 0 \quad \Rightarrow \quad k = -\frac{13}{24}$$

Substituting this value of k in (iii), we get

$$\left(2 - \frac{13}{8}\right)x + \left(3 - \frac{13}{12}\right)y + \left(1 - \frac{13}{6}\right) = 0 \Rightarrow \frac{3}{8}x + \frac{23}{12}y - \frac{7}{6} = 0$$

Q.12 Show that all chords of the curve $3x^2 - y^2 - 2x + 4y = 0$ subtending right angles at the origin pass through a fixed point.find also the coordinates of the fixed point.

Sol. $3x^2 - y^2 - 2x + 4y = 0$

Let the point through which it passes is (h, k)

Equation of line through (h, k)

$$y - k = m\,(x - h)$$

$$y - mx = k - mh$$

Homogenising the curve with the line, then

$$3x^2 - y^2 - 2x\left(\frac{y - mx}{k - mh}\right) + 4y\left(\frac{y - mx}{k - mh}\right) = 0$$

$$x^2\left(3 + \frac{2m}{k - mh}\right) + y^2\left(\frac{4}{k - mh} - 1\right) + xy\left(\frac{-2}{k - mh}\frac{-4m}{k - mh}\right) = 0$$

If they subtend a right angle at origin then, coeffi. of x^2 + coeffi. of $y^2 = 0$

$$3 + \frac{2m}{k - mh} + \frac{4}{k - mh} - 1 = 0$$

$$2 + \frac{2m}{k - mh} + \frac{4}{k - mh} = 0$$

$k - mh + m + 2 = 0$

$k + 2 + m(1 - h) = 0$

$k = -2$ and $h = 1$

Hence the fixed point is $(1, -2)$

Q.13 A straight line is drawn from the point (1, 0) to intersect the curve $x^2 + y^2 + 6x - 10y + 1 = 0$ such that the intercept made by it on the curve subtend a right angle at the origin. Find the equation of the line.

Sol. Let the equation of line is $y = m(x - 1)$

Equation of curve $x^2 + y^2 + 6x - 10y + 1 = 0$

Homogenising the curve with the line, then

$$x^2 + y^2 + 6x\left(\frac{mx - y}{m}\right) - 10y\left(\frac{mx - y}{m}\right) + \left(\frac{mx - y}{m}\right)^2 = 0$$

coeffi. of $x^2 = 1 + 6 + 1 = 8$

coeffi. of $y^2 = 1 + \frac{10}{m} + \frac{1}{m^2}$

If it subtends a right angle at origin the

$$8 + 1 + \frac{10}{m} + \frac{1}{m^2} = 0$$

$$\frac{1}{m^2} + \frac{10}{m} + 9 = 0$$

$9m^2 + 10m + 1 = 0$

$9m^2 + 9m + m + 1 = 0$

$(m + 1)(9m + 1) = 0$

$m = -1$ or $m = \frac{-1}{9}$

Equation of lines are

$y = -1(x - 1)$ or $y = \frac{-1}{9}(x - 1)$

$x + y = 1$ or $x + 9y = 1$

RELATION & FUNCTION

1. ORDERED PAIRS :

An ordered pair consisting of two elements in a given fixed order.
Eg. (a, b).
An order pair is not a set consisting of two elements. The position of a point in two dimentional plane is Eg. of an ordered pair (1, 2), (2, 2) ......

1.1 Equality of ordered pairs :

Two ordered pairs (a_1, b_1) and (a_2, b_2) are said to be equal if $a_1 = a_2$ & $b_1 = b_2$
Eg. Find the values of a and b if $(3a - 2, b + 3) = (2a - 1, 3)$

$b + 3 = 3,\ 3a - 2b = 2a - 1 \Rightarrow a = 1, b = 0$

1.2 Cartesian Product of two sets :

Let A and B be any two non empty sets. The set of all ordered pairs (a, b) such that $a \in A$ and $b \in B$ is called as cartesian product of sets A and B and is denotes by $A \times B$.
The cartesian product of two sets A, B is a non-void set of all ordered pairs (a, b).

$$A = \{1, 2, 3\};\quad B = \{p, q, r\}$$
$$A \times B = \{(a, b) / a \in A \text{ and } b \in B\}$$
$$= \{(1, p), (1, q), (1, r), (2, p), (2, q), (2, r), (3, p), (3, q), (3, r)\}$$

2. RELATION :

Every non-zero subset of $A \times B$ defined a relation from set A to set B.

Defination :
Relation is a linear operation which establishes relationship between the element's of two set's according to some definite rule of relationship.

$R : \{(a, b) \mid (a, b) \in A \times B \text{ and } a\,R\,b\}$

Eg:1 A is {2, 3, 5}
B is {1, 4, 9, 25, 30}
If a R b $\rightarrow$ b is square of a
Discreet element of relation are {(2, 4), (3, 9), (5, 25)}

Eg:2 A = {Jaipur, Patna, Kanpur, Lucknow}
B = {Rajasthan, Uttar Pradesh, Bihar}
aRb $\rightarrow$ a is capital of b,
$A \times B$ = {(Jaipur, Rajasthan), (Patna, Bihar), (Lucknow, Uttar Pradesh) }

2.1 Total number of Relation from A to B :

Let number of relations from A to B be x.
Let A contain 'm' elements and B contain 'n' element's.
Number of element's in $A \times B \rightarrow m \times n$
Number of non void subset's $= {}^{mn}C_1 + {}^{mn}C_2 + \ldots\ldots + {}^{mn}C_{mn} = 2^{mn} - 1$

2.2 Domain and Range of Relation:

If R be a relation from a set A to set B. Then set of all first component's or coordinates of ordered pairs is called the domain of R, while the set of all second component's or coordinates of the ordered pairs is called as range of relation.

Let $R : A \to B$ (R is a relation defined from set A to set B) then domain of this relation is

Domain : Set of all the first entries in R

$\{a \mid (a, b) \in R\}$

Range : Set of all the second entries in R

$\{b \mid (a, b) \in R\}$

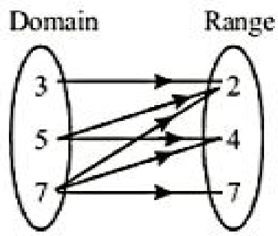
E.g. $A = \{1, 3, 5, 7\}$; $B = \{2, 4, 6, 8\}$

Relation is $aRb \Rightarrow a > b;\quad a \in A,\ b \in B$

$R = \{(3, 2), (5, 2), (5, 3), (7, 2), (7, 4), (7, 6)\}$

Domain = $\{3, 5, 7\}$

Range = $\{2, 4, 6\}$



2.3 Inverse of a Relation:

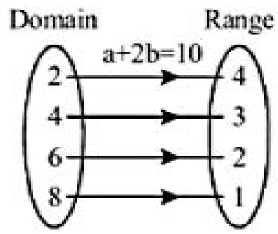
If R is a relation defined from $A \to B$ then R^{-1} is a relation defined from $B \to A$ as

$R^{-1} = \{(b, a) \mid (a, b) \in R\}$

i.e. domain is converted in to range element's and range is converted into domain elements.

i.e. Doman of R = Range of R^{-1}

Range of R = Domain of R^{-1}



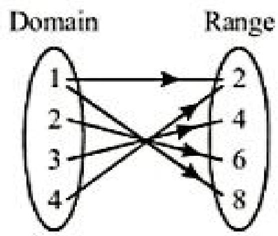
E.g. N is a set of first 10 natural nos.

$aRb \Rightarrow a + 2b = 10$

$N = \{1, 2, 3, ..., 10\}$ & $a, b \in N$

$R = \{(2, 4), (4, 3), (6, 2), (8, 1)\}$

Inverse relation is $R^{-1} \to \{(1, 8), (2, 6), (3, 4), (4, 2)\}$



2.4 Types of Relation :

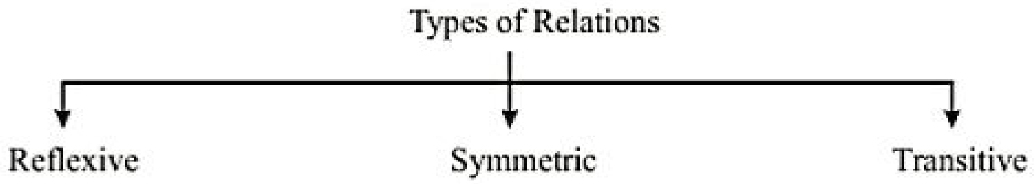
(i) Identity Relation

A relation defined on a set A is said to be an Identity relation if every element of A is related to itself and only to itself.

Eg.1 A relation defined on the set of natural number's as with rule $aRb \Leftrightarrow a = b$ is an identity relation

$R = \{(1, 1), (2, 2), (3, 3), ...\}$

Eg: 2 The relation $I_A = \{(1, 1), (2, 2), (3, 3), ...\}$ is the identity relation on set $A = \{1, 2, 3\}$ but $\{(1, 1), (2, 2), (1, 3)\}$ are not identity relation



(ii) Reflexive:

A relation defined on a set A is said to be an Identity relation if each & every element of A is related to itself.

i.e. if $(a, b) \in R$ then $(a, a) \in R$. However if there is a single ordered pair of $(a, b) \in R$ such $(a, a) \notin R$ then R is not reflexive.

Eg. 1 : Let $A = \{1, 2, 3\}$ be a set then $R = \{(1, 1), (2, 2), (3, 3), (1, 3), (2, 1)\}$ is a reflexive relation on A.
$R_1 = \{(1, 1), (3, 3), (2, 1), (6, 2)\}$ is not a reflexive relation on A, because $2 \in A$ but $(2, 2) \in R$.

E.g. 2: A relation defined on (set of natural numbers)
aRb $\Rightarrow$ 'a' divides 'b' $a, b \in N$

R would always contain (a, a) because every natural number divides itself and hence it is a reflexive relation.

Note :

Every identity relation is a reflexive relation but every reflexive relation need not be an identity.

(iii) Symmetric Relation:

A relation defined on a set is said to be symmetric

if $a R b \Rightarrow bRa$. If $(a, b) \in R$ then (b, a) must be necessarily there in the same relation.

Eg:

(i) a R b $\Rightarrow$ a is parallel to b
It is a symmetric relation because if a is parallel to b then the line b is parallel to a.

(ii) $L_1 R L_2$ ...... L_1 is perpendicular to L_2 is a symmetric relation.

(iii) a R b $\Rightarrow$ a is brother of b is not necessarily brother a.

(iv) a R b $\Rightarrow$ a is a cousin of b. This is a symmetric relation.

Note : If R is symmetric

(i) $R = R^{-1}$

(ii) Rangle of R = Domain of R

(iv) Transitive relation :

A relation on set A is said to be transitive if a R b and b R c implies a R c then it is transitive.

$(a, b) \in R$ & $(b, c) \in R \Rightarrow (a, c) \in R$ and (a, b, c) need not be distinct.

Eg. 1 : a R b (a – b) is even
$(6, 4), (4, 20) \Rightarrow (6, 20) \in R$

Eg. 2 : On the set of natural numbers, the relation R defined by $x R y \Rightarrow x < y$ because for any
$x, y, z \in N$ $x < y, y < z \Rightarrow x < z$.

(v) Equivalence Relation:

If a relation is Reflexive, Symmetric and Transitive then it is said to be an equivalence relation.

Eg. 1 : A relation defined on N
xRy $\Rightarrow$ $x = y$
R is an equivalence relation.

Eg. 2 : A relation defined on a set of || lines in a plane
aRb $\Rightarrow$ a || b
It is an equivalence relation.

Eg. 3 : Relation defined on the set of integer (I)
xRy $\Rightarrow$ (x – y) is even is an equivalence relation.

Illustration :

Check the following relations for being reflexive, symmetric, transitive and thus choose the equivalence relations if any.

(i) a R b if $|a| \le b$; $a, b \in$ set of real numbers.

(ii) $a\ R\ b$ iff $a < b$; $a, b \in N$.

(iii) $a\ R\ b$ iff $|a - b| > \frac{1}{2}$; $a, b \in R$.

(iv) $a\ R\ b$ iff a divides b; $a, b \in N$.

(v) $a\ R\ b$ iff $(a - b)$ is divisible by n; $a, b \in I$, n is a fixed positive integer.

Sol.

(i) *Not reflexive, not symmetric but transitive*

Let $a = -2$ and $b = 3$; $(-2, 3) \in R$. Since $|-2| \le 3$ is true

Since $|-2| = 2 \nleq -2$ hence relation is not Reflexive

Since $|3| \le -2$ is wrong hence relation is not symmetric

Now Let a, b, c be three real numbers such that $|a| \le b$ and $|b| \le c$

$|a| \le b \Rightarrow b \ge 0$, so $|b| \le c \Rightarrow b \le c$

Hence $|a| \le c$ is true so the given relation is transitive.

(ii) *Not reflexive, not symmetric but transitive.*
Since no natural number is less than itself hence not reflexive,
If $a < b$ then $b < a$ is false. Hence not symmetric.
If $a < b$ then $b < c$ clearly $a < c$. Hence transitive

(iii) *Not reflexive, symmetric, not transitive.*

$|a - a| = 0 \ngtr \frac{1}{2}$ hence it is not reflexive.

$\sqrt{x^2} = |x|$ hence symmetric.

Let $a = 1$, $b = -1$ and $c = \frac{3}{2}$, $|a - b| = 2 > \frac{1}{2}$ so $(a, b) \in R$; $|b - c| = \frac{5}{2} > \frac{1}{2}$ so $(b, c) \in R$

But $|a - c| = \left|1 - \frac{3}{2}\right| = \frac{1}{2} \ngtr \frac{1}{2}$ so $(a, c) \notin R$. Hence R is not a transitive relation.

(iv) *Reflexive, not symmetric, transitive*

Since $\frac{a}{a} = 1$ *i.e. every number divides itself, hence R is reflexive.*

If a divides b then b does not divide a (unless (a = b) hence the relation is not symmetric (but anti-symmetric).

If a divides b and b divides c then it is clear that a will divide c. Hence transitive.

(v) *Relfexive, symmetric as well as transitive, hence it is an equivalence relation.*

Since 0 is divisible by n $\left(\frac{0}{n} = 0\right)$ *so given relation is reflexive*

If $a - b$ *is divisible by n, then* $(b - a)$ *will also be divisible by n. Hence, symmetric.*

If $a - b = nI_1$ *and* $b - c = nI_2$*, where* I_1*,* I_2 *are integer.*

Then, $a - c = (a - b) + (b - c) = n(I_1 + I_2)$ *so* $a - c$ *is also divisible by n, hence transitive.*

Practice Problem

Q.1 If R is a relation from a finite set A having m elements to a finite set B having n elements, then the number of relations from A to B is-

(A) 2^{mn} (B) $2^{mn} - 1$ (C) 2mn (D) m^n

Q.2 Let L denote the set of all straight lines in a plane. Let a relation R be defined by $\alpha R \beta \Leftrightarrow \alpha \perp \beta, \alpha, \beta \in L$. Then R is -

(A) Reflexive (B) Symmetric (C) Transitive (D) None of these

Q.3 Two points A and B in a plane are related if OA = OB, where O is a fixed point. This relation is -

(A) Reflexive but not symmetric (B) Symmetric but not transitive

(C) An equivalence relation (D) None of these

Answer key

Q.1 B Q.2 B Q.3 C

FUNCTION

3. INTRODUCTION :

A function is like a machine which gives unique output for each input that is fed into it. But every machine is designed for certain defined inputs for eg. a juicer is designed for fruits & not for wood. Similarly functions are defined for certain inputs which are called as its "domain and corresponding outputs are called "Range".

3.1 General Definition :

Definition-1 :

Let A and B be two sets and let there exist a rule or manner or correspondence 'f' which associates to each element of A to a unique element in B,then f is called a <u>function</u> or <u>mapping</u> from A to B. It is denoted by the symbol

$$f : A \to B \text{ or } A \xrightarrow{f} B$$

which reads 'f' is a function from A to B' or 'f maps A to B,

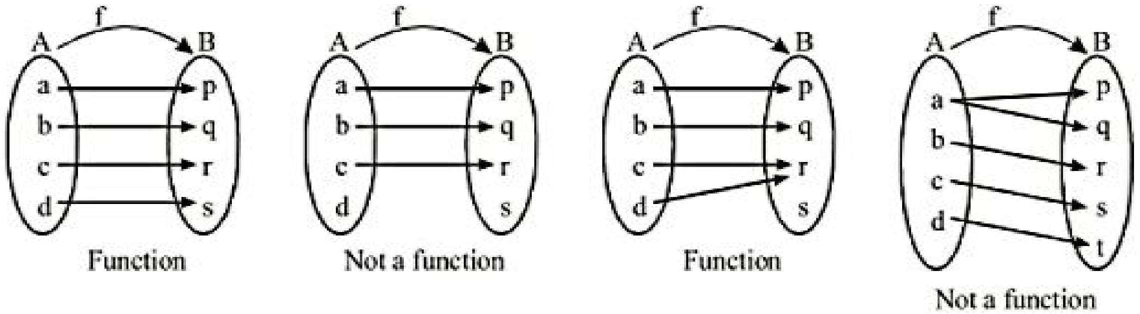
If an element $a \in A$ is associated with an element $b \in B$ then b is called 'the f image of a' or 'image of a under f' or 'the value of the function f at a'. Also a is called the pre-image of b or argument of b under the function f. We write it as

$$b = f(a) \text{ or } f : a \to b \text{ or } f : (a, b)$$

3.2 Function as a set of ordered pairs :

A function $f : A \to B$ can be expressed as a set of ordered pairs in which each ordered pair is such that its first element belongs to A and second element is the corresponding element of B.

As such a function $f : A \to B$ can be considered as a set of ordered pairs $(a, f(a))$ where $a \in A$ and $f(a) \in B$ which is the f image of a. Hence f is a subset of $A \times B$.



As a particular type of relation, we can define a function as follows :

Definition-2 :

A relation R from a set A to a set B is called a function if

(i) each element of A is associated with some element of B.

(ii) each element of A has unique image in B.

Thus a function 'f' from a set A to a set B is a subset of A × B in which each 'a' belonging to A appears in one and only one ordered pair belonging to f. Hence a function f is a relation from A to B satisfying the following properties :

Every function from $A \to B$ satisfies the following conditions.

(i) $f \subset A \times B$ (ii) $\forall\ a \in A \Rightarrow (a, f(a)) \in f$ and

(iii) $(a, b) \in f\ \&\ (a, c) \in f \Rightarrow b = c$.

Thus the ordered pairs of f must satisfy the property that each element of A appears in some ordered pair and no two ordered pairs have same first element.

Note : Every function is a relation but every relation is not necessarily a function.

3.3 Domain, Co-domain & Range of A Function :

Let $f: A \to B$, then the set A is known as the domain of f & the set B is known as co–domain of f. The set of all f images of elements of A is known as the range of f. Thus :

$$\text{Domain of } f = \{a \mid a \in A, (a, f(a)) \in f\}$$

$$\text{Range of } f = \{f(a) \mid a \in A, f(a) \in B, (a, f(a)) \in f\}$$

It should be noted that range is a subset of co–domain . If only the rule of function is given then the domain of the function is the set of those real numbers, where function is defined. For a continuous function, the interval from minimum to maximum value of a function gives the range.

Let f and g be function with domain D_1 and D_2 then the functions

Note :

$f+g, f-g, fg, f/g$ are defined as

$(f+g)(x) = f(x) + g(x);$ Domain $D_1 \cap D_2$

$(f-g)(x) = f(x) - g(x);$ Domain $D_1 \cap D_2$

$(fg)(x) = f(x) \cdot g(x);$ Domain $D_1 \cap D_2$

$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)};$ Domain $= \{x \in D_1 \cap D_2 \mid g(x) \neq 0\}$

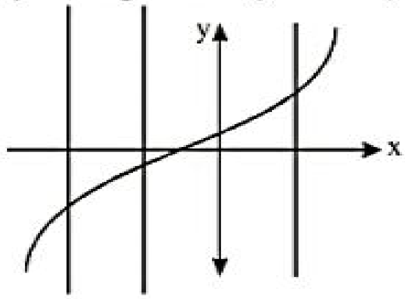
e.g. $f(x) = x^3 + 2x^2$ and $g(x) = 3x^2 - 1$. Find $f \pm g$, fg and f/g.

3.4 Graphical Representation of function :

Let f be a mapping with domain D such that y = f(x) should assume angle value for each x,(i.e. the straight line drawn parallel to y-axis in its domain should cut at only one point.

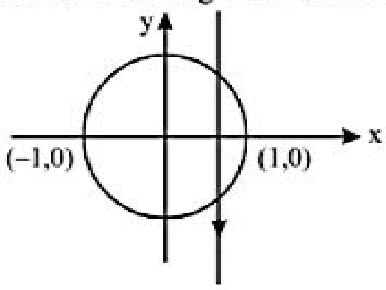
Eg. $y = x^3$

Here all the straight lines parallel to y-axis cut $y = x^3$ only at one print.



Eg. $x^2 + y^2 = 1^2$

Here line parallel to y-axis is intersecting the circle at two points hence it is not a function.



3.5 Domain :

Rule for finding Domain :

(i) Expression under even root (i.e. square root, fourth root etc) ≥ 0.

(ii) Denominator $\neq 0$

(iii) If domain of $y = f(x)$ & $y = g(x)$ are D_1 & D_2 respectively then the domain of $f(x) \pm g(x)$ or $f(x) \cdot g(x)$ is $D_1 \cap D_2$

(iv) Domain of $\frac{f(x)}{g(x)}$ is $D_1 \cap D_2 - \{g(x) = 0\}$.

3.6 Classification of Functions:

(i) Polynomial Function:

If a function f is defined by $f(x) = a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + ... + a_{n-1} x + a_n$ where n is a non negative integer and $a_0, a_1, a_2, ..., a_n$ are real numbers and $a_0 \neq 0$, then f is called a polynomial function of degree *n*. A polynomial function is always continuous.

NOTE: **(A)** A polynomial of degree one with no constant term is called an odd linear function i.e. $f(x) = ax,\ a \neq 0$

(B) There are two polynomial functions, satisfying the relation ; $f(x).f(1/x) = f(x) + f(1/x)$. They are :
(a) $f(x) = x^n + 1$ & **(b)** $f(x) = 1 - x^n$, where n is a positive integer.

(C) A polynomial of degree odd has its range $(-\infty, \infty)$ put a polynomial of degree even has a range which is always subset of R.

(ii) Algebraic Function:

A function f is called an algebraic function if it can be constructed using algebraic operations such as addition, substraction, multiplication, division and taking roots, started with polynomials.

e.g. $f(x) = \sqrt{x^2+1}$; $g(x) = \dfrac{x^4 - 16x^2}{x + \sqrt{x}} + (x-2) \times \sqrt[3]{x+1}$

Note that all polynomial are algebraic but converse is not true. Functions which are not algebraic, are known as Transcidential function.

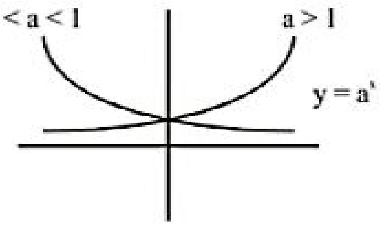
(iii) Fractional Rational Function:

A rational function is a function of the form. $y = f(x) = \dfrac{g(x)}{h(x)}$, where $g(x)$ & $h(x)$ are polynomials & $h(x) \neq 0$. The domain of $f(x)$ is set of realx such that $h(x) \neq 0$.

e.g. $f(x) = \dfrac{2x^4 - x^2 + 1}{x^2 - 4}$; $D = \{x \mid x \neq \pm 2\}$

(iv) Exponential Function:

A function $f(x) = a^x = e^{x \ln a}$ $(a > 0,\ a \neq 1,\ x \in R)$ is called an exponential function. $f(x) = a^x$ is called an exponential function because the variable x is the exponent. It should not be confused with power function. $g(x) = x^2$ in which variable x is the base.

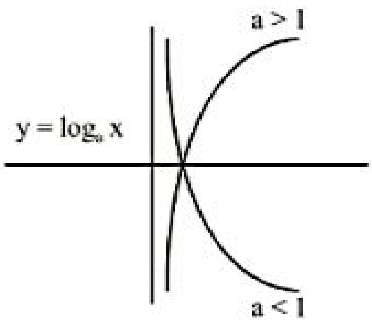


For $f(x) = e^x$ domain is R and range is R^+.

For $f(x) = e^{1/x}$ domain is $R - \{0\}$ and range is $R^+ - \{1\}$. i.e. $(0, 1) \cup (1, \infty)$

$f(x) = \dfrac{1}{\ln x}$ with domain $R^+ - \{1\}$, range is $R - \{0\}$

(v) Logarithmic function: A function of the form $y = \log_a x,\ x > 0,\ a > 0,\ a \neq 1$, is called Logarithmic function.



(vi) Absolute Value Function:

A function $y = f(x) = |x|$ is called the absolute value function or Modulus function. It is defined as:

$$y = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

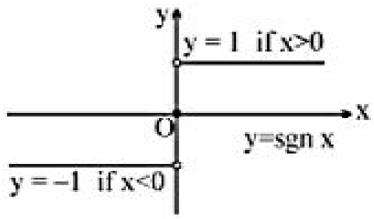
For $f(x) = |x|$, domain is R and range is $R^+ \cup \{0\}$.

For $f(x) = \dfrac{1}{|x|}$ or $\dfrac{|x|}{x^2}$, domain is $R - \{0\}$ and range is R^+.

(vii) Signum Function:

A function $y = f(x) = \text{Sgn}(x)$ is defined as follows :

$$y = f(x) = \begin{cases} 1 & \text{for} \quad x > 0 \\ 0 & \text{for} \quad x = 0 \\ -1 & \text{for} \quad x < 0 \end{cases}$$



It is also written as $\text{Sgn}\, x = |x|/x$ or $\dfrac{x}{|x|}$

$x \neq 0$; $f(0) = 0$

Note that $\text{Sgn}(\text{Sgn}\, x) = \text{Sgn}\, x$;

$$y = \text{Sgn}(x^2 - 1) = \begin{cases} 1, & |x| > 1 \\ 0, & |x| = 1 \\ -1, & |x| < 1 \end{cases}$$

(viii) Greatest Integer Or Step Up Function :

The function $y = f(x) = [x]$ is called the greatest integer function where $[x]$ denotes the greatest integer less than or equal to x . Note that for :

$-1 \le x < 0$; $[x] = -1$ $\quad$ $0 \le x < 1$; $[x] = 0$

$1 \le x < 2$; $[x] = 1$ $\quad$ $2 \le x < 3$; $[x] = 2$ and so on.

For $f(x) = [x]$, domain is R and range is I.

For $f(x) = \dfrac{1}{[x]}$ domain is $R - [0, 1)$ and range is $\left\{ \dfrac{1}{n} \,\middle|\, n \in I - \{0\} \right\}$.

Properties of greatest integer function :

(a) $[x] \le x < [x] + 1$ and
$x - 1 < [x] \le x$, $0 \le x - [x] < 1$

(b) $[x + m] = [x] + m$, if m is an integer.

(c) $[x] + [y] \le [x + y] \le [x] + [y] + 1$

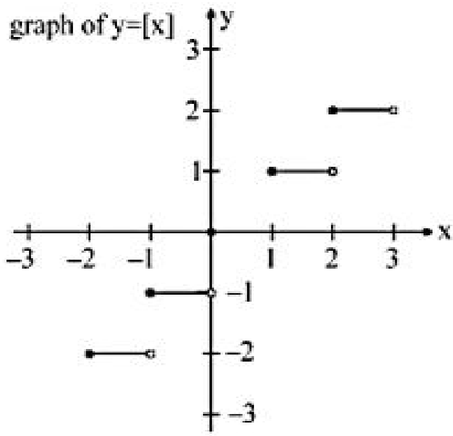
(d) $[x] + [-x] = \begin{cases} 0 & \text{if } x \text{ is an integer} \\ -1 & \text{otherwise.} \end{cases}$

(e) $[x] \ge n \Rightarrow x \in [n, \infty) \ \forall \ n \in I$

(f) $[x] > n \Rightarrow x \in [n + 1, \infty) \ \forall \ n \in I$

(g) $[x] \le n \Rightarrow x \in (-\infty, n + 1) \ \forall \ n \in I$

(h) $[x] < n \Rightarrow x \in (-\infty, n) \ \forall \ n \in I$



(ix) Fractional Part Function :

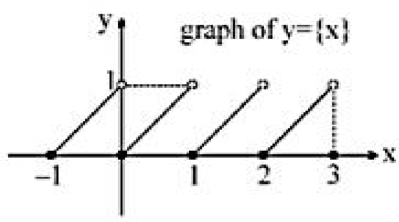
It is defined as :

$g(x) = \{x\} = x - [x]$.

e.g. the fractional part of the number 2.1 is $2.1 - 2 = 0.1$ and the fractional part of -3.7 is 0.3. The period of this function is 1 and graph of this function is as shown.

For $\quad f(x) = \{x\}$, domain is R and range is $[0, 1)$

For $\quad f(x) = \dfrac{1}{\{x\}}$, domain is $R - I$, range is $(1, \infty)$



Properties of fractional part :

(a) $0 \le \{x\} < 1$

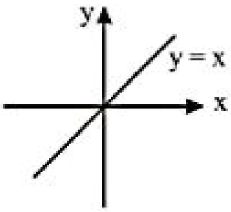
(b) $\{x+n\} = \{x\},\ n \in I$

(c) $\{x\}+\{-x\} = \begin{cases} 0, & x \in I \\ 1, & x \notin I \end{cases}$

(x) Identity function :

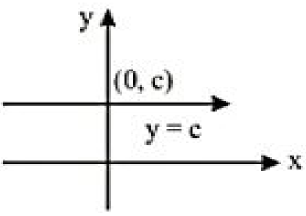
The function $f: A \to A$ defined by $f(x) = x\ \forall\ x \in A$ is called the identity of A and is denoted by I_A. It is easy to observe that identity function defined on R is a bijection.

$f: R \to R, \qquad f(x) = x$



(xi) Constant function :

A function $f: A \to B$ is said to be a constant function if every element of A has the same f image in B. Thus $f: A \to B$; $f(x) = c,\ \forall\ x \in A$, $c \in B$ is a constant function. Note that the range of a constant function is a singleton and a constant function may be one–one or many–one, onto or into.



e.g. $f(x) = [\{x\}]$; $g(x) = \sin^2x + \cos^2x$; $h(x) = \text{sgn}(x^2 - 3x + 4)$ etc, all are constant functions.

Illustration :

Find the domain of following function

(i) $f(x) = \sqrt{x^2 - 5x + 6}$

(ii) $f(x) = \sqrt{x^2 - 3x + 2} + \dfrac{1}{\sqrt{x^2 - 3x - 4}}$

(iii) $f(x) = \dfrac{2}{x^2 - 4} + \log_{10}(x^3 - x)$

(iv) $f(x) = \dfrac{1}{\sqrt{|x| - x}}$

(v) $f(x) = \dfrac{1}{\sqrt{[x] - x}}$

(vi) $f(x) = \sqrt{\log_{\frac{1}{2}}\left(\dfrac{5x - x^2}{4}\right)}$

(vii) $f(x) = \sqrt{\cos(\sin x)} + \sin^{-1}\left(\dfrac{1 + x^2}{2x}\right)$

(viii) $f(x) = \log_4 \log_2 \log_{1/2}(x)$

Sol.

(i) $f(x) = \sqrt{x^2 - 5x + 6}$

$\Rightarrow \quad x^2 - 5x + 6 \ge 0$

$\Rightarrow \quad (x - 2)(x - 3) \ge 0$

$\Rightarrow \quad x \in (-\infty, 2] \cup [3, \infty)$

(ii) $f(x) = \sqrt{x^2 - 3x + 2} + \dfrac{1}{\sqrt{x^2 - 3x - 4}}$

$x^2 - 3x + 2 \geq 0$ and $x^2 - 3x - 4 > 0$

$x^2 - 3x + 2 \geq 0$

$\Rightarrow \quad (x - 2)(x - 1) \geq 0$

$\Rightarrow \quad x \in (-\infty, 1] \cup [2, \infty)$ ...(i)

and $x^2 - 3x - 4 > 0$

$(x - 4)(x + 1) > 0$

$x \in (-\infty, -1) \cup (4, \infty)$ ...(ii)

Taking union of (i) & (ii)

$x \in (-\infty, -1) \cup (4, \infty)$

(iii) $f(x) = \dfrac{2}{x^2 - 4} + \log_{10}(x^3 - x)$

Following conditions should be followed

$x^2 - 4 \neq 0$ & $x^3 - x > 0$

$x \neq \pm 2$

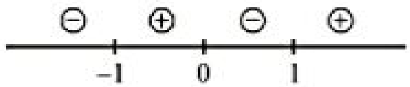
$x \in R - \{-2, 2\}$ ...(i)

$x^3 - x > 0$

$\Rightarrow \quad x(x^2 - 1) > 0$

$\Rightarrow \quad x(x - 1)(x + 1) > 0$

$x \in (-1, 0) \cup (1, \infty)$ ...(ii)



Taking union of (i) & (ii)

$x \in (-1, 0) \cup (1, 2) \cup (2, \infty)$

(iv) $f(x) = \dfrac{1}{\sqrt{|x| - x}}$

$|x| - x > 0$

$|x| > x$

This is possible only when x is negative i.e. $x < 0$, hence

$x \in (-\infty, 0)$

(v) $f(x) = \dfrac{1}{\sqrt{[x] - x}}$

$[x] - x > 0$

$[x] > x$

but we know that $[x] \leq x$

Hence domain is ϕ

(vi) $f(x) = \sqrt{\log_{\frac{1}{2}}\left(\frac{5x-x^2}{4}\right)}$

$\frac{5x-x^2}{4} > 0$

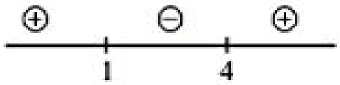
$\Rightarrow \quad x(5-x) > 0 \qquad\qquad \Rightarrow \quad x(x-5) < 0$

$\quad x \in (0, 5) \qquad\qquad ...(i)$

Also $\quad \log_{\frac{1}{2}}\left(\frac{5x-x^2}{4}\right) \geq 0$

$\Rightarrow \quad \frac{5x-x^2}{4} \leq \left(\frac{1}{2}\right)^0 \qquad \Rightarrow \quad 5x - x^2 \leq 4$

$\Rightarrow \quad x^2 - 5x + 4 \geq 0$



$\quad x \in (-\infty, 1] \cup [4, \infty) \quad ...(ii)$

Using (i) and (ii)

$\quad x \in (0, 1] \cup [4, 5)$

(vii) $f(x) = \sqrt{\cos(\sin x)} + \sin^{-1}\left(\frac{1+x^2}{2x}\right)$

$\cos(\sin x) \geq 0$

$-1 \leq \sin x \leq 1 \;\; \forall x \in R$

$\therefore \cos\theta > 0$ for $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$

hence $x \in R \qquad\qquad ...(i)$

$\sin^{-1}\left(\frac{1+x^2}{2x}\right)$

$\Rightarrow \quad \left|\frac{1+x^2}{2x}\right| \leq 1 \qquad\qquad \Rightarrow |1+x^2| \leq |2x|$

$\Rightarrow \quad |x|^2 - 2|x| + 1 \leq 0$

$\Rightarrow \quad (|x| - 1)^2 \leq 0$

Only at $x = \pm 1 \qquad\qquad ...(ii)$

Using (i) and (ii)

$x \in \{-1, 1\}$

(viii) $f(x) = \log_4 \log_2 \log_{1/2}(x)$

$\Rightarrow \log_2 \log_{1/2}(x) > 0 \qquad\qquad \Rightarrow \log_{1/2}(x) > (2^0)$

$\Rightarrow \log_{1/2} x > 1 \qquad\qquad \Rightarrow x < \left(\frac{1}{2}\right)^1$

Also $0 < x$

$\Rightarrow x \in \left(0, \frac{1}{2}\right)$

RANGE :

Range of y = f(x) is the collection of all outputs corresponding to each real number of the domain.

To find the range of function

(i) First of all find the domain of y = f(x).

(ii) If domain is a set having only finite number of points, then range is the set of corresponding f(x) values.

(iii) If domain of y = f(x) is R or R– {Some finite points}, then express x in terms of y. From this find y for x to be defined or real or form an equation in terms of x & apply the condition for real roots.

Illustration :

Find the range of the following function

(i) $f(x) = a \sin x + b.\ a > 0\ ,\ b \in R$ *(ii)* $f(x) = 4 \tan x \cos x$

(iii) $y = \dfrac{x^2}{1+x^2}$ *(iv)* $y = \log_e (3x^2 - 4x + 5)$

(v) $y = \dfrac{x^2 - x}{x^2 + 2x} = \dfrac{x(x-1)}{x(x+2)}$ *(vi)* $y = 3 - 2^x$

Sol.

(i) $f(x) = a \sin x + b.\ a > 0\ ,\ b \in R$

$f()x = a \sin x + b$

$\because -1 \le \sin x \le 1$

$\therefore -a + b \le f(x) \le a + b$

Range $\in [b - a,\ b + a]$

(ii) $f(x) = 4 \tan x \cos x$

$f(x) = 4 \sin x$ *for* $\cos x \ne 0$

$-1 \le \sin x \le 1$

but at $\sin x = \pm 1,\ \cos x = 0$

hence points with $\sin x = \pm 1$ *will not be included in range.*

Range $\in (-4, 4)$.

(iii) $y = \dfrac{x^2}{1+x^2}$

y is defined $\forall x \in R$, *domain is R*

from $y = \dfrac{x^2}{1-x^2} \quad \Rightarrow \quad x^2 = \dfrac{y}{1-y}$

$\Rightarrow x = \sqrt{\dfrac{y}{1-y}} \ge 0 \quad \Rightarrow \quad \dfrac{y}{1-y} \ge 0$

$\Rightarrow 0 \le y < 1$

Range $[0, 1)$

(iv) $y = \log_e(3x^2 - 4x + 5)$
y is defined if $3x^2 - 4x + 5 > 0$
$D < 0$ and coefficeint of $x^2 > 0$
hence domain is R and log is increaisng function.

Minimum value of $3x^2 - 4x + 5$ is $-\dfrac{D}{4a}$

$$\Rightarrow \quad = \frac{-(-44)}{4(3)} = \frac{11}{3} \quad \Rightarrow \quad y \geq \log_e\left(\frac{11}{3}\right)$$

$$\text{Range} \in \left[\log_e\left(\frac{11}{3}\right), \infty\right)$$

(v) $y = \dfrac{x^2 - x}{x^2 + 2x} = \dfrac{x(x-1)}{x(x+2)}$
Domain is $x \in R - \{-2, 0\}$

$$y = \frac{x(x-1)}{x(x+2)}$$

when $x \neq 0$, $y = \dfrac{x-1}{x+2}$ $\quad \Rightarrow \quad x = \dfrac{1+2y}{1-y}$
$\therefore$ If x is real $y - 1 \neq 0 \Rightarrow y \neq 1$

Also for $x = \dfrac{1+2y}{1-y}$; $x \neq 0$

hence $y \neq -\dfrac{1}{2}$ $\quad \Rightarrow \quad$ Hence range $y \in R - \left\{\dfrac{-1}{2}, 1\right\}$

(vi) $y = 3 - 2^x$
Domain is $x \in R$
$0 \leq 2^x < \infty$
Range $\in (-\infty, 3)$

Illustration :

Find the domain of the folloiwng function

(i) $f(x) = \ln(3x^2 - 4x + 5) + \sqrt{2\sin^2 x - 5\sin x + 2}$

(ii) $f(x) = \ln\{x\} + \sqrt{x - 2\{x\}}$

(iii) $f(x) = \sqrt{\log_{0.3}\left|\dfrac{x-2}{x}\right|}$

(iv) $f(x) = \dfrac{1}{[\,|x-1| + |\,|7-x|\,|\,] - 6}$

(v) $f(x) = \log\sqrt{-\left(\cos x + \dfrac{1}{2}\right)}$

Sol.

(i) $f(x) = \ln(3x^2 - 4x + 5) + \sqrt{2\sin^2 x - 5\sin x + 2}$

$3x^2 - 4x + 5$

Coefficient of $x^2 = 3$

Discriminant $(D) = 16 - 4 \times 5 \times 3 = -44 < 0$

hence $3x^2 - 4x + 5 > 0,\ \forall x \in R$

$2\sin^2 x - 5\sin x + 2 \geq 0$

$$\Rightarrow 2\left(\sin^2 x - \frac{5}{2}\sin x\right) + 2 \geq 0 \qquad \Rightarrow \qquad 2\left[\left(\sin^2 x - \frac{5}{4}\right)^2 - \frac{25}{16}\right] + 2 \geq 0$$

$$\Rightarrow \left(\sin x - \frac{5}{4}\right)^2 \geq \frac{9}{16} \qquad \Rightarrow \qquad \sin x \geq 2 \quad or \quad \sin x - \frac{5}{4} \leq \frac{-3}{4}$$

$$\sin x \geq 2 \qquad \sin x \leq \frac{1}{2} \qquad \Rightarrow \qquad x \in \left[2n\pi - \frac{7\pi}{6}, 2n\pi + \frac{\pi}{6}\right], n \in I$$

(ii) $f(x) = \ln\{x\} + \sqrt{x - 2\{x\}}$

for $\ln\{x\}$ to be defined

$\{x\} > 0$

$x \in R - I$ ... (i)

$x - 2\{x\} > 0$

$\Rightarrow \quad [x] - \{x\} > 0 \quad \Rightarrow \quad [x] > \{x\}$

$x \geq 1$ ... (ii)

Using (i) and (ii)

$x \in (1, \infty) - I^+$

(iii) $f(x) = \sqrt{\log_{0.3}\left|\frac{x-2}{x}\right|}$

for $f(x)$ to be defined

$$0 < \left|\frac{x-2}{x}\right| \leq 1$$

$$\Rightarrow -1 \leq \frac{x-2}{x} \leq 1 \ \ and \ \ \frac{x-2}{x} \neq 0 \qquad \text{... (i)}$$

Solving LHS

$$\frac{x-2}{x} + 1 \geq 0$$

$\Rightarrow x < 0, x \geq 1$ ... (ii)

Solving RHS

$\frac{x-2}{x} \quad 1 \le 1$

$x > 0$ ... (iii)

hence from (i), (ii) and (iii)

$x \in [1, \infty) - \{2\}$

(iv) $f(x) = \dfrac{1}{[|x-1|]+[|7-x|] - 6}$

$[|x-1|] + [|7-x|] - 6 \neq 0$

Case I :

$1 < x < 7$

$[x-1] + [7-x] - 6 \neq 0$

$[x] - 1 + [-x] + 7 \neq 0$

$[x] + [-x] \neq 0$

$x \notin I$

$x \in (1, 7) - \{2, 3, 4, 5, 6\}$

Case II :

$x \le 1$

$[1-x] + [7+x] - 6 \neq 0$

$2 + 2[-x] \neq 0$

$[-x] \neq -1$

$-x \notin -1$

$\Rightarrow x\ (0, 1] \Rightarrow x \in (-\infty, 0]$

Case III :

$[x-1] + [x-7] - 6 \neq 0$

$2[x] \neq 14$

$[x] \neq 7$

$x \notin [7, 8)$

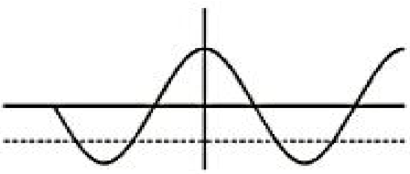
using can I, II and III we get

$x \in R - (0, 1] - [7, 8) - \{2, 3, 4, 5, 6)$

(v) $f(x) = \log \sqrt{-\left(\cos x + \frac{1}{2}\right)}$

$-\left(\cos x + \frac{1}{2}\right) > 0$

$\Rightarrow \cos x < -\frac{1}{2}$



$x \in \left(2n\pi + \frac{2\pi}{3}, 2n\pi + \frac{4\pi}{3}\right)$

Practice Problem

Q.1 Find the domain of the following functions

(i) $f(x)=\dfrac{\sqrt{\cos x-\frac{1}{2}}}{\sqrt{6+35x-6x^2}}$ (ii) $f(x)=\sqrt{(x^2-3x-10)\ln^2(x-3)}$

(iii) $f(x)=\log_{2\{x\}-3}(x^2-5x+13)$ (iv) $f(x)=\left(\log_{\frac{x-2}{x+3}}2\right)+\sqrt{9-x^2}$

Q.2 Find the range of the following functions

(i) $y=\dfrac{x^2+x-1}{x^2-x+2}$ (ii) $y=3\sin\sqrt{\dfrac{9\pi^2}{16}-x^2}$

(iii) $y=3\sin x+4\sin\left(x+\dfrac{\pi}{3}\right)+7$ (iv) $y=\dfrac{e^x-e^{-|x|}}{e^x+e^{|x|}}$

(v) $y=[x^2]-[x]^2$

4. EQUAL OR IDENTICAL FUNCTION :

Two functions f & g are said to be equal if

(i) The domain of f = the domain of g.

(ii) The range of f = the range of g and

(iii) $f(x)=g(x)$, for every x belonging to their common domain.

e.g. $f(x)=\dfrac{1}{x}$ & $g(x)=\dfrac{x}{x^2}$ are identical functions.

Note : Functions are also equal if their graphs are same

Illustration :

Find the domain of x for which the function $f(x) = \ln x^2$ and $g(x) = 2\ln x$ are identical.

Sol. $f(x) = \ln x^2 = 2\ln|x|$

$g(x) = 2\ln x$

if $f(x) = g(x)$

$2\ln|x| = 2\ln x$

function are equal only if $x \in (0, \infty)$

Illustration :

Find out which of the following functions are identical.

(i) $f(x) = \operatorname{cosec} x,\ g(x) = \dfrac{1}{\sin x}$

(ii) $f(x) = \tan x,\ g(x) = \dfrac{1}{\cot x}$

(iii) $f(x) = \ln e^x,\ g(x) = e^{\ln x}$

(iv) $f(x) = \sqrt{\dfrac{1-\cos 2x}{2}},\ g(x) = \sin x$

(v) $f(x) = \dfrac{1}{|x|},\ g(x) = \sqrt{x^{-2}}$

Sol.

(i) $f(x) = \operatorname{cosec} x,\ g(x) = \dfrac{1}{\sin x}$

Domain of $f(x) \Rightarrow x \notin n\pi$

Domain of $g(x) \Rightarrow x \notin n\pi$

Since domain and range are same hence identical fucntion

(ii) $f(x) = \tan x,\ g(x) = \dfrac{1}{\cot x}$

$f(x) = \tan x$, $x = 0$ is domain of $f(x)$

$$g(x) = \frac{1}{\cot x}$$

$x = 0$ is not in the domain of $g(x)$

hence $f(x)$ and $g(x)$ are not identical.

(iii) $f(x) = \ln e^x,\ g(x) = e^{\ln x}$

$f(x) = \ln e^x$ Domain $= R$

$g(x) = e^{\ln x}$ Domain $= R^+$

hence not identical funciton

(iv) $f(x) = \sqrt{\dfrac{1-\cos 2x}{2}},\ g(x) = \sin x$

$f(x) = |\sin x|$ Range $[0, 1]$

$g(x) = \sin x$ Range $[0, 1]$

hence not identical.

(v) $f(x) = \dfrac{1}{|x|},\ g(x) = \sqrt{x^{-2}}$

$$f(x) = \frac{1}{|x|}$$

$$g(x) = \frac{1}{\sqrt{x^2}} = \frac{1}{|x|}$$

hence identical functions.

Practice Problem

Q.1 Identify the equal function

(i) $f(x) = \log_x e\ ;\ g(x) = \dfrac{1}{\log_e x}$

(ii) $f(x) = \log_e x\ ;\ g(x) = \dfrac{1}{\log_x e}$

(iii) $f(x) = \sqrt{x^2 - 1}\ ;\ g(x) = \sqrt{x-1}\ \sqrt{x+1}$

(iv) $f(x) = \log(x+2) + \log(x-3)\ ;\ g(x) = (x^2 - x - 6)$

(v) $f(x) = x\,|\,x\,|\ ;\ g(x) = x^2 \operatorname{sgn} x$

(vi) $f(x) = \dfrac{1}{1+\frac{1}{x}}\ ;\ g(x) = \dfrac{x}{1+x}$

(vii) $f(x) = [\{x\}]\ ;\ g(x) = \{[x]\}$

5. CLASSIFICATION OF FUNCTIONS:

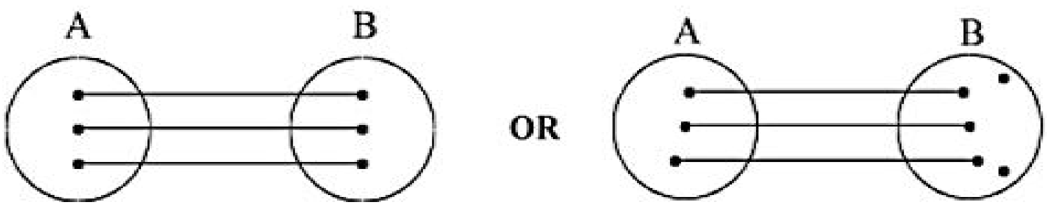
5.1 One-One Function (Injective mapping):

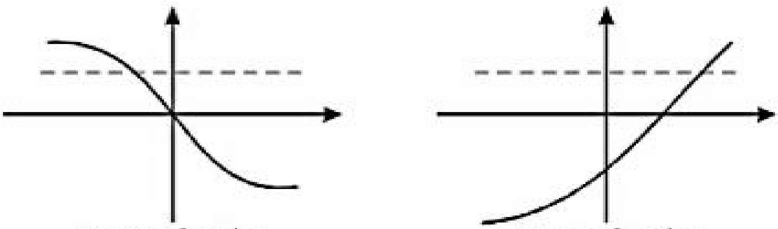
A function $f: A \to B$ is said to be a one–one function or injective mapping if different elements of A have different f images in B. Thus for $x_1, x_2 \in A$ & $f(x_1), f(x_2) \in B$,
$f(x_1) = f(x_2) \Leftrightarrow x_1 = x_2$ or $x_1 \neq x_2 \Leftrightarrow f(x_1) \neq f(x_2)$.

Examples: $R \to R$ $f(x) = x^3 + 1$; $f(x) = e^{-x}$; $f(x) = \ln x$

Remember that a linear function is always one-one.

Diagramatically an injective mapping can be shown as





one-one function   one-one function

Note:

(i) A continuous function which is always increasing or decreasing in whole domain, then f(x) is one–one.

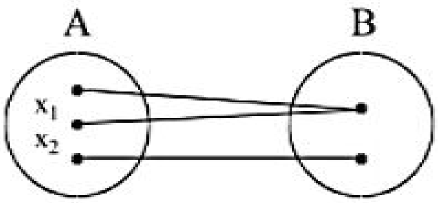
(ii) A function is one to one if and only if a horizontal line intersects its graph at most once.

5.2 Many-one function (not injective) :

A function $f: A \to B$ is said to be a many one function if two or more elements of A have the same f image in B. Thus $f: A \to B$ is many one if for;
$x_1, x_2 \in A$, $f(x_1) = f(x_2)$ but $x_1 \neq x_2$.

Examples : $R \to R$ $f(x) = [x]$; $f(x) = |x|$; $f(x) = ax^2 + bx + c$; $f(x) = \sin x$

Diagramatically a many one mapping can be shown as



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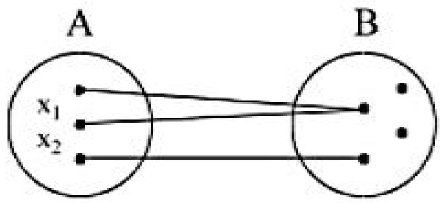
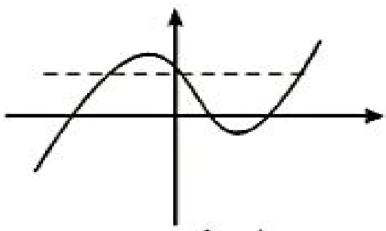
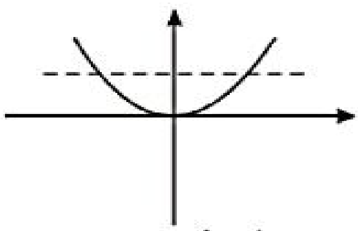



many-one function


many-one function

Note:

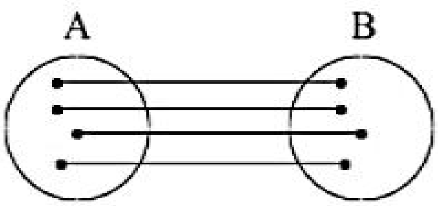
(i) Any continuous function which has atleast one local maximum or local minimum in its domain, then f(x) is many–one. In other words, if a line parallel to x–axis cuts the graph of the function atleast at two points, then f is many–one.

(ii) If a function is one–one, it cannot be many–one and vice versa.
One One + Many One = Total number of mappings.

5.3 Onto function (Surjective mapping) :

If the function $f: A \to B$ is such that each element in B (co–domain) is the f image of atleast one element in A, then we say that f is a function of A 'onto' B. Thus $f: A \to B$ is surjective iff $\forall\, b \in B$, $\exists$ some $a \in A$ such that $f(a) = b$.
$f: R \to R$ $f(x) = 2x + 1$; $f: R \to R^+$ $f(x) = e^x$; $f: R^+ \to R$ $f(x) = \ln x$

Diagramatically surjective mapping can be shown as



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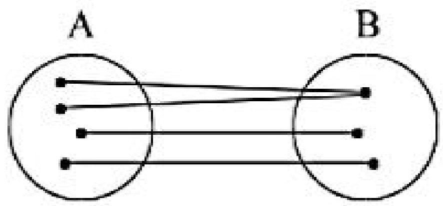


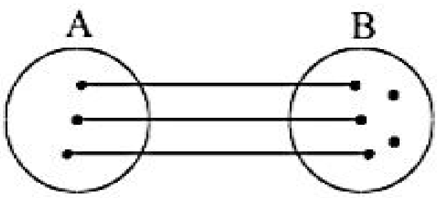
Note that: If range = co–domain, then f (x) is onto. Any polynomial of degree odd , $f: R \to R$ is onto.

5.4 Into function:

If $f : A \to B$ is such that there exists atleast one element in co–domain which is not the image of any element in domain, then $f(x)$ is into.

e.g. $\quad f : R \to R \quad f(x) = [x], |x|, \text{sgn}\, x, \; f(x) = ax^2 + bx + c$

Diagramatically into function can be shown as



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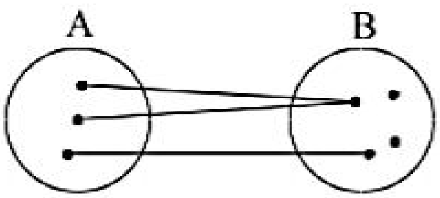
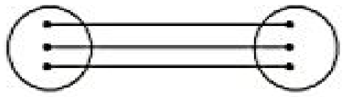


Note :

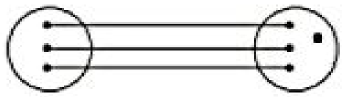
(i) If a function is onto, it cannot be into and vice versa . A polynomial of degree even define from $R \to R$ will always be into & a polynomial of degree odd defined from $R \to R$ will always be onto.

(ii) A function can be one of these four types :

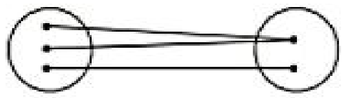
(a) one–one onto (injective & surjective) $(I \cap S)$



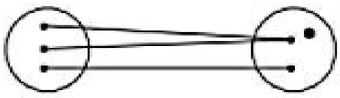
(b) one–one into (injective but not surjective) $(I \cap \overline{S})$



(c) many–one onto (surjective but not injective) $(S \cap \overline{I})$



(d) many–one into (neither surjective nor injective) $(\overline{I} \cap \overline{S})$



(iii) If f is both injective & surjective, then it is called a **Bijective** mapping. The bijective functions are also named as invertible, non singular or biuniform functions.

Illustration :

Classify the following functions as many-one, one-one, onto or into functions.

(i) $f(x) = e^x + e^{-x}$ $\qquad$ *(ii)* $f(x) = x^3$

(iii) $f(x) = \sqrt{1+x^2}$ $\qquad$ *(iv)* $f : [-1, 1] \to [-1, 1], f(x) = \sin 2x$

Sol.

(i) $f(x) = e^x + e^{-x}$

Domain $\in R$

$$y = e^x + \frac{1}{e^x} \quad \Rightarrow \quad y = e^x + \frac{1}{e^x} \geq 2$$

Range $[2, \infty)$

also $f(x) = f(-x)$

hence function is many one into

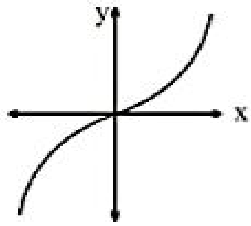
(ii) $f(x) = x^3$

Domain $\in R$

Range $\in R$

we know that ($y = x^3$) cubic, equation has a solution for all $x \in R$.

$f(x)$ is one-one onto $y = x^3$.



(iii) $f(x) = \sqrt{1+x^2}$

Domain $x \in R$

Range $y \in [1, \infty)$

$f(x) = f(-x)$

$f(x)$ is many one-into

(iv) $f : [-1, 1] \to [-1, 1], f(x) = \sin 2x$

From graph we can say that

$f(x)$ is many one onto.

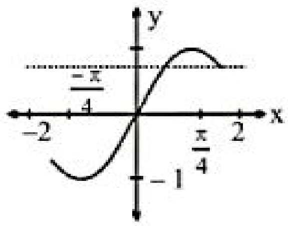


Illustration :

The function $f : [2, \infty) \to y$ defined by $f(x) = x^2 - 4x + 5$ is both one-one and onto if

(A) $y = R$ (B) $y = [1, \infty)$ (C) $y = [4, \infty]$ (D) $y = [5, \infty]$

Sol. $f(x) = x^2 - 4x + 5$

Minima at $x = 2$

at $x = 2, y = 4 - 8 + 5 = 1$

For function to be one-one it should be monotonic.

Hence, for $x \in [2, \infty)$, $f(x)$ is increasing.

at $x = 2, y = 1$. Hence $y \in [1, \infty)$

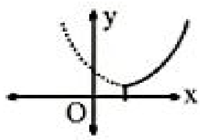


Illustration :

If $f(x) = x^2 + bx + 3$ is not injective for values of x in the interval, $0 \le x \le 1$. Find the interval in which b lies.

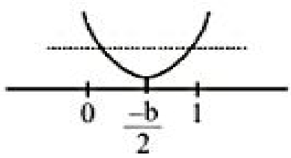
(A) $(-\infty, \infty)$ (B) $(-2, \infty)$ (C) $(-2, 0)$ (D) $(-\infty, 2)$

Sol. If $f(x)$ is not one-one then atleast one horizontal line should intersect it at two points

$$0 < \frac{-b}{2} < 1$$

$\Rightarrow$ $0 < -b < 2$

$\Rightarrow$ $b \in (-2, 0)$



5.6 Permutation and Combinations Problems :

Illustration :

A function $f: A \to B$, such that set "A" and "B" contain four elements each then find

(i) Total number of functions

(ii) Number of one-one functions

(iii) Number of many one functions

(iv) Number of onto functions

(v) Number of into functions

Sol.

(i) I^{st} *element of A can have its image in 4 ways.*
Similarly II, III and IV can have 4 options for their image each.
hence number of functions $= 4^4$

(ii) *4 different elements can be matched in 4! ways*

(iii) *Number of many one functions*
= Total number of functions – number of one-one function
$= 4^4 - 4!$

(iv) *Since 4 elements in B are given hence each should be image of atleast one.*
So number of onto function = 4!

(v) *Number of into functions* $= 4^4 - 4!$.

Illustration :

A function $f: A \to B$, such that set "A" contains five element and "B" contains four elements then find

(i) Total number of functions

(ii) Number of one-one functions

(iii) Number of onto function

(iv) Number of many one function

(v) Number of into functions

Sol.

(i) *Total number of functions*
Hence number of functions $= 4 \times 4 \times 4 \times 4 \times 4 = 4^5$

(ii) *Number of one-one functions*
Since A contains five elements hence one-one function is not possible.

(iii) *Number of onto function*
Divide 5 elements into 4 groups of size = 1, 1, 1, 2

Number of ways mapping 4 groups with four images $= \left(\dfrac{5!}{1!\,1!\,1!\,2!} \times \dfrac{1}{3!}\right) \times 4! = 240$

(iv) *Number of many one function*
All the possible functions are many-one.
$= 4^5 = 1024$

(v) *Number of into functions*

Number of into function = Total number of functions – number of onto functions

= 1024 – 240 =784

Illustration :

A function $f: A \to B$ such that set A contains 4 elements and set B contains 5 elements, then find the

(i) *Total number of functions* (ii) *Number of injective (one-one) mapping.*

(iii) *Number of many-one functions* (iv) *Number of onto function.*

(v) *Number of into functions*

Sol.

(i) *Total number of functions*

Every element in A has 5 options for image, hence

Total number of functions = 5^4 = 625.

(ii) *Number of injective (one-one) mapping.*

4 elements in A needs four images hence number of one one functions = ${}^5C_4 \times 4! = 120$.

(iii) *Number of many-one functions*

Number of many-one mapping

= Total number of mapping – number of one-one mapping

= $5^4 - {}^5C_4 \times 4! = 505$

(iv) *Number of onto function = 0*

(v) *Number of into functions = 5^4 = 625*

Practice Problem

Q.1 Show that there are exactly two distinct linear function which map$[-2, 0]$ onto $[1, 3]$.

Q.2 Let f be a one-one function with domain $\equiv \{x, y, z\}$ and range $\equiv \{1, 2, 3\}$. It is given that exactly one of the following statements is true & the remaining two are false. $f(x) = 1$, $f(y) \neq 1$, $f(z) \neq 2$. Find f(x), f(y) & f(z).

Q.3 If $f: R - \{3\} \to R - \{1\}$, where $f(x) = \dfrac{x-2}{x-3}$. Find out if f(x) is bijective or not.

Q.4 $f: R \to R$ is defined as $f(x) = \begin{bmatrix} x^2 + 2mx - 1 & \text{for } x \le 0 \\ mx - 1 & \text{for } x > 0 \end{bmatrix}$. If f(x) is one-one then m must lie in the interval

(A) $(-\infty, 0)$ (B) $(-\infty, 0]$ (C) $(0, \infty)$ (D) $[0, \infty)$

Answer key

Q.2 $f(x) = 2, f(y) = 1, f(z) = 3$ Q.3 Bijective Q.4 A

Some important points to remember :

If x, y are independent variables, then :

(i) $f(xy) = f(x) + f(y) \Rightarrow f(x) = k \ln x$ or $f(x) = 0$.

(ii) $f(xy) = f(x) \cdot f(y) \Rightarrow f(x) = x^n, \; n \in R$

(iii) $f(x+y) = f(x) \cdot f(y) \Rightarrow f(x) = a^{kx} \Rightarrow f(x) = A^x, A > 0$.

(iv) $f(x+y) = f(x) + f(y) \Rightarrow f(x) = kx$, where k is a constant.

(v) If P(x) is a polynomial function of degree n and $P(x) \cdot P\left(\frac{1}{x}\right) = P(x) + P\left(\frac{1}{x}\right)$ for $x \neq 0$. Then $P(x) = 1 + x^n$ or $1 - x^n$.

6. FUNCTIONAL EQUATIONS :

Illustration :

If $f(0) = 1, f(1) = 2$ & $f(x) = \frac{1}{2}[f(x+1) + f(x+2)]$, *find the value of* $f(5)$.

Sol. $f(x+2) = 2f(x) - f(x+1)$

thus $f(0+2) = f(2) = 2f(0) - f(1)$

$= 2(1) - 2 = 0$

$f(3) = 2f(1) - f(2) = 2(2) - 0 = 4$

$f(4) = 2f(2) - f(3) = 0 - 4 = -4$

$f(5) = 2f(3) - f(4) = 2(4) - (-4) = 12$

Illustration :

If $f(x) + 2f(1-x) = x^2 + 2 \;\; \forall \; x \in R$, *find* $f(x)$.

Sol. $f(x) + 2f(1-x) = x^2$ ...(i)

Replacing x by $1 - x$

$f(1-x) + 2f(x) = (1-x)^2$ ...(ii)

Solving (i) & (ii), we get

$3f(x) = 2x^2 - (1-x)^2$

$$f(x) = \frac{x^2 + 2x - 1}{3}$$

Illustration :

If $2f(x^2) + 3f\left(\frac{1}{x^2}\right) = x^2 - 1 \; (x \neq 0)$ *then find* $f(x^2)$.

Sol. $2f(x^2) + 3f\left(\frac{1}{x^2}\right) = x^2 - 1$ ...(i)

Replace x by $\frac{1}{x}$

$$2f\left(\frac{1}{x^2}\right) + 3f(x^2) = \frac{1}{x^2} - 1 \qquad \text{...(ii)}$$

Solving (i) & (ii) we get

$$9f(x)^2 - 4f(x^2) = 3\left(\frac{1}{x^2} - 1\right) - 2(x^2 - 1)$$

$$5f(x^2) = \frac{3}{x^2} - 2x^2 - 1$$

$$f(x^2) = -\left(\frac{2x^4 + x^2 - 3}{5x^2}\right)$$

Illustration :

Let f(x) & g(x) be functions which take integers as arguments let $f(x + y) = f(x) + g(y) + 8$ for all integer x & y. Let $f(x) = x$ for all negative integers x let $g(8) = 17$, find $f(0)$.

Sol. $f(x) = x$ *for integers less than zero*

$\therefore\ f(-8) = -8$

$f(x + y) = f(x) + g(y) + 8$

$f(-8 + 8) = f(-8) + g(8) + 8$

$f(0) = -8 + g(8) + 8$

$f(0) = 17$

Practice Problem

Q.1 Let $f(x) = ax^5 + bx^3 + cx - 5$, where a, b & c are constants. If $f(-7) = 7$, then find $f(7)$.

Q.2 The function $f : R \to R$ satisfies the condition $mf(x-1) + nf(-x) = 2|x| + 1$. If $f(-2) = 5$ and $f(1) = 1$, then find $(m + n)$.

Q.3 If $f(x + y) = f(x)\, f(y)\ \forall\ x, y \in N$, $f(1) = 2$ and $\sum_{k=1}^{n} f(a+k) = 16(2^n - 1)$. Find a.

Q.4 Solve the inequality $|f(x) - g(x)| < |f(x)| + |g(x)|$ where $f(x) = x - 3$ and $g(x) = 4 - x$.

Answer key

Q.1	-17	Q.2	$\frac{4}{3}$	Q.3	$a = 3$	Q.4	$x \in (3, 4)$

7. COMPOSITE FUNCTION :

Let $f : A \to B$ & $g : B \to C$ be two functions. Then the function $gof : A \to C$ defined by

$$(gof)(x) = g(f(x)) \ \forall \ x \in A$$

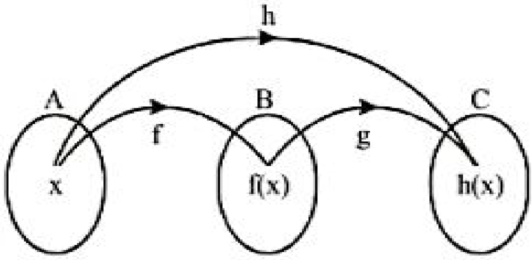
is called the composite of the two functions f & g. Diagramatically

$$\xrightarrow{x} \boxed{f} \xrightarrow{f(x)} \boxed{g} \longrightarrow g(f(x)).$$

Thus the image of every $x \in A$ under the function gof is the g–image of the f–image of x.

Note that gof is defined only if $\forall \ x \in A$, $f(x)$ is an element of the domain of g so that we can take its g–image. Hence for gof of two functions f & g, the range of f must be a subset of the domain of g.

Note that gof in general not equal to fog.



$g\{f(x)\} = h(x)$

Illustration :

$f : R \to R$ be the function defined by $f(x) = ax^2 - \sqrt{2}$ for some positive a. If $(fof)\sqrt{2} = -\sqrt{2}$, then find a.

Sol. $f(x) = ax^2 - \sqrt{2}$

$$(fof)(x) = a(ax^2 - \sqrt{2})^2 - \sqrt{2}$$

$$= 4a\left(a - \frac{1}{\sqrt{2}}\right)^2 = 0$$

$$= a = 0, \ \frac{1}{\sqrt{2}}$$

Illustration :

Let $f(x) = \sqrt{x}$; $g(x) = \sqrt{2-x}$ find the domain of

(i) (fog) (x) (ii) (gof) (x) (iii) (fof) (x) (iv) (gog) (x)

Sol.

(i) $(fog)(x) = f(g(x)) = f\left(\sqrt{2-x}\right) = \sqrt{\sqrt{2-x}}$

Domain $2 - x \geq 0$

$x \leq 2$

$\Rightarrow \quad x \in (-\infty, 2]$

(ii) $(gof)(x) = g(f(x)) = g\left(\sqrt{x}\right) = \sqrt{2-\sqrt{x}}$

$\therefore \ 2 - \sqrt{x} \geq 0 \ \Rightarrow \ 0 \leq \sqrt{x} \leq 2 \ \Rightarrow \ x \in [0, 4]$

(iii) $(fof)(x) = f(f(x)) = f\left(\sqrt{x}\right) = \sqrt{\sqrt{x}}$

Domain $\Rightarrow x \geq 0$

$x \in [0, \infty)$

(iv) $(gog)(x) = g(g(x)) = g\left(\sqrt{2-x}\right) = \sqrt{2-\sqrt{2-x}}$

$\therefore\ 0 \le \sqrt{2-x} \le 2 \Rightarrow 0 \le 2-x \le 4 \Rightarrow -2 \le x \le 4$

$x \in [-2, 2]$

Illustration :

Let $f(x) = x^x$ *&* $g(x) = x^{2x}$*, then find* $f(g(x))$.

Sol. $f(g(x)) = f\left(x^{2x}\right) = \left(x^{2x}\right)^{x^{2x}} = x^{2x.x^{2x}} = x^{2x^{2x+1}}$

7.1 Properties Of Composite Functions :

(i) The composite of functions is not commutative i.e. $gof \ne fog$.

(ii) The composite of functions is associative i.e. if f, g, h are three functions such that $fo(goh)$ & $(fog)oh$ are defined, then $fo(goh) = (fog)oh$.

Associativety : $f : (N) \to I_0 \quad f(x) = 2x$

$$g : I_0 \to Q \quad g(x) = \frac{1}{x}$$

$$h : Q \to R \quad h(x) = e^{\frac{1}{x}}$$

$$(hog)of = ho(gof) = e^{2x}$$

(iii) The composite of two bijections is a bijection i.e. if f and g are two bijections such that gof is defined, then gof is also a bijection.

Proof: Let $f : A \to B$ and $g : B \to C$ be two bijections. Then gof exists such that $gof : A \to C$

We have to prove that gof is one-one and onto.

One-one : Let $a_1, a_2 \in A$ such that $(gof)(a_1) = (gof)(a_2)$, then

$(gof)(a_1) = (gof)(a_2) \Rightarrow g[f(a_1)] = g[f(a_2)]$

$\Rightarrow f(a_1) = f(a_2)$ [$\because$ g is one-one]

$\Rightarrow a_1 = a_2$ [$\because$ f is one-one]

$\therefore$ gof is also one-one function.

Onto : Let $c \in C$, then

$c \in C \Rightarrow \exists\, b \in B$ s.t. $g(b) = c$ [$\because$ g is onto]

and $b \in B \Rightarrow \exists\, a \in A$ s.t. $f(a) = b$ [$\because$ f is onto]

Therefore, we see that

$c \in C \Rightarrow \{\exists\, a \in A \text{ s.t. } (gof)(a) = g[f(a)] = g(b) = c\}$

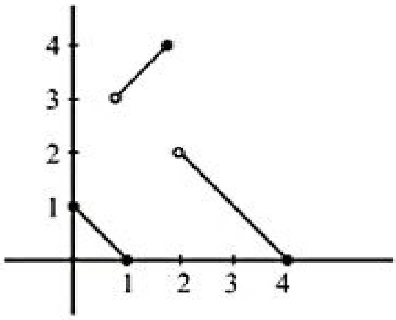
i.e. every element of C is the gof image of some element of A. As such gof is onto function. Hence gof befing one-one and onto, is a bijection.

Illustration :

Evaluate $f(f(x))$, *where*

$$f(x) = (1-x),\quad 0 \le x \le 1$$
$$= (x+2),\quad 1 < x \le 2$$
$$= (4-x),\quad 2 < x \le 4$$

Sol. $$f(x)=\begin{cases}1-x, & 0\le x\le 1\\ x+2, & 1<x\le 2\\ 4-x, & 2<x\le 4\end{cases}$$



graph of $f(x)$

$$f(f(x))=\begin{cases}1-f(x), & 0\le f(x)\le 1\\ f(x)+2, & 1<f(x)\le 2\\ 4-f(x), & 2<f(x)\le 4\end{cases}$$

from the graph we can see that

$0 \le f(x) \le 1$ when $x \in [0, 1] \cup [3, 4]$

$1 < f(x) \le 2$ when $x \in (2, 3]$

$2 < f(x) \le 4$ when $x \in [1, 2]$

$$f(f(x))=\begin{cases}1-(1-x), & f(x)=1-x,\ 0\le x\le 1\\ 1-(4-x), & f(x)=4-x,\ 3\le x\le 4\\ (4-x)+2, & f(x)=4-x,\ 2<x<3\\ 4-(x+2), & f(x)=x+2,\ 1<x\le 2\end{cases}$$

$$f(f(x))=\begin{cases}x, & 0\le x\le 1\\ x-3, & 3\le x\le 4\\ 6-x, & 2<x<3\\ 2-x, & 1<x\le 2\end{cases}$$

Alternative method :

We have

$$f\{f(x)\} = 1-f \qquad 0\le f\le 1 \qquad \ldots(i)$$
$$= f+2 \qquad 1<f\le 2 \qquad \ldots(ii)$$
$$= 4-f \qquad 2<f\le 4 \qquad \ldots(iii)$$

Putting the values of $f(x)$ *in (i)*

$$1-f = 1-(1-x),\quad 0\le f\le 1,\quad 0\le x\le 1$$
$$= 1-(x+2),\quad 0\le x+2\le 1,\quad 1<x\le 2$$
$$= 1-(4-x),\quad 0\le 4-x\le 1,\quad 2<x\le 4$$

On solving $0\le 1-x\le 1$ & $0<x\le 1$ *we get* $0\le x\le 1$

Solving $0\le x+2\le 1$ & $2<x\le 4 \Rightarrow$ *null set*

Solving $0\le 4-x\le 1$ & $2<x\le 4 \Rightarrow$ $3<x\le 4$

thus $\Rightarrow$ $$1-f=\begin{cases}x, & 0\le x\le 1\\ x-3, & 3<x\le 4\end{cases} \qquad \ldots(iv)$$

$$f+2 = (1-x)+2,\quad 1<(1-x)\le 2,\quad 0\le x\le 1$$
$$= (x+2)+2,\quad 1<(x+2)\le 2,\quad 1<x\le 2$$
$$= (4-x)+2,\quad 1<(4-x)\le 2,\quad 2<x\le 4$$

Solving $1 < (1-x) \le 2$, *we have* $-1 \le x \le 0$ *& its intersection with* $0 \le x \le 1$ *gives null set*
Solving $1 < (x+2) \le 2$, *we get* $-1 \le x \le 0$ *& intersection with* $1 < x \le 2$ *gives null set*
Solving $1 < (4-x) \le 2$, *we get* $2 \le x < 3$ *& its intersection with* $2 < x \le 4$ *gives* $2 < x < 3$
thus $f+2 = 6-x, \quad 2 < x < 3$ ...(v)
Putting the values of f(x) in (iii), we have

$$\begin{aligned} 4-f &= 4-(1-x), & 2 < 1-x \le 4, & \quad 0 \le x \le 1 \\ &= 4-(x+2), & 2 < x+2 \le 4, & \quad 1 < x \le 2 \\ &= 4-(4-x), & 2 < 4-x \le 4, & \quad 2 < x \le 4 \end{aligned}$$

Solving $2 \le (1-x) \le 4$, *we get* $-3 \le x \le -1$ *& its intersection with* $0 \le x \le 1$ *gives null set*
Solving $2 < (x+2) \le 4$, *we get* $0 < x \le 2$ *& intersection with* $1 < x \le 2$ *gives* $1 < x \le 2$
Solving $2 < (4-x) \le 4$, *we get* $0 \le x < 3$ *& its intersection with* $2 < x \le 4$ *gives null set*
thus $4-f = 2-x, \quad 1 < x \le 2$ ...(vi)
Using (iv),(v) & (vi)

$$f(f(x)) = \begin{cases} x, & 0 \le x \le 1 \\ 2-x, & 1 < x \le 2 \\ 6-x, & 2 < x < 3 \\ x-3, & 3 \le x \le 4 \end{cases}$$

Practice Problem

Q.1 If $f(x) = \begin{cases} 1-x & \text{if} \quad 0 \le x \le 1 \\ x+2 & \text{if} \quad 1 < x < 2 \\ 4-x & \text{if} \quad 2 \le x \le 4 \end{cases}$. Find fof(x)

Q.2 If $g(x) = 2x+1$ & $h(x) = 4x^2+4x+7$, find a function f such that fog = h.

Q.3 If $f(x) = \dfrac{2x-7}{x+3}$, find a function g such that g[f(x)] = x for all x in the domain of f and find its domain & range.

Q.4 Evaluate g{f(x)}, where

$$\begin{aligned} f(x) &= 1+x^3, & x < 0 \\ &= x^2-1, & x \ge 0 \\ g(x) &= (x-1)^{1/3}, & x < 0 \\ &= (x+1)^{1/2}, & x \ge 0 \end{aligned}$$

Answer key

Q.1 $f(f(x)) = \begin{cases} x, & 0 \le x \le 1 \\ 2-x, & 1 < x \le 2 \\ 6-x, & 2 < x < 3 \\ x-3, & 3 \le x \le 4 \end{cases}$

Q.2 $f(x) = x^2+6$

Q.3 Domain = R – {–3}, Range = R – {2}

Q.4 $g(f(x)) = \begin{cases} x, & x < -1 \\ (x^3+2)^{\frac{1}{2}}, & -1 \le x < 0 \\ (x^2-2)^{\frac{1}{3}}, & 0 \le x < 1 \\ x, & x \ge 1 \end{cases}$

8.1 Homogeneous Functions :

A function is said to be homogeneous with respect to any set of variables when each of its terms is of the same degree with respect to those variables.

For example $5x^2+3y^2-xy$ is homogeneous in x & y.
$f(x, y)$ is a homogeneous function iff
$f(tx, ty) = t^n f(x, y)$

or $f(x, y) = x^n g\left(\dfrac{y}{x}\right) = y^n h\left(\dfrac{x}{y}\right)$, where n is the degree of homogenity

$f(x, y) = \dfrac{x - y\cos x}{y\sin x + x}$ is not a homogeneous function and

$f(x, y) = \dfrac{x}{y}\ln\dfrac{y}{x} + \dfrac{y}{x}\ln\dfrac{x}{y}$; $\sqrt{x^2-y^2}+x$; $x + y\cos\dfrac{y}{x}$ are homogeneous functions of degree one.

8.2 Bounded Function :

A function is said to be bounded if $|f(x)| \le M$, where M is a finite quantity.
e.g. $f(x) = \sin x$ is bounded in $[-1, 1]$

8.3 Implicit & Explicit Function :

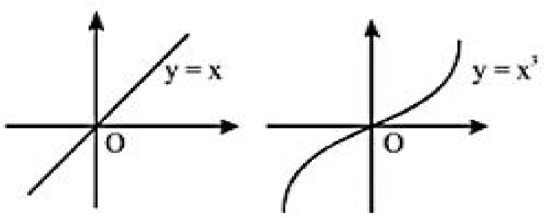
A function defined by an equation not solved for the dependent variable is called an **IMPLICIT FUNCTION**. For eg. the equation $x^3+y^3=1$ defines y as an implicit function. If y has been expressed in terms of x alone then it is called an **EXPLICIT FUNCTION**.

8.4 Odd & Even Functions :

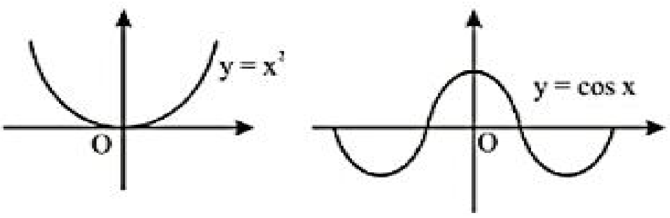
A function f(x) defined on the symmetric interval (–a, a)

If $f(-x) = f(x)$ for all x in the domain of 'f' then f is said to be an even function.
e.g. $f(x) = \cos x$; $g(x) = x^2+3$.

If $f(-x) = -f(x)$ for all x in the domain of 'f' then f is said to be an odd function.
e.g. $f(x) = \sin x$; $g(x) = x^3+x$.



Odd functions (Symmetric about origin)



Even functions (Symmetric about y-axis)

NOTE :

(a) $f(x) - f(-x) = 0 \Rightarrow f(x)$ is even & $f(x) + f(-x) = 0 \Rightarrow f(x)$ is odd.

(b) A function may neither be odd nor even.

(c) Inverse of an even function is not defined and an even function can not be strictly monotonic

(d) Every even function is symmetric about the y–axis & every odd function is symmetric about the origin.

(e) Every function can be expressed as the sum of an even & an odd function.

$$\text{e.g. } f(x) = \underbrace{\frac{f(x)+f(-x)}{2}}_{\text{EVEN}} + \underbrace{\frac{f(x)-f(-x)}{2}}_{\text{ODD}} \qquad 2^x = \underbrace{\frac{2^x+2^{-x}}{2}}_{\text{EVEN}} + \underbrace{\frac{2^x-2^{-x}}{2}}_{\text{ODD}}$$

(f) The only function which is defined on the entire number line & is even and odd at the same time is f(x) = 0. Any non zero constant is even.

(g) If f and g both are even or both are odd then the function f.g will be even but if any one of them is odd then f.g will be odd.

f(x)	g(x)	f(x) + g(x)	f(x) - g(x)	f(x).g(x)	f(x) / g(x)	(gof)(x)	(fo g) (x)
odd	odd	odd	odd	even	even	odd	odd
even	even	even	even	even	even	even	even
odd	even	neither odd nor even	neither odd nor even	odd	odd	even	even
even	odd	neither odd nor even	neither odd nor even	odd	odd	even	even

Illustration :

Identity the functions, as even, odd or neither nor odd.

(i) $f(x) = \left(\ln x + \sqrt{1+x^2}\right)$ *(ii)* $f(x) = x \cdot \left(\frac{2^x+1}{2^x-1}\right)$

(iii) $f(x) = 2x^3 - x + 1$ *(iv)* $f(x) = 3$

(v) $f(x) = x^2 - |x|$

Sol.

(i) $f(x) = \left(\ln + \sqrt{1+x^2}\right)$

$$f(-x) = \ln\left(-x + \sqrt{1+x^2}\right) = \ln\left(\sqrt{1+x^2} - x\right)$$

$$= \ln\left(\frac{\left(\sqrt{1+x^2}-x\right)\left(\sqrt{1+x^2}+x\right)}{\left(\sqrt{1+x^2}+x\right)}\right) = \ln\left(\sqrt{1+x^2}\right) = -f(x)$$

Hence odd function.

(ii) $f(x) = x\left(\dfrac{2^x+1}{2^x-1}\right)$

$$f(-x) = (-x)\left(\frac{2^{-x}+1}{2^{-x}-1}\right)$$

$$= (-x)\left(\frac{1+2^x}{1-2^x}\right) = x\left(\frac{2^x+1}{2^x-1}\right) = f(x)$$

hence even function

(iii) $f(x) = 2x^3 - x + 1$
$f(-x) = -2x^3 + x + 1 \neq f(x)$ *or* $-f(x)$
Hence neither even nor odd function

(iv) $f(x) = 3$
$f(-x) = 3 = f(x)$
Hence even function

(v) $f(x) = x^2 - |x|$
$f(-x) = x^2 - |-x| = f(x)$
even function

Practice Problem

Q.1 Let $f(x) = [x], x \geq 0$
$= g(x), x < 0$
Find $g(x)$ if $f(x)$ is even

Q.2 Let $f: [-2, 2] \to R$, where $f(x) = x^3 + \sin x + \left[\dfrac{x^2+1}{a}\right]$ be an odd function. Then find the values of the parameter a.

Q.3 Identify whether the given function is even odd or neither even nor odd where

$$f(x) = \begin{cases} x|x|, & x \leq -1 \\ [1+x]+[1-x], & -1 < x < 1 \\ -x|x|, & x \geq 1 \end{cases}$$

where $|\,|$ & $[\cdot]$ represents modulus and greatest integral function

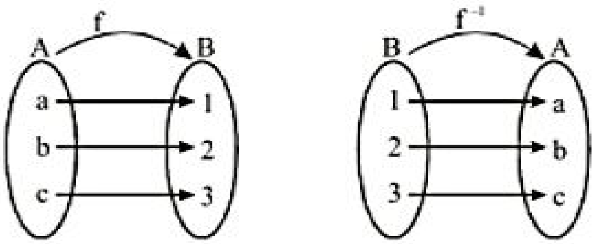
Answer key

Q.1	$g(x) = -[x]$	Q.2	$a > 5$	Q.3	$f(x)$ is even

9. INVERSE OF A FUNCTION :

Let $f: A \to B$ be a one–one & onto function, then their exists a unique function $g: B \to A$ such that $f(x) = y \Leftrightarrow g(y) = x,\ \forall\ x \in A\ \&\ y \in B$. Then g is said to be inverse of f. Thus $g = f^{-1}: B \to A = \{(f(x), x) \mid (x, f(x)) \in f\}$.

Consider a one-one onto function with domain A= {a, b, c} & range B = {1, 2, 3}



Domain of $f = \{a, b, c\}$ = Range of f^{-1}
Range of $f = \{1, 2, 3\}$ = Domain of f^{-1}

Note: (a) Only one-one onto functions (i.e., Bijections) are invertible.

(b) To find the inverse
Step-1: write $y = f(x)$
Step-2: solve this equation for x in terms of y (if possible)
Step-3: To express f^{-1} as a function of x, interchange x and y.

Illustration :

Find the inverse of the following bijective function
(i) $f: R \to R^+$, $f(x) = 10^{x+1}$ *(ii)* $f(x) = 3x - 5$

(iii) $f: [1, \infty) \to [2, \infty), f(x) = x + \dfrac{1}{x}$ *(iv)* $f: R \to (0, 1),\ f(x) = \dfrac{2^x}{1+2^x}$

Sol.

(i) $y = 10^{x+1}$
$x + 1 = \log_{10} y$
$x = -1 + \log_{10} y$
$\Rightarrow f^{-1} = y = -1 + \log_{10} x,\quad f^{-1}: R^+ \to R$

(ii) $f(x) = 3x - 5$
$y = 3x - 5$

$$x = \frac{y+5}{3}$$

$$\Rightarrow f^{-1}(x) = y = \frac{x+5}{3}$$

(iii) $f: [1, \infty) \to [2, \infty)$

$$y = f(x) = x + \frac{1}{x}$$

$$\Rightarrow \quad x^2 - xy + 1 = 0$$

$$\Rightarrow \quad x = \frac{y \pm \sqrt{y^2-4}}{2}$$

$$\Rightarrow \quad f^{-1}(x) = \frac{x \pm \sqrt{x^2-4}}{2}$$

Since range is [1, ∞), hence

$$f^{-1}(x) = \frac{x + \sqrt{x^2-y}}{2}$$

(iv) $f: R \to (0, 1),\ f(x) = \dfrac{2^x}{1+2^x}$

$$y = \frac{2^x}{1+2^x} \quad \Rightarrow \quad y + 2^x y = 2^x$$

$$\Rightarrow \quad 2^x = \frac{y}{1-y} \quad \Rightarrow \quad x = \log_2\left(\frac{y}{1-y}\right)$$

$$\Rightarrow \quad f^{-1}(x) = y = \log_2\left(\frac{x}{1-x}\right)$$

9.1 Properties of inverse of a function :

(i) The inverse of Bijection is unique.

(ii) The inverse of Bijection is also bijection.

(iii) If $f: A \to B$ is Bijection & $g: B \to A$ is inverse of f, then $fog = I_B$ & $gof = I_A$, where I_A, I_B are the identical function on the set A and B respectively

(iv) If $f: A \to B$ and $g: B \to C$ are two bijections, then $gof: A \to C$ is bijections and $(gof)^{-1} = f^{-1}og^{-1}$.

(v) In general $fog \neq gof$ but if $fog = gof$ then either $f^{-1} = g$ or $g^{-1} = f$ also $(fog)(x) = (gof)(x) = x$.

(vi) The graphs of f & g are the mirror images of each other in the line y = x. As shown in the figure given below a point (x', y') corresponding to $y = x^2 (x \geq 0)$ changes to (y', x') corresponding to $y = +\sqrt{x}$, the changed form of $x = \sqrt{y}$.

fig. 1

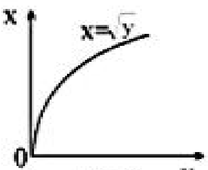


fig. 2

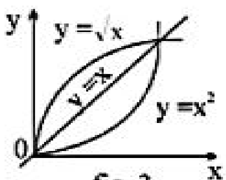


fig. 3

Illustration :

If $f: R \to R$ is defined by $f(x) = x^3 + 1$, then find value of $f^{-1}(28)$.

Sol. $f^{-1}(28) = x \quad \Rightarrow f(x) = 28 \quad \Rightarrow x^3 + 1 = 28 \quad \Rightarrow x = 3$

Illustration :

If the function f & g *be defined as* $f(x) = e^x$ *and* $g(x) = 3x - 2$ *where* $f : R \to R$ *and* $g : R \to R$ *then find the function* fog *and* gof. *Also find the domain of* $(fog)^{-1}$ *and* $(gof)^{-1}$.

Sol. $(fog)(x) = f\{g(x)\}$

$f\{g(x)\} = f(3x - 2)$

$= e^{3x-2}$

$(gof)(x) = g\{f(x)\} = g(e^x) = 3e^x - 2$

To find $(fog)^{-1}$ & $(gof)^{-1}$

$(fog)(x) = y = e^{3x-2}$

$\Rightarrow \quad 3x - 2 = \log y$

$\Rightarrow \quad x = \dfrac{\log y + 2}{3} \Rightarrow \quad (fog)^{-1} x = \dfrac{\log x + 2}{3}$

Domain of $(fog)^{-1}$ *is* $x > 0$ *i.e.* $x \in (0, \infty)$

Again $gof(x) = y = 3e^x - 2$

$\Rightarrow \quad e^x = \dfrac{y+2}{3} \quad \Rightarrow \quad x = \log\left(\dfrac{y+2}{3}\right) \quad \Rightarrow \quad (gof)^{-1} x = \log\left(\dfrac{x+2}{3}\right)$

Domain of $(gof)^{-1}$ *is* $\dfrac{x+2}{3} > 0$

$x > -2$

$\Rightarrow \quad x \in (-2, \infty)$

Illustration :

If $f: [0, \infty) \to [1, \infty)$, $f(x) = \dfrac{e^x + e^{-x}}{2}$. *Find* $f^{-1}(x)$.

Sol. $f(x) = \dfrac{e^x + e^{-x}}{2}$

$\Rightarrow \quad 2y = e^x + \dfrac{1}{e^x} \qquad \Rightarrow \quad e^{2x} - 2e^x y + 1 = 0$

$\Rightarrow \quad e^{2x} - 2e^x y + y^2 = y^2 - 1 \qquad \Rightarrow \quad (e^x - y)^2 = y^2 - 1$

$\Rightarrow \quad e^x = y \pm \sqrt{y^2 - 1} \qquad \Rightarrow \quad x = \log\left(y \pm \sqrt{y^2 - 1}\right)$

$\Rightarrow \quad f^{-1}(x) = y = \log\left(x \pm \sqrt{x^2 - 1}\right)$

Since range is $[0, \infty)$ *hence*

$\Rightarrow \quad f^{-1}(x) = y = \log\left(x + \sqrt{x^2 - 1}\right)$

Illustration :

Find the inverse of the function $f : N \to N$, $f(x) = x + (-1)^{x-1}$.

Sol. $f(x) = x + (-1)^{x-1}$, $x \in N$

Then we have $f(1) = 1 + 1 = 2,\quad f(2) = 1$

$f(3) = 4,\quad f(4) = 3$

$f(5) = 6,\quad f(6) = 5$

The points on graph are (1, 2), (2, 1), (3, 4) (4, 3), (5, 6), (6, 5) .... etc. Thus if (a, b) is a point on the graph then (b, a) is also a point on the graph. Hence f is the inverse of itself.

i.e. $f^{-1}(x) = x + (-1)^{x-1}$, $x \in N$

Practice Problem

Q.1 If $y = ax + b$ and the equation $f(x) = f^{-1}(x)$ is satisfied by every real value of x then

(A) $a = 2, b = -1$ (B) $a = -1, b \in R$

(C) $a = 1, b \in R$ (D) $a = 1, b = -1$

Q.2 Find the inverse of following functions

(i) $f(x) = 5^{\log_e x}$, $x > 0$

(ii) $f(x) = \begin{cases} x & , \ x < 1 \\ x^2 & , \ 1 \le x \le 4 \\ 8\sqrt{x} & , \ x > 4 \end{cases}$

(ii) $f(x) = \log_e(x^2 + 3x + 1)$, $x \in [1, 3]$

Answer key

Q.1 B **Q.2** (i) $f^{-1}(x) = e^{\log_5 x}$ (ii) $f^{-1}(x) = \begin{cases} x, & -\infty < x < 1 \\ \sqrt{x}, & 1 \le x \le 16 \\ \log_2 x, & 16 < x < \infty \end{cases}$ (ii) $f^{-1}(x) = \dfrac{\sqrt{5 + 4e^x} - 3}{2}$

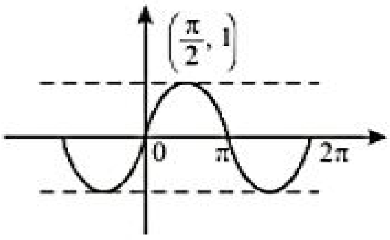
10. PERIODIC FUNCTION:

A function $f(x)$ is called periodic if there exists a positive number $T (T > 0)$ called the period of the function such that $f(x + T) = f(x)$, for all values of x within the domain of x.

e.g. The function $\sin x$ & $\cos x$ both are periodic over 2π & $\tan x$ is periodic over π.

Graphically :

If the graph repeats at fixed interval then function is said to be periodic and its period is the width of that interval. For example graph of $\sin x$ repeats itself at an interval of 2π



$f(x) = \sin x$

10.1 Properties of periodic function :

(i) $f(T) = f(0) = f(-T)$, where 'T' is the period.
(ii) Inverse of a periodic function does not exist.
(iii) Every constant function is always periodic, with no fundamental period.
(iv) If f(x) has a period p, then $\frac{1}{f(x)}$ and $\sqrt{f(x)}$ also has a period p.
(v) if f(x) has a period T then f(ax + b) has a period $\frac{T}{|a|}$.
(vi) If f(x) has a period T & g(x) also has a period T then it does not mean that f(x) + g(x) must have a period T. e.g. $f(x) = |\sin x| + |\cos x|$; $\sin^4 x + \cos^4 x$ has fundamental period equal to $\frac{\pi}{2}$.
(vii) If f(x) and g(x) are periodic then f(x) + g(x) need not be periodic.
e.g. $f(x) = \cos x$ and $g(x) = \{x\}$

Illustration :

Find the period of the following functions.

(i) $f(x) = \cos\frac{2x}{3} - \sin\frac{4x}{5}$;
(ii) $f(x) = \cos(\sin x)$
(iii) $f(x) = \sin(\cos x)$;
(iv) $f(x) = [x] + [2x] + [3x] + \ldots\ldots + [nx] - \frac{n(n+1)}{2}x$

where $n \in N$ *& [ ] denotes greatest integer function*

Sol.(i) $f(x) = \cos\left(\frac{2x}{3}\right) - \sin\left(\frac{4x}{5}\right)$

Period of $\cos\left(\frac{2x}{3}\right) = \frac{2\pi(3)}{2} = 3\pi$

Period of $\sin\left(\frac{4x}{5}\right) = \frac{2\pi}{4} \times 5 = \frac{5}{2}\pi$

L.C.M. of 3π *&* $\frac{5}{2}\pi = 15\pi$

(ii) $f(x) = \cos(\sin x)$
Since cos is even functions $f(\pi + x) = \cos(\sin(\pi + x)) = \cos(-\sin x) = \cos(\sin x) = f(x)$
Hence π *is period.*

(iii) $f(x) = \sin(\cos x)$
Period is 2π

(iv) $f(x) = [x] + [2x] + [3x] + \ldots\ldots + [nx] - \frac{n(n+1)}{2}x$

$= -\{x\} - \{2x\} - \ldots\ldots - \{nx\}$

Period of $\{x\} = 1$

period of $\{2x\} = \frac{1}{2}$

period of $\{3x\} = \frac{1}{3}$

....

....

L.C.M. of $\left(1, \frac{1}{2}, \frac{1}{3}, \ldots\ldots, \frac{1}{n}\right) = 1$

Illustration :

If $f(x) = \dfrac{\sin nx}{\sin \frac{x}{n}}$ has its period as 4π, then find the integral values of n.

Sol. Period of $\sin nx = \frac{2\pi}{n}$

Period of $\sin \frac{x}{n} = 2n\pi$

L.C.M. of $\left(\frac{2\pi}{n}, 2n\pi\right) = 2n\pi$

$2n\pi = 4\pi$

$n = 2, -2$

Illustration :

Find the period of $f(x) = |\sin x| + |\cos x|$.

Sol. $|\sin x|$ has period π

$|\cos x|$ has period π

$f(x)$ is an even function & $\sin x$, $\cos x$ are complementary then period of

$f(x) = \frac{1}{2}\{\text{LCM of } \pi \text{ \& } \pi\} = \frac{\pi}{2}$

Illustration :

Prove that if $f(x) = \sin x + \cos ax$ is a periodic function then a must be rational.

Sol. $f(x) = \sin x + \cos ax$

Period of $\sin x = 2\pi$

Period of $\cos ax = \frac{2\pi}{a}$

LCM of 2π & $\frac{2\pi}{a}$ is possible only when a is rational, hence a must be rational.

Practice Problem

Q.1 Find the period of the function $f(x)=\frac{1}{2}\left(\frac{|\sin x|}{\cos x}+\frac{\sin x}{|\cos x|}\right)$

Q.2 Find the period of following functions

(i) $f(x)=\cos 2\pi x$ (ii) $f(x)=2\sin(3x-\pi)$

Q.3 Let a function satisfying $f(x+4)+f(x-4)=f(x)$ for all real x is periodic, then period p for them is

(A) 8 (B) 12 (C) 16 (D) 24

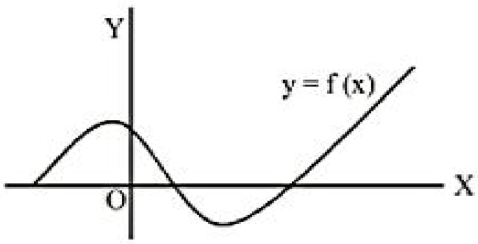
Q.4 If $f(x)=(a+3)x+5a,\ x\in R$ is periodic then find the value of a.

Answer key

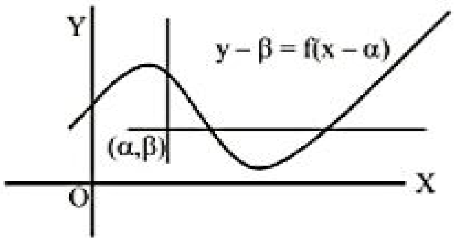
Q.1 2π	Q.2 (i) 1, (ii) $\frac{2\pi}{3}$	Q.3 D	Q.4 $a=-3$

11. SOME GRAPHICAL TRANSFORMATION :

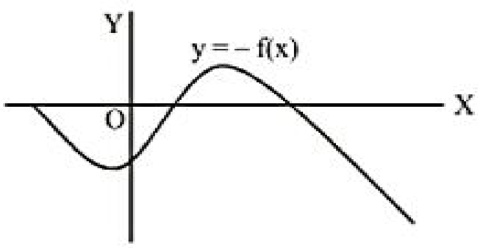
Consider the graph $y=f(x)$ shown alongside.



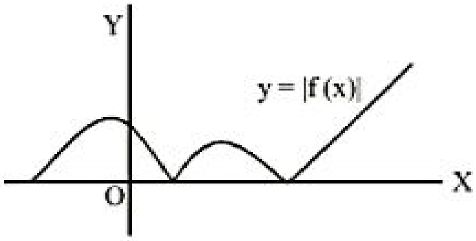
(i) Graph of $y-\beta=f(x-\alpha)$ is drawn by shifting the origin to (α,β) & then translating the graph of $y=f(x)$ w.r.t. new axes



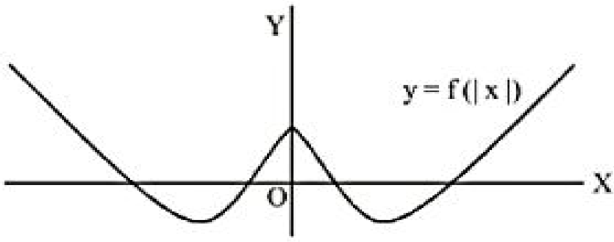
(ii) The graph of $y=-f(x)$ is the mirror image of f(x) in X-axis.



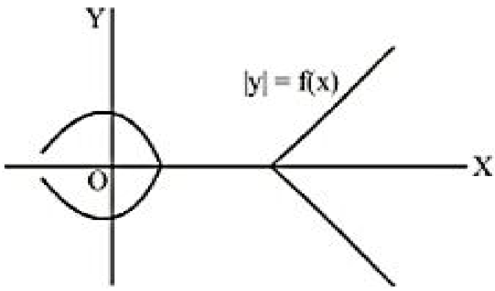
(iii) $y = |f(x)|$ is mirror image of negative portion of $y = f(x)$ in X-axis.



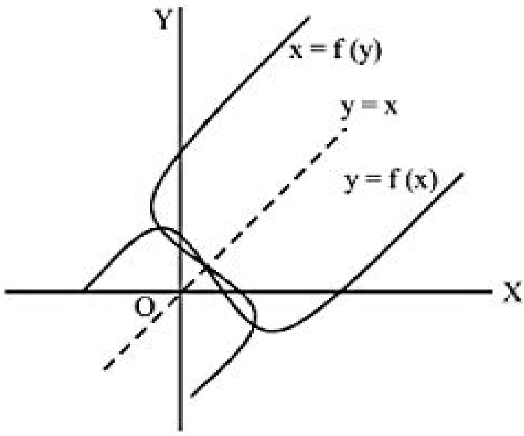
(iv) $y = f(|x|)$ is drawn by taking the mirror image of positive x-axis graph in y-axis.



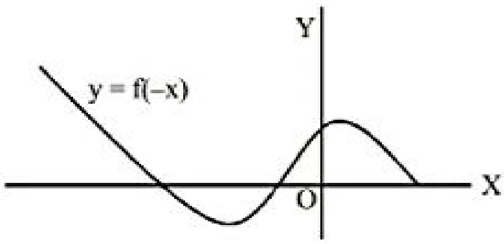
(v) The graph of $|y| = f(x)$ is drawn by deleting those portions of the graph $y = f(x)$ which lie below the X-axis and then taking the mirror image of the remaining portion in the X-axis, as shown alongside.



(vi) $x = f(y)$ is drawn by taking mirror image of $y = f(x)$ in the line $y = x$.



(vii) $y = f(-x)$ is drawn by taking the mirror image of $y = f(x)$ in Y-axis.



Solved Examples

Q.1 Evaluate $g\{f(x)\}$, where $f(x)=\begin{cases}\cos x, & x\neq n\pi\\ 3, & \text{otherwise}\end{cases}$ and $g(x)=\begin{cases}x^2+1, & x\neq 0,3\\ 3, & x=0\\ 5, & x=3\end{cases}$.

Sol. We have $g\{f(x)\}=\begin{cases}f^2+1, & f\neq 0,3 & \text{.......(1)}\\ 3, & f=0 & \text{.......(2)}\\ 5, & f=3 & \text{.......(3)}\end{cases}$

Putting values of f(x) in (1)

$$f^2+1=\begin{cases}\cos^2 x+1, & \cos x\neq 0,3;\ x\neq n\pi\\ 3^2+1, & 3\neq 0,\ 3\neq 3,\ x=n\pi\end{cases}$$

i.e., $f^2+1=\cos^2 x+1,\ x\neq(2n+1)\dfrac{\pi}{2},\ x\neq n\pi$ ....(4)

Putting the values of f(x) in (2),

$g\{f(x)\}=3,\quad \cos x=0,\ x\neq n\pi$

$\quad\quad\quad\ =3,\quad 3=0,\ x=n\pi$

On solving above equation $g\{f(x)\}=3,\ x=(2n+1)\dfrac{\pi}{2}$ ....(5)

Putting the values of f(x) in (3)

$g\{f(x)\}=5,\quad \cos x=3,\ x\neq n\pi$

$\quad\quad\quad\ =3,\quad 3=3,\ x=n\pi$

On solving above equation $g\{f(x)\}=n\pi$ ....(6)

Using (4), (5) and (6), we get

$$g\{f(x)\}=\begin{cases}1+\cos^2 x, & x\neq(2n+1)\dfrac{\pi}{2},\ x\neq n\pi\\ 3, & x=(2n+1)\dfrac{\pi}{2}\\ 5 & x=n\pi\end{cases}.$$

Q.2 Consider the function $f(x)=\begin{cases}x|x|, & 0\le x<1\\ 2x, & x\ge 1\end{cases}$. Find the extension of the function $\forall\ x\in R$ if

(i) f(x) is even (ii) f(x) is odd.

Sol.

(i) We have $f(x)=\begin{cases}x^2, & 0\le x<1\\ 2x, & x\ge 1\end{cases}$

If f is even $\forall\ x\in R$, then $f(-x)=f(x)$

Hence $f(-x)=\begin{cases}x^2, & -1<x\le 0\\ -2x, & x\le 1\end{cases}$; $f(x)=\begin{cases}-2x, & x\le 1\\ x^2, & -1<x<1\\ 2x, & 1\le x\end{cases}$

(ii) For f(x) to be odd function

$-f(-x) = -x^2, -1 < x \le 0$

$= 2x, \quad x \le -1$

$$f(x) = \begin{cases} 2x, & x \le -1 \\ -x^2, & -1 < x \le 0 \\ x^2, & 0 \le x < 1 \\ 2x, & 1 \le x \end{cases} \qquad \text{or}$$

$$f(x) = \begin{cases} x|x|, & |x| < 1 \\ 2x, & |x| \ge 1 \end{cases}.$$

Q.3 Let $f(x) = \dfrac{x-1}{x+1}$, $f^2(x) = f\{f(x)\}$, $f^3(x) = f\{f^2(x)\}$ .... $f^{k+1}(x) = f\{f^k(x)\}$, for k = 1, 2, 3, ...

Find $f^{1998}(x)$.

Sol. $f(x) = \dfrac{x-1}{x+1}$

$f^2(x) = f\{f(x)\} = \dfrac{f-1}{f+1} = \dfrac{-1}{x}$

$f^3(x) = \dfrac{x+1}{1-x}$

$f^4(x) = x$

$f^5(x) = f\{f^4(x)\} = f(x)$

$f^{1998}(x) = f^2(x) = \dfrac{-1}{x}$.

Q.4 Find the domain of the function

(i) $f(x) = \log_3 \log_{(1/3)}(x^2 + 10x + 25) + \dfrac{1}{[x]+5}$ (where [·] denotes greatest integer function.)

(ii) $f(x) = \dfrac{1}{\sqrt{[x]^2 - [x] - 6}}$

Sol.

(i) $f(x) = \log_3 \log_{(1/3)}(x^2 + 10x + 25) + \dfrac{1}{[x]+5}$

$x^2 + 10x + 25 > 0 \Rightarrow (x+5)^2 > 0 \Rightarrow x \ne -5$ ....(1)

Also $\log_{(1/3)}(x^2 + 10x + 25) > 0$

$x^2 + 10x + 25 < 1$

$(x+6)(x+4) < 0$

$\Rightarrow x \in (-6, -4)$ ....(2)

From (1) and (2) $\quad x \in (-6, -5) \cup (-5, -4)$

Also $[x] + 5 \ne 0$; $[x] \ne -5 \Rightarrow x \notin [-5, -4)$

Hence, domain of $f(x) \in (-6, -5)$.

(ii) $f(x) = \dfrac{1}{\sqrt{[x]^2 - [x] - 6}}$

$[x]^2 - [x] - 6 > 0 \Rightarrow ([x] - 3)([x] + 2) > 0 \Rightarrow [x] \in (3, \infty) \cup (-\infty, -2)$

$x \in [4, \infty)$ also $x \in (-\infty, -2)$

$x \in (-\infty, -2) \in [4, \infty)$.

Q.5 Find the range of following functions

(i) $f(x) = \log_3\{\log_{1/2}(x^2 + 4x + 4)\}$ (ii) $f(x) = \sin^2 x - 5\sin x - 6$.

Sol.

(i) $f(x) = \log_3\{\log_{1/2}(x^2 + 4x + 4)\}$

firstly finding the domain

$\log_{1/2}(x^2 + 4x + 4) > 0$

$x^2 + 4x + 4 < 1 \Rightarrow x^2 + 4x + 3 < 0 \Rightarrow (x + 1)(x + 3) < 0 \Rightarrow -3 < x < -1$

Also $x^2 + 4x + 4 > 0$

$(x + 2)^2 > 0 \Rightarrow x \neq -2$

Hence, $x \in (-3, -1) - \{2\}$

Since $0 < \log_{1/2}(x^2 + 4x + 4) < \infty \ \forall \ x \in$ Domain thus

Range $\in R$

(ii) $f(x) = \sin^2 x - 5\sin x - 6 = \sin^2 x - 2\left(\dfrac{5}{2}\right)\sin x + \dfrac{25}{4} - 6 - \dfrac{25}{4}$

$= \left(\sin x - \dfrac{5}{2}\right)^2 - \dfrac{49}{4}$

where $\dfrac{9}{4} \leq \left(\sin x - \dfrac{5}{2}\right)^2 \leq \dfrac{49}{4}$

Hence, $f(x) \in [-10, 0]$. **Ans.**

Q.6 Find the period of f(x) satisfying the condition

(i) $f(x-1) + f(x+3) = f(x+1) + f(x+5)$

(ii) $f(x) = \{x\} + \cos \pi x$

where $\{\cdot\}$ denotes fraction part.

Sol.

(i) $f(x-1) + f(x+3) = f(x+1) + f(x+5)$ ....(1)

Replacing x by x + 2

$f(x+1) + f(x+5) = f(x+3) + f(x+7)$ ....(2)

Adding (1) and (2), we get

$f(x-1) = f(x+7)$ i.e. $f(x) = f(x+8)$

Hence, period is 8.

(ii) $f(x) = \{x\} + \cos \pi x$

Period of $\{x\} = 1$

Period of $\cos \pi x = \dfrac{2\pi}{\pi} = 2$

Hence period of $f(x) = 2$.

Q.7 Let $f : R \to R$, $f(x) = \dfrac{\alpha x^2 + 6x - 8}{\alpha + 6x - 8x^2}$. Find the value of α for f(x) to be onto.

Sol. $y = \dfrac{\alpha x^2 + 6x - 8}{\alpha + 6x - 8x^2}$

$\Rightarrow (\alpha + 8y)\, x^2 + 6\,(1 - y)\, x - (\alpha y + 8) = 0$

According to condition, y takes all real values for all real x,

i.e. $D \geq 0\ \forall\ y \in R$.

$\Rightarrow 36\,(1 - y)^2 + 4\,(\alpha y + 8)\,(\alpha + 8y) \geq 0\ \forall\ y \in R$

$\Rightarrow (9 + 8\alpha)y^2 + (\alpha^2 + 46)y + (9 + 8\alpha) \geq 0\ \forall\ y \in R$

i.e. $D \leq 0$ and coefficient of $y^2 > 0$

$\Rightarrow (\alpha^2 + 46)^2 \leq 4\,(9 + 8\alpha)^2$ and $9 + 8\alpha > 0$

$\Rightarrow \alpha^2 - 16\alpha + 28 \leq 0$ and $\alpha > \dfrac{-9}{8}$

$\Rightarrow 2 \leq \alpha \leq 14$ and $\alpha > \dfrac{-9}{8}$

Hence, $\alpha \in [2, 14]$. **Ans.**

Q.8 If $f(x) = ax^7 + bx^3 + cx - 5$, where a, b and c are constants. If $f(-7) = 7$, then find f(7).

Sol. $f(-7) = a\,(-7)^7 + b(-7)^3 + c(-7) - 5$

$7 = -(a \cdot 7^7 + b \cdot 7^3 + c \cdot 7) - 5$

$a \cdot 7^7 + b \cdot 7^3 + c \cdot 7 = -12$

$f(7) = a \cdot 7^7 + b \cdot 7^3 + c \cdot 7 - 5 = -12 - 5 = -17$. **Ans.**

Q.9 Let f be a real valued function of real and positive argument such that $f(x) + 3x\, f\left(\dfrac{1}{x}\right) = 2\,(x+1)\ \forall\ x > 0$, find the value of f(10099).

Sol. $f(x) + 3x\, f\left(\dfrac{1}{x}\right) = 2\,(x+1)$ ......(1)

replacing x by $\dfrac{1}{x}$

$f\left(\dfrac{1}{x}\right) + \dfrac{3}{x} f(x) = 2\left(\dfrac{1}{x} + 1\right)$

$x\ f\left(\dfrac{1}{x}\right) + 3f(x) = 2\,(1 + x)$ ......(2)

On solving (1) and (2)

$8f(x) = 4\,(1 + x) \Rightarrow f(x) = \dfrac{x+1}{2}$

$f(10099) = \dfrac{10100}{2} = 5050$. **Ans.**

Q.10 Find the inverse of the function $f(x) = \log_a\left(x + \sqrt{x^2+1}\right), a > 0, a \neq 1.$

Sol. Since $\sqrt{x^2+1} > \sqrt{x^2} \quad \forall\, x \in R$

Hence $x + \sqrt{x^2+1} > 0 \quad \forall\, x \in R$

f(x) is one-one onto hence invertible

$$y = \log_a\left(x + \sqrt{x^2+1}\right)$$

$$a^y = x + \sqrt{x^2+1} \qquad \ldots(i)$$

$$a^{-y} = \frac{1}{x + \sqrt{x^2+1}} = -x + \sqrt{x^2+1} \qquad \ldots(ii)$$

(i) − (ii)

$$a^y - a^{-y} = 2x \quad \Rightarrow \quad x = \frac{1}{2}\left(a^y - a^{-y}\right)$$

Hence $f^{-1}(x) = \frac{1}{2}\left(a^x - a^{-x}\right)$

INVERSE TRIGONOMETRIC FUNCTIONS

(A) GENERAL INTRODUCTION :

$\sin^{-1}x$, $\cos^{-1}x$, $\tan^{-1}x$ etc. denote angles or real numbers whose sine is x, whose cosine is x and whose tangent is x, provided that the answers given are numerically smallest available. These are also written as arc sinx, arc cosx etc.

If there are two angles one positive & the other negative having same numerical value, then positive angle should be taken.

(B) PRINCIPAL VALUES AND DOMAINS OF INVERSE CIRCULAR FUNCTIONS :

S.No.	Function	Domain	Range
(i)	$y = \sin^{-1}x$	$-1 \le x \le 1$	$-\frac{\pi}{2} \le y \le \frac{\pi}{2}$
(ii)	$y = \cos^{-1}x$	$-1 \le x \le 1$	$0 \le y \le \pi$
(iii)	$y = \tan^{-1}x$	$x \in R$	$-\frac{\pi}{2} < x < \frac{\pi}{2}$
(iv)	$y = \cot^{-1}x$	$x \in R$	$0 < y < \pi$
(v)	$y = \operatorname{cosec}^{-1}x$	$x \le -1$ or $x \ge 1$	$-\frac{\pi}{2} \le y \le \frac{\pi}{2}$, $y \ne 0$
(vi)	$y = \sec^{-1}x$	$x \le -1$ or $x \ge 1$	$0 \le y \le \pi$; $y \ne \frac{\pi}{2}$

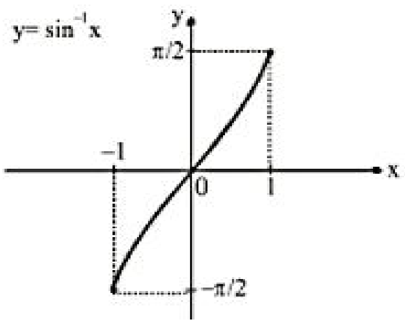
Note That : **(a)** 1st quadrant is common to all the inverse functions.

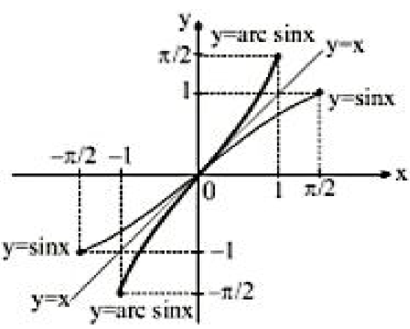
(b) 3rd quadrant is **not used** in inverse functions.

(c) 4th quadrant is used in the CLOCKWISE DIRECTION i.e. $-\frac{\pi}{2} \le y \le 0$.

Graphs of all 6 inverse circular functions :

(1) $y = \sin^{-1}x$, $|x| \le 1$, $y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$





Note : Graph of y = sin⁻¹x and y = sin x are mirror image of each other about the line y = x.

Highlights : -

(i) $\sin^{-1}x$ is bounded in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

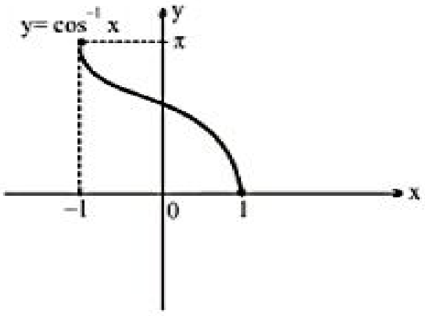
(ii) $\sin^{-1}x$ is an odd function. (symmetric about origin)

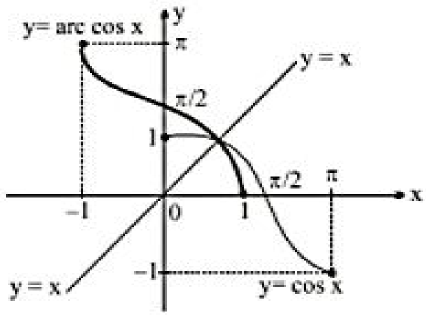
(iii) $\sin^{-1}x$ is an increasing function in its domain.

(iv) Maximum value of $\sin^{-1}x = \frac{\pi}{2}$, occurs at $x = 1$ and minimum value of $\sin^{-1}x = -\frac{\pi}{2}$, occurs at $x = -1$.

(v) $\sin^{-1}x$ is an aperiodic function.

(2) $y = \cos^{-1}x$, $|x| \le 1$, $y \in [0, \pi]$





Note : Graph of $y = \cos^{-1}x$ and $y = \cos x$ are mirror image of each other about the line $y = x$.

Highlights : -

(i) $\cos^{-1}x$ is bounded in $[0, \pi]$.

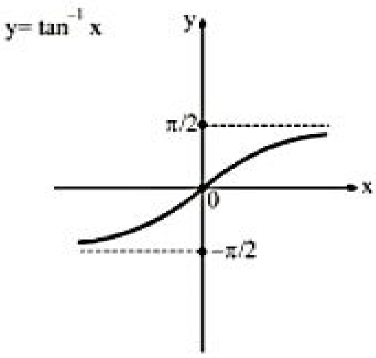
(ii) $\cos^{-1}x$ is a neither odd nor even function.

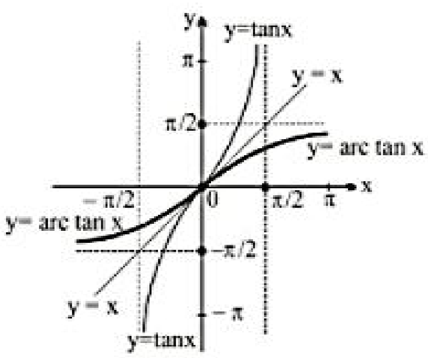
(iii) $\cos^{-1}x$ is a decreasing function in its domain.

(iv) Maximum value of $\cos^{-1}x = \pi$, occurs at $x = -1$ and minimum value of $\cos^{-1}x = 0$, occurs at $x = 1$.

(v) $\cos^{-1}x$ is an aperiodic function.

(3) $y = \tan^{-1}x$, $x \in R$, $y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$





Note : Graph of $y = \tan^{-1}x$ and $y = \tan x$ are mirror image of each other about the line $y = x$.

Highlights : -

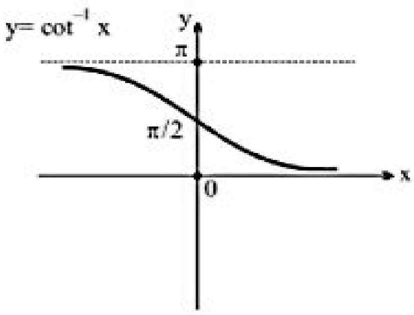
(i) $\tan^{-1}x$ is bounded in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

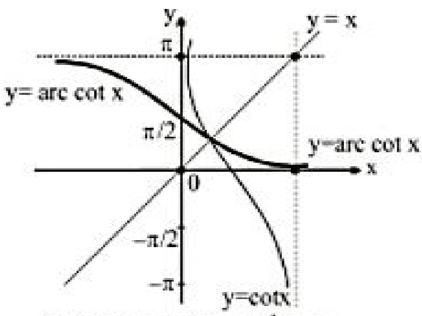
(ii) $\tan^{-1}x$ is an odd function. (symmetric about origin)

(iii) $\tan^{-1}x$ is an increasing function in its domain.

(iv) $\tan^{-1}x$ is an aperiodic function.

(4) $y = \cot^{-1} x$, $x \in R$, $y \in (0 , \pi)$





Note : Graph of $y = \cot^{-1}x$ and $y = \cot x$ are mirror image of each other about the line $y = x$.

Highlights : -

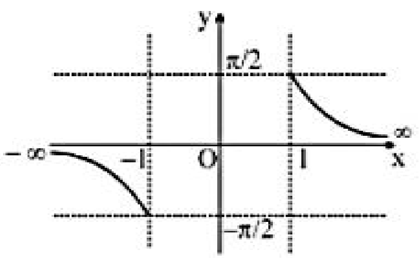
(i) $\cot^{-1}x$ is bounded in $(0, \pi)$.

(ii) $\cot^{-1}x$ is a neither odd nor even function.

(iii) $\cot^{-1}x$ is a decreasing function in its domain.

(iv) $\cot^{-1}x$ is an aperiodic function.

(5) $y = \operatorname{cosec}^{-1}x, |x| \geq 1,\ y \in \left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$



Highlights : -

(i) $\operatorname{cosec}^{-1}x$ is bounded in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

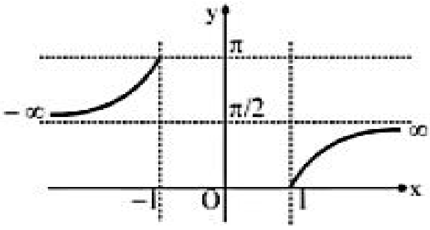
(ii) $\operatorname{cosec}^{-1}x$ is an odd function. (symmetric about origin)

(iii) Maximum value of $\operatorname{cosec}^{-1}x = \frac{\pi}{2}$, occurs at $x = 1$ and minimum value of $\operatorname{cosec}^{-1}x = -\frac{\pi}{2}$, occurs at $x = -1$.

(iv) $\operatorname{cosec}^{-1}x$ is a decreasing function.

(v) $\operatorname{cosec}^{-1}x$ is an aperiodic function.

(6) $y = \sec^{-1} x, |x| \geq 1, \qquad y \in \left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$



Highlights : -

(i) $\sec^{-1}x$ is bounded in $[0, \pi]$.

(ii) $\sec^{-1}x$ is a neither odd nor even function.

(iii) Maximum value of $\sec^{-1}x = \pi$, occurs at $x = -1$ and minimum value of $\sec^{1} x = 0$, occurs at $x = 1$.

(iv) $\sec^{-1}x$ is an increasing function.

(v) $\sec^{-1}x$ is an aperiodic function.

Note :

(a) $\tan^{-1}(x)$ and $\cot^{-1}(x)$ are continuous and monotonic on R $\Rightarrow$ that their range is R

(b) If $f(x)$ is continuous and has a range R $\not\Rightarrow$ it is monotonic. ***e.g.*** $y = x^3 - 3x$.

Illustration :

Find domain and range of the following

(a) $\sin^{-1}[x]$ *(b)* $\cos^{-1}\{x\}$

(c) $\sin^{-1}(e^x)$ *(d)* $f(x) = \tan^{-1}(\log_{4/5}(5x^2 - 8x + 4))$

(where [x] denotes the greatest integer function and {x} denotes the fractional part function.)

Sol.

(a) $\sin^{-1}[x]$ *defined when*

$-1 \leq [x] \leq 1 \quad \Rightarrow \quad -1 \leq x < 2$

domain : $x \in [-1, 2)$

In this domain [x] takes the values –1, 0, 1

$\Rightarrow$ *range of* $\sin^{-1}[x] = \{\sin^{-1} -1, \sin^{-1} 0, \sin^{-1} 1\}$

$$Range = \left\{-\frac{\pi}{2}, 0, \frac{\pi}{2}\right\}$$

(b) $\cos^{-1}\{x\}$ *defined when*

$-1 \leq \{x\} \leq 1$

$\Rightarrow$ *domain :* $x \in R \quad (\because \{x\} \in [0, 1))$

$Range = \cos^{-1}[0, 1)$

$= (\cos^{-1} 1, \cos^{-1} 0]$

$$Range = \left(0, \frac{\pi}{2}\right]$$

(c) $\sin^{-1} e^x$ defined when

$-1 \le e^x \le 1 \Rightarrow e^x \ge -1$ holds always true

So $e^x \le 1 \Rightarrow x \le 0$

domain $x \in (-\infty, 0]$

In this domain $e^x \in (0, 1]$

$\Rightarrow$ Range of $\sin^{-1} e^x = \sin^{-1}(0, 1]$

$= \sin^{-1} (0, 1]$

$= (\sin^{-1} 0, \sin^{-1} 1]$

$$\text{Range } = \left(0, \frac{\pi}{2}\right]$$

(d) $f(x)$ is defined when $5x^2 - 8x + 4 > 0$

$\because a > 0, D < 0 \Rightarrow 5x^2 - 8x + 4 > 0$ is true for all $x \in R$

$\Rightarrow$ domain : $x \in R$

$$\text{Now, } \quad 5x^2 - 8x + 4 = 5\left[\left(x - \frac{4}{5}\right)^2 + \frac{4}{25}\right]$$

$$\Rightarrow 5x^2 - 8x + 4 \in \left[\frac{4}{5}, \infty\right) \text{ for } x \in R$$

$$\Rightarrow \text{Range of } f(x) = \tan^{-1}\left(\log_{4/5}\left[\frac{4}{5}, \infty\right)\right)$$

$$\text{Range } = \tan^{-1}(-\infty, 1] = \left(-\frac{\pi}{2}, \frac{\pi}{4}\right]$$

Illustration :

Find the value of

(a) $\sin\left(2\sin^{-1}\frac{3}{5}\right)$

(b) $\cos(2\tan^{-1}2) + \sin(2\tan^{-1}3)$

(c) $\cos\left(\text{arc}\sin\frac{4}{5} - \text{arc}\cos\frac{4}{5}\right)$

(d) $\tan\left(2\cot^{-1}5 - \frac{\pi}{4}\right)$

Sol.

(a) $\sin^{-1}\frac{3}{5} = \theta \Rightarrow \sin\theta = \frac{3}{5}, \cos\theta = \frac{4}{5}$

$\sin(2\theta) = 2\sin\theta.\cos\theta$

$= 2\left(\frac{3}{5}\right)\left(\frac{4}{5}\right) = \frac{24}{25}$ $\qquad \left[\text{Ans. } \frac{24}{25}\right]$

(b) Let $\tan^{-1}2 = \theta \Rightarrow \tan\theta = 2$

$\tan^{-1}3 = \phi \Rightarrow \tan\phi = 3$

Now

$$\cos(2\theta) + \sin(2\phi) = \frac{1-\tan^2\theta}{1+\tan^2\theta} + \frac{2\tan\phi}{1+\tan^2\phi}$$

$$= \frac{1-(2)^2}{1+(2)^2} + \frac{2(3)}{1+(3)^2}$$

$$= \frac{-3}{5} + \frac{3}{5} = 0 \qquad [Ans.\ 0]$$

(c) Let $\sin^{-1}\frac{4}{5} = \theta \Rightarrow \sin\theta = \frac{4}{5}, \cos\theta = \frac{3}{5}$

$\cos^{-1}\frac{4}{5} = \phi \Rightarrow \cos\phi = \frac{4}{5}, \sin\phi = \frac{3}{5}$

Now

$\cos(\theta - \phi) = \cos\theta\cos\phi + \sin\theta\sin\phi$

$= \frac{4}{5}\cdot\frac{3}{5} + \frac{4}{5}\cdot\frac{3}{5} = \frac{24}{25}$ $\qquad \left[Ans.\ \frac{24}{25}\right]$

(d) let $\cot^{-1}5 = \theta \Rightarrow \cot\theta = 5, \tan\theta = \frac{1}{5}$

Now

$$\tan\left(2\theta - \frac{\pi}{4}\right) = \frac{\tan 2\theta - 1}{1+\tan 2\theta}$$

$$= \frac{\left(\frac{2\tan\theta}{1-\tan^2\theta}\right)-1}{1+\left(\frac{2\tan\theta}{1-\tan^2\theta}\right)} = \frac{\left(\frac{2\left(\frac{1}{5}\right)}{1-\left(\frac{1}{5}\right)^2}\right)-1}{1+\left(\frac{2\left(\frac{1}{5}\right)}{1-\left(\frac{1}{5}\right)^2}\right)} = \frac{\frac{5}{12}-1}{1+\frac{5}{12}} = \frac{-7}{17} \qquad \left[Ans.\ \frac{-7}{17}\right]$$

Illustration :

Find the domain of definition of following functions.

(a) $f(x) = \text{arc}\cos\frac{2x}{1+x}$

(b) $f(x) = \sin^{-1}\left(\frac{x-3}{2}\right) - \log_{10}(4-x)$

(c) $f(x) = \sqrt{3-x} + \cos^{-1}\left(\frac{3-2x}{5}\right) + \log_6(2|x|-3) + \sin^{-1}(\log_2 x)$

Sol.

(a) $f(x) = \cos^{-1} \dfrac{2x}{1+x}$

$f(x)$ defined when $-1 \le \dfrac{2x}{1+x} \le 1$

Now $\dfrac{2x}{1+x} \ge -1 \Rightarrow \dfrac{2x}{1+x} + 1 \ge 0 \Rightarrow \dfrac{3x+1}{1+x} \ge 0$

$\Rightarrow x \in (-\infty, -1) \cup \left[-\dfrac{1}{3}, \infty\right)$ ... (1)

and $\dfrac{2x}{1+x} \le 1 \Rightarrow \dfrac{2x}{1+x} - 1 \le 0 \Rightarrow \dfrac{x-1}{1+x} \le 0$

$\Rightarrow x \in (-1, 1]$ ... (ii)

from (i) and (ii)

$x \in \left[-\dfrac{1}{3}, 1\right]$ **Ans.** $x \in \left[-\dfrac{1}{3}, 1\right]$

(b) $f(x) = \sin^{-1}\left(\dfrac{x-3}{2}\right) - \log_{10}(4-x)$

$f(x)$ defined when $-1 \le \dfrac{x-3}{2} \le 1$ and $4 - x > 0$

Now $-1 \le \dfrac{x-3}{2} \le 1$

$\Rightarrow -2 \le x - 3 \le 2$

$\Rightarrow 1 \le x \le 5$

$\Rightarrow x \in [1, 5]$ ... (i)

and $4 - x > 0$

$\Rightarrow x < 4 \Rightarrow x \in (-\infty, 4)$ ... (ii)

$\Rightarrow$ from (i) and (ii)

$x \in [1, 4)$ **Ans.** $x \in [1, 4)$

(c) $f(x) = \sqrt{3-x} + \cos^{-1}\left(\dfrac{3-2x}{5}\right) + \log_6(2|x| - 3) + \sin^{-1}(\log_2 x)$

Now $3 - x \ge 0 \Rightarrow x \le 3 \Rightarrow x \in (-\infty, 3]$ ... (i)

$-1 \le \dfrac{3-2x}{5} \le 1 \Rightarrow -5 \le 3 - 2x \le 5$

$\Rightarrow -2 \le 2x \le 8 \Rightarrow -1 \le x \le 4$

$\Rightarrow x \in [-1, 4]$ ... (ii)

$$2|x| - 3 > 0 \Rightarrow |x| > \frac{3}{2} \Rightarrow x \in \left(-\infty, -\frac{3}{2}\right) \cup \left(\frac{3}{2}, \infty\right) \quad \text{... (iii)}$$

$$-1 \le \log_2 x \le 1 \Rightarrow \frac{1}{2} \le x \le 2 \Rightarrow x \in \left[\frac{1}{2}, 2\right] \quad \text{... (iv)}$$

from (i), (ii), (iii) and (iv)

$$x \in \left(\frac{3}{2}, 2\right] \qquad \text{Ans. } \left(\frac{3}{2}, 2\right]$$

Illustration :

If $\cos^{-1} x + \cos^{-1} y + \cos^{-1} z = 3\pi$, *then the value of* $x^{2012} + y^{2012} + z^{2012} + \dfrac{6}{x^{2011} + y^{2011} + z^{2011}}$ *is equal to*

(A) 0 (B) 1 (C) –1 (D) 2

Sol. $\cos^{-1} x + \cos^{-1} y + \cos^{-1} z = 3\pi$ *is possible only when* $\cos^{-1} x = \cos^{-1} y = \cos^{-1} z = \pi$ *(all take their maximum value)*

$\Rightarrow \quad x = -1, y = -1, z = -1$

Now

$$x^{2012} + y^{2012} + z^{2012} + \frac{6}{x^{2011} + y^{2011} + z^{2011}}$$

$$= (-1)^{2012} + (-1)^{2012} + (-1)^{2012} + \frac{6}{(-1)^{2011} + (-1)^{2011} + (-1)^{2011}}$$

$$= 3 + \frac{6}{-3} = 3 - 2 = \mathbf{1} \qquad \textbf{Ans.}$$

Illustration :

Equation of the image of the line $x + y = \sin^{-1}(a^6 + 1) + \cos^{-1}(a^4 + 1) - \tan^{-1}(a^2 + 1)$, $a \in R$ *about x axis is given by*

(A) $x - y = 0$ (B) $x - y = \dfrac{\pi}{2}$ (C) $x - y = \pi$ (D) $x - y = \dfrac{\pi}{4}$

Sol. $\because \quad \sin^{-1}$ *is defined for* $[-1, 1]$

$\therefore \quad a = 0$

$\therefore \quad x + y = \sin^{-1} 1 + \cos^{-1} 1 - \tan^{-1} 1 = \dfrac{\pi}{4}$

Clearly image about x axis will be $x - y = \dfrac{\pi}{4}$ **Ans.**

Illustration :

If $\sin^{-1}\left(x^2-\frac{x^4}{3}+\frac{x^6}{9}\ldots\ldots\ldots\right)+\cos^{-1}\left(x^4-\frac{x^8}{3}+\frac{x^{12}}{9}\ldots\ldots\ldots\right)=\frac{\pi}{2}$, *where* $0 \le |x| < \sqrt{3}$, *then number of values of 'x' is equal to*

(A) 1 (B) 2 (C) 3 (D) 4

Sol. $\sin^{-1}\underbrace{\left(x^2-\frac{x^4}{3}+\frac{x^6}{9}\ldots\ldots\right)}_{X}+\cos^{-1}\underbrace{\left(x^4-\frac{x^8}{3}+\frac{x^{12}}{9}\ldots\ldots\right)}_{Y}=\frac{\pi}{2}$

Now $X = Y$

$$\frac{x^2}{1+\frac{x^2}{3}}=\frac{x^4}{1+\frac{x^4}{3}} \Rightarrow \frac{3}{3+x^2}=\frac{3x^2}{3+x^4} \Rightarrow 9+3x^4=9x^2+3x^4 \Rightarrow x^2=1$$

$\Rightarrow$ $x = 0, 1$ *or* -1

$\therefore$ *Number of values is equal to* **3.** ***Ans.***

Illustration :

If $0 < \cos^{-1}x < 1$ *and* $1+\cos^{-1}x+(\cos^{-1}x)^2+\ldots\ldots\ \infty = 2$ *then* x *is equal to*

(A) $\frac{\pi}{4}$ (B) $\cos\frac{1}{2}$ (C) $\cos\frac{1}{\sqrt{2}}$ (D) $\frac{\pi}{6}$

Sol. *We have*

$1+\cos^{-1}x+(\cos^{-1}x)^2+\ldots\ldots\ \infty = 2$, *(which is an infinite geometric progression)*

$\Rightarrow \frac{1}{1-\cos^{-1}x}=2 \Rightarrow \cos^{-1}x=\frac{1}{2} \Rightarrow x=\cos\frac{1}{2}$ ***Ans.***

Illustration :

Let $f(x)=\frac{\left|\cos^{-1}(\operatorname{sgn}x)\right|}{\cos^{-1}(\operatorname{sgn}x)}$ *then which of the following is (are) not correct?*

*[**Note:** sgn x denotes signum function of x.]*

(A) Range of f(x) contains no integer.

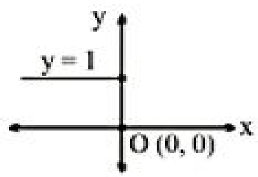
(B) Graph of f(x) is symmetric about y-axis.

(C) The equation f(x) = 0 has two distinct real solutions.

(D) Inverse of f(x) is not defined.

Sol. *Clearly,* $D_f = (-\infty, 0]$

Now, $f(x) = 1\ \forall x \in (-\infty, 0]$ ***Ans.***



Practice Problem

Q.1 Find domain and range of the following

(a) $\cos^{-1}[x]$ (b) $\sin^{-1}\{x\}$ (c) $\cot^{-1}(\text{sgn}\, x)$ (d) $\cot^{-1}\log_{\frac{4}{5}}(5x^2-8x+4)$

(where [x] denotes the greatest integer function and {x} denotes the fractional part function.)

Q.2 Find the value of

(a) $\tan\left(\frac{1}{2}\cos^{-1}\frac{\sqrt{5}}{3}\right)$ (b) $\sin(\tan^{-1}2)+\cos(\tan^{-1}2)$

Q.3 State which of the statements are True or False ?

(i) $y=\text{sgn}(\cot^{-1}x)$ and $y=1$ identical. (ii) $e^{\ln(\tan^{-1}x)}$ and $\tan^{-1}x$ identical.

(iii) $e^{\ln(\cot^{-1}x)}$ and $\cot^{-1}x$ identical.

Q.4 Find domain of definition the functions $f(x)=\sqrt{\cos(\sin x)}+\sin^{-1}\frac{1+x^2}{2x}$

Q.5 The range of the function $y=\left(\frac{\cos^{-1}(3x-1)}{\pi}+1\right)^2$ is

(A) [1,4] (B) $[0,\pi]$ (C) $[1,\pi]$ (D) $[0,\pi^2]$

Q.6 If $\sin^{-1}x+\sin^{-1}y+\sin^{-1}z=\frac{3\pi}{2}$, then possible value(s) of a satisfying the equation

$x^{100}+y^{100}+z^{100}+\frac{a^2}{x^{50}+y^{50}+z^{50}}=\frac{10a}{3}$ are

(A) 1 (B) 4 (C) 9 (D) 16

Answer key

Q.1 (a) $D: x\in[-1,2)$ and $R\in\left\{0,\frac{\pi}{2},\pi\right\}$ (b) $D: x\in R;\ R:[0,\pi/2)$

(c) $D: x\in R;\ R:\left\{\frac{\pi}{2},\frac{\pi}{4},\frac{3\pi}{4}\right\}$ (d) $D: x\in R;\ R:\left[\frac{\pi}{4},\pi\right)$

Q.2 (a) $\frac{3-\sqrt{5}}{2}$, (b) $\frac{3}{\sqrt{5}}$ **Q.3** (i) True ; (ii) False ; (iii) True

Q.4 $\{-1, 1\}$ **Q.5** A **Q.6** A,C

PROPERTIES OF INVERSE TRIGONOMETRIC FUNCTION :

Property-1 :

(i) $\sin(\sin^{-1}x) = x\ ,\ -1 \le x \le 1$

(ii) $\cos(\cos^{-1}x) = x\ ,\ -1 \le x \le 1$

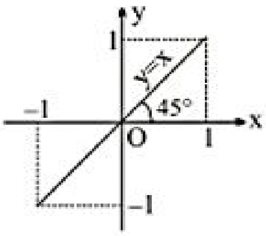
(iii) $\tan(\tan^{-1}x) = x\ ,\ x \in R$

(iv) $\cot(\cot^{-1}x) = x\ ,\ x \in R$

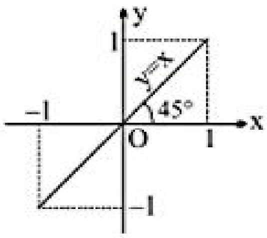
(v) $\operatorname{cosec}(\operatorname{cosec}^{-1}x) = x\ ,\ |x| \ge 1$

(vi) $\sec(\sec^{-1}x) = x\ ,\ |x| \ge 1$

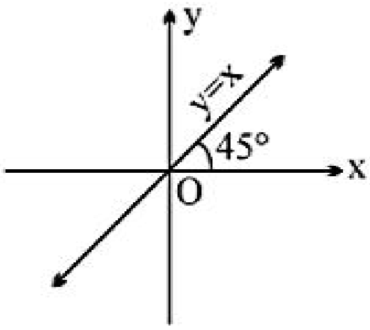
(1) $y = \sin(\sin^{-1}x) = x, x \in [-1, 1], y \in [-1, 1]$, y is aperiodic.



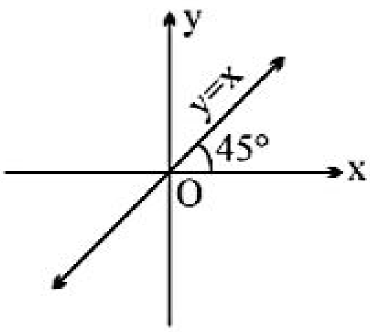
(2) $y = \cos(\cos^{-1}x) = x, x \in [-1, 1], y \in [-1, 1]$, y is aperiodic.



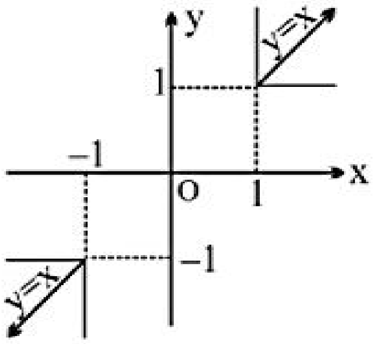
(3) $y = \tan(\tan^{-1}x) = x, x \in R, y \in R$, y is aperiodic.



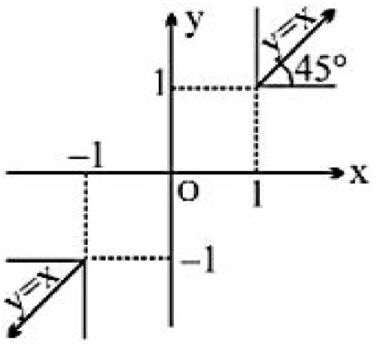
(4) $y = \cot(\cot^{-1}x) = x, x \in R, y \in R$, y is aperiodic.



(5) $y = \text{cosec}(\text{cosec}^{-1}x) = x, |x| \geq 1, |y| \geq 1,$ y is aperiodic.



(6) $y = \sec(\sec^{-1}x) = x, |x| \geq 1, |y| \geq 1,$ y is aperiodic.



Note that: (1, 2); (3, 4) and (5, 6) are identical function.

(vii) $\sin^{-1}(\sin x) = x\ ,\ -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$

(vii) $\cos^{-1}(\cos x) = x\ ;\ 0 \leq x \leq \pi$

(ix) $\tan^{-1}(\tan x) = x\ ;\ -\frac{\pi}{2} < x < \frac{\pi}{2}$

(x) $\cot^{-1}(\cot x) = x\ ,\ 0 < x < \pi$

(xi) $\text{cosec}^{-1}(\text{cosec}\ x) = x\ ;\ -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}, x \neq 0$

(xii) $\sec^{-1}(\sec x) = x\ ;\ 0 \leq x \leq \pi, x \neq \frac{\pi}{2}$

(7) $y = \sin^{-1}(\sin x), x \in R, y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, Periodic with period 2π

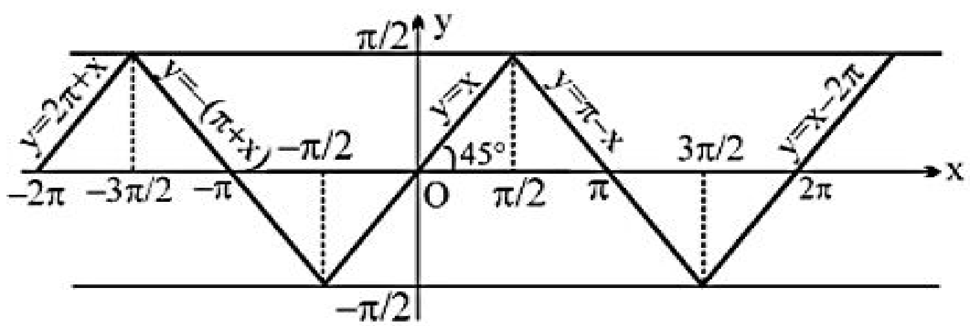


Illustration :

Find the value of following

(a) $\sin^{-1}(\sin 1)$ (b) $\sin^{-1}(\sin 2)$ (c) $\sin^{-1}(\sin 3)$

(d) $\sin^{-1}(\sin 4)$ (e) $\sin^{-1}(\sin 5)$ (f) $\sin^{-1}(\sin 10)$

Sol. (a) $\sin^{-1}(\sin 1) = 1 \quad (\because 1 \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right])$

(b) $\sin^{-1}(\sin 2) \neq 2 \quad (\because 2 \notin \left[-\frac{\pi}{2}, \frac{\pi}{2}\right])$

$\Rightarrow \sin^{-1}(\sin 2) = \sin^{-1}(\sin(\pi - 2)) = \pi - 2 \quad (\because \pi - 2 \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right])$

(c) $\sin^{-1}(\sin 3) = \sin^{-1}(\sin(\pi - 3)) = \pi - 3 \quad (\because \pi - 3 \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right])$

(d) $\sin^{-1}(\sin 4) = \sin^{-1}(\sin(\pi - 4)) = \pi - 4 \quad (\because \pi - 4 \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right])$

(e) $\sin^{-1}(\sin 5) = \sin^{-1}(\sin(5 - 2\pi)) = 5 - 2\pi \quad (\because 5 - 2\pi \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right])$

(f) $\sin^{-1}(\sin 10) = \sin^{-1}(\sin(3\pi - 10)) = 3\pi - 10 \quad (\because 3\pi - 10 \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right])$

(8) $y = \cos^{-1}(\cos x) = x, x \in \mathbb{R}, y \in [0, \pi]$, periodic with period 2π

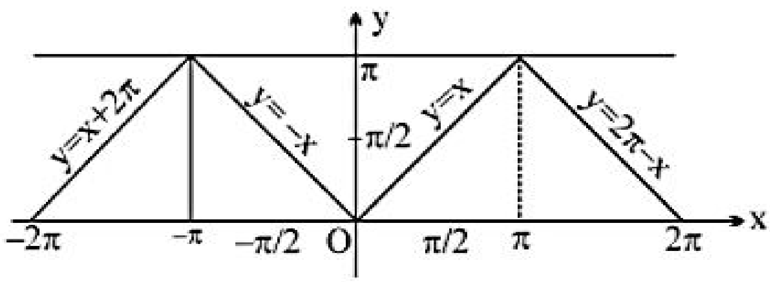


Illustration :

Find the value of following

(a) $\cos^{-1}(\cos 1)$ (b) $\cos^{-1}(\cos 2)$ (c) $\cos^{-1}(\cos 3)$

(d) $\cos^{-1}(\cos 4)$ (e) $\cos^{-1}(\cos 5)$ (f) $\cos^{-1}(\cos 10)$

Sol. (a) $\cos^{-1}(\cos 1) = 1$; $(\because 1 \in [0, \pi])$

(b) $\cos^{-1}(\cos 2) = 2$; $(\because 2 \in [0, \pi])$

(c) $\cos^{-1}(\cos 3) = 3$; $(\because 3 \in [0, \pi])$

(d) $\cos^{-1}(\cos 4) = \cos^{-1}(\cos(2\pi - 4)) = 2\pi - 4$; $(\because 2\pi - 4 \in [0, \pi])$

(e) $\cos^{-1}(\cos 5) = \cos^{-1}(\cos(2\pi - 5)) = 2\pi - 5$; $(\because 2\pi - 5 \in [0, \pi])$

(f) $\cos^{-1}(\cos 10) = \cos^{-1}(\cos(10 - 3\pi)) = 10 - 3\pi$. $(\because 10 - 3\pi \in [0, \pi])$

(9) $y = \tan^{-1}(\tan x) = x, x \in R - \left\{(2n-1)\frac{\pi}{2} n \in I\right\}, \; y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right),$
periodic with period π

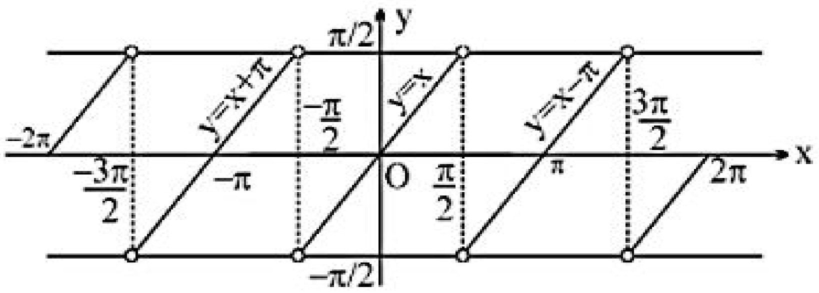


Illustration :

Find the value of following

(a) $\tan^{-1}(\tan 1)$ *(b)* $\tan^{-1}(\tan 2)$ *(c)* $\tan^{-1}(\tan 3)$

(d) $\tan^{-1}(\tan 4)$ *(e)* $\tan^{-1}(\tan 5)$ *(f)* $\tan^{-1}(\tan 10)$

Sol. *(a)* $\tan^{-1}(\tan 1) = 1 \quad (\because 1 \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right))$

(b) $\tan^{-1}(\tan 2) \neq 2 \quad (\because 2 \notin \left(-\frac{\pi}{2}, \frac{\pi}{2}\right))$

$\Rightarrow \tan^{-1}(\tan 2) = \tan^{-1}(\tan(\pi - 2)) = \pi - 2 \qquad (\because \pi - 2 \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right))$

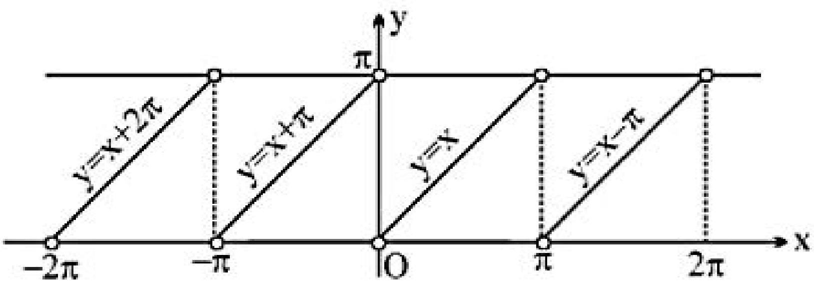
(c) $\tan^{-1}(\tan 3) = \tan^{-1}(\tan(\pi - 3)) = \pi - 3 \qquad (\because \pi - 3 \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right))$

(d) $\tan^{-1}(\tan 4) = \tan^{-1}(\tan(\pi - 4)) = \pi - 4 \qquad (\because \pi - 4 \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right))$

(e) $\tan^{-1}(\tan 5) = \tan^{-1}(\tan(5 - 2\pi)) = 5 - 2\pi \qquad (\because 5 - 2\pi \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right))$

(f) $\tan^{-1}(\tan 10) = \tan^{-1}(\tan(3\pi - 10)) = 3\pi - 10 \qquad (\because 3\pi - 10 \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right))$

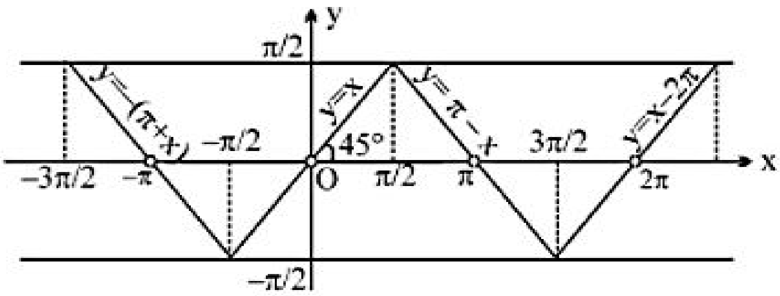
(10) $y = \cot^{-1}(\cot x) = x, x \in R - \{n\pi\}, \; y \in (0, \pi)$, periodic with π



(11) y = $\text{cosec}^{-1}(\text{cosec}\,x)$, $x \in R - \{n\pi, n \in I\}$,

$$y \in \left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$$

y is periodic with period 2π



(12) $y = \sec^{-1}(\sec x) = x$, y is periodic ;

$x \in R - \left\{(2n-1)\frac{\pi}{2}\ n \in I\right\}$, $y \in \left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$

with period 2π

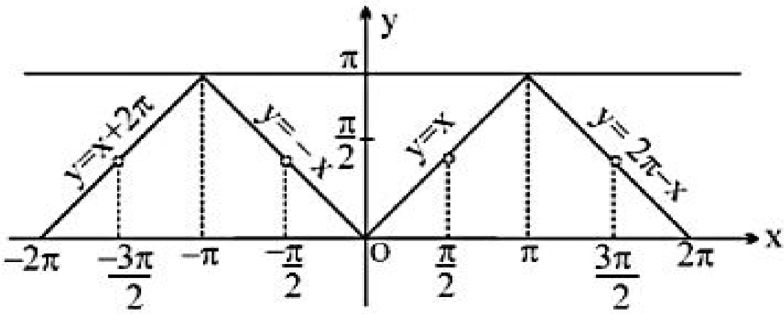


Illustration :

Find the integral solution of inequality $6x^2 - 5x < \cos^{-1}(\cos 5) - 2\sin^{-1}(\sin 3)$.

Sol. $6x^2 - 5x < \cos^{-1}(\cos(2\pi - 5)) - 2\sin^{-1}(\sin(\pi - 3))$

$6x^2 - 5x < 2\pi - 5 - 2\pi + 6$

$6x^2 - 5x < 1$

$6x^2 - 5x - 1 < 0$

$(6x + 1)(x - 1) < 0$

$\Rightarrow x \in \left(-\frac{1}{6}, 1\right)$

Integral solution is $x = 0$ *Ans.* [0]

Illustration :

If $\sin^{-1}(\sin 9) - \cos^{-1}(\cos 15)$ *can be written in the form* $a\pi - b$, *then find the value of* $a + b$. $(a, b \in N)$.

Sol. $\sin^{-1}(\sin 9) = \sin^{-1}\sin(3\pi - 9) = 3\pi - 9$ $\quad \left(\because 3\pi - 9 \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]\right)$

$\cos^{-1}(\cos 15) = \cos^{-1}(\cos(15 - 4\pi)) = 15 - 4\pi$ $\quad (\because 15 - 4\pi \in [0, \pi])$

$\Rightarrow \sin^{-1}(\sin 9) - \cos^{-1}(\cos 15) = (3\pi - 9) - (15 - 4\pi)$
$= 7\pi - 24 \Rightarrow a = 7, b = 24$
$a + b = 7 + 24 = 31.$ *Ans. 31*

Illustration :

Find the value of following

(a) $\sin^{-1}\sin\left(\frac{13\pi}{11}\right)$ (b) $\cos^{-1}\left(\sin\left(-\frac{\pi}{4}\right)\right)$ (c) $\sin^{-1}\left(\cos\frac{33\pi}{10}\right)$

Sol. (a) $\sin^{-1}\sin\left(\frac{13\pi}{11}\right) = \sin^{-1}\sin\left(\pi + \frac{2\pi}{11}\right) = \sin^{-1}\left(-\sin\left(\frac{2\pi}{11}\right)\right) = \sin^{-1}\sin\left(-\frac{2\pi}{11}\right) = -\frac{2\pi}{11}$ **Ans.**

(b) $\cos^{-1}\sin\left(-\frac{\pi}{4}\right) = \cos^{-1}\cos\left(\frac{\pi}{2} + \frac{\pi}{4}\right) = \cos^{-1}\cos\left(\frac{3\pi}{4}\right) = \frac{3\pi}{4}$ **Ans.**

(c) $\sin^{-1}\cos\left(\frac{33\pi}{10}\right) = \sin^{-1}\cos\frac{13\pi}{10} = \sin^{-1}\left(-\cos\frac{3\pi}{10}\right) = \sin^{-1}\left(-\sin\left(\frac{5\pi}{10} - \frac{3\pi}{10}\right)\right)$

$$= \sin^{-1}\left(-\sin\frac{\pi}{5}\right) = \sin^{-1}\left(\sin\left(-\frac{\pi}{5}\right)\right) = -\frac{\pi}{5} \text{ Ans.}$$

Illustration :

If $\sin^{-1}\left(\sin\frac{33\pi}{7}\right) + \cos^{-1}\left(\cos\frac{46\pi}{7}\right) + \tan^{-1}\left(-\tan\frac{13\pi}{8}\right) + \cot^{-1}\left(\cot\left(-\frac{19\pi}{8}\right)\right)$

can be written in the from of $\frac{a\pi}{b}$ *(where a, b ∈ N) then find the minimum value of a + b.*

Sol. $\sin^{-1}\left(\sin\left(\frac{33\pi}{7}\right)\right) = \sin^{-1}\left(\sin\left(5\pi - \frac{2\pi}{7}\right)\right) = \sin^{-1}\left(\sin\left(\frac{2\pi}{7}\right)\right) = \frac{2\pi}{7}$

$\cos^{-1}\left(\cos\left(\frac{46\pi}{7}\right)\right) = \cos^{-1}\left(\cos\left(6\pi + \frac{4\pi}{7}\right)\right) = \cos^{-1}\left(\cos\left(\frac{4\pi}{7}\right)\right) = \frac{4\pi}{7}$

$\tan^{-1}\left(-\tan\left(\frac{13\pi}{8}\right)\right) = \tan^{-1}\left(\tan\left(2\pi - \frac{13\pi}{8}\right)\right) = \tan^{-1}\left(\tan\left(\frac{3\pi}{8}\right)\right) = \frac{3\pi}{8}$

$\cot^{-1}\left(\cot\left(\frac{-19\pi}{8}\right)\right) = \cot^{-1}\left(\cot\left(3\pi - \frac{19\pi}{8}\right)\right) = \cot^{-1}\left(\cot\left(\frac{5\pi}{8}\right)\right) = \frac{5\pi}{8}$

$\Rightarrow \sin^{-1}\left(\sin\frac{33\pi}{7}\right) + \cos^{-1}\left(\cos\frac{46\pi}{7}\right) + \tan^{-1}\left(-\tan\frac{13\pi}{8}\right) + \cot^{-1}\left(\cot\left(-\frac{19\pi}{8}\right)\right)$

$= \frac{2\pi}{7} + \frac{4\pi}{7} + \frac{3\pi}{8} + \frac{5\pi}{8} = \frac{6\pi}{7} + \pi = \frac{13\pi}{7}$
$\Rightarrow a = 13, b = 7 \Rightarrow a + b = 13 + 7 = 20$ ***Ans. 20***

Illustration :

The smallest positive integral value of n for which

$$(n-2)x^2 + 8x + n + 4 > \sin^{-1}(\sin 12) + \cos^{-1}(\cos 12)\ \forall x \in R, \text{ is}$$

(A) 4 (B) 5 (C) 6 (D) 7

Sol. We have

$\sin^{-1}(\sin 12) + \cos^{-1}(\cos 12) = -(4\pi - 12) + (4\pi - 12) = 0$

$\therefore \quad (n-2)x^2 + 8x + n + 4 > 0 \quad \forall x \in R$

$\Rightarrow \quad (n-2) > 0 \Rightarrow n \geq 3$ and $(8)^2 - 4(n-2)(n+4) < 0$ or $n^2 + 2n - 24 > 0$

$\Rightarrow \quad n > 4 \Rightarrow n \geq 5$

So, $\quad n_{smallest} = 5$. **Ans.**

Illustration :

The product of all real values of x satisfying the equation

$$\sin^{-1}\cos\left(\frac{2x^2+10|x|+4}{x^2+5|x|+3}\right) = \cot\left(\cot^{-1}\left(\frac{2-18|x|}{9|x|}\right)\right) + \frac{\pi}{2} \text{ is}$$

(A) 9 (B) – 9 (C) – 3 (D) – 1

Sol. $$\frac{\pi}{2} - \cos^{-1}\cos\left(\underbrace{\frac{2(x^2+5|x|+3)-2}{x^2+5|x|+3}}_{0<\downarrow<2}\right) = \cot\cot^{-1}\left(\frac{2}{9|x|}-2\right) + \frac{\pi}{2}$$

$$\frac{\pi}{2} - 2 + \frac{2}{x^2+5|x|+3} = \frac{2}{9|x|} - 2 + \frac{\pi}{2}$$

$\Rightarrow \quad |x|^2 - 4|x| + 3 = 0$

$\quad |x| = 1, 3 \quad \Rightarrow \quad x = \pm 1, \pm 3$

$\Rightarrow \quad$ Product $= (1)(-1)(3)(-3) = 9$ **Ans.**

Illustration :

Which of the following is/are correct?

(A) $\cos(\cos(\cos^{-1}1)) < \sin(\sin^{-1}(\sin(\pi-1))) < \sin(\cos^{-1}(\cos(2\pi-2)))$

(B) $\cos(\cos(\cos^{-1}1)) < \sin(\cos^{-1}(\cos(2\pi-2))) < \sin(\sin^{-1}(\sin(\pi-1))) < \tan(\cot^{-1}(\cot 1))$

(C) $\displaystyle\sum_{t=1}^{5000}\cos^{-1}(\cos(2t\pi-1)) = \sum_{t=1}^{2500}\cot^{-1}(\cot(t\pi+2))$ where $t \in I$

(D) $\cot^{-1}\cot\ \mathrm{cosec}^{-1}\mathrm{cosec}\ \sec^{-1}\sec\ \tan\ \tan^{-1}\cos\ \cos^{-1}\sin^{-1}\sin 4 = 4 - \pi$

Sol. ***for (A) and (B)***

$\cos(\cos^{-1}1) = 1 \quad \Rightarrow \quad \cos(\cos(\cos^{-1}1)) = \cos 1$

$\sin^{-1}(\sin(\pi-1)) = \pi - (\pi - 1) = 1 \quad \Rightarrow \quad \sin(\sin^{-1}(\sin(\pi-1))) = \sin 1$

$\cos^{-1}(\cos(2\pi-2)) = \cos^{-1}(\cos 2) = 2 \quad \Rightarrow \quad \sin(\cos^{-1}(\cos(2\pi-2))) = \sin 2$

$\tan(\cot^{-1}(\cot 1)) = \tan 1$

It is easy to compare cos 1, sin 1, sin 2, tan 1

$\cos 1 < \sin 1 < \sin 2 < \tan 1 \Rightarrow$ ***(A) is correct***

for (C)

$\cos^{-1} \cos x$ *is periodic and even*

$\cos^{-1} \cos(2t\pi - 1) = \cos^{-1}(\cos 1) = 1 \ (t \in I)$

$$\sum_{t=1}^{5000} \cos^{-1} \cos(2t\pi - 1) = 5000$$

now $\cot^{-1} \cot(t\pi + 2) = 2$ [$\cot^{-1} \cot x$ *is periodic with period* π]

$$\therefore \quad \sum_{t=1}^{2500} \cot^{-1} \cot(t\pi + 2) = 5000 \Rightarrow$$ ***(C) is correct***

(D) $\sin^{-1} \sin 4 = \pi - 4$

$\cos \cos^{-1}(\pi - 4) = \pi - 4$

$\tan \tan^{-1}(4 - \pi) = \pi - 4$

$\sec^{-1} \sec(\pi - 4) = 4 - \pi$

$\operatorname{cosec}^{-1}\operatorname{cosec}(4 - \pi) = 4 - \pi$

$\cot^{-1}\cot(4 - \pi) = 4 - \pi \Rightarrow$ ***(D) is correct***

Practice Problem

Q.1 Find the value of following

(i) $\sin^{-1}[\cos 2\cot^{-1}(\sqrt{2} - 1)]$ (ii) $\sin^{-1}(\sin 7) + \cos^{-1}\cos(13)$

(iii) $\sin^{-1}\left(\sin \frac{10\pi}{7}\right)$ (iv) $\cos^{-1}\left(\sin\left(-\frac{\pi}{9}\right)\right)$

Q.2 If $3 \le a < 4$ then the value of $\sin^{-1}(\sin [a]) + \tan^{-1}(\tan [a]) + \sec^{-1}(\sec [a])$, where [x] denotes greatest integer function less than or equal to x, is equal to

(A) 3 (B) $2\pi - 9$ (C) $2\pi - 3$ (D) $9 - 2\pi$

Q.3 The value of $\sin^{-1}(\cos 2) - \cos^{-1}(\sin 2) + \tan^{-1}(\cot 4) - \cot^{-1}(\tan 4) + \sec^{-1}(\operatorname{cosec} 6) - \operatorname{cosec}^{-1}(\sec 6)$ is

(A) 0 (B) 3π (C) $8 - 3\pi$ (D) $5\pi - 16$

Paragraph for question nos. 4 to 6

For $x \in \left(0, \frac{\pi}{4}\right)$,

Let $S_n = \sum_{r=1}^{2n} \sin\left(\sin^{-1} x^{3r-2}\right)$, $C_n = \sum_{r=1}^{2n} \cos\left(\cos^{-1} x^{3r-1}\right)$ and $T_n = \sum_{r=1}^{2n} \tan\left(\tan^{-1} x^{3r}\right)$

where $n \in N$ and $n \ge 3$.

Q.4 The correct order of S_n, C_n and T_n is given by

(A) $S_n > T_n > C_n$ (B) $S_n < C_n < T_n$ (C) $S_n < T_n < C_n$ (D) $S_n > C_n > T_n$

Q.5 The value of $\lim\limits_{n\to\infty}(S_n + C_n + T_n)$ is equal to

(A) $\frac{1}{1-x}$ (B) $\frac{x}{1-x}$ (C) $\frac{1}{1-x}$ (D) $\frac{x}{1+x}$

Q.6 The value of 'x' for which $S_n = C_n + T_n$, is

(A) $\sin\frac{\pi}{5}$ (B) $2\sin\frac{\pi}{5}$ (C) $2\sin\frac{\pi}{10}$ (D) $\sin\frac{\pi}{10}$

Answer key

Q.1 (i) $-\frac{\pi}{4}$; (ii) $20-6\pi$; (iii) $-\frac{3\pi}{7}$; (iv) $\frac{11\pi}{18}$

Q.2 A Q.3 D Q.4 D Q.5 B Q.6 C

Property-2 :

(1) $\operatorname{cosec}^{-1}x = \sin^{-1}\frac{1}{x}$; $|x| \geq 1$

(2) $\sin^{-1}x = \operatorname{cosec}^{-1}\frac{1}{x}$, $|x| \leq 1$, $x \neq 0$

(3) $\sec^{-1}x = \cos^{-1}\frac{1}{x}$; $|x| \geq 1$

(4) $\cos^{-1}x = \sec^{-1}\frac{1}{x}$, $|x| \leq 1$, $x \neq 0$

(5) $\cot^{-1}x = \tan^{-1}\frac{1}{x}$; $x > 0$

$= \pi + \tan^{-1}\frac{1}{x}$; $x < 0$

Note : (i) $\operatorname{cosec}^{-1}x$ and $\sin^{-1}\frac{1}{x}$ are identical function.

(ii) $\sin^{-1}x$ and $\operatorname{cosec}^{-1}\frac{1}{x}$ are not identical because domain of $\sin^{-1}x$ and $\operatorname{cosec}^{-1}\frac{1}{x}$ is not equal.

(iii) $\sec^{-1}x$ and $\cos^{-1}\frac{1}{x}$ are identical function.

(iv) $\cos^{-1}x$ and $\sec^{-1}\frac{1}{x}$ are not identical because domain of $\cos^{-1}x$ and $\sec^{-1}\frac{1}{x}$ is not equal.

Illustration :

Are $tan(cot^{-1}x)$ and $cot(tan^{-1}x)$ are identical ?

Sol. *[True], as both functions have same graph.*

Illustration :

Find the value(s) of x satisfying the equation

$$cot^{-1}\frac{x^2-1}{2x}+tan^{-1}\frac{2x}{x^2-1}=\frac{2x}{3}$$

Sol. *Case-(i)* $\frac{x^2-1}{2x}>0$

$$cot^{-1}\frac{x^2-1}{2x} + tan^{-1}\frac{2x}{x^2-1} = \frac{2\pi}{3} \qquad (\because cot^{-1}x = tan^{-1}\frac{1}{x};\ x>0)$$

$$\Rightarrow tan^{-1}\frac{2x}{x^2-1} + tan^{-1}\frac{2x}{x^2-1} = \frac{2\pi}{3} \quad \Rightarrow \quad tan^{-1}\frac{2x}{x^2-1} = \frac{\pi}{3}$$

$$\Rightarrow \frac{2x}{x^2-1} = \sqrt{3} \quad \Rightarrow \sqrt{3}\,x^2-2x-\sqrt{3} = 0 \Rightarrow x = \frac{2\pm 4}{2\sqrt{3}} \quad \Rightarrow \quad x = \frac{-1}{\sqrt{3}},\ \sqrt{3}$$

Case-(ii)

$$\frac{x^2-1}{2x}<0$$

$$cot^{-1}\frac{x^2-1}{2x} + tan^{-1}\frac{2x}{x^2-1} = \frac{2\pi}{3}$$

$$\Rightarrow \quad \pi + tan^{-1}\frac{2x}{x^2-1} + tan^{-1}\frac{2x}{x^2-1} = \frac{2\pi}{3} \qquad (\because cot^{-1}x = \pi + tan^{-1}\frac{1}{x};\ x<0)$$

$$\Rightarrow \quad tan^{-1}\frac{2x}{x^2-1} = \frac{-\pi}{6} \quad \Rightarrow \quad \frac{2x}{x^2-1} = \frac{-1}{\sqrt{3}} \quad \Rightarrow \quad x^2+2\sqrt{3}\,x-1=0$$

$$\Rightarrow \quad x = -\sqrt{3}\ \pm 2 \quad \Rightarrow \quad x = -(2+\sqrt{3}),\ 2-\sqrt{3}$$

From case (i) and (ii)

$$\Rightarrow \quad x = \sqrt{3}\ ,\ -\frac{1}{\sqrt{3}}\ ,\ -(2+\sqrt{3}),\ (2-\sqrt{3}) \quad \textit{Ans.}$$

Property-3 :

(i) $\sin^{-1}(-x) = -\sin^{-1}x \quad , \quad -1 \le x \le 1$

(ii) $\tan^{-1}(-x) = -\tan^{-1}x \quad , \quad x \in R$

(iii) $\cos^{-1}(-x) = \pi - \cos^{-1}x \quad , \quad -1 \le x \le 1$

(iv) $\cot^{-1}(-x) = \pi - \cot^{-1}x \quad , \quad x \in R$

(v) $\operatorname{cosec}^{-1}(-x) = -\operatorname{cosec}^{-1}x \quad , \quad |x| \ge 1$

(vi) $\sec^{-1}(-x) = \pi - \sec^{-1}x \quad , \quad |x| \ge 1$

Property-4 :

(i) $\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}, \quad -1 \le x \le 1$ **(ii)** $\tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}, \quad x \in R$

(iii) $\operatorname{cosec}^{-1}x + \sec^{-1}x = \frac{\pi}{2}, \quad |x| \ge 1$

Illustration :

Find the value of x if

(a) $4\sin^{-1}x + \cos^{-1}x = \frac{3\pi}{4}$; *(b)* $5\tan^{-1}x + 3\cot^{-1}x = \frac{7\pi}{4}$

Sol.

(a) $4\sin^{-1}x + \frac{\pi}{2} - \sin^{-1}x = \frac{3\pi}{4}$

$\Rightarrow 3\sin^{-1}x = \frac{\pi}{4} \quad \Rightarrow \quad \sin^{-1}x = \frac{\pi}{12} \quad \Rightarrow \quad x = \sin\frac{\pi}{12} \quad \Rightarrow \quad x = \frac{\sqrt{3}-1}{2\sqrt{2}}$ ***Ans.***

(b) $5\tan^{-1}x + 3\left(\frac{\pi}{2} - \tan^{-1}x\right) = \frac{7\pi}{4}$

$2\tan^{-1}x = \frac{7\pi}{4} - \frac{3\pi}{2} \quad \Rightarrow \quad 2\tan^{-1}x = \frac{\pi}{4} \quad \Rightarrow \quad \tan^{-1}x = \frac{\pi}{8}$

$\Rightarrow x = \tan\frac{\pi}{8} \quad \Rightarrow \quad x = \sqrt{2} - 1$ ***Ans.***

Illustration :

Find the maximum and minimum values of $(\sin^{-1}x)^3 + (\cos^{-1}x)^3$

Sol. $(\sin^{-1}x)^3 + (\cos^{-1}x)^3 = (\sin^{-1}x + \cos^{-1}x)\,((\sin^{-1}x)^2 + (\cos^{-1}x)^2 - \sin^{-1}x \cdot \cos^{-1}x)$

$= \frac{\pi}{2}((\sin^{-1}x) + (\cos^{-1}x))^2 - 3\sin^{-1}x \cdot \cos^{-1}x)$

$= \frac{\pi}{2}\left[\left(\frac{\pi}{2}\right)^2 - 3\sin^{-1}x\left(\frac{\pi}{2} - \sin^{-1}x\right)\right] = \frac{\pi}{2}\left[\frac{\pi^2}{4} - \frac{3\pi}{2}\sin^{-1}x + 3(\sin^{-1}x)^2\right]$

$$= \frac{3\pi}{2}\left[(\sin^{-1}x)^2 - \frac{\pi}{2}\sin^{-1}x + \frac{\pi^2}{12}\right] \quad = \frac{3\pi}{2}\left[\left(\sin^{-1}x - \frac{\pi}{4}\right)^2 + \frac{\pi^2}{48}\right]$$

Maximum value occurs when $\sin^{-1}x = -\frac{\pi}{2}$

$$\Rightarrow \text{Maximum value} = \frac{3\pi}{2}\left[\left(\frac{-\pi}{2} - \frac{\pi}{4}\right)^2 + \frac{\pi^2}{48}\right] = \frac{3\pi}{2} \cdot \frac{7\pi^2}{12} = \frac{7\pi^3}{8}$$

Minimum value occurs when $\sin^{-1}x = \frac{\pi}{4}$

$$\Rightarrow \text{Minimum value} = \frac{3\pi}{2} \cdot \left[\frac{\pi^2}{48}\right] = \frac{\pi^3}{32}$$

Illustration :

Find the range of $f(x) = \sin^{-1}x + \cos^{-1}x + \tan^{-1}x$.

Sol. $f(x) = \sin^{-1}x + \cos^{-1}x + \tan^{-1}x$

domain : $x \in [-1, 1]$

$$\Rightarrow f(x) = \frac{\pi}{2} + [\tan^{-1} -1, \tan^{-1} 1] = \frac{\pi}{2} + \left[-\frac{\pi}{4}, \frac{\pi}{4}\right] = \left[\frac{\pi}{4}, \frac{3\pi}{4}\right]$$

Property-5 :

(1) $$\tan^{-1}x + \tan^{-1}y = \begin{cases} \tan^{-1}\left(\dfrac{x+y}{1-xy}\right), & \text{if } xy < 1 \\ \pi + \tan^{-1}\left(\dfrac{x+y}{1-xy}\right), & \text{if } x > 0, y > 0 \text{ and } xy > 1 \\ -\pi + \tan^{-1}\left(\dfrac{x+y}{1-xy}\right), & \text{if } x < 0, y < 0 \text{ and } xy > 1 \end{cases}$$

Proof :

Let $\tan^{-1}x = A$ and $\tan^{-1}y = B$, where $A, B \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

Now, $\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B} = \dfrac{x+y}{1-xy}$

$$\Rightarrow \tan^{-1}\left(\frac{x+y}{1-xy}\right) = \tan^{-1}\tan(A+B)$$

$= \tan^{-1}\tan\alpha$, where $\alpha \in (-\pi, \pi)$

$$\tan^{-1}\left(\frac{x+y}{1-xy}\right) = \tan^{-1}(\tan\alpha)$$

$$= \begin{cases} \alpha+\pi, & -\pi<\alpha<-\dfrac{\pi}{2} \\ \alpha, & -\dfrac{\pi}{2}\le\alpha\le\dfrac{\pi}{2} \\ \alpha-\pi, & \dfrac{\pi}{2}<\alpha<\pi \end{cases} = \begin{cases} \tan^{-1}x+\tan^{-1}y+\pi, & -\pi<\tan^{-1}x+\tan^{-1}y<-\dfrac{\pi}{2} \\ \tan^{-1}x+\tan^{-1}y, & -\dfrac{\pi}{2}\le\tan^{-1}x+\tan^{-1}y\le\dfrac{\pi}{2} \\ \tan^{-1}x+\tan^{-1}y-\pi, & \dfrac{\pi}{2}<\tan^{-1}x+\tan^{-1}y<\pi \end{cases}$$

Case-I :

$$-\pi<\tan^{-1}x+\tan^{-1}y<-\frac{\pi}{2} \quad \Rightarrow \quad x<0, y<0$$

Also, $\tan^{-1}x < -\dfrac{\pi}{2} - \tan^{-1}y$

$$\Rightarrow \quad \tan^{-1}x < -\left(\frac{\pi}{2} - \tan^{-1}(-y)\right) \Rightarrow x < -\left(-\frac{1}{y}\right) \Rightarrow x < \frac{1}{y} \Rightarrow xy > 1$$

Case-II :

$$\frac{\pi}{2} < \tan^{-1}x + \tan^{-1}y < \pi \;\Rightarrow\; x, y > 0$$

Also, $\tan^{-1}x > \dfrac{\pi}{2} - \tan^{-1}y \quad \Rightarrow \tan^{-1}x > \tan^{-1}\dfrac{1}{y} \Rightarrow x > \dfrac{1}{y} \;\Rightarrow\; xy > 1$

Case-III :

$$-\frac{\pi}{2} \le \tan^{-1}x + \tan^{-1}y \le \pi/2 \;\Rightarrow\; xy < 1$$

(2) $x>0$ and $y>0$, $\tan^{-1}x - \tan^{-1}y = \tan^{-1}\dfrac{x-y}{1+xy}$ (with no other restriction)

(Remember)

(i) $\tan^{-1}1 + \tan^{-1}2 + \tan^{-1}3 = \pi$

(ii) $\tan^{-1}1 + \tan^{-1}\dfrac{1}{2} + \tan^{-1}\dfrac{1}{3} = \dfrac{\pi}{2}$

(iii) $\dfrac{\tan^{-1}1 + \tan^{-1}2 + \tan^{-1}3}{\cot^{-1}1 + \cot^{-1}2 + \cot^{-1}3} = 2$

Sol.(i) $\tan^{-1}1 + \tan^{-1}2 + \tan^{-1}3 = \tan^{-1}1 + \left(\pi + \tan^{-1}\dfrac{2+3}{1-2.3}\right)$

$$= \tan^{-1}1 + (\pi + \tan^{-1}(-1))$$

$$= \frac{\pi}{4} + \pi - \frac{\pi}{4} = \pi$$

(ii) $$\tan^{-1}1+\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{3} = \tan^{-1}1+\tan^{-1}\left(\frac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{2}\cdot\frac{1}{3}}\right)$$

$$= \tan^{-1}1+\tan^{-1}\left(\frac{5}{5}\right) = \tan^{-1}1+\tan^{-1}1 = \frac{\pi}{4}+\frac{\pi}{4} = \frac{\pi}{2}$$

(iii) $$\frac{\tan^{-1}1+\tan^{-1}2+\tan^{-1}3}{\cot^{-1}1+\cot^{-1}2+\cot^{-1}3} = \frac{\tan^{-1}1+\tan^{-1}2+\tan^{-1}3}{\tan^{-1}1+\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{3}} = \frac{\pi}{\left(\frac{\pi}{2}\right)} = 2$$

Illustration :

If $\tan^{-1}2+\tan^{-1}4=\cot^{-1}(\lambda)$ *then find* λ.

Sol. $$\tan^{-1}2+\tan^{-1}4=\pi+\tan^{-1}\left(\frac{2+4}{1-2.4}\right) = \pi+\tan^{-1}\left(\frac{6}{-7}\right)$$

$$= \pi-\tan^{-1}\frac{6}{7} = \pi-\cot^{-1}\frac{7}{6} = \cot^{-1}\left(-\frac{7}{6}\right) \Rightarrow \lambda=-\frac{7}{6}\ \textbf{Ans.}$$

Illustration :

If $\alpha=\tan^{-1}5-\tan^{-1}3+\tan^{-1}\frac{7}{9}$ *and* $\beta=\tan^{-1}\frac{2}{11}+\cot^{-1}\frac{24}{7}+\tan^{-1}\frac{1}{3}$, *then*

(A) $\alpha=\beta$ *(B)* $\alpha>\beta$ *(C)* $\alpha<\beta$ *(D)* $\alpha+\beta=\pi/2$

Sol. $$\alpha=\tan^{-1}5-\tan^{-1}3+\tan^{-1}\frac{7}{9}$$

$$=\tan^{-1}\left(\frac{5-3}{1+5.3}\right)+\tan^{-1}\frac{7}{9} = \tan^{-1}\frac{1}{8}+\tan^{-1}\frac{7}{9}$$

$$=\tan^{-1}\left(\frac{\frac{1}{8}+\frac{7}{9}}{1-\frac{1}{8}\cdot\frac{7}{9}}\right)=\tan^{-1}\left(\frac{65}{65}\right)=\tan^{-1}1=\frac{\pi}{4}$$

$$\beta=\tan^{-1}\frac{2}{11}+\cot^{-1}\frac{24}{7}+\tan^{-1}\frac{1}{3}$$

$$=\tan^{-1}\frac{2}{11}+\tan^{-1}\frac{7}{24}+\tan^{-1}\frac{1}{3} = \tan^{-1}\frac{2}{11}+\tan^{-1}\left(\frac{\frac{7}{24}+\frac{1}{3}}{1-\frac{7}{24}\cdot\frac{1}{3}}\right)$$

$$= \tan^{-1}\frac{2}{11} + \tan^{-1}\left(\frac{45}{65}\right) \quad = \tan^{-1}\frac{2}{11} + \tan^{-1}\frac{9}{13}$$

$$= \tan^{-1}\left(\frac{\frac{2}{11}+\frac{9}{13}}{1-\frac{2}{11}\cdot\frac{9}{13}}\right) = \tan^{-1}\left(\frac{125}{125}\right) = \tan^{-1}\frac{\pi}{4} \Rightarrow \quad \alpha = \beta$$

Illustration :

Find the value of $\cos^{-1}\sqrt{\frac{2}{3}} - \cos^{-1}\frac{\sqrt{6}+1}{2\sqrt{3}}$.

Sol. $\cos^{-1}\sqrt{\frac{2}{3}} - \cos^{-1}\frac{\sqrt{6}+1}{2\sqrt{3}} = \tan^{-1}\frac{1}{\sqrt{2}} - \tan^{-1}\left(\frac{\sqrt{3}-\sqrt{2}}{1+\sqrt{6}}\right) = \tan^{-1}\frac{1}{\sqrt{2}} - \tan^{-1}\left(\frac{\sqrt{3}-\sqrt{2}}{1+\sqrt{3}\cdot\sqrt{2}}\right)$

$$= \tan^{-1}\frac{1}{\sqrt{2}} - (\tan^{-1}\sqrt{3} - \tan^{-1}\sqrt{2}) = \cot^{-1}\sqrt{2} - \tan^{-1}\sqrt{3} + \tan^{-1}\sqrt{2}$$

$$= \frac{\pi}{2} - \tan^{-1}\sqrt{3} = \cot^{-1}\sqrt{3} = \frac{\pi}{6} \textbf{ Ans.}$$

Illustration :

Find the value of $\tan^{-1}\left(\frac{1}{2}\tan 2A\right) + \tan^{-1}(\cot A) + \tan^{-1}(\cot^3 A)$, for $0 < A < \frac{\pi}{4}$.

Sol. For $0 < A < \frac{\pi}{4}$, $\cot A > 1 \Rightarrow (\cot A)(\cot^3 A) > 1$

Then $\tan^{-1}\left(\frac{1}{2}\tan 2A\right) + \tan^{-1}(\cot A) + \tan^{-1}(\cot^3 A)$

$$= \tan^{-1}\left(\frac{\tan A}{1-\tan^2 A}\right) + \pi + \tan^{-1}\left(\frac{\cot A + \cot^3 A}{1-\cot^4 A}\right)$$

$$= \tan^{-1}\left(\frac{\tan A}{1-\tan^2 A}\right) + \pi + \tan^{-1}\left(\frac{\cot A}{1-\cot^2 A}\right)$$

$$= \tan^{-1}\left(\frac{\tan A}{1-\tan^2 A}\right) + \pi + \tan^{-1}\left(\frac{\tan A}{\tan^2 A - 1}\right) = \pi \textbf{ Ans.}$$

Property-6 :

(I) $\sin^{-1}x+\sin^{-1}y=\begin{cases}\sin^{-1}\left(x\sqrt{1-y^2}+y\sqrt{1-x^2}\right) \text{if } x\geq 0; y\geq 0 \text{ and } x^2+y^2\leq 1\\ \pi-\sin^{-1}\left(x\sqrt{1-y^2}+y\sqrt{1-x^2}\right) \text{if } x\geq 0; y\geq 0 \text{ and } x^2+y^2>1\end{cases}$

note that $x^2+y^2\leq 1 \Rightarrow 0\leq \sin^{-1}x+\sin^{-1}y\leq \frac{\pi}{2}$

Let $\sin^{-1}x=\alpha$ and $\sin^{-1}y=\beta$; $\alpha, \beta \in\left[0,\frac{\pi}{2}\right]$

now $x^2+y^2\leq 1$

$\sin^2\alpha+\sin^2\beta\leq 1 \quad\Rightarrow\quad \sin^2\alpha\leq\cos^2\beta$

$\sin^2\alpha\leq\sin^2\left(\frac{\pi}{2}-\beta\right) \quad\Rightarrow\quad \alpha\leq\frac{\pi}{2}-\beta \quad\Rightarrow\quad \alpha+\beta\leq\frac{\pi}{2}$

$0 \leq \sin^{-1}x+\sin^{-1}y\leq\frac{\pi}{2}$]

and $x^2+y^2>1\Rightarrow \frac{\pi}{2}<\sin^{-1}x+\sin^{-1}y<\pi$

This formula should normally be used in establishing the identities.

e.g. find whether $\sin^{-1}\frac{3}{5}+\sin^{-1}\frac{12}{13}$ is acute / obtuse will be unduly difficult using the above.

However if we convert it into $\tan^{-1}\frac{3}{4}+\tan^{-1}\frac{12}{5}$ it becomes simple.

(II) ||| ly we have $\sin^{-1}x-\sin^{-1}y=\sin^{-1}\left(x\sqrt{1-y^2}-y\sqrt{1-x^2}\right), x>0; y>0$

and $\cos^{-1}x\pm\cos^{-1}y=\cos^{-1}\left(xy\mp\sqrt{1-x^2}\sqrt{1-y^2}\right), x>0, y>0, x<y$

Illustration :

Solve the equation $\sin^{-1}x+\sin^{-1}2x=\frac{\pi}{3}$.

Sol. $\sin^{-1}x+\sin^{-1}2x=\frac{\pi}{3}$.

$\sin^{-1}2x=\sin^{-1}\frac{\sqrt{3}}{2}-\sin^{-1}x=\sin^{-1}\left[\frac{\sqrt{3}}{2}\sqrt{1-x^2}-x\sqrt{1-\frac{3}{4}}\right]$

$$\Rightarrow 2x = \frac{\sqrt{3}}{2}\sqrt{1-x^2} - \frac{x}{2}$$

$$\Rightarrow \left(\frac{5x}{2}\right)^2 = \frac{3}{4}(1-x^2) \Rightarrow 28x^2 = 3$$

$$\Rightarrow x = \sqrt{\frac{3}{28}} = \frac{1}{2}\sqrt{\frac{3}{7}} \text{ Ans.} \qquad \left(\because x = -\frac{1}{2}\sqrt{\frac{3}{7}} \text{ makes L.H.S. of (1) negative}\right)$$

Illustration :

If $\cos^{-1}x + \cos^{-1}y + \cos^{-1}z = \pi$ *then value of* $x^2 + y^2 + z^2 + 2xyz$ *is equal to*

(A) 1 *(B) –1* *(C) 0* *(D) 3*

Sol. $\cos^{-1}x + \cos^{-1}y + \cos^{-1}z = \pi \Rightarrow \cos^{-1}x + \cos^{-1}y = \pi - \cos^{-1}z$

$$\Rightarrow \cos^{-1}\left(xy - \sqrt{1-x^2}\sqrt{1-y^2}\right) = \cos^{-1}(-z)$$

$$\Rightarrow xy - \sqrt{1-x^2}\sqrt{1-y^2} = -z$$

$$\Rightarrow xy + z = \sqrt{1-x^2}\sqrt{1-y^2}$$

squaring both sides

$$\Rightarrow x^2y^2 + z^2 + 2xyz = 1 - x^2 - y^2 + x^2y^2$$

$$\Rightarrow x^2 + y^2 + z^2 + 2xyz = \mathbf{1}.$$

Property-7 :

$$\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \tan^{-1}\left[\frac{x+y+z-xyz}{1-(xy+yz+zx)}\right]$$

where $x > 0, y > 0, z > 0$ and $xy + yz + zx < 1$ and $xy < 1, yz < 1, zx < 1$

Solution

$$\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \tan^{-1}\left(\frac{x+y}{1-xy}\right) + \tan^{-1}z$$

$$= \tan^{-1}\left(\frac{\frac{x+y}{1-xy}+z}{1-\left(\frac{x+y}{1-xy}\right)z}\right) = \tan^{-1}\left(\frac{x+y+z-xyz}{1-xy-(x+y)z}\right) = \tan^{-1}\left(\frac{x+y+z-xyz}{1-(xy+yz+zx)}\right)$$

Practice Problem

Q.1 Find the minimum value of $(\sec^{-1} x)^2 + (\text{cosec}^{-1} x)^2$.

Q.2 If two angles of a triangle are $\tan^{-1}(2)$ and $\tan^{-1}(3)$, then find the third angle.

Q.3 Find x satisfying $\sin^{-1}\left(\frac{1}{\sqrt{5}}\right) + \cos^{-1}(x) = \frac{\pi}{4}$.

Q.4 Value of $\sin^{-1}\left(\frac{3}{\sqrt{73}}\right) + \cos^{-1}\left(\frac{11}{\sqrt{146}}\right) + \cot^{-1}\left(\sqrt{3}\right)$ is equal to

(A) π (B) $\pi/2$ (C) $5\pi/12$ (D) $\pi/3$

Q.5 $\cos^{-1} x + \cos^{-1}\left(\frac{x}{2} + \frac{1}{2}\sqrt{3-3x^2}\right)$ is equal to

(A) $\frac{\pi}{3}$ for $x \in \left[\frac{1}{2}, 1\right]$ (B) $\frac{\pi}{3}$ for $x \in \left[0, \frac{1}{2}\right]$

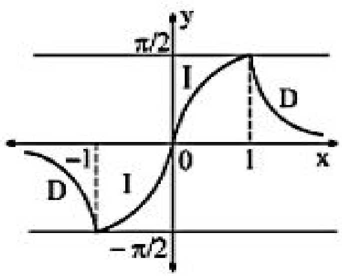
(C) $2\cos^{-1} x - \cos^{-1}\frac{1}{2}$ for $x \in \left[\frac{1}{2}, 1\right]$ (D) $2\cos^{-1} x - \cos^{-1}\frac{1}{2}$ for $x \in \left[0, \frac{1}{2}\right]$

Answer key

Q.1 $\frac{\pi^2}{8}$ Q.2 $\frac{\pi}{4}$ Q.3 $x = \frac{3}{\sqrt{10}}$ Q.4 C Q.5 A, D

SIMPLIFICATION & TRANSFORMATION OF INVERSE FUNCTIONS BY ELEMENTRY SUBSTITUTION AND THEIR GRAPHS :

(1) $$\sin^{-1}\frac{2x}{1+x^2} = \begin{cases} 2\tan^{-1} x & -1 \le x \le 1 \\ \pi - 2\tan^{-1} x & \text{if} \quad x \ge 1 \\ -\pi - 2\tan^{-1} x & x \le -1 \end{cases}$$



Proof:

Let $x = \tan\theta$, $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \Rightarrow \theta = \tan^{-1} x$

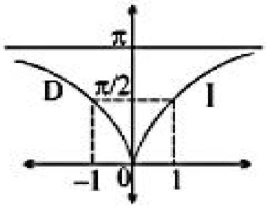
Now, $\sin^{-1}\left(\frac{2x}{1+x^2}\right)=\sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right)=\sin^{-1}(\sin 2\theta)=\sin^{-1}(\sin\alpha)$, where $\alpha\in(-\pi,\pi)$

$$\sin^{-1}\left(\frac{2x}{1+x^2}\right)=\sin^{-1}(\sin\alpha)$$

$$=\begin{cases}-\alpha-\pi, & -\pi<\alpha<-\frac{\pi}{2}\\ \alpha, & -\frac{\pi}{2}\le\alpha\le-\frac{\pi}{2}\\ -\alpha+\pi, & \frac{\pi}{2}<\alpha<\pi\end{cases}=\begin{cases}-2\tan^{-1}x-\pi, & -\pi<2\tan^{-1}x<-\frac{\pi}{2}\\ 2\tan^{-1}x, & -\frac{\pi}{2}\le 2\tan^{-1}x\le\frac{\pi}{2}\\ -2\tan^{-1}x+\pi, & \frac{\pi}{2}<2\tan^{-1}x<\pi\end{cases}$$

$$=\begin{cases}-2\tan^{-1}x-\pi, & -\frac{\pi}{2}<\tan^{-1}x<-\frac{\pi}{4}\\ 2\tan^{-1}x, & -\frac{\pi}{4}\le\tan^{-1}x\le\frac{\pi}{4}\\ -2\tan^{-1}x+\pi, & \frac{\pi}{4}<\tan^{-1}x<\frac{\pi}{2}\end{cases}=\begin{cases}-2\tan^{-1}x-\pi, & x<-1\\ 2\tan^{-1}x, & -1\le x\le 1\\ -2\tan^{-1}x+\pi, & x>1\end{cases}$$

(2) $\cos^{-1}\frac{1-x^2}{1+x^2}=\begin{cases}2\tan^{-1}x & x\ge 0\\ -2\tan^{-1}x & x<0\end{cases}$



Proof :

Let $x=\tan\theta,\ \theta\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)\Rightarrow\theta=\tan^{-1}x$

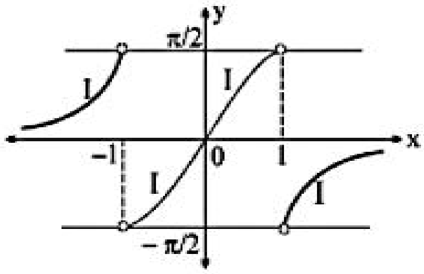
Now, $\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)=\cos^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right)=\cos^{-1}(\cos 2\theta)=\cos^{-1}(\cos\alpha)$, where $\alpha\in(-\pi,\pi)$

$$\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)=\cos^{-1}(\cos\alpha)$$

$$=\begin{cases}-\alpha, & -\pi<\alpha<0\\ \alpha, & 0\le\alpha<\pi\end{cases}=\begin{cases}-2\tan^{-1}x, & -\pi<2\tan^{-1}x<0\\ 2\tan^{-1}x, & 0\le 2\tan^{-1}x<\pi\end{cases}$$

$$=\begin{cases}-2\tan^{-1}x, & -\frac{\pi}{2}<\tan^{-1}x<0\\ 2\tan^{-1}x, & 0\le\tan^{-1}x<\frac{\pi}{2}\end{cases}=\begin{cases}-2\tan^{-1}x, & x<0\\ 2\tan^{-1}x, & x\ge 0\end{cases}$$

(3) $\tan^{-1}\dfrac{2x}{1-x^2} = \begin{cases} \pi + 2\tan^{-1}x & x < -1 \\ 2\tan^{-1}x & -1 < x < 1 \\ 2\tan^{-1}x - \pi & x > 1 \end{cases}$



Proof:

Let $x = \tan\theta,\ \theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \Rightarrow \theta = \tan^{-1}x$

Now, $\tan^{-1}\left(\dfrac{2x}{1-x^2}\right) = \tan^{-1}\left(\dfrac{2\tan\theta}{1-\tan^2\theta}\right) = \tan^{-1}(\tan 2\theta) = \tan^{-1}(\tan\alpha)$, where $\alpha \in (-\pi, \pi)$

$\tan^{-1}\left(\dfrac{2x}{1-x^2}\right) = \tan^{-1}(\tan\alpha)$

$$= \begin{cases} \alpha + \pi, & -\pi < \alpha < -\frac{\pi}{2} \\ \alpha, & -\frac{\pi}{2} \le \alpha \le \frac{\pi}{2} \\ \alpha - \pi, & \frac{\pi}{2} < \alpha < \pi \end{cases} \qquad = \begin{cases} 2\tan^{-1}x + \pi, & -\pi < 2\tan^{-1}x < -\frac{\pi}{2} \\ 2\tan^{-1}x, & -\frac{\pi}{2} \le 2\tan^{-1}x \le \frac{\pi}{2} \\ 2\tan^{-1}x - \pi, & \frac{\pi}{2} < 2\tan^{-1}x < \pi \end{cases}$$

$$= \begin{cases} 2\tan^{-1}x + \pi, & -\frac{\pi}{2} < \tan^{-1}x < -\frac{\pi}{4} \\ 2\tan^{-1}x, & -\frac{\pi}{4} \le \tan^{-1}x \le \frac{\pi}{4} \\ 2\tan^{-1}x - \pi, & \frac{\pi}{4} < \tan^{-1}x < \frac{\pi}{2} \end{cases} \qquad = \begin{cases} \pi + 2\tan^{-1}x & x < -1 \\ 2\tan^{-1}x & -1 \le x \le 1 \\ 2\tan^{-1}x - \pi & x > 1 \end{cases}$$

Highlights :-

(a) $f(x) = \sin^{-1}\dfrac{2x}{1+x^2} + 2\tan^{-1}x = \pi$ if $x \ge 1$

(b) $f(x) = \sin^{-1}\dfrac{2x}{1+x^2} + 2\tan^{-1}x = -\pi$ if $x \le -1$

Illustration :

If $f(x) = \sin^{-1}\dfrac{2x}{1+x^2} + 2\tan^{-1}x$ *then find*

(a) $f(100)$ *(b)* $\cos(f(-10))$

Sol. *We know that*

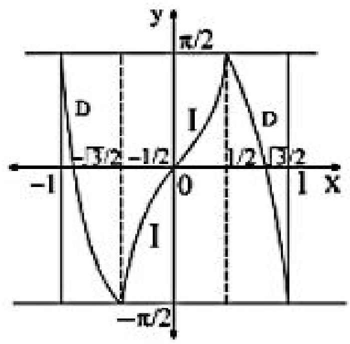
$$f(x) = \sin^{-1}\frac{2x}{1+x^2} + 2\tan^{-1}x = \pi \text{ if } x \ge 1$$

$$f(x) = \sin^{-1}\frac{2x}{1+x^2} + 2\tan^{-1}x = -\pi \text{ if } x \le -1$$

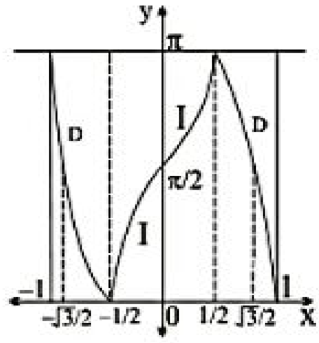
$\Rightarrow$ *(a)* $f(100) = \pi$

(b) $\cos(f(-10)) = \cos(-\pi) = \mathbf{-1}$

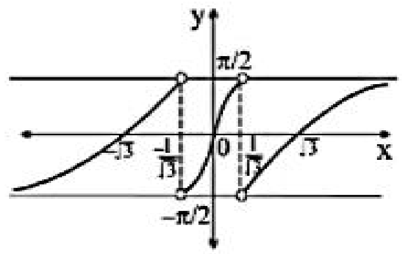
(4) $\sin^{-1}(3x-4x^3) = \begin{cases} -(\pi+3\sin^{-1}x) & \text{if } -1 \le x \le -1/2 \\ 3\sin^{-1}x & \text{if } -1/2 \le x \le 1/2 \\ \pi - 3\sin^{-1}x & \text{if } 1/2 \le x \le 1 \end{cases}$;



(5) $\cos^{-1}(4x^3-3x) = \begin{cases} 3\cos^{-1}x - 2\pi & \text{if } -1 \le x \le -1/2 \\ 2\pi - 3\cos^{-1}x & \text{if } -1/2 \le x \le 1/2 \\ 3\cos^{-1}x & \text{if } 1/2 \le x \le 1 \end{cases}$;



(6) $\tan^{-1}\dfrac{3x-x^3}{1-3x^2} = \begin{cases} 3\tan^{-1}x & \text{if } -\dfrac{1}{\sqrt{3}} < x < \dfrac{1}{\sqrt{3}} \\ -\pi + 3\tan^{-1}x & \text{if } x > \dfrac{1}{\sqrt{3}} \\ \pi + 3\tan^{-1}x & \text{if } x < -\dfrac{1}{\sqrt{3}} \end{cases}$



* **(4, 5, 6 to be proved similarly as 1, 2, 3)**

(C) IDENTITIES INVOLVING INVRSE TRIGONOMETRIC FUNCTIONS:

(I) $2\tan^{-1}\left(\tan\left(\dfrac{\pi}{4}-\alpha\right)\tan\dfrac{\beta}{2}\right) = \cos^{-1}\left(\dfrac{\sin 2\alpha + \cos\beta}{1+\sin 2\alpha\cos\beta}\right)$

Proof: Let $x = \tan\left(\dfrac{\pi}{4}-\alpha\right)\tan\dfrac{\beta}{2}$

$$x = \frac{1-\tan\alpha}{1+\tan\alpha}\tan\frac{\beta}{2} \Rightarrow x = \left(\frac{\cos\alpha-\sin\alpha}{\cos\alpha+\sin\alpha}\right)\frac{\sin\frac{\beta}{2}}{\cos\frac{\beta}{2}}$$

$$x^2 = \left(\frac{1-\sin 2\alpha}{1+\sin 2\alpha}\right)\frac{\sin^2\frac{\beta}{2}}{\cos^2\frac{\beta}{2}}$$

$$x^2 = \frac{(1-\sin 2\alpha)(1-\cos\beta)}{(1+\sin 2\alpha)(1+\cos\beta)} = \frac{1-\sin 2\alpha-\cos\beta+\sin 2\alpha\cdot\cos\beta}{1+\sin 2\alpha+\cos\beta+\sin 2\alpha\cdot\cos\beta}$$

$$\frac{x^2-1}{x^2+1}=\frac{-(\sin 2\alpha+\cos\beta)}{(1+\sin 2\alpha.\cos\beta)}$$

(By applying componendo and dividendo)

$$\frac{1-x^2}{1+x^2}=\frac{\sin 2\alpha+\cos\beta}{1+\sin 2\alpha.\cos\beta}$$

We know that

$$2\tan^{-1}x=\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$$

$$\Rightarrow \quad 2\tan^{-1}\left(\tan\left(\frac{\pi}{4}-\alpha\right)\tan\frac{\beta}{2}\right)=\cos^{-1}\left(\frac{\sin 2\alpha+\cos\beta}{1+\sin 2\alpha\cos\beta}\right)$$

(II) $\tan^{-1}x=2\tan^{-1}[\text{cosec}(\tan^{-1}x)-\tan(\cot^{-1}x)]$ $\qquad (x\neq 0)$

Sol. R.H.S. $2\tan^{-1}[\text{cosec}(\tan^{-1}x)-\tan(\cot^{-1}x)]$

$$=2\tan^{-1}\left[\text{cosec}(\tan^{-1}x)-\tan\left(\frac{\pi}{2}-\tan^{-1}x\right)\right]$$

$=2\tan^{-1}[\text{cosec}(\tan^{-1}x)-\cot\tan^{-1}x]$

let $\tan^{-1}x=\theta$

$\Rightarrow 2\tan^{-1}[\text{cosec}\,\theta-\cot\theta]$

$$=2\tan^{-1}\left[\frac{1-\cos\theta}{\sin\theta}\right]=2\tan^{-1}\left[\frac{2\sin^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}.\cos\frac{\theta}{2}}\right]=2\tan^{-1}\tan\frac{\theta}{2}=2\left(\frac{\theta}{2}\right)$$

$=\theta=\tan^{-1}x$ $\qquad$ L.H.S.

(D) EQUATIONS INVOLVING INVERSE TRIGONOMETRIC FUNCTIONS :

Illustration :

Solve the equation $\tan^{-1}\frac{x-1}{x+2}+\tan^{-1}\frac{x+1}{x+2}=\frac{\pi}{4}$.

Sol. $\tan^{-1}\frac{x-1}{x+2}+\tan^{-1}\frac{x+1}{x+2}=\frac{\pi}{4}$

$$\Rightarrow \tan^{-1}\left[\frac{\frac{x-1}{x+2}+\frac{x+1}{x+2}}{1-\left(\frac{x-1}{x+2}\right)\left(\frac{x+1}{x+2}\right)}\right]=\frac{\pi}{4} \quad \Rightarrow \left[\frac{2x(x+2)}{x^2+4+4x-x^2+1}\right]=\tan\frac{\pi}{4}$$

$$\Rightarrow \quad \frac{2x(x+2)}{4x+5}=1 \Rightarrow 2x^2+4x=4x+5 \quad \Rightarrow x=\pm\sqrt{\frac{5}{2}}$$

But for $x=-\sqrt{\frac{5}{2}}$, *L.H.S. is negative. Hence* $x=\sqrt{\frac{5}{2}}$.

Illustration :

Find the x *satisfying the equation* $2cot^{-1}2 - cos^{-1}\frac{4}{5} = cosec^{-1}x$.

Sol. $2cot^{-1}2 - cos^{-1}\frac{4}{5} = 2tan^{-1}\frac{1}{2} - cos^{-1}\frac{4}{5}$

$$= tan^{-1}\frac{2\left(\frac{1}{2}\right)}{1-\left(\frac{1}{2}\right)^2} - tan^{-1}\frac{3}{4} \quad = tan^{-1}\left(\frac{4}{3}\right) - tan^{-1}\frac{3}{4}$$

$$= tan^{-1}\left(\frac{\frac{4}{3}-\frac{3}{4}}{1+\frac{4}{3}\cdot\frac{3}{4}}\right) = tan^{-1}\left(\frac{7}{24}\right) = cosec^{-1}\left(\frac{25}{7}\right)$$

$$= cosec^{-1}x = cosec^{-1}\left(\frac{25}{7}\right)$$

$x = \frac{25}{7}$ *Ans.*

Illustration :

Find the x *satisfying the equation* $sin[2\,cos^{-1}\{cot(2tan^{-1}x)\}] = 0$

Sol. $sin[2\,cos^{-1}\{cot(2tan^{-1}x)\}] = 0$

$\Rightarrow 2\,cos^{-1}\{cot(2tan^{-1}x)\} = n\pi,\ n \in I$

$\Rightarrow cos^{-1}\{cot(2tan^{-1}x)\} = \frac{n\pi}{2} \quad \Rightarrow cot(2tan^{-1}x) = cos\frac{n\pi}{2}$

$cos\frac{n\pi}{2}$ *can take the values* $\pm 1,\ 0$ *for* $n \in I$.

Case-I : *When* $cos\frac{n\pi}{2} = \pm 1$

$\Rightarrow cot(2tan^{-1}x) = \pm 1 \quad \Rightarrow 2tan^{-1}x = n\pi \pm \frac{\pi}{4}$

$\Rightarrow tan^{-1}x = \frac{n\pi}{2} \pm \frac{\pi}{8} \quad \Rightarrow x = tan\left(\frac{n\pi}{2} \pm \frac{\pi}{8}\right) \quad \Rightarrow x = \pm\, tan\frac{\pi}{8},\ \pm cot\frac{\pi}{8}$

$x = \pm(\sqrt{2} - 1), \pm(\sqrt{2} + 1)$

$\Rightarrow x = \pm(\sqrt{2} \pm 1)$

Case-II : *When* $\cos\dfrac{n\pi}{2} = 0$

$$\Rightarrow \cot(2\tan^{-1}x) = 0 \quad \Rightarrow \quad 2\tan^{-1}x = n\pi + \frac{\pi}{2}$$

$$\Rightarrow \tan^{-1}x = \frac{n\pi}{2} + \frac{\pi}{4} \quad \Rightarrow x = \tan\left(\frac{n\pi}{2}+\frac{\pi}{4}\right) \quad \Rightarrow x = \tan\frac{\pi}{4}, -\cot\frac{\pi}{4}$$

$x = \pm 1$

Case-I and Case-II

$x = \pm 1, \pm(\sqrt{2} \pm 1)$ *Ans.*

(E) SIMULTANEOUS EQUATIONS AND INEQUATIONS INVOLVING I.T.F. :

Illustration :

Find the x *satisfying the inequality* $\cos^{-1}x > \cos^{-1}x^2$.

Sol. $\cos^{-1}x > \cos^{-1}x^2$

$\Rightarrow x^2 - x > 0 \Rightarrow x(x-1) > 0 \qquad \Rightarrow x \in (-\infty, 0) \cup (1, \infty)$

$\because \cos^{-1}x$ *defined for* $x \in [-1, 1]$

$\Rightarrow x \in [-1, 0)$ *Ans.*

Illustration :

Solve the inequality satisfying

$$\text{arc}\tan^2 x - 3\,\text{arc}\tan x + 2 > 0$$

where [ ] denotes the greatest integer function.

Sol. $(\tan^{-1}x)^2 - 3\tan^{-1}x + 2 > 0$

$\Rightarrow (\tan^{-1}x - 1)(\tan^{-1}x - 2) > 0$

$$\because \tan^{-1}x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

$\Rightarrow (\tan^{-1}x - 2)$ *is always negative*

$\Rightarrow (\tan^{-1}x - 1)(\tan^{-1}x - 2) > 0$ *holds true only when* $\tan^{-1}x - 1 < 0$

$\Rightarrow \tan^{-1}x < 1 \quad \Rightarrow x < \tan 1$

$\Rightarrow x \in (-\infty, \tan 1)$ ***Ans.***

(F) SUMMATION OF SERIES :

Illustration :

Prove that

$$\tan^{-1}\frac{2}{2+1^2+1^4}+\tan^{-1}\frac{4}{2+2^2+2^4}+\tan^{-1}\frac{6}{2+3^2+3^4}+\text{.....upto } n \text{ terms} = \tan^{-1}(n\,(n+1)+1)-\frac{\pi}{4}$$

Sol. $$T_r = \tan^{-1}\left(\frac{2r}{2+r^2+r^4}\right) = \tan^{-1}\left(\frac{2r}{1+(r^2+1)^2-r^2}\right)$$

$$= \tan^{-1}\left[\frac{(r^2+r+1)-(r^2-r+1)}{1+(r^2+r+1)(r^2-r+1)}\right]$$

$$= \tan^{-1}(r^2+r+1) - \tan^{-1}(r^2-r+1)$$

$$S_n = \sum_{r=1}^{n} T_r = (\tan^{-1}3-\tan^{-1}1)+(\tan^{-1}7-\tan^{-1}3)+(\tan^{-1}13-\tan^{-1}7)$$

$$+ \,\ldots\ldots + (\tan^{-1}(n^2+n+1)-\tan^{-1}(n^2-n+1))$$

$$S_n = \tan^{-1}(n^2+n+1)-\tan^{-1}1 = \tan^{-1}(n\,(n+1)+1)-\frac{\pi}{4}$$

Illustration :

Prove that

$$\tan^{-1}\frac{x}{1+(1\times 2)x^2}+\tan^{-1}\frac{x}{1+(2\times 3)x^2}+\,\ldots\ldots\ldots+\tan^{-1}\frac{x}{1+n(n+1)x^2} = \tan^{-1}(n+1)x-\tan^{-1}x.$$

Sol. $$T_r = \tan^{-1}\frac{x}{1+r(r+1)x^2} = \tan^{-1}\left(\frac{(r+1)x-rx}{1+r(r+1)x^2}\right) = \tan^{-1}(r+1)x-\tan^{-1}rx$$

$$S_n = \sum_{r=1}^{n} T_r = (\tan^{-1}2x-\tan^{-1}x)+(\tan^{-1}3x-\tan^{-1}2x)+\,\ldots\ldots+(\tan^{-1}(n+1)x-\tan^{-1}nx)$$

$$S_n = \tan^{-1}(n+1)x-\tan^{-1}x$$

Illustration :

The value of $cosec^{-1}\sqrt{5} + cosec^{-1}\sqrt{65} + cosec^{-1}\sqrt{325} + \ldots\ldots\ldots\ldots\infty$ is equal to

(A) 0 (B) $\frac{\pi}{4}$ (C) $\frac{\pi}{2}$ (D) π

Sol. $cosec^{-1}\sqrt{5} + cosec^{-1}\sqrt{65} + cosec^{-1}\sqrt{325} + \ldots\ldots\ldots\ldots\infty.$

$$= tan^{-1}\frac{1}{2} + tan^{-1}\frac{1}{8} + tan^{-1}\frac{1}{18} + \ldots\infty$$

$$T_r = tan^{-1}\frac{1}{2r^2} = tan^{-1}\frac{2}{4r^2} = tan^{-1}\left(\frac{(2r+1)-(2r-1)}{1+(2r-1)(2r+1)}\right)$$

$$T_r = tan^{-1}(2r + 1) - tan^{-1}(2r - 1)$$

$$S_n = \sum_{r=1}^{n} T_r = (tan^{-1}3 - tan^{-1}1) + (tan^{-1}5 - tan^{-1}3) + \ldots\ldots + (tan^{-1}(2n + 1)x - tan^{-1}(2n - 1))$$

$$= tan^{-1}(2n + 1)x - tan^{-1}1$$

when $n \to \infty$

$$S_\infty = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}\ \textbf{Ans.}$$

Illustration :

The value of $S = \sum_{n=1}^{\infty} tan^{-1}\left(\frac{4n}{n^4 - 2n^2 + 2}\right)$ is equal to

(A) $3tan^{-1}1$ (B) $cot^{-1}sgn(x^2 + 1)$ (C) $\frac{\pi}{4}$ (D) $tan^{-1}2 + tan^{-1}3$

Sol. $S = \sum_{n=1}^{\infty} tan^{-1}\left(\frac{4n}{n^4 - 2n^2 + 2}\right) = \sum_{n=1}^{\infty} tan^{-1}\left(\frac{4n}{1+(n^2-1)^2}\right)$

$$= \sum_{n=1}^{\infty} tan^{-1}\left(\frac{(n+1)^2-(n-1)^2}{1+(n+1)^2(n-1)^2}\right) = \sum_{n=1}^{\infty}\left[tan^{-1}(n+1)^2 - tan(n-1)^2\right]$$

$$= (tan^{-1}2^2 - tan^{-1}0) + (tan^{-1}3^2 - tan^{-1}1^2) + \ldots\ldots + (tan^{-1}n^2 - tan^{-1}(n-2)^2) + (tan^{-1}(n+1)^2 - tan^{-1}(n-1)^2)$$

$$\Rightarrow S = tan^{-1}(n+1)^2 + tan^{-1}n^2 - (tan^{-1}1^2 + tan^{-1}0)$$

$$= \left(\frac{\pi}{2}+\frac{\pi}{2}\right)-\left(\frac{\pi}{4}+0\right)\ (\because n \to \infty)\ = \frac{3\pi}{4}\ \textbf{Ans.}$$

Practice Problem

Q.1 If $(x-1)(x^2+1)>0$, then find the value of $\sin\left(\frac{1}{2}\tan^{-1}\frac{2x}{1-x^2}-\tan^{-1}x\right)$

Q.2 If $\cos^{-1}\frac{6x}{1+9x^2}=-\frac{\pi}{2}+2\tan^{-1}3x$, then find the value of x.

Q.3 Solve the equation $[\sin^{-1}x]>[\cos^{-1}x]$
where [ ] denotes the greatest integer function.

Q.4 Value of x satisfying the equation $2\cot^{-1}2+\cos^{-1}(3/5)=\mathrm{cosec}^{-1}x$ is
(A) ϕ (B) 1 (C) 25/7 (D) 25/24

Q.5 Sum of the series $\cot^{-1}(2a^{-1}+a)+\cot^{-1}(2a^{-1}+3a)+\cot^{-1}(2a^{-1}+6a)+\cot^{-1}(2a^{-1}+10a)+.......$ to ∞ is equal to $(a>0)$
(A) $\tan^{-1}\left(\frac{a}{2}\right)$ (B) $\cot^{-1}\left(\frac{a}{2}\right)$ (C) $a+\frac{1}{a}$ (D) 2a

Q.6 If $\cos^{-1}x-\sin^{-1}x=\cos^{-1}x\sqrt{3}$ then value(s) of x satisfying
(A) 0 (B) $\frac{1}{2}$ (C) 1 (D) $-\frac{1}{2}$

Answer key

Q.1 -1 Q.2 $x\in\left(\frac{1}{3},\infty\right)$ Q.3 $x\in(\sin 1, 1)$ Q.4 A

Q.5 B Q.6 A,B,D

Solved Examples

Q.1 Domain of definition of the function $f(x) = \sqrt{3\cos^{-1}(4x) - \pi}$ is equal to

(A) $\left[\frac{-1}{4}, \frac{1}{8}\right]$ (B) $\left[\frac{1}{8}, 1\right]$ (C) $\left[\frac{1}{8}, \frac{1}{4}\right]$ (D) $\left[-1, \frac{1}{8}\right]$

Sol. For domain of $f(x) = \sqrt{3\cos^{-1}(4x) - \pi}$, we must have

$$\cos^{-1} 4x \geq \frac{\pi}{3} \Rightarrow 4x \leq \frac{1}{2} \quad \Rightarrow x \leq \frac{1}{8} \quad ....(1)$$

Also $-1 \leq 4x \leq 1 \Rightarrow \frac{-1}{4} \leq x \leq \frac{1}{4}$ .... (2)

$\therefore$ From (1) and (2), we get $x \in \left[\frac{-1}{4}, \frac{1}{8}\right]$

Q.2 If $a\sin^{-1} x - b\cos^{-1} x = c$, then the value of $a\sin^{-1} x + b\cos^{-1} x$ (whenever exists) is equal to

(A) 0 (B) $\frac{\pi ab + c(b-a)}{a+b}$ (C) $\frac{\pi}{2}$ (D) $\frac{\pi ab + c(a-b)}{a+b}$

Sol. We have $b\sin^{-1} x + b\cos^{-1} x = \frac{b\pi}{2}$ ...... (1)

and $a\sin^{-1} x - b\cos^{-1} x = c$ ...... (2) (given)

$\therefore$ On adding (1) and (2), we get $(a+b)\sin^{-1} x = \frac{b\pi}{2} + c$

$$\Rightarrow \quad \sin^{-1} x = \frac{\frac{b\pi}{2} + c}{a+b}. \quad \text{Similarly } \cos^{-1} x = \frac{\frac{a\pi}{2} - c}{a+b}$$

Hence $(a\sin^{-1} x + b\cos^{-1} x) = \frac{\pi ab + c(a-b)}{a+b}$

Q.3 If $0 < \cos^{-1} x < 1$ and $1 + \sin(\cos^{-1} x) + \sin^2(\cos^{-1} x) + \sin^3(\cos^{-1} x) + ... \infty = 2$, then x equals

(A) $\frac{1}{2}$ (B) $\frac{1}{\sqrt{2}}$ (C) $\frac{\sqrt{3}}{2}$ (D) $\frac{1}{2\sqrt{3}}$

Sol. We have $1 + \sin(\cos^{-1} x) + \sin^2(\cos^{-1} x) + ... \infty = 2$

$$\Rightarrow \frac{1}{1 - \sin\left(\cos^{-1} x\right)} = 2 \Rightarrow \frac{1}{2} = 1 - \sin\left(\cos^{-1} x\right) \Rightarrow \sin(\cos^{-1} x) = \frac{1}{2}$$

$$\Rightarrow \cos^{-1} x = \frac{\pi}{6} \qquad \Rightarrow \quad x = \frac{\sqrt{3}}{2} \quad \textbf{Ans.}$$

Q.4 If $\tan^{-1}\left(x+\frac{3}{x}\right)-\tan^{-1}\left(x-\frac{3}{x}\right)=\tan^{-1}\frac{6}{x}$, then the value of $5x^8-4x^4+7$ equals

(A) 397 (B) 393 (C) 376 (D) 379

Sol. We have $\tan^{-1}\left(x+\frac{3}{x}\right)-\tan^{-1}\left(x-\frac{3}{x}\right)=\tan^{-1}\frac{6}{x}$

$$\Rightarrow \quad \tan^{-1}\left(\frac{\left(x+\frac{3}{x}\right)-\left(x-\frac{3}{x}\right)}{1+\left(x+\frac{3}{x}\right)\left(x-\frac{3}{x}\right)}\right)=\tan^{-1}\frac{6}{x} \quad \Rightarrow \quad x^2-\frac{9}{x^2}=0 \quad \Rightarrow \quad x^4=9$$

Hence $(5x^8-4x^4+7)=5(81)-4(9)+7=405-36+7=412-36=376$.

Q.5 The value of $\tan^{-1}\frac{4}{7}+\tan^{-1}\frac{4}{19}+\tan^{-1}\frac{4}{39}+\tan^{-1}\frac{4}{67}+........\infty$ equals

(A) $\tan^{-1}1+\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{3}$ (B) $\tan^{-1}1+\cot^{-1}3$

(C) $\cot^{-1}1+\cot^{-1}\frac{1}{2}+\cot^{-1}\frac{1}{3}$ (D) $\cot^{-1}1+\tan^{-1}3$

Sol. Let

$$S=7+19+39+67+......+T_n$$
$$S=0+\ 7+19+39+..........+T_{n-1}+T_n$$

(Subtracting) $-\quad - \quad - \quad - \quad - \quad - \quad -$

$$T_n=7+12+20+28+......+(T_n-T_{n-1})$$
$$=7+\frac{(n-1)}{2}\,[24+8(n-2)]=4n^2+3$$

$$\therefore\ T_n'=\tan^{-1}\frac{4}{4n^2+3}=\tan^{-1}\frac{1}{n^2+\frac{3}{4}}=\tan^{-1}\frac{1}{1+\left(n^2-\frac{1}{4}\right)}$$

$$=\tan^{-1}\left[\frac{\left(n+\frac{1}{2}\right)-\left(n-\frac{1}{2}\right)}{1+\left(n+\frac{1}{2}\right)\left(n-\frac{1}{2}\right)}\right]=\tan^{-1}\left(n+\frac{1}{2}\right)-\tan^{-1}\left(n-\frac{1}{2}\right)$$

Hence $S_\infty=\sum_{n=1}^{\infty}T_n'=\frac{\pi}{2}-\tan^{-1}\frac{1}{2}=\tan^{-1}1+\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{3}-\tan^{-1}\frac{1}{2}=\tan^{-1}1+\cot^{-1}3$

Q.6 Number of values of x satisfying the equation

$$\tan^{-1}\left(x-\frac{x^3}{4}+\frac{x^5}{16}-......\right)+\cot^{-1}\left(x+\frac{x^2}{2}+\frac{x^3}{4}+......\right)=\frac{\pi}{2} \quad \text{for } 0<|x|<2, \text{ is}$$

(A) 0 (B) 1 (C) 2 (D) 3

Sol. We must have $x-\frac{x^3}{4}+\frac{x^5}{16}-........=x+\frac{x^2}{2}+\frac{x^3}{4}+......$

$$\Rightarrow \frac{x}{1+\frac{x^2}{4}}=\frac{x}{1-\frac{x}{2}} \Rightarrow \frac{4x}{4+x^2}=\frac{2x}{2-x} \Rightarrow 2x^2(x+2)=0$$

$\therefore$ $x=0,-2$ (As $0<|x|<2$)

Clearly, no value of x satisfies given equation.

Q.7 Number of integral ordered pairs (x, y) satisfying the equation $\text{arc tan}\frac{1}{x}+\text{arc tan}\frac{1}{y}=\text{arc tan}\frac{1}{10}$, is

(A) 1 (B) 2 (C) 3 (D) 4

Sol. Since $\tan(\text{arc tan } a)=a\ \forall\ a\in R\ = a\ \forall\ a\in R$,

Take tan both side

$$\tan\left(\text{arc tan}\frac{1}{x}+\text{arc tan}\frac{1}{y}\right)=\tan\left(\text{arc tan}\frac{1}{10}\right)$$

$$\Rightarrow \frac{\frac{1}{x}+\frac{1}{y}}{1-\frac{1}{xy}}=\frac{1}{10}=(x-10)(y-10)=101$$

The following four ordered pair of integer numbers are solutions of this equation :

(11, 111) ; (111, 11), (9, –91) $\Rightarrow$ ordered pairs **Ans.**

Q.8 For $n\in N$, if $\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{4}+\tan^{-1}\frac{1}{5}+\tan^{-1}\frac{1}{n}=\frac{\pi}{4}$ then n is equal to

(A) 43 (B) 47 (C) 49 (D) 51

Sol. We have, $\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{4}=\tan^{-1}\left(\frac{\frac{1}{3}+\frac{1}{4}}{1-\frac{1}{12}}\right)=\tan^{-1}\left(\frac{7}{11}\right)$

Again, $\tan^{-1}\frac{7}{11}+\tan^{-1}\frac{1}{5}=\tan^{-1}\left(\frac{\frac{7}{11}+\frac{1}{5}}{1-\frac{7}{55}}\right)=\tan^{-1}\left(\frac{46}{48}\right)=\tan^{-1}\frac{23}{24}$

$\therefore$ $\tan^{-1}\frac{1}{n}=\tan^{-1}1-\tan^{-1}\frac{23}{24}=\tan^{-1}\left(\frac{1-\frac{23}{24}}{1+\frac{23}{24}}\right)=\tan^{-1}\left(\frac{1}{47}\right)\Rightarrow n=47$ **Ans.**

Q.9 If $0 < \cos^{-1}x < 1$ and $1 + \cos^{-1}x + (\cos^{-1}x)^2 + \ldots\ldots \infty = 2$ then x is equal to

(A) $\frac{\pi}{4}$ (B) $\cos\frac{1}{2}$ (C) $\cos\frac{1}{\sqrt{2}}$ (D) $\frac{\pi}{6}$

Sol. We have

$1 + \cos^{-1}x + (\cos^{-1}x)^2 + \ldots\ldots \infty = 2$, (which is an infinite geometric progression)

$\Rightarrow \frac{1}{1-\cos^{-1}x} = 2 \Rightarrow \cos^{-1}x = \frac{1}{2} \Rightarrow x = \cos\frac{1}{2}$ **Ans.**

Q.10 The domain of definition of function $f(x) = \sqrt{\cos^{-1}x - 2\sin^{-1}x}$ is equal to

(A) $\left[\frac{-1}{2}, \frac{1}{2}\right]$ (B) $\left[\frac{1}{2}, 1\right]$ (C) $\left[-1, \frac{1}{2}\right]$ (D) $\left[-1, \frac{\sqrt{3}}{2}\right]$

Sol. We have $f(x) = \sqrt{\cos^{-1}x - 2\sin^{-1}x}$

Clearly, for domain of $f(x)$, $\cos^{-1}x - 2\sin^{-1}x \geq 0$

$\Rightarrow \frac{\pi}{2} \geq 3\sin^{-1}x \Rightarrow \sin^{-1}x \leq \frac{\pi}{6} \Rightarrow x \leq \frac{1}{2}$

So, $D_f = \left[-1, \frac{1}{2}\right]$ **Ans.**

Paragraph for question nos. 11 to 13

In ΔABC, if $\angle B = \sec^{-1}\left(\frac{5}{4}\right) + \text{cosec}^{-1}\sqrt{5}$, $\angle C = \text{cosec}^{-1}\left(\frac{25}{7}\right) + \cot^{-1}\left(\frac{9}{13}\right)$ and $c = 3$.

(All symbols used have their usual meaning in a triangle.)

Q.11 tan A, tan B, tan C are in

(A) A.P. (B) G.P. (C) H.P. (D) neither A.P, G.P. nor H.P.

Q.12 The distance between orthocentre and centroid of triangle with sides a^2, $b^{\frac{4}{3}}$ and c is equal to

(A) $\frac{5}{2}$ (B) $\frac{5}{3}$ (C) $\frac{10}{3}$ (D) $\frac{7}{2}$

Q.13 Which of the following is rational with respect to ΔABC?

(A) r_1 (B) r_2 (C) r_3 (D) Δ

Sol. $\angle B = \sec^{-1}\left(\frac{5}{4}\right) + \text{cosec}^{-1}\sqrt{5} = \tan^{-1}\left(\frac{3}{4}\right) + \tan^{-1}\left(\frac{1}{2}\right) = \tan^{-1}\frac{\frac{3}{4}+\frac{1}{2}}{1-\frac{3}{4}\cdot\frac{1}{2}} = \tan^{-1}2$

$\angle C = \text{cosec}^{-1}\left(\frac{25}{7}\right) + \cot^{-1}\left(\frac{9}{13}\right) = \tan^{-1}\left(\frac{7}{24}\right) + \tan^{-1}\left(\frac{13}{9}\right) = \tan^{-1}\left(\frac{\frac{7}{24}+\frac{13}{9}}{1-\frac{7}{24}\cdot\frac{13}{9}}\right) = \tan^{-1}3$

$\therefore \angle A = \pi - \angle B - \angle C = \pi - \tan^{-1}2 - \tan^{-1}3 = \tan^{-1}1$

$\therefore \sin A = \frac{1}{\sqrt{2}}, \sin B = \frac{2}{\sqrt{5}}$ and $\sin C = \frac{3}{\sqrt{10}}$

$$\therefore \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} \Rightarrow \sqrt{2}a = \frac{\sqrt{5}b}{2} = \frac{c\sqrt{10}}{3}$$

Hence $a = \sqrt{5}$ and $b = 2\sqrt{2}$, $c = 3$

(i) $\tan A = 1$, $\tan B = 2$, $\tan C = 3$ are in A.P.

(ii) The triangle with sides a^2, $b^{\frac{4}{3}}$ and c will have side-length 5, 4 and 3 respectively

$$\therefore \text{Distance between orthocentre and centroid} = \frac{2}{3}\text{(circumradius)} = \frac{\text{hypotenuse}}{3} = \frac{5}{3} \text{ Ans.}$$

(iii) Area of ΔABC, $\Delta = \frac{1}{2} ab \sin C = \frac{1}{2}(\sqrt{5})(2\sqrt{2})\left(\frac{3}{\sqrt{10}}\right) = 3$

Also $s = \frac{1}{2}(a+b+c) = \frac{1}{2}\left(\sqrt{5}+2\sqrt{2}+3\right)$

$\therefore$ $s-a = \frac{1}{2}\left(-\sqrt{5}+2\sqrt{2}+3\right)$, $s-b = \frac{1}{2}\left(\sqrt{5}-2\sqrt{2}+3\right)$ and $s-c = \frac{1}{2}\left(\sqrt{5}+2\sqrt{2}-3\right)$

$\because$ Δ is rational

$\therefore$ Each of values $\frac{\Delta}{s-a}$, $\frac{\Delta}{s-b}$ and $\frac{\Delta}{s-c}$ i.e. r_1, r_2 and r_3 (respectively) will be irrational.

Q.14 If $\left(\sin^{-1}x\right)^2 + \left(\sin^{-1}y\right)^2 + \left(\sin^{-1}z\right)^2 = \frac{3\pi^2}{4}$, then the value of $(x-y+z)$ can be

(A) 1 (B) – 1 (C) 3 (D) – 3

Sol. As $\frac{-\pi}{2} \le \sin^{-1}x \le \frac{\pi}{2}$ $\forall -1 \le x \le 1$

$\therefore$ $\left(\sin^{-1}x\right)^2 + \left(\sin^{-1}y\right)^2 + \left(\sin^{-1}z\right)^2 = \frac{\pi^2}{4} + \frac{\pi^2}{4} + \frac{\pi^2}{4}$ is possible if $x, y, z \in \{-1, 1\}$

$\therefore$ Possible values of $x-y+z$ from the ordered triplet (x, y, z) are as follows :

(x, y, z)	(x – y + z)
(– 1, –1, –1)	– 1
(– 1, 1, 1)	– 1
(1, – 1, 1)	3
(1, 1, – 1)	– 1
(1, 1, 1)	1
(1, – 1, – 1)	– 1
(– 1, 1, – 1)	–3
(– 1, – 1, 1)	1

Hence set of values of $(x-y+z)$ is $\{\pm 1, \pm 3\}$

Q.15 Let $f(x) = \sin^{-1}|\sin x| + \cos^{-1}(\cos x)$. Which of the following statement(s) is/are **TRUE** ?

(A) $f(f(3)) = \pi$

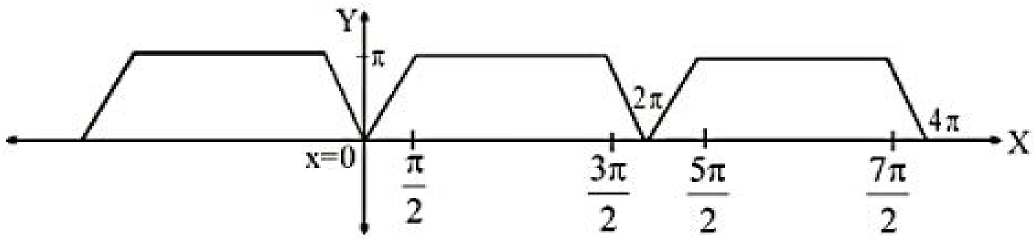
(B) $f(x)$ is periodic with fundamental period 2π.

(C) $f(x)$ is neither even nor odd.

(D) Range of $f(x)$ is $[0, 2\pi]$

Sol. $$f(x) = \begin{cases} 2x & ; 0 \le x \le \frac{\pi}{2} \\ \pi & ; \frac{\pi}{2} < x \le \frac{3\pi}{2} \\ 4\pi - 2x & ; \frac{3\pi}{2} < x \le 2\pi \end{cases}$$

Clearly f(x) is periodic function with period 2π. The graph of $f(x)$ is shown below.



Q.16 If $f(x) = \sin^{-1}x \cdot \cos^{-1}x \cdot \tan^{-1}x \cdot \cot^{-1}x \cdot \sec^{-1}x \cdot \operatorname{cosec}^{-1}x$, then which of the following statement(s) **hold(s) good** ?

(A) The graph of $y = f(x)$ does not lie above x axis.

(B) The non-negative difference between maximum and minimum value of the function $y = f(x)$ is $\frac{3\pi^6}{64}$.

(C) The function $y = f(x)$ is not injective.

(D) Number of non-negative integers in the domain of f(x) is two.

Sol. Domain of $\sin^{-1}x$ and $\cos^{-1}x$, each is $[-1, 1]$ and that of $\sec^{-1}x$ and $\operatorname{cosec}^{-1}x$, each is $(-\infty, -1] \cup [1, \infty)$

$\therefore$ Domain of $f(x)$ must be $\{-1, 1\}$ $\quad\therefore$ Range of f(x) will be $\{f(-1), f(1)\}$

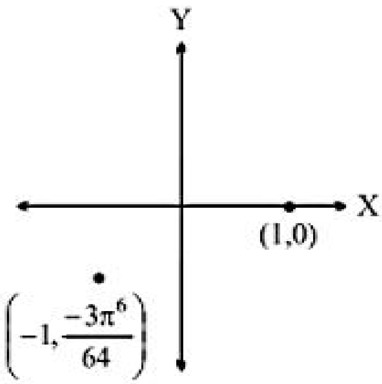
where $f(-1) = \sin^{-1}(-1) \cdot \cos^{-1}(-1) \cdot \tan^{-1}(-1) \cdot \cot^{-1}(-1) \cdot \sec^{-1}(-1) \cdot \operatorname{cosec}^{-1}(-1)$

$$= \left(\frac{-\pi}{2}\right)\cdot(\pi)\cdot\left(\frac{-\pi}{4}\right)\cdot\left(\frac{3\pi}{4}\right)\cdot(\pi)\cdot\left(\frac{-\pi}{2}\right) = \frac{-3\pi^6}{64} \text{ and } f(1) = 0 \ \{\text{as } \cos^{-1}1 = 0\}$$

(i) Thus, the graph of f(x) is a two point graph which doesn't lie above x - axis.

(ii) $f(x)_{max} = 0$ and $f(x)_{min} = \frac{-3\pi^6}{64}$

Hence $\left|f(x)_{max} - f(x)_{min}\right| = \frac{3\pi^6}{64}$



(iii) f(x) is one-one hence injective.

(iv) Domain is $\{-1, 1\}$

$\therefore$ Number of non-negative integers in the domain of f(x) is one.

Q.17 Consider $f(x) = \tan^{-1}\left(\dfrac{(\sqrt{12}-2)x^2}{x^4+2x^2+3}\right)$ and m and M are respectively minimum and maximum values of f(x) and x = a (a > 0) is the point in the domain of f(x) where f(x) attains its maximum value.

	Column I		Column II
(A)	If $\sin^{-1} 2\sqrt{x} = 3\tan^{-1}(\tan(m+M))$ then 8x equals	(P)	0
(B)	If $\cos^{-1} x + \cos^{-1} y = 3\left\{\tan^{-1}\left(\tan\dfrac{7M}{2}\right) + \tan^{-1}\left(m + \tan\dfrac{3\pi}{8}\right)\right\}$ then $(x+y)$ equals	(Q)	2
(C)	The value of $\tan\left(\sec^{-1}\left(\dfrac{2}{a^2}\right)+M\right)$ equals	(R)	– 2
(D)	If α and β are roots of the equation $x^2 - (\tan(3\sin^{-1}(\sin M)))x + a^4 = 0$, then $\alpha\beta - (\alpha+\beta)$ equals	(S)	1
		(T)	–1

Sol. We have $f(x) = \tan^{-1}\left(\dfrac{2(\sqrt{3}-1)}{x^2+\dfrac{3}{x^2}+2}\right)$

As $x^2 + \dfrac{3}{x^2} \ge 2\sqrt{3}$ (Using A.M. – G.M. inequality)

$\Rightarrow x^2 + \dfrac{3}{x^2} + 2 \ge 2 + 2\sqrt{3}$

$\therefore$ $f(x)|_{max} = \tan^{-1}\left(\dfrac{2(\sqrt{3}-1)}{2(\sqrt{3}+1)}\right) = \dfrac{\pi}{12} = M$, which occurs at $x^2 = \dfrac{3}{x^2} \Rightarrow x = 3^{\frac{1}{4}} = a$

$f(x)|_{min} = 0 = m$, which occurs at $x = 0$

(A) $\sin^{-1}(2\sqrt{x}) = 3\tan^{-1}\left(\tan\dfrac{\pi}{12}\right) = \dfrac{\pi}{4}$

$2\sqrt{x} = \dfrac{1}{\sqrt{2}} \Rightarrow 8x = 1$

(B) $\cos^{-1}x + \cos^{-1}y = 3\left[\tan^{-1}\left(\tan\dfrac{7\pi}{24}\right) + \tan^{-1}\left(0+\tan\dfrac{3\pi}{8}\right)\right] = 3\left[\dfrac{7\pi}{24}+\dfrac{3\pi}{8}\right] = 3\left(\dfrac{16\pi}{24}\right) = 2\pi$

$\therefore$ $x = y = -1 \Rightarrow x + y = -2$

(C) $\tan\left(\sec^{-1}\left(\dfrac{2}{\sqrt{3}}\right)+\dfrac{\pi}{12}\right) = \tan\left(\dfrac{\pi}{6}+\dfrac{\pi}{12}\right) = 1$

(D) $x^2 - \tan\left(3\sin^{-1}\left(\sin\frac{\pi}{12}\right)\right)x + 3 = 0$

$x^2 - x + 3 = 0 \quad \begin{matrix} \alpha \\ \beta \end{matrix}$

$\therefore \quad \alpha + \beta = 1$

$\therefore \quad \alpha\beta = 3$

Hence $\alpha\beta - (\alpha + \beta) = 2$

Q.18 Let $\alpha = 3\cos^{-1}\left(\frac{5}{\sqrt{28}}\right) + 3\tan^{-1}\left(\frac{\sqrt{3}}{2}\right)$ and $\beta = 4\sin^{-1}\left(\frac{7\sqrt{2}}{10}\right) - 4\tan^{-1}\left(\frac{3}{4}\right)$

then which of the following does not hold(s) good?

(A) $\alpha < \pi$ but $\beta > \pi$.
(B) $\alpha > \pi$ but $\beta < \pi$.
(C) Both α and β are equal.
(D) $\cos(\alpha + \beta) = 0$.

Sol. $\alpha = 3\tan^{-1}\left(\frac{\sqrt{3}}{5}\right) + 3\tan^{-1}\left(\frac{\sqrt{3}}{2}\right) = 3\left[\tan^{-1}\frac{\frac{\sqrt{3}}{5} + \frac{\sqrt{3}}{2}}{1 - \frac{3}{10}}\right] = 3\tan^{-1}\left(\frac{7\sqrt{3}}{7}\right) = \pi.$

$\beta = 4\left[\tan^{-1}7 - \tan^{-1}\frac{3}{4}\right] = 4\left[\tan^{-1}\frac{7 - \frac{3}{4}}{1 + \frac{21}{4}}\right] = 4\tan^{-1}\left(\frac{25}{25}\right) = \pi.$

Q.19 If range of the function $f(x) = \sin^{-1}x + 2\tan^{-1}x + x^2 + 4x + 1$ is $[p, q]$ then find the value of $(p + q)$.

Sol. We have $f(x) = \sin^{-1}x + 2\tan^{-1}x + x^2 + 4x + 1$

Clearly domain of $f(x)$ is $[-1, 1]$.

Also $f(x)$ is increasing function in its domain.

$\therefore \quad p = f_{min.}(x) = f(-1) = -\frac{\pi}{2} + 2\left(\frac{-\pi}{4}\right) + 1 - 4 + 1 = -\pi - 2.$

$q = f_{max.}(x) = f(1) = \frac{\pi}{2} + 2\left(\frac{\pi}{4}\right) + 1 + 4 + 1 = \pi + 6.$

$\therefore \quad$ Range of $f(x)$ is $[-\pi - 2, \pi + 6]$

Hence $(p + q) = 4$

Note : Vertex of $y = x^2 + 4x + 1$ is at $x = -2$ and hence in the domain $(x^2 + 4x + 1)$ is increasing.

Q.20 Let α, β, γ and δ be the roots of equation $x^4 - 3x^3 + 5x^2 - 7x + 9 = 0$. If the value of $|\tan(\tan^{-1}\alpha + \tan^{-1}\beta + \tan^{-1}\gamma + \tan^{-1}\delta)| = \frac{a}{b}$ where a and b are coprime to each other, then find the value of $(a^b + b^a + a^a + b^b + ab)$.

Sol. From given equation, we have

$S_1 = \Sigma\alpha = 3, \quad S_2 = \Sigma\alpha\beta = 5$

$S_3 = \Sigma\alpha\beta\gamma = 7$ and $S_4 = \alpha\beta\gamma\delta = 9$

Let $\tan^{-1}\alpha = A$, $\tan^{-1}\beta = B$, $\tan^{-1}\gamma = C$ & $\tan^{-1}\delta = D$

Now | tan ($\tan^{-1}\alpha + \tan^{-1}\beta + \tan^{-1}\gamma + \tan^{-1}\delta$) |

$$= |\tan(A+B+C+D)| = \left|\frac{S_1 - S_3}{1 - S_2 + S_4}\right| = \left|\frac{3-7}{1-5+9}\right| = \frac{4}{5} = \frac{a}{b}$$

Hence a = 4 and b = 5

So $(a^b + b^a + a^a + b^b + ab) = 4^5 + 5^4 + 4^4 + 5^5 + 4.5 = 1024 + 625 + 256 + 3125 + 20 = 5050$ **Ans.**

Q.21 How many terms of the sequence $\cot^{-1} 3, \cot^{-1} 7, \cot^{-1} 13, \cot^{-1} 21$, ... must be taken to have their sum equal to $\frac{1}{2}\cos^{-1}\left(\frac{24}{145}\right)$.

Sol. $T_1 = \tan^{-1}\frac{1}{3} = \tan^{-1}2 - \tan^{-1}1$; $T_2 = \tan^{-1}\frac{1}{7} = \tan^{-1}3 - \tan^{-1}2$; $T_3 = \tan^{-1}\frac{1}{13} = \tan^{-1}4 - \tan^{-1}3$

Clearly $T_n = \tan^{-1}(n+1) - \tan^{-1}(n)$

Hence $S_n = \tan^{-1}(n+1) - \tan^{-1}1 = \tan^{-1}\left(\frac{n+1-1}{1+(n+1)\cdot 1}\right) = \left(\tan^{-1}\frac{n}{n+2}\right) = \frac{1}{2}\cos^{-1}\left(\frac{24}{145}\right)$

$$\Rightarrow 2\left(\tan^{-1}\frac{n}{n+2}\right) = \cos^{-1}\left(\frac{24}{145}\right) \qquad \left(\text{Using } 2\tan^{-1}x = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) \forall x \ge 0\right)$$

$$\Rightarrow \cos^{-1}\left(\frac{2(n+1)}{n^2+2n+2}\right) = \cos^{-1}\left(\frac{24}{145}\right) \Rightarrow \left(\frac{2(n+1)}{n^2+2n+2}\right) = \left(\frac{24}{145}\right)$$

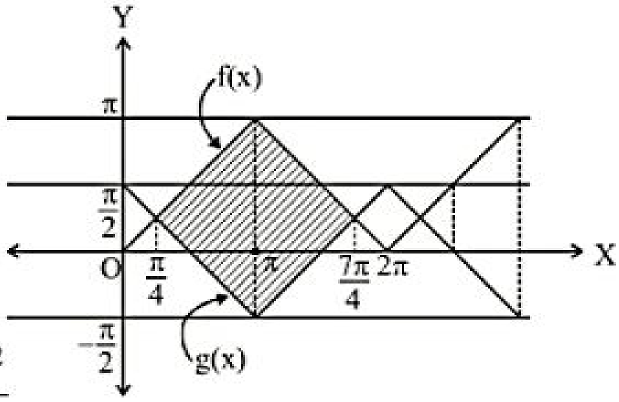
$$\Rightarrow 12(n+1)^2 - 144(n+1) - (n+1) + 12 = 0 = ((n+1)-12)(12(n+1)-1) = 0$$

$\therefore n+1 = 12, \frac{1}{12}$ $\qquad \therefore n = 11, \frac{-11}{12}$ $\qquad \because n \in N$ $\quad \therefore n \ne \frac{-11}{12}$

Hence, n = 11 **Ans.**

Q.22 If the area enclosed by the curves $f(x) = \cos^{-1}(\cos x)$ and $g(x) = \sin^{-1}(\cos x)$ in $x \in \left[\frac{9\pi}{4}, \frac{15\pi}{4}\right]$ is $\frac{a\pi^2}{b}$ (where a and b are coprime), then find (a + b).

Sol. We have $g(x) = \sin^{-1}(\cos x) = \frac{\pi}{2} - \cos^{-1}(\cos x)$



Both the curves bound the regions of same area in $\left[\frac{\pi}{4}, \frac{7\pi}{4}\right], \left[\frac{9\pi}{4}, \frac{15\pi}{4}\right]$ and so on

$\therefore$ Required area = area of shaded square $= \frac{9\pi^2}{8} = \frac{a\pi^2}{b}$

$\therefore$ a = 9 and b = 8

Hence a + b = 17 **Ans.**

DETERMINANT

HISTORICAL DEVELOPMENT :

Development of determinants took place while mathematicians were trying to solve a system of simultaneous linear equations.

e.g. $\left.\begin{array}{l} a_1x + b_1y = c_1 \\ a_2x + b_2y = c_2 \end{array}\right] \Rightarrow \quad x = \dfrac{b_2c_1 - b_1c_2}{a_1b_2 - a_2b_1}$ and $y = \dfrac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$

Mathematicians defined the symbol $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$ as determinant of order 2 and the four numbers arranged in row and column were called its elements. Its value was taken as $a_1b_2 - a_2b_1$ which is the same as denominator.

This kind of definition helped then to state the solution of the simultaneous equation as

$$x = \frac{D_1}{D} \text{ and } y = \frac{D_2}{D} \text{ where } D = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}; D_1 = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}; D_2 = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$$

Note : A determinant of order 1 is the number itself.

The symbol $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ is called the determinant of order 3. Its value can be found as

$$D = a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} \qquad \text{or}$$

$$= a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - a_2\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + a_3\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}$$

In the way we can expand a determinant in 6 ways using elements of $R_1, R_2, R_3, C_1, C_2, C_3$.

COFACTOR AND MINORS OF AN ELEMENT :

Minors :

Minors of an element is defined as the minor determinant obtained by deleting a particular row or column in which that element lies. e.g. in the determinant

$$D = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \quad \text{minor of } a_{12} \text{ denoted as } M_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} \text{ and so on}$$

Cofactor :

It has no separate identity and is related to the cofactor as

$C_{ij} = (-1)^{i+j} M_{ij}$ where 'i' denotes the row and 'j' denotes the column.

Hence the value of a determinant of order three in terms of 'Minor' and 'Cofactor' can be written as

$$D = a_{11} M_{11} - a_{12} M_{12} + a_{13} M_{13} \text{ or}$$
$$= a_{11} C_{11} + a_{12} C_{12} + a_{13} C_{13}$$

Note : Determinant of order 3 will have 9 minors and each minor will be a determinant of order 2 and a determinant of order 4 will have 16 minors and each minor will be determinant of order 3.

Illustration :

The value of $\begin{vmatrix} a+1 & a-2 \\ a+2 & a-1 \end{vmatrix}$ *is*

(A) $2a^2$ *(B)* 0 *(C)* -3 *(D)* 3

Sol. $\begin{vmatrix} a+1 & a-2 \\ a+2 & a-1 \end{vmatrix}$

$= (a+1)(a-1) - (a+2)(a-2)$

$= (a^2-1) - (a^2-4) = 3$ *Ans. [D]*

Illustration :

The value of $\begin{vmatrix} 1+\cos\theta & \sin\theta \\ \sin\theta & 1-\cos\theta \end{vmatrix}$ *is*

(A) 2 *(B)* -1 *(C)* 0 *(D)* $\cos 2\theta$

Sol. $\begin{vmatrix} 1+\cos\theta & \sin\theta \\ \sin\theta & 1-\cos\theta \end{vmatrix}$

$= (1+\cos\theta)(1-\cos\theta) - (\sin\theta)(\sin\theta)$

$= 1 - \cos^2\theta - \sin^2\theta = 0$ *Ans. [C]*

Illustration :

The value of $\begin{vmatrix} 1 & 2 & 3 \\ -4 & 3 & 6 \\ 2 & -7 & 9 \end{vmatrix}$ *is*

(A) 213 *(B)* -231 *(C)* 231 *(D)* 39

Sol. $\begin{vmatrix} 1 & 2 & 3 \\ -4 & 3 & 6 \\ 2 & -7 & 9 \end{vmatrix} = 1\begin{vmatrix} 3 & 6 \\ -7 & 9 \end{vmatrix} - 2\begin{vmatrix} -4 & 6 \\ 2 & 9 \end{vmatrix} + 3\begin{vmatrix} -4 & 3 \\ 2 & 7 \end{vmatrix}$

$= 1(3 \times 9 - 6(-7)) - 2(-4 \times 9 - 2 \times 6) + 3[(-4)(-7) - 3 \times 2]$

$= (27 + 42) - 2(-36 - 12) + 3(28 - 6) = 231$ *Ans. [C]*

PROPERTIES OF DETERMINANTS :

P-1: The value of a determinant remains unaltered, if the rows & columns are inter changed. e.g. if

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = D'$$

D & D' are transpose of each other. If $D' = -D$ then it is **Skew symmetric** determinant but $D' = D \Rightarrow 2D = 0 \Rightarrow D = 0 \Rightarrow$ Skew symmetric determinant of third order has the value zero.

Remember: Without expanding prove that the value of the determinant

$$D = \begin{vmatrix} 0 & b & -c \\ -b & 0 & a \\ c & -a & 0 \end{vmatrix} = 0$$

Note : The value of a skew symmetric determinant of odd order is zero.

P-2: If any two rows (or columns) of a determinant be interchanged, the value of determinant is changed in sign only. e.g.

Let $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ & $D' = \begin{vmatrix} a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix}$ Then $D' = -D$.

P-3: If a determinant has any two rows (or columns) identical, then its value is zero.

e.g. Let $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix}$ then it can be verified that $D = 0$.

P-4: If all the elements of any row (or column) be multiplied by the same number, then the determinant is multiplied by that number.

e.g. If $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ and $D' = \begin{vmatrix} Ka_1 & Kb_1 & Kc_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ Then $D' = KD$

P-5: If each element of any row (or column) can be expressed as a sum of two terms then the determinant can be expressed as the sum of two determinants. e.g.

$$\begin{vmatrix} a_1 + x & b_1 + y & c_1 + z \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} x & y & z \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

P-6: The value of a determinant is not altered by adding to the elements of any row (or column) the same multiples of the corresponding elements of any other row (or column).

e.g. Let $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ and $D' = \begin{vmatrix} a_1 + ma_2 & b_1 + mb_2 & c_1 + mc_2 \\ a_2 & b_2 & c_2 \\ a_3 + na_1 & b_3 + nb_1 & c_3 + nc_1 \end{vmatrix}$. Then $D' = D$.

Note : that while applying this property **atleast one row (or column)** must remain unchanged.

P-7: If by putting $x = a$ the value of a determinant vanishes then $(x - a)$ is a factor of the determinant.

P-8: In a determinant the sum of the product's of the element's of any row (column) with their corresponding cofactor's is equal to the value of determinant.

Let $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

Let A_i, B_i, C_i be the cofactor's of the element's a_i, b_i, c_i $(i = 1, 2, 3)$

Then $a_1A_1 + b_1B_1 + c_1C_1 = D$

$a_2A_2 + b_2B_2 + c_2C_2 = D$

Similarly,

In a determinant the sum of the product's of the element's of any row(column) with the cofactor's of corresponding element's of any other row (column) is zero.

i.e. $a_1A_2 + b_1B_2 + c_1C_2 = 0$ or $a_2A_1 + b_2B_1 + c_2C_1 = 0.$

Remember:

Factorisation in respect the following determinants are very useful and should be remembered.

SOME IMPORTANT DETERMINANTS TO REMEMBER :

(1) $\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} = (x-y)(y-z)(z-x)$

Proof :

Let $D = \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}$

$R_1 \to R_1 - R_2, R_2 \to R_2 - R_3$

$\Rightarrow \quad D = \begin{vmatrix} 0 & x-y & x^2-y^2 \\ 0 & y-z & y^2-z^2 \\ 1 & z & z^2 \end{vmatrix}$

$D = (x-y)(y-z)\begin{vmatrix} 0 & 1 & x+y \\ 0 & 1 & y+z \\ 1 & z & z^2 \end{vmatrix} = (x-y)(y-z)(z-x)$

$D = (x-y)(y-z)(z-x).$

Hence proved.

(2) $$\begin{vmatrix} 1 & x & x^3 \\ 1 & y & y^3 \\ 1 & z & z^3 \end{vmatrix} = (x-y)(y-z)(z-x)(x+y+z)$$

Proof :

Let $$D = \begin{vmatrix} 1 & x & x^3 \\ 1 & y & y^3 \\ 1 & z & z^3 \end{vmatrix}$$

Apply $R_1 \to R_1 - R_2$ and $R_2 \to R_2 - R_3$. Given

$$D = \begin{vmatrix} 0 & x-y & x^3-y^3 \\ 0 & y-z & y^3-z^3 \\ 1 & z & z^3 \end{vmatrix} = (x-y)(y-z)\begin{vmatrix} 0 & 1 & x^2+xy+y^2 \\ 0 & 1 & y^2+yz+z^2 \\ 1 & z & z^3 \end{vmatrix}$$

$$D = (x-y)(y-z)[y^2+yz+z^2-x^2-xy-y^2]$$
$$D = (x-y)(y-z)[y(z-x)+z^2-x^2]$$
$$= (x-y)(y-z)(z-x)(x+y+z).$$

(3) $$\begin{vmatrix} x & x^2 & yz \\ y & y^2 & zx \\ z & z^2 & xy \end{vmatrix} = (x-y)(y-z)(z-x)(xy+yz+zx)$$

Proof :

Let $$D = \begin{vmatrix} x & x^2 & yz \\ y & y^2 & zx \\ z & z^2 & xy \end{vmatrix} = \begin{vmatrix} 1 & x^2 & x^3 \\ 1 & y^2 & y^3 \\ 1 & z^2 & y^4 \end{vmatrix}$$

Apply $R_1 \to xR_1$; $R_2 \to yR_2$, $R_3 \to zR_3$ divide by xyz balancing.

$$D = \frac{1}{xyz}\begin{vmatrix} x^2 & x^3 & xyz \\ y^2 & y^3 & xyz \\ z^2 & z^3 & xyz \end{vmatrix} = \frac{xyz}{xyz}\begin{vmatrix} x^2 & x^3 & 1 \\ y^2 & y^3 & 1 \\ z^2 & z^3 & 1 \end{vmatrix} = \begin{vmatrix} 1 & x^2 & x^3 \\ 1 & y^2 & y^3 \\ 1 & z^2 & z^3 \end{vmatrix}$$

Applying $R_1 \to R_1 - R_2$ and $R_2 \to R_2 - R_3$.

$$= \begin{vmatrix} 0 & x^2-y^2 & x^3-y^3 \\ 0 & y^2-z^2 & y^2-z^3 \\ 1 & z^2 & z^3 \end{vmatrix} = (x-y)(y-z)(xy+yz+zy)$$

(4) $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = -(a^3 + b^3 + c^3 - 3abc) < 0$ if a, b, c are different and positive

Proof:

$$\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = a\,[bc - a^2] - [b^2 - ac] + c(ab - c^2)$$
$$= 3abc - (a^3 + b^3 + c^3).$$

Illustration :

Prove that $\begin{vmatrix} b+c & c & b \\ c & c+a & a \\ b & a & a+b \end{vmatrix} = 4abc$

Sol. $L.H.S. = \begin{vmatrix} 0 & c & b \\ -2a & c+a & a \\ -2a & a & a+b \end{vmatrix}$ $\quad [C_1 \to C_1 - (C_2 + C_3)]$

$$= -2a\begin{vmatrix} 0 & c & b \\ 1 & c+a & a \\ 1 & a & a+b \end{vmatrix} = -2a\begin{vmatrix} 0 & c & b \\ 0 & c & -b \\ 1 & a & a+b \end{vmatrix} \quad [R_2 \to R_2 - R_3]$$

$$= -2a\begin{vmatrix} c & b \\ c & -b \end{vmatrix} \quad [\text{expanding along } C_1]$$

$$= -(-2a)\,(-2bc) = 4abc = R.H.S.$$

Illustration :

Show that $\begin{vmatrix} b+c & a+b & a \\ c+a & b+c & b \\ a+b & c+a & c \end{vmatrix} = a^3 + b^3 + c^3 - 3abc.$

Sol. *We have*

$$L.H.S. = \begin{vmatrix} b+c & a+b & a \\ c+a & b+c & b \\ a+b & c+a & c \end{vmatrix} \quad [C_1 \to C_1 + C_3]$$

$$= (a+b+c)\begin{vmatrix} 1 & a+b & a \\ 1 & b+c & b \\ 1 & c+a & c \end{vmatrix} = (a+b+c)\begin{vmatrix} 1 & a+b & a \\ 0 & c-a & b-a \\ 0 & c-b & c-a \end{vmatrix} \quad \begin{bmatrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{bmatrix}$$

$$= (a+b+c)\begin{vmatrix} c-a & b-a \\ c-b & c-a \end{vmatrix} \quad [\text{expending along } C_1]$$

$$= (a + b + c)\,[(c - a)^2 - (c - b)\,(b - a)]$$
$$= (a + b + c)\,[(c^2 + a^2 - 2ac)^2 - (cb - ca - b^2 + ab)]$$
$$= (a + b + c)\,[a^2 + b^2 + c^2 - ab - bc - ca]$$
$$= a^3 + b^3 + c^3 - 3abc = R.H.S.$$

Illustration :

Show that $\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} = (a+b+c)^3.$

Sol. L.H.S. $= \begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$ $\qquad [R_1 \to R_1 + R_2 + R_3]$

$$= (a+b+c)\begin{vmatrix} 1 & 1 & 1 \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$$

$$= (a+b+c)\begin{vmatrix} 1 & 0 & 0 \\ 2b & -b-c-a & 0 \\ 2c & 0 & -c-a-b \end{vmatrix} \qquad \begin{bmatrix} C_2 \to C_2 - C_1 \\ C_3 \to C_3 - C_1 \end{bmatrix}$$

$$= (a+b+c)\begin{vmatrix} -b-c-a & 0 \\ 0 & -c-a-b \end{vmatrix} \qquad [\text{expanding along } C_1]$$

$$= (a+b+c)(a+b+c)^2 = (a+b+c)^3 = R.H.S.$$

Illustration :

Show that $\begin{vmatrix} a^2+1 & ab & ac \\ ab & b^2+1 & bc \\ ac & bc & c^2+1 \end{vmatrix} = 1 + a^2 + b^2 + c^2.$

Sol. L.H.S. $= \dfrac{1}{abc}\begin{vmatrix} a(a^2+1) & ab^2 & ac^2 \\ a^2b & b(b^2+1) & bc^2 \\ a^2c & b^2c & c(c^2+1) \end{vmatrix}$ $\qquad \begin{bmatrix} C_1 \to aC_1 \\ C_2 \to bC_2 \\ C_3 \to cC_3 \end{bmatrix}$

$$= \frac{abc}{abc}\begin{vmatrix} a^2+1 & b^2 & c^2 \\ a^2 & b^2+1 & c^2 \\ a^2 & b^2 & c^2+1 \end{vmatrix} \qquad \begin{bmatrix} \text{taking } a,b,c \text{ common from} \\ C_1, C_2, C_3 \text{ respectively} \end{bmatrix}$$

$$= \begin{vmatrix} 1+a^2+b^2+c^2 & b^2 & c^2 \\ 1+a^2+b^2+c^2 & b^2+1 & c^2 \\ 1+a^2+b^2+c^2 & b^2 & c^2+1 \end{vmatrix} \qquad [C_1 \to C_1 + C_2 + C_3]$$

$$=(1+a^2+b^2+c^2)\begin{vmatrix} 1 & b^2 & c^2 \\ 1 & b^2+1 & c^2 \\ 1 & b^2 & c^2+1 \end{vmatrix}$$

$$=(1+a^2+b^2+c^2)\begin{vmatrix} 1 & b^2 & c^2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} \qquad \begin{bmatrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{bmatrix}$$

$$= (1 + a^2 + b^2 + c^2) \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \qquad [\text{expanding along } C_1]$$

$$= 1 + a^2 + b^2 + c^2 = R.H.S.$$

Illustration :

If none of a, b, c is zero, show that $\Delta = \begin{vmatrix} -bc & b^2+bc & c^2+bc \\ a^2+ac & -ac & c^2+ac \\ a^2+ab & b^2+ab & -ab \end{vmatrix} = (ab + bc + ca)^3.$

Sol. *We have*

$$\Delta = \frac{1}{abc}\begin{vmatrix} -abc & ab^2+abc & ac^2+abc \\ a^2b+abc & -abc & bc^2+abc \\ a^2c+abc & b^2c+abc & -abc \end{vmatrix} \qquad \begin{bmatrix} R_1 \to aR_1 \\ R_2 \to bR_2 \\ R_3 \to aR_3 \end{bmatrix}$$

$$=\frac{abc}{abc}\begin{vmatrix} -bc & ab+bc & ac+ab \\ ab+bc & -ac & bc+ab \\ ac+bc & bc+ac & -ab \end{vmatrix} \qquad \begin{bmatrix} \text{taking } a, b, c, \text{ common from} \\ C_1, C_2, C_3 \text{ respectively} \end{bmatrix}$$

$$= (ab + bc + ca)\begin{vmatrix} 1 & 1 & 1 \\ ab+bc & -ac & bc+ab \\ ac+bc & bc+ac & -ab \end{vmatrix} \qquad [R_1 \to R_1 + R_2 + R_3]$$

$$= (ab + bc + ca)\begin{vmatrix} 1 & 0 & 0 \\ ab+bc & -(ab+bc+ca) & 0 \\ ac+bc & 0 & -(ab+bc+ca) \end{vmatrix}$$

$$[C_1 \to C_2 - C_1 \text{ and } C_3 \to C_3 - C_1]$$

$$= (ab + bc + ca)\begin{vmatrix} -(ab+bc+ca) & 0 \\ 0 & -(ab+bc+ca) \end{vmatrix} \qquad [\text{expanding along } R_1]$$

$$= (ab + bc + ca)^3$$

Illustration :

Prove that $\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix} = abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$.

Sol. *Since, the answer contains abc, therefore, taking a, b, c common from R_1, R_2, R_3 respectively, we have*

$$\Delta = abc\begin{vmatrix} \frac{1}{a}+1 & \frac{1}{a} & \frac{1}{a} \\ \frac{1}{b} & \frac{1}{b}+1 & \frac{1}{b} \\ \frac{1}{c} & \frac{1}{c} & \frac{1}{c}+1 \end{vmatrix}$$

$$= abc\begin{vmatrix} 1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c} & 1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c} & 1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \\ \frac{1}{b} & \frac{1}{b}+1 & \frac{1}{b} \\ \frac{1}{c} & \frac{1}{c} & \frac{1}{c}+1 \end{vmatrix} \quad [R_1 \to R_2 + R_2 + R_3]$$

$$= abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\begin{vmatrix} 1 & 1 & 1 \\ \frac{1}{b} & \frac{1}{b}+1 & \frac{1}{b} \\ \frac{1}{c} & \frac{1}{c} & \frac{1}{c}+1 \end{vmatrix}$$

$$= abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\begin{vmatrix} 1 & 0 & 1 \\ \frac{1}{b} & 1 & \frac{1}{b} \\ \frac{1}{c} & 0 & \frac{1}{c}+1 \end{vmatrix} \quad [C_2 \to C_2 - C_1]$$

$$= abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\begin{vmatrix} 1 & 1 \\ \frac{1}{c} & \frac{1}{c}+1 \end{vmatrix} \quad [\text{expanding along } C_2]$$

$$= abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right).$$

Illustration :

Show that

$$\begin{vmatrix} ax-by-cz & ay+bx & cx+cz \\ ay+bx & by-cz-ax & bz+cy \\ cx+az & bz+cy & cz-ax-by \end{vmatrix} = (x^2+y^2+z^2)(a^2+b^2+c^2)(ax+by+cz).$$

Sol. *We have*

$$\Delta = \frac{1}{a}\begin{vmatrix} x(a^2+b^2+c^2) & ay+bx & cx+cz \\ y(a^2+b^2+c^2) & by-cz-ax & bz+cy \\ z(a^2+b^2+c^2) & bz+cy & cz-ax-by \end{vmatrix} \qquad [C_1 \to aC_1 + bC_2 + cC_3]$$

$$= \frac{1}{a}(a^2+b^2+c^2)\begin{vmatrix} x & ay+bx & cx+cz \\ y & by-cz-ax & bz+cy \\ z & bz+cy & cz-ax-by \end{vmatrix}$$

$$= \frac{1}{ax}(a^2+b^2+c^2)\begin{vmatrix} x^2+y^2+z^2 & b(x^2+y^2+z^2) & c(x^2+y^2+z^2) \\ y & by-cz-ax & bz+cy \\ z & bz+cy & cz-ax-by \end{vmatrix}$$

$$[R_1 \to xR_1 + yR_2 + zR_3]$$

$$= \frac{1}{ax}(a^2+b^2+c^2)(x^2+y^2+z^2)\begin{vmatrix} 1 & b & c \\ y & by-cz-ax & bz+cy \\ z & bz+cy & cz-ax-by \end{vmatrix}$$

$$= \frac{1}{ax}(a^2+b^2+c^2)(x^2+y^2+z^2)\begin{vmatrix} 1 & b & c \\ 0 & -cz-ax & bz \\ 0 & cy & -ax-by \end{vmatrix}$$

$$[R_2 \to R_2 - yR_1,\ R_3 \to R_3 - zR_1]$$

$$= \frac{1}{ax}(a^2+b^2+c^2)(x^2+y^2+z^2)\begin{vmatrix} 1 & b & c \\ 0 & -cz-ax & bz \\ 0 & cy & -ax-by \end{vmatrix}$$

$$= \frac{1}{ax}(a^2+b^2+c^2)(x^2+y^2+z^2)[(cz+ax)(ax+by) - bcyz]$$

$$= \frac{1}{ax}(a^2+b^2+c^2)(x^2+y^2+z^2)[acxz + a^2x^2 + bcyz + abxy - bcyz]$$

$$= (a^2+b^2+c^2)(x^2+y^2+z^2)(ax+by+cz).$$

Illustration :

If A, B, C are the angle of a triangle and $\begin{vmatrix} 1 & 1 & 1 \\ 1+\sin A & 1+\sin B & 1+\sin C \\ \sin A+\sin^2 A & \sin B+\sin^2 B & \sin C+\sin^2 C \end{vmatrix} = 0$

prove that ΔABC *must be isosceles*

Sol. *Let*

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 1+\sin A & 1+\sin B & 1+\sin C \\ \sin A+\sin^2 A & \sin B+\sin^2 B & \sin C+\sin^2 C \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 & 0 \\ 1+\sin A & \sin B-\sin A & \sin C-\sin A \\ \sin A+\sin^2 A & (\sin B-\sin A)(\sin B+\sin A+1) & (\sin C-\sin A)(\sin C+\sin A+1) \end{vmatrix}$$

$[C_2 \to C_2 - C_1,\ C_3 \to C_3 - C_1]$

$= (\sin B - \sin A)\ (\sin C - \sin A)\ (\sin C - \sin B)$

Now, since Δ *is given to be zero, therefore we have*

$$(\sin B - \sin A)\ (\sin C - \sin A)\ (\sin C - \sin B) = 0$$

i.e.	$\sin B - \sin A = 0$	*or*	$\sin C - \sin A = 0$	*or*	$\sin C - \sin B = 0$
i.e.	$\sin B = \sin A$	*or*	$\sin C = \sin A$	*or*	$\sin C = \sin B$
i.e.	$B = A$	*or*	$C = A$	*or*	$C = B$

In all the three cases, the triangle will be isosceles.

Illustration :

Without expanding the determinant at any stage show that $\begin{vmatrix} x^2+x & x+1 & x-2 \\ 2x^2+3x-1 & 3x & 3x-3 \\ x^2+2x+3 & 2x-1 & 2x-1 \end{vmatrix} = Ax + B.$

Sol. $L.H.S. = \begin{vmatrix} x^2+x & x+1 & x-2 \\ 2x^2+3x-1 & 3x & 3x-3 \\ x^2+2x+3 & 2x-1 & 2x-1 \end{vmatrix}$ $\quad [R_1 \to R_1 + R_3 - R_2]$

$$= \begin{vmatrix} 4 & 0 & 0 \\ 2x^2+2 & 3 & 3x+3 \\ x^2+4 & 0 & 2x-1 \end{vmatrix} \qquad \begin{bmatrix} C_1 \to C_1 - C_3 \\ C_2 \to C_2 - C_3 \end{bmatrix}$$

$$= \begin{vmatrix} 4 & 0 & 0 \\ 2 & 3 & 3x-3 \\ 4 & 0 & 2x-1 \end{vmatrix} \qquad \begin{bmatrix} R_2 \to R_2 - \dfrac{x^2}{2} R_1 \\ R_3 \to R_3 - \dfrac{x^2}{4} R_1 \end{bmatrix}$$

$$= \begin{vmatrix} 4 & 0 & 0 \\ 2 & 3 & 3x \\ 4 & 0 & 2x \end{vmatrix} + \begin{vmatrix} 4 & 0 & 0 \\ 2 & 3 & -3 \\ 4 & 0 & -1 \end{vmatrix}$$

$$= x\begin{vmatrix} 4 & 0 & 0 \\ 2 & 3 & 3 \\ 4 & 0 & 2 \end{vmatrix} + \begin{vmatrix} 4 & 0 & 0 \\ 2 & 3 & -3 \\ 4 & 0 & -1 \end{vmatrix} = xA + B = R.H.S.$$

Illustration :

If $f(x) = \begin{vmatrix} x+c_1 & x+a & x+a \\ x+b & x+c_2 & x+a \\ x+b & x+b & x+c_3 \end{vmatrix}$ *then show that* $f(x)$ *is linear in x.*

Hence, deduce that $f(0) = \dfrac{bg(a)-ag(b)}{(b-a)}$ *where* $g(x) = (c_1 - x)(c_2 - x)(c_3 - x)$.

Sol. *We have*

$$f(x) = \begin{vmatrix} x+c_1 & x+a & x+a \\ x+b & x+c_2 & x+a \\ x+b & x+b & x+c_3 \end{vmatrix} \qquad \text{......(i)}$$

$$= \begin{vmatrix} x+c_1 & a-c_1 & 0 \\ x+b & c_2-b & a-c_2 \\ x+b & 0 & c_2-b \end{vmatrix} \qquad \begin{bmatrix} C_3 \to C_3 - C_2 \\ C_2 \to C_2 - C_1 \end{bmatrix}$$

$$= x\begin{vmatrix} 1 & a-c_1 & 0 \\ 1 & c_2-b & a-c_2 \\ 1 & 0 & c_3-b \end{vmatrix} + \begin{vmatrix} c_1 & a-c_1 & 0 \\ b & c_2-b & a-c_2 \\ b & 0 & c_3-b \end{vmatrix}$$

which proves that f(x) is linear.

Let $f(x) = Px + Q$

Then $f(-a) = -aP + Q$ ......(ii)

$f(-b) = -bP + Q$ ......(iii)

and $f(0) = Q = \dfrac{bf(-a)-af(-b)}{(b-a)}$ *[from results (ii) and (iii)]* ......(iv)

From equation (i), we have

$$f(-a) = \begin{vmatrix} c_1-a & 0 & 0 \\ b-a & c_2-a & 0 \\ b-a & b-a & c_3-a \end{vmatrix} = (c_1 - a)(c_2 - a)(c_3 - a)$$

Similarly, $f(-b) = (c_1 - b)(c_2 - b)(c_3 - b)$

Since, $g(x) = (c_1 - x)(c_3 - x)(c_2 - x)$, *therefore, we can see that*

$g(a) = f(-a)$ *and* $g(b) = f(-b)$

Hence, from result (iv), we have

$$f(0) = \frac{bg(a)-ag(b)}{(b-a)}$$

Illustration :

$$\begin{vmatrix} a_1-b_1+x & a_1-b_2 & a_1-b_3 \\ a_2-b_1 & a_2-b_2+x & a_2-b_3 \\ a_3-b_1 & a_3-b_2 & a_3-b_3+x \end{vmatrix} = x^3 + x^2\sum_{i=1}^{3}(a_i-b_i) + x\sum_{1\le i<j\le 3}(a_i-a_j)(b_i-b_j).$$

Sol. $$\Delta = \begin{vmatrix} a_1-b_1+x & a_1-b_2 & a_1-b_3 \\ a_2-b_1 & a_2-b_2+x & a_2-b_3 \\ a_3-b_1 & a_3-b_2 & a_3-b_3+x \end{vmatrix} =$$

$$\begin{vmatrix} a_1 & a_1-b_2 & a_1-b_3 \\ a_2 & a_2-b_2+x & a_2-b_3 \\ a_3 & a_3-b_2 & a_3-b_1+x \end{vmatrix} - \begin{vmatrix} b_1 & a_1-b_2 & a_1-b_3 \\ b_1 & a_2-b_2+x & a_2-b_3 \\ b_1 & a_3-b_2 & a_3-b_1+x \end{vmatrix} + \begin{vmatrix} x & a_1-b_2 & a_1-b_3 \\ 0 & a_2-b_2+x & a_2-b_3 \\ 0 & a_3-b_2 & a_3-b_3+x \end{vmatrix}$$

$= \Delta_1 - \Delta_2 + \Delta_3$ *(say)* ......(i)

Now, we have

$$\Delta_1 = \begin{vmatrix} a_1 & -b_2 & -b_3 \\ a_2 & -b_2+x & -b_3 \\ a_3 & -b_2 & -b_3+x \end{vmatrix} \qquad \begin{bmatrix} C_2 \to C_2 - C_1 \\ C_3 \to C_3 - C_1 \end{bmatrix}$$

$$= \begin{vmatrix} a_1 & -b_2 & -b_3 \\ a_2-a_1 & x & 0 \\ a_3-a_1 & 0 & x \end{vmatrix} \qquad \begin{bmatrix} R_3 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{bmatrix}$$

$= xb_3(a_3 - a_1) + x^2a_1 + xb_2(a_2 - a_1)$ *[expanding along C_3]*

$$\Delta_2 = b_1\begin{vmatrix} 1 & a_1-b_2 & a_1-b_3 \\ 1 & a_2-b_2+x & a_2-b_3 \\ 1 & a_3-b_2 & a_3-b_3+x \end{vmatrix}$$

$$= b_1\begin{vmatrix} 1 & a_1 & a_1 \\ 1 & a_2+x & a_2 \\ 1 & a_3 & a_3+x \end{vmatrix} \qquad \begin{bmatrix} C_2 \to C_2 + b_2C_1 \\ C_3 \to C_3 + b_3C_1 \end{bmatrix}$$

$$= b_1\begin{vmatrix} 1 & a_1 & a_1 \\ 0 & a_2-a_1+x & a_2 \\ 0 & a_3-a_1 & a_3-a_1+x \end{vmatrix} \qquad \begin{bmatrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{bmatrix}$$

$$= b_1\begin{vmatrix} a_2-a_1+x & a_2-a_1 \\ a_3-a_1 & a_2-a_1+x \end{vmatrix} \qquad [\text{expanding along } C_1]$$

$$= b_1\begin{vmatrix} a_2-a_1+x & -x \\ a_3-a_1 & x \end{vmatrix} \qquad [C_2 \to C_2 - C_1]$$

$= b_1x(a_2 - a_1 + x + a_3 - a_1)$

$= x^2b_1 + xb_1(a_2 + a_3 + x - 2a_1)$

and $\quad \Delta_3 = x[(a_2 - b_2 + x)(a_3 - b_3 + x) - (a_3 - b_2)(a_2 - b_3)]$ $\quad$ [expanding along C_1]

$$= x[x^2 + x(a_2 - b_2 + a_3 - b_3) + (a_2 - b_2)(a_3 - b_3) - (a_3 - b_2)(a_2 - b_3)]$$

$$= x^3 + x^2(a_2 - b_2 + a_3 - b_3) + x(a_2 - a_3)(b_2 - b_3)$$

Putting the values of Δ_1, Δ_2, Δ_3 in equation (i), we have

$$\Delta = [xb_3(a_3 - a_1) + x^2 a_1 + xb_2(a_2 - a_1)] - [x^2 b_1 + xb_1(a_2 + a_3 - 2a_1)]$$
$$+ [x^3 + x^2(a_2 - b_2 + a_3 - b_3) + x(a_2 - a_3)(b_2 - b_3)]$$

$$= x[(a_1 - a_2)(b_1 - b_2) + (a_1 - a_3)(b_1 - b_3) + (a_2 - a_3)(b_2 - b_3)]$$
$$+ x^2 (a_1 - b_1) + (a_2 - b_2) + (a_3 - b_3)] + x^3$$

$$= x^3 + x^2 \sum_{i=1}^{3}(a_i - b_j) + x \sum_{1 \le i \le j \le 3}(a_i - a_j)(b_i - b_j)$$

Illustration :

If a, b, c are distinct, solve the equation $\begin{vmatrix} x^2 - a^2 & x^2 - b^2 & x^2 - c^2 \\ (x-a)^3 & (x-b)^3 & (x-c)^3 \\ (x+a)^3 & (x+b)^3 & (x+c)^3 \end{vmatrix} = 0.$

Sol. $\quad \Delta = \begin{vmatrix} x^2 - a^2 & x^2 - b^2 & x^2 - c^2 \\ (x-a)^3 & (x-b)^3 & (x-c)^3 \\ (x+a)^3 & (x+b)^3 & (x+c)^3 \end{vmatrix}$ $\quad$ [$R_3 \to R_3 - R_2$]

$$= 2\begin{vmatrix} x^2 - a^2 & x^2 - b^2 & x^2 - c^2 \\ x^3 + 3xa^2 & x^3 + 3xb^2 & x^3 + 3xc^3 \\ 3x^2 a + a^3 & 3x^2 b + b^3 & 3x^2 c + c^3 \end{vmatrix} \quad \text{[taking 2 common from } R_1 \text{ and applying } R_2 \to R_2 + R_3]$$

$$= 2x\begin{vmatrix} x^2 - a^2 & x^2 - b^2 & x^2 - c^2 \\ x^2 + 3a^2 & x^2 + 3b^2 & x^2 + 3c^2 \\ 3x^2 a + a^3 & 3x^2 b + b^3 & 3x^2 c + c^3 \end{vmatrix} \quad \text{[taking } x \text{ common from } R_2]$$

$$= 2x\begin{vmatrix} x^2 - a^2 & a^2 - b^2 & a^2 - c^2 \\ x^2 + 3a^2 & 3(b^2 - a^2) & 3(c^2 - a^2) \\ 3x^2 a + a^3 & 3x^2(a-b) + b^3 - a^3 & 3x^2(c-a) + c^3 - a^3 \end{vmatrix} \quad [C_2 \to C_2 - C_1,\ C_3 \to C_3 - C_1]$$

$$= 2x(b-a)(c-a)\begin{vmatrix} x^2 - a^2 & -(a+b) & -(a+c) \\ x^2 + 3a^2 & 3(b+a) & 3(a+c) \\ 3x^2 a + a^3 & 3x^2 + b^2 + a^2 + ba & 3x^2 + c^2 + a^2 + ac \end{vmatrix}$$

$$= 2x(b-a)(c-a)\begin{vmatrix} x^2 - a^2 & -(a+b) & -(a+c) \\ 4x^2 & 0 & 0 \\ 3x^2 a + a^3 & 3x^2 + b^2 + a^2 + ba & 3x^2 + c^2 + a^2 + ac \end{vmatrix} \quad [R_2 \to R_2 + 3R_1]$$

$$= 8x^3(b-a)(c-a)\begin{vmatrix} a+b & a+c \\ 3x^2+b^2+a^2+ba & 3x^2+c^2+a^2+ac \end{vmatrix} \qquad [\text{expanding along } R_3]$$

$$= 8x^3(b-a)(c-a)\begin{vmatrix} a+b & c-b \\ 3x^2+b^2+a^2+ba & c^2-b^2+ca-ab \end{vmatrix} \qquad [C_2 \to C_2 - C_1]$$

$$= 8x^3(b-a)(c-b)(c-a)\begin{vmatrix} a+b & 1 \\ 3x^2+b^2+a^2+ba & a+b+c \end{vmatrix}$$

$= 8x^3 (b-a)(c-a)(c-b)[(a+b)(a+b+c) - 3x^2 - b^2 - a^2 - ba]$

$= 8x^3 (b-a)(c-a)(c-b)[(ab+bc+ca) - 3x^2]$

Now, since, Δ is given to be zero, therefore, we have

$8x^3 (b-a)(c-a)(c-b)[(ab+bc+ca) - 3x^2] = 0$

gives $x = 0, \pm\sqrt{\frac{1}{3}(ab+bc+ca)}$ $\qquad [\because$ *a, b, c are distinct*, $\therefore (b-a)(c-a)(c-b) \neq 0]$

Illustration :

Find the coefficient of x in the determinant $\begin{vmatrix} (1+x)^{a_1b_1} & (1+x)^{a_1b_2} & (1+x)^{a_1b_3} \\ (1+x)^{a_2b_1} & (1+x)^{a_2b_2} & (1+x)^{a_2b_3} \\ (1+x)^{a_3b_1} & (1+x)^{a_3b_2} & (1+x)^{a_3b_3} \end{vmatrix} = 0.$

Sol.

> *If f(x) be a polynomial in x, then coefficient of* x^n *in* $f(x) = \dfrac{f^n(0)}{n!}$
> *(from differential calculus)*

Let the given determinant be denoted by f (x), then

$$f'(x) = \begin{vmatrix} a_1b_1(1+x)^{a_1b_1-1} & (1+x)^{a_1b_2} & (1+x)^{a_1b_3} \\ a_2b_1(1+x)^{a_2b_1-1} & (1+x)^{a_2b_2} & (1+x)^{a_2b_3} \\ a_3b_1(1+x)^{a_3b_1-1} & (1+x)^{a_3b_2} & (1+x)^{a_3b_3} \end{vmatrix} + \begin{vmatrix} (1+x)^{a_1b_1} & a_1b_2(1+x)^{a_1b_2-1} & (1+x)^{a_1b_3} \\ (1+x)^{a_2b_1} & a_2b_2(1+x)^{a_2b_2-1} & (1+x)^{a_2b_3} \\ (1+x)^{a_3b_1} & a_3b_2(1+x)^{a_3b_2-1} & (1+x)^{a_3b_3} \end{vmatrix}$$

$$+ \begin{vmatrix} (1+x)^{a_1b_1} & (1+x)^{a_1b_2} & a_1b_3(1+x)^{a_1b_3-1} \\ (1+x)^{a_2b_1} & (1+x)^{a_2b_2} & a_2b_3(1+x)^{a_2b_3-1} \\ (1+x)^{a_3b_1} & (1+x)^{a_3b_2} & a_3b_3(1+x)^{a_3b_3-1} \end{vmatrix}$$

Thus, we have

$$f'(0) = \begin{vmatrix} a_1b_1 & 1 & 1 \\ a_2b_1 & 1 & 1 \\ a_3b_1 & 1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & a_1b_2 & 1 \\ 1 & a_2b_2 & 1 \\ 1 & a_3b_2 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 1 & a_1b_3 \\ 1 & 1 & a_2b_3 \\ 1 & 1 & a_3b_3 \end{vmatrix} = 0$$

Hence, we have

coeff. of x in $f(x) = \dfrac{f'(0)}{1!} = 0$

Practice Problem

Q.1 If x, y, z are postive and none of them is 1, then the value of the following determinant is

$$\begin{vmatrix} 1 & \log_x y & \log_x z \\ \log_y x & 1 & \log_y z \\ \log_z x & \log_z y & 1 \end{vmatrix} \text{ is}$$

(A) 1 (B) 0 (C) 2 (D) –2

Q.2 If $\Delta_k = \begin{vmatrix} 1 & n & n \\ 2k & n^2+n+1 & n^2+n \\ 2k-1 & n^2 & n^2+n+1 \end{vmatrix}$ and $\sum_{k=1}^{n} \Delta_k = 56$, then n is equal to

(A) 4 (B) 6 (C) 8 (D) 7

Q.3 If $\begin{vmatrix} a^2 & bc & ac+c^2 \\ a^2+ab & b^2 & ac \\ ab & b^2+bc & c^2 \end{vmatrix} = ma^n b^n c^n$ then

(A) $m+n=6$ (B) $m+n=4$ (C) $m-n=0$ (D) $m-n=2$

Q.4 If $\begin{vmatrix} x+y & x & x \\ 5x+4y & 4x & 2x \\ 10x+8y & 8x & 3x \end{vmatrix} = x^n$ (where $n \in N$), find the value of n.

Q.5 If $x \neq y \neq z$ and $\begin{vmatrix} x & x^2 & 1+x^3 \\ y & y^2 & 1+y^3 \\ z & z^2 & 1+z^3 \end{vmatrix} = 0$ then, find the value of xyz.

Answer key

Q.1	B	Q.2	D	Q. 3	A, D	Q.4	3	Q.5	–1

MULTIPLICATION OF TWO DETERMINANTS :

1. Row by Row multiplication :

(i^{th} row of Δ_1) × (j^{th} row Δ_2)

$$\Delta_1\Delta_2 = \begin{vmatrix} a_1\alpha_1 + b_1\beta_1 + c_1\gamma_1 & a_1\alpha_2 + b_1\beta_2 + c_1\gamma_3 & a_1\alpha_3 + b_1\beta_3 + c_1\gamma_3 \\ a_2\alpha_1 + b_2\beta_1 + c_2\gamma_1 & a_2\alpha_2 + b_2\beta_2 + c_2\gamma_2 & a_2\alpha_3 + b_2\beta_3 + c_2\gamma_3 \\ a_3\alpha_1 + b_3\beta_1 + c_3\gamma_1 & a_3\alpha_2 + b_3\beta_2 + c_3\gamma_2 & a_3\alpha_3 + b_3\beta_3 + c_3\gamma_3 \end{vmatrix}$$

2. Row by column Multiplication :

(i^{th} row of Δ_1) × (j^{th} column Δ_2)

$$\Delta_1\Delta_2 = \begin{vmatrix} a_1\alpha_1 + b_1\alpha_2 + c_1\alpha_3 & a_1\beta_1 + b_1\beta_2 + c_1\beta_3 & a_1\gamma_1 + b_1\gamma_2 + c_1\gamma_3 \\ a_2\alpha_1 + b_2\alpha_2 + c_2\alpha_3 & a_2\beta_1 + b_2\beta_2 + c_2\beta_3 & a_2\gamma_1 + b_2\gamma_2 + c_2\gamma_3 \\ a_3\alpha_1 + b_3\alpha_2 + c_3\alpha_3 & a_3\beta_1 + b_3\beta_2 + c_3\beta_3 & a_3\gamma_1 + b_3\gamma_2 + c_3\gamma_3 \end{vmatrix}$$

3. Column by Row Multiplication :

(i^{th} column of Δ_1) × (j^{th} row of Δ_2)

$$\Delta_1\Delta_2 = \begin{vmatrix} a_1\alpha_1 + a_2\beta_1 + a_3\gamma_1 & a_1\alpha_2 + a_2\beta_2 + a_3\gamma_2 & a_1\alpha_3 + a_2\beta_3 + a_3\gamma_3 \\ b_1\alpha_1 + b_2\beta_1 + b_3\gamma_1 & b_1\alpha_2 + b_2\beta_2 + b_2\gamma_2 & b_1\alpha_3 + b_2\beta_3 + b_3\gamma_3 \\ c_1\alpha_1 + c_2\beta_1 + c_3\gamma_1 & c_1\alpha_2 + c_2\beta_2 + c_3\gamma_2 & c_1\alpha_3 + c_2\beta_3 + c_3\gamma_3 \end{vmatrix}$$

4. Column by column Multiplication :

(i^{th} column of Δ_1) × (j^{th} column of Δ_2)

$$\Delta_1\Delta_2 = \begin{vmatrix} a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3 & a_1\beta_1 + a_2\beta_2 + a_3\beta_3 & a_1\gamma_1 + a_2\gamma_2 + a_3\gamma_3 \\ b_1\alpha_1 + b_2\alpha_2 + b_3\alpha_3 & b_1\beta_1 + b_2\beta_2 + b_3\beta_3 & b_1\gamma_1 + b_2\gamma_2 + b_3\gamma_3 \\ c_1\alpha_1 + c_2\alpha_2 + c_3\alpha_3 & c_1\beta_1 + c_2\beta_2 + c_3\beta_3 & c_1\gamma_1 + c_2\gamma_2 + c_3\gamma_3 \end{vmatrix}$$

But we prefer row by column multiplication.

To express a determinants as a product of two determinants :

To express a determinant as product of two determinants one requires a lot of practice and this can be done only by inspection and trial. It can be understood by the following examples.

Illustration :

Let $\Delta = \begin{vmatrix} 2bc-a^2 & c^2 & b^2 \\ c^2 & 2ca-b^2 & a^2 \\ b^2 & a^2 & 2ab-c^2 \end{vmatrix}$, then Δ can be expressed as

(A) $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}^2$ (B) $\begin{vmatrix} c & b & a \\ a & b & c \\ c & a & b \end{vmatrix}^2$ (C) $\begin{vmatrix} a & b & c \\ c & b & a \\ c & a & b \end{vmatrix}^2$ (D) None

Sol. $\Delta = \begin{vmatrix} 2bc-a^2 & c^2 & b^2 \\ c^2 & 2ca-b^2 & a^2 \\ b^2 & a^2 & 2ab-c^2 \end{vmatrix} = \begin{vmatrix} b & c & a \\ c & a & b \\ a & b & c \end{vmatrix}\begin{vmatrix} c & a & b \\ b & c & a \\ -a & -b & -c \end{vmatrix}$

$= \begin{vmatrix} b & c & a \\ c & a & b \\ a & b & c \end{vmatrix}\begin{vmatrix} b & c & a \\ c & a & b \\ a & b & c \end{vmatrix}$ $\left(\because \text{ by properties } \begin{vmatrix} c & a & b \\ b & c & a \\ -a & -b & -c \end{vmatrix} = \begin{vmatrix} b & c & a \\ c & a & b \\ a & b & c \end{vmatrix}\right)$

$= \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}^2$

SYSTEM OF LINEAR EQUATIONS :

Definition-1 :

A system of linear equations in n unknowns $x_1, x_2, x_3, \ldots\ldots x_n$ is of the form :

$$\left\{\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \ldots\ldots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots\ldots + a_{2n}x_n = b_2 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{n1}x_1 + a_{n2}x_2 + \ldots\ldots + a_{nn}x_n = b_n \end{array}\right\} \quad \ldots\ldots (A)$$

If $b_1, b_2, \ldots\ldots, b_n$ are all zero, the system is called **homogeneous** and non-homogeneous if at least one b_i is non-zero.

Definition-2 :

The solution set of the system (A) is an n tupe $(\alpha_1, \alpha_2, \ldots\ldots, \alpha_n)$ of real numbers (or complex numbers if the coefficients are complex) which satisfy each of the equations of the system.

Definition-3 :

A system of equations is called **consistent** if it has at least one solution; **inconsistent** if it does not have any solution; **determinate** if it has a unique solution; **indeterminate** if it has more than one solution.

(A) Non-homogeneous Equations in two unknowns :

Consider the system of equations

$$\begin{Bmatrix} a_1x + b_1y = c_1 \\ a_2x + b_2y = c_2 \end{Bmatrix} \qquad \ldots(i)$$

We consider the following cases.

(1) **a_i, b_i, c_i (i = 1, 2) are all zero :**
Then any pair of numbers (x, y) is a solution of the system (i) since in this case equation reduces to an identity.
So, in this case equations are always **consistent and indeterminate.**

(2) **a_i, b_i (i = 1, 2)** are all zero, but at least one **c_1 and c_2** is non-zero. Then the system has solution i.e. the equation are **inconsistent.**

(3) **Al least one of $a_i b_i$ (i = 1, 2) is non-zero**
Suppose $b_2 \neq 0$. Then system (i), is equivalent to the system.

$$\begin{Bmatrix} a_1x + b_1y = c_1 \\ \dfrac{a_2}{b_2}x + y = \dfrac{c_2}{b_2} \end{Bmatrix} \qquad \ldots(ii)$$

i.e., if the pair (x_0, y_0) is a solution of system (i) then it is also a solution of system (ii), and vice-versa.
Multiplying the second equation of system (ii) by b_1 and subtracting from first, we get

$$\left(a_1 - \frac{a_2}{b_2}b_1\right)x = c_1 - \frac{c_2}{b_2}\cdot b_1 \qquad \ldots(iii)$$

Now replacing the first equation of system (ii) by equation (iii), we obtain the system

$$\begin{Bmatrix} \left(a_1 - \dfrac{a_2}{b_2}b_1\right)x = c_1 - \dfrac{c_2}{b_2}\cdot b_1 \\ \dfrac{a_2}{b_2}x + y = \dfrac{c_2}{b_2} \end{Bmatrix} \qquad \ldots(iv).$$

(a) If $a_1 - \dfrac{a_2}{b_2}\ b_1 \neq 0$ i.e., if $a_1b_2 - a_2b_1 \neq 0$.

then we find from the first equation of system (iv) that

$$x = \frac{c_1b_2 - c_2b_1}{a_1b_2 - a_2b_1} \qquad \ldots(v)$$

Substituting this value of x into the second equation of system (iv), we obtain

$$y = \frac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$$

For convenience, we write

$$\Delta = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}, \quad \Delta_x = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}, \quad \Delta_y = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} \qquad \ldots(vi)$$

[Note that Δ_x and Δ_y are obtained by replacing the first and second columns in Δ by the column of c_1 and c_2 respectively].

Then (v) and (vi) can be written as

$$x = \frac{\Delta_x}{\Delta}, \qquad y = \frac{\Delta_y}{\Delta} \qquad \ldots(vii)$$

This is known as **Cramer's rule**. If $a_1b_2 - a_2b_1 \neq 0$ then the system (iv) or system (i) has the unique solution given by (vii). Hence in this case, the equations are **consistent and determinate.**

(b) Now let $\Delta = a_1b_2 - a_2b_1 = 0$.

Then the system (iv) has the form

$$\begin{Bmatrix} 0.x = c_1b_2 - c_2b_1 \\ \dfrac{a_2}{b_2}x + y = \dfrac{c_2}{b_2} \end{Bmatrix} \qquad \ldots(ix)$$

Obviously this system has no solution if

$$c_1b_2 - c_2b_1 = \Delta_x \neq 0$$

thus in this case, the equations are inconsistent.

But if $\Delta_x = 0$, then any pair of numbers (x, y),

where $y = \dfrac{c_2}{b_2} - \dfrac{a_2}{b_2}x$, $x \in R$, is a solution of system (9).

So in this case, the equations are consistent and indeterminate.

We summarize the whole discussion given in (A) as follows :

(i) If $\Delta \neq 0$, then the system is consistent and determinant and its solution is given by

$$x = \frac{\Delta_x}{\Delta}, \; y = \frac{\Delta_y}{\Delta} \qquad \text{(i.e., unique solution)}$$

(ii) If $\Delta = 0$, but at least one of the numbers Δ_x, Δ_y is non-zero, then the system is inconsistent i.e., it has no solution.

(iii) If $\Delta = 0$, and $\Delta_x = \Delta_y = 0$ but at least one of the numbers a_1, b_1, a_2, b_2 is non-zero, then the system has infinite number of solutions and hence it is consistent and indeterminante.

(iv) If $a_i = b_i = c_i = 0$ (i = 1, 2), then then system has infinite number of solutions and so it is consistent and indeterminate.

(B) Homogenous linear equations in two unknowns :

Consider the system of equations

$$\begin{Bmatrix} a_1x + b_1y = 0 \\ a_2x + b_2y = 0 \end{Bmatrix} \qquad \text{......(10)}$$

The system always has the solution x = 0, y = 0. If follows from the discussion in part (A) that if $\Delta \neq 0$, then the system (10) has the unique solution $x = 0, y = 0$.

And if $\Delta = 0$, and at least one of a_1, a_2, b_1, b_2 is non-zero then system (1) reduced to the single equation so that any pair of numbers (x, y) is a solution.
Thys system (10) is always consistent.

(C) Non-homogeneous linear equations in three unknowns :

Consider the system of equations

$$\begin{Bmatrix} a_1x + b_1y + c_1z = d_1 \\ a_2x + b_2y + c_2z = d_2 \\ a_3x + b_3y + c_3z = d_3 \end{Bmatrix} \qquad \text{.........(1)}$$

Let us introduce the following notations

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}, \Delta_x = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$$

$$D_y = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}, D_z = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$$

Without going into detials, we give the following rule for testing the consistency of the system (1).

(1) Let $a_i = b_i = c_i = d_i = 0, i = 1, 2, 3$
In this case any triplet (x, y, z) is a solution of the system.
Hence equations are consistent and indeterminate.

(2) If $a_i = b_i = c_i = 0, i = 1, 2, 3$ and at least one d_i $(i = 1, 2, 3)$ is non-zero, then the system has no solution, i.e., the equations in this case are inconsistent.

(3) Let $\Delta \neq 0$. In this case the system (1) has the unique solution

$$x = \frac{\Delta_x}{\Delta}, y = \frac{\Delta_y}{\Delta}, z = \frac{\Delta_z}{\Delta} \qquad \text{........(2)}$$

This is known as Crammer's rule. So equations in this case are consistent and determinate.

(4) If $\Delta = 0$, $\Delta_x \neq 0$ (or $\Delta_y \neq 0$ or $\Delta_z \neq 0$), then the system has no solution so the equations are inconsistent.

(5) If $\Delta = \Delta_x = \Delta_y = \Delta_z = 0$ and at least one of the cofactors of Δ is non-zero, then the system will have an infinite number of solutions. In this case, any one of the variables can be given arbitrary value and other variables can be expressed in terms of that variable.

In such cases, the three equations reduce to two equations

If all the cofactors $\Delta, \Delta_x, \Delta_y, \Delta_z$ are zero but elements of Δ are not all zero, then in this case the system will reduced to single equation and any two variables can be given arbitrary values. So equations are consistent and indeterminate.

(D) Homogeneous linear equations :

If in (1), we take $d_i = 0$ $(i = 1, 2, 3)$ then the system is called the homogenous system of equations. For such a system if $\Delta \neq 0$, then it has the unique solution $x = 0, y = 0, z = 0$. **(Trivial)**
So such system of equations is always consistent.

(1) Three equations in two unknowns :

Consider the equations

$$\begin{Bmatrix} a_1x + b_1y = c_1 \\ a_2x + b_2y = c_2 \\ a_3x + b_3y = c_2 \end{Bmatrix} \qquad \text{.......(3)}$$

The system (3) will be consistent if the solutions set of any satisfies the third equations, i.e., if

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0.$$

Note : The factors of the following two determinants be remembered.

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (a-b)(b-c)(c-a)$$

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} = \begin{vmatrix} 1 & a & a^3 \\ 1 & b & b^3 \\ 1 & c & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(a+b+c).$$

(2) Gist of discussion in simple language :

(i) Consistent : Solution exists whether unique infinite number of solutions.
(ii) Inconsistent : Solution does not exist.
(iii) Homogeneous Equations : constant terms zero.
(iv) Trivial solution : All variables zero i.e., $x = 0, y = 0, z = 0$.
(v) Non-trivial solution : Infinite number of solutions.

Illustration :

$a_1x + b_1y = c_1$
$a_2x + b_2y = c_2$

$$\Delta = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}, \Delta_1 \text{ or } \Delta_x = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix},$$

$$\Delta_2 \text{ or } \Delta_y = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$$

(3) **Case-I :** Intersecting lines

$2x + 3y = 10$ and $x + y = 4$

$\therefore$ $x = 2, y = 2$

$$\Delta = \begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix} = -1, \Delta \neq 0.$$

(4) **Case II :** $2x + 3y = 10$
$4x + 6y = 20$

Here $\Delta = \begin{vmatrix} 2 & 3 \\ 4 & 6 \end{vmatrix} = 0,$

but $\Delta_1 = \begin{vmatrix} 10 & 2 \\ 20 & 4 \end{vmatrix} = 0, \Delta_2 = 0$

As a matter of fact on division by 2 the second equation reduces to first. Thus we have got only one line $2x + 3y = 10$ on which lie infinite number of points. Thus there are infinite number of solutions and the system is consistent. $\left(k, \dfrac{10-3k}{2}\right)$ are infinite number of solutions by giving different values to k.

Case-III $\begin{matrix} 2x+3y=10 \\ 4x+16y=15 \end{matrix}$ or $\begin{matrix} 2x+3y=10 \\ 2x+3y=15/2 \end{matrix}$

i.e. parallel lines which we know do not intersect and hence no solution.
i.e. inconsistent. Here $\Delta = 0$ but $\Delta_1 \neq 0, \Delta_2 \neq 0$

***Summary :**

(i) $\Delta \neq 0$ Unique (Intersecting lines) Consistent
(ii) $\Delta = 0, \Delta_1 = 0, \Delta_2 = 0$ (Identical lines) Consistent, Infinite solution.
(iii) $\Delta = 0, \Delta_1 \neq 0$ (Parallel lines)
Inconsistent. No solution.
Homogeneous : $a_1x + b_1y = 0$
$a_2x + b_2y = 0$
$\Delta \neq 0$, Unique $x = 0, y = 0$, Trivial.
$\Delta = 0$, Identical line through origin, Non-trivial solution.

(5) Concurrent lines : Two variable, three equations :

$a_1x + b_1y = c$, $a_2x + b_2y = c_2$, $a_3x + b_3y = c_3$
The point of intersection of any two lines should satisfy the third.

$$\therefore \quad \Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$$

is the required condition.

Illustration :

For what value of λ the equations

$$2x + 3y = 8,\ 7x - 5y + 3 = 0 \text{ and } 4x - 6y + \lambda = 0$$

are consistent ? Also find the solution of the system of equations for the values of λ.

Sol. *Here the equations are linear. We have 3 equations in 2 unknowns.*

$$\therefore \quad \text{they are consistent if } \begin{vmatrix} 2 & 3 & -8 \\ 7 & -5 & 3 \\ 4 & -6 & \lambda \end{vmatrix} = 0$$

or $\quad 2(-5\lambda + 18) - 3(7y - 12) - 8(-42 + 20) = 0$

or $\quad -10\lambda + 36 - 21\lambda + 36 + 176 = 0$

or $\quad -31\lambda + 248 = 0\ ;\ \therefore \lambda = 8$

$\therefore$ *for $\lambda = 8$ the system has a solution which can be obtained by solving any two of the three equations.*

Solving $\quad 2x + 3y - 8 = 0$

$\quad 7x - 5y + 3 = 0$ *by Cramer's rule,*

$$\frac{x}{\begin{vmatrix} 3 & -8 \\ -5 & 3 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} 2 & -8 \\ 7 & 3 \end{vmatrix}} = \frac{1}{\begin{vmatrix} 2 & 3 \\ 7 & -5 \end{vmatrix}}$$

or $$\frac{x}{9-40} = \frac{-y}{6+56} = \frac{1}{-10-21}$$

or $$\frac{x}{-31} = \frac{-y}{62} = \frac{1}{-31}, \quad \therefore x = 1, y = 2$$

Illustration :

For what values of p and q the system of equations

$$2x + py + 6z = 8$$
$$x + 2y + qz = 5$$
$$x + y + 3z = 4$$

has (i) unique solution (ii) no solution (iii) infinite number of solutions?

Sol. *Here the system of linear equations in x, y, z are*

$$2x + py + 6z - 8 = 0$$
$$x + 2y + qz - 5 = 0$$
$$x + y + 3z - 4 = 0$$

$$\therefore \quad \Delta = \begin{vmatrix} 2 & p & 6 \\ 1 & 2 & q \\ 1 & 1 & 3 \end{vmatrix} = \begin{vmatrix} 2 & p-2 & 0 \\ 1 & 1 & q-3 \\ 1 & 0 & 0 \end{vmatrix}, \ C_2 \to C_2 - C_1, \ C_3 \to C_3 - 3 \times C_1$$

$$= \begin{vmatrix} p-2 & 0 \\ 1 & q-3 \end{vmatrix} = (p-2)(q-3)$$

$\therefore$ *If* $p \neq 2, q \neq 3$ *then* $D \neq 0$

and so the system will have unique solution, i.e., the system will be independent/solvable/consistent.

If $p = 2$ *or* $q = 3$ *then* $\Delta = 0$.

and so the system cannot have unique solution.

When $p = 2$,

$$\Delta_x = \begin{vmatrix} p & 6 & -8 \\ 2 & q & -5 \\ 1 & 3 & -4 \end{vmatrix} = \begin{vmatrix} 2 & 6 & -8 \\ 2 & q & -5 \\ 1 & 3 & -4 \end{vmatrix} = 2\begin{vmatrix} 1 & 3 & -4 \\ 2 & 1 & -5 \\ 1 & 3 & -4 \end{vmatrix} = 0 \ (\because R_1 \equiv R_3)$$

$$\Delta_y = \begin{vmatrix} 2 & 6 & -8 \\ 1 & q & -5 \\ 1 & 3 & -4 \end{vmatrix} = 2\begin{vmatrix} 1 & 3 & -4 \\ 1 & q & -5 \\ 1 & 3 & -4 \end{vmatrix} = 0 \ (\because R_1 \equiv R_3)$$

$$\Delta_z = \begin{vmatrix} 2 & p & -8 \\ 1 & 2 & -5 \\ 1 & 1 & -4 \end{vmatrix} = \begin{vmatrix} 2 & 2 & -8 \\ 1 & 2 & -5 \\ 1 & 1 & -4 \end{vmatrix} = 2\begin{vmatrix} 1 & 1 & -4 \\ 1 & 2 & -5 \\ 1 & 1 & -4 \end{vmatrix} = 0 \ (\because R_1 \equiv R_3)$$

$\therefore$ *when* $p = 2$, $\Delta = 0$, $\Delta_x = \Delta_y = \Delta_z$.

$\therefore$ *the system of equaitons will have infinite number of solutions (the system of equations will be dependent) for* $p = 2$ *and any real value of* q.

$$\text{When } q = 3, \ \Delta_x = \begin{vmatrix} p & 6 & -8 \\ 2 & q & -5 \\ 1 & 3 & -4 \end{vmatrix} = \begin{vmatrix} p & 6 & -8 \\ 2 & 3 & -5 \\ 1 & 3 & -4 \end{vmatrix} = 2\begin{vmatrix} p-2 & 0 & 0 \\ 2 & 3 & -5 \\ 1 & 3 & -4 \end{vmatrix}, \ R_1 \to R_1 - 2R_3$$

$$= (p-2)3$$

$\therefore p \neq 2$, $\Delta x \neq 0$ *and so the system of equaitons will have no solutions, i.e., the system is solvable/ inconsistent when* $q = 3$ *but* $p \neq 2$.

Thus we find that the system of equations will have

(i) *unique solution if* $p \neq 2$ *and* $q \neq 3$

(ii) *no solution if* $p \neq 2$ *and* $q = 3$

(iii) *infinite number of solutions if* $p = 2$.

Practice Problem

Q.1 If $\Sigma\cos^2\alpha_1 = \Sigma\cos^2\beta_1 = \Sigma\cos^2\gamma_1 = 1$;
$\Sigma\cos\alpha_1\cos\beta_1 = \Sigma\cos\beta_1\cos\gamma_1 = \Sigma\cos\gamma_1\cos\alpha_1 = 0$

Then the value of $\begin{vmatrix} \cos\alpha_1 & \cos\alpha_2 & \cos\alpha_3 \\ \cos\beta_1 & \cos\beta_2 & \cos\beta_3 \\ \cos\gamma_1 & \cos\gamma_2 & \cos\gamma_3 \end{vmatrix}^2$

(A) 1 (B) –1 (C) 0 (D) None

Q.2 If $u = ax + by + cz$, $v = ay + bz + cx$, $w = az + bx + cy$, then the value of $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} \times \begin{vmatrix} x & y & z \\ y & z & x \\ z & x & y \end{vmatrix}$ is

(A) $u^2 + v^2 + w^2 - 2uvw$ (B) $u^3 + v^3 + w^3 - 3uvw$
(C) 0 (D) None of these

Q.3 If $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = (lx + my + n)(l'x + m'y + n')$

and $\Delta_1 = \begin{vmatrix} l & l' & 0 \\ m & m' & 0 \\ n & n' & 0 \end{vmatrix}$ and $\Delta_2 = \begin{vmatrix} l' & l & 0 \\ m' & m & 0 \\ n' & n & 0 \end{vmatrix}$, then the product $\Delta_1\Delta_2$ is

(A) $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$ (B) $\begin{vmatrix} h & g & f \\ a & f & c \\ b & c & h \end{vmatrix} = 0$ (C) 1 (D) None

Q.4 The determinant $\Delta = \begin{vmatrix} (a-x)^2 & (b-x)^2 & (c-x)^2 \\ (a-y)^2 & (b-y)^2 & (c-y)^2 \\ (x-z)^2 & (b-z)^2 & (c-z)^2 \end{vmatrix}$ can be expressed as

(A) $2\begin{vmatrix} 1 & x & y \\ 1 & z & z^2 \\ 1 & x^2 & y^2 \end{vmatrix}\begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix}$ (B) $4\begin{vmatrix} 1 & x & y \\ 1 & z & y^2 \\ 1 & x^2 & x \end{vmatrix}\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$

(C) $2\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$ (D) None of these

Q.5 Find the values of c for which the following system of equations in x, y is consistent
$2x + 3y = 3$
$(c + 2)x + (c + 4)y = c + 6$
$(c + 2)^2x + (c + 4)^2y = (c + 6)^2$
(A) 0 (B) –4 (C) 10 (D) –10

Answer key

Q.1	A	Q.2	B	Q.3	A	Q.4	C	Q.5	A, D

MATRICES

Introduction :

Elementary matrix already has now becomes as integral part of the mathematical background necessary in field of electrical / computer engineering / chemistry.

A matrix is any rectangular array of numbers written within brakets. A matrix is usually represented by a capital letter and classified by its dimensions. The dimension of the matrices are the number of rows and columns.

A $m \times n$ matrix is usually written as

$$A_{m\times n} = \begin{bmatrix} a_{11} & a_{12} & \cdots\cdots & a_{1n} \\ a_{21} & a_{22} & \cdots\cdots & a_{2n} \\ a_{m1} & a_{m2} & \cdots\cdots & a_{mn} \end{bmatrix}$$

(where a_{ij} represents any number which lies i^{th} row (from top) & j^{th} column form left)

(i) The matrix is not a number. It has got no numerical value.

(ii) The determinant of matrix $A_{m\times m} = |A_{m\times m}| = \begin{vmatrix} a_{11}\cdots\cdots\cdots a_{1m} \\ \cdots\cdots\cdots\cdots\cdots \\ a_{m1}\cdots\cdots\cdots a_{mm} \end{vmatrix}$

Abbreviated as :

$A = [a_{ij}]$ $1 \le i \le m$; $1 \le j \le n$, i denotes the row and j denotes the column is called a matrix of order $m \times n$. The elements of a matrix may be real or complex numbers. If all the elements of a matrix are real, the matrix is called real matrix.

SPECIAL TYPE OF MATRICES :

(A) Row Matrix :

$A = [\, a_{11}, a_{12}, \ldots\ldots a_{1n} \,]$ having one row . $(1 \times n)$ matrix. (or row vectors)

(B) Column Matrix :

$$A = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}$$ having one column. $(m \times 1)$ matrix (or column vectors)

(C) Zero or Null Matrix :

$(A = O_{m \times n})$
An $m \times n$ matrix all whose entries are zero.

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \text{ is a } 3 \times 2 \text{ null matrix } \& \ B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ is } 3 \times 3 \text{ null matrix}$$

(D) Horizontal Matrix :

A matrix of order $m \times n$ is a horizontal matrix if $n > m$.

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 5 & 1 & 1 \end{bmatrix}$$

(E) Verical Matrix :

A matix of order $m \times n$ is a vertical matrix if $m > n$. $\begin{bmatrix} 2 & 5 \\ 1 & 1 \\ 3 & 6 \\ 2 & 4 \end{bmatrix}$

Note: Every row matrix is also a Horizontal but not the converse.
|||ly every column matrix is also a vertical matrix but not the converse.

(F) Square Matrix : (Order n)

If number of rows = number of columns $\Rightarrow$ a square matrix. A real square matrix all whose elements are positive is called a positive matrices. Such matrices have application in mechanics and economics.

Note :

(i) In a square matrix the pair of elements a_{ij} & a_{ji} are called **Conjugate Elements.**

e.g. in the matrix $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, a_{21} and a_{12} are conjugate elements.

(ii) The elements a_{11}, a_{22}, a_{33}, ... a_{nn} are called **Diagonal Elements** . The line along which the diagonal elements lie is called **"Principal or Leading"** diagonal.
The quantity Σa_{ii} = trace of the matrix written as, $(t_r)A = t_r(A)$

- Square Matrix
  - Triangular Matrix
    - Upper triangular matrix, if $a_{ij} = 0 \ \forall\, i > j$
      $$A = \begin{bmatrix} x & x & x \\ 0 & x & x \\ 0 & 0 & x \end{bmatrix} \;;$$
    - Lower triangular matrix, if $a_{ij} = 0 \ \forall\, i < j$
      $$A = \begin{bmatrix} x & 0 & 0 \\ x & x & 0 \\ x & x & x \end{bmatrix}$$
  - Diagonal Matrix, atleast one, $a_{ii} \neq 0$ and $a_{ij} = 0$ if $i \neq j$
    $$\begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix}$$
    abbreviated as dia $(d_1, d_2, d_3 \,.....d_n)$
    - Scalar matrix, if $d_1 = d_2 = d_3 ..... = a \neq 0$
      $$\begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix}$$
    - Unit matrix, if $d_1 = d_2 = d_3 ..... = 1$
      $$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3$$

Note:

(i) Minimum number of zeros in an upper or lower triangular matrix of order n

$$= 1 + 2 + 3 + ... + (n - 1) = \frac{n(n-1)}{2}$$

(ii) Minimum number of cyphers in a diagonal/scalar/unit matrix of order $n = n(n-1)$ and maximum number of cyphers $= n^2 - 1$.

"It is to be noted that with every square matrix there is a corresponding determinant formed by the elements of A in the same order." If $|A| = 0$ then A is called a **singular matrix** and if $|A| \neq 0$ then a is called a **non singular matrix.**

Note: If $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ then det. A = 0 but not conversely.

ALGEBRA OF MATRICES :

ADDITION :

$A + B = [a_{ij} + b_{ij}]$ where A & B are of the same type . (same order)
If A and B are square matrices of the same type then, $t_r(A + B) = t_r(A) + t_2(B)$

(a) Addition of matrices is commutative :

i.e. $A + B = B + A$ where A and B must have the same order

(b) Addition of matrices is associative :

$(A + B) + C = A + (B + C)$ Provided A , B & C have the same order.

(c) Additive inverse :

If $A+B=O=B+A$ $[A=m\times n]$
and both A and B have the same order then A and B are said to be the to be the additive inverse of each other where **O** is the null matrix of the same order as that of A and B. '**O**' is the additive identity element.

$$\left.\begin{array}{l}\text{If } A+B=A+C\Rightarrow B=C\\ \text{and If } B+A=C+A\Rightarrow B=C\end{array}\right\}\text{ cancellation laws hold good.}$$

MULTIPLICATION OF A MATRIX BY A SCALAR :

If $A=\begin{bmatrix} a & b & c\\ b & c & a\\ c & a & b\end{bmatrix}$; $kA=\begin{bmatrix} ka & kb & kc\\ kb & kc & ka\\ kc & ka & kb\end{bmatrix}$ i.e. $k(A+B)=kA+kB$

Note:

(i) If A is a square matrix then $t_r(kA)=k[t_r(A)]$

(ii) $A=\begin{bmatrix}1 & -2\\ 2 & 3\end{bmatrix}$ then $A+A+A=\begin{bmatrix}1 & -2\\ 2 & 3\end{bmatrix}+\begin{bmatrix}1 & -2\\ 2 & 3\end{bmatrix}+\begin{bmatrix}1 & -2\\ 2 & 3\end{bmatrix}=\begin{bmatrix}3 & -6\\ 6 & 9\end{bmatrix}=3A$

Illustration :

Solve the equation, $[x \quad 2y \quad 3z]-2[y \quad z \quad -x]+3[-z \quad x \quad y]=[-12,\ 1,\ 17]$
on adding 3 matrices of LHS
$[x-2y-3z \quad 2y-2z+3x \quad 3z+2x+3y]=[-12 \quad 1 \quad 17]$

Sol. *Solving we get* $x=1,\ y=2,\ z=3$

MULTIPLICATION OF MATRICES :

(ROW BY COLUMN)

AB exists if, $\underset{2\times 3}{A=m\times n}$ & $\underset{3\times 3}{B=n\times p}$

A B is matrix of 2×3
Note that, AB exists, but BA does not $\Rightarrow AB\neq BA$
(number of columns in the pre multiplier = number of rows in post multiplier)

Note : In the product AB, $\begin{cases}A = \text{pre factor}\\ B = \text{post factor}\end{cases}$

$$\underset{1\times n}{A=(a_1, a_2, \ldots\ldots a_n)} \qquad \& \qquad \underset{n\times 1}{B=\begin{bmatrix}b_1\\ b_2\\ \vdots\\ b_n\end{bmatrix}}$$

$AB=[a_1b_1+a_2b_2+\ldots\ldots a_nb_n]$

If $A=[a_{ij}]$ be an $m\times n$ matrix & $B=[b_{ij}]$ be an $n\times p$ matrix,

then $(AB)_{ij}=\sum_{r=1}^{n} a_{ir}.b_{rj}$ is a matrix of order $m\times p$.

PROPERTIES OF MATRIX MULTIPLICATION :

Matrix multiplication is not commutative
i.e. $AB \neq BA$ (in general)

e.g. $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$; $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$; $AB = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $BA = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$

In fact if AB is defined it is possible that AB is not defined or may have different order

	A	B	
(1)	3×2	2×3	then AB is of order 3×3 and BA is of order 2×2
(2)	2×2	2×3	

Note :

(i) If $AB = 0 \not\Rightarrow$ that one of the matrices is zero however if any one of either A or B is null matrix then $AB = 0$ provided the product exist.

e.g. $A = \begin{bmatrix} 0 & -1 \\ 0 & 2 \end{bmatrix}$; $B = \begin{bmatrix} 5 & 5 \\ 0 & 0 \end{bmatrix}$

A and B are two square matrix of the same order such that $AB = 0$

$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ and det. $A \neq 0$ then B must be a null matrix.

Verification: det. $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \cdot$ det. $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = 0$

at least one of $|A|$ or $|B|$ must be if det. $(A) \neq 0$ $\Rightarrow$ det. $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = 0$

$$\begin{matrix} b = 0 \\ d = 0 \end{matrix}\left[\begin{matrix} a + 2b = 0 \\ b + 2d = 0 \\ 3a + 4c = 0 \\ 3b + 4d = 0 \end{matrix}\right] \Rightarrow a = 0 \Rightarrow c = 0$$

(ii) If $AB = AC \not\Rightarrow B = C$ but if $B = C \Rightarrow AB = AC$

(iii) In case $AB = BA$ is restrict of matrices A and B the two matrices are said to commute each other one if $AB = -BA$ then they are said to anticommute each other.

e.g. (i) $A = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ and $B = \begin{bmatrix} c & 0 \\ 0 & d \end{bmatrix}$ [AB = BA]

Note that multiplication of diagonal matrices of the same order will be commutative.

(iv) For every square matrix A, there exist an identity matrix of the same order such that $IA = AI = A$ where I is the unit matrix of the same order.

(v) If $A = 0$ then det. $A = 0$, however if det. $A = 0 \not\Rightarrow A = 0$

MATRIX MULTIPLICATION IS ASSOCIATIVE :

If A, B & C are conformable for the product AB & BC, then

$$(A . B) . C = A . (B . C)$$

$A = [a_{ij}]$ is $m \times n$; $B = [b_{ij}]$ is $n \times p$; $C = [c_{ij}]$ is $p \times q$

Note : $(A \cdot B) \cdot C$ & $A \cdot (B \cdot C)$ have the same order $\Rightarrow$ comparable.

$$[(AB) . C]_{ij} = \sum_{r=1}^{p} (AB)_{ir} C_{rj}$$

$$= \sum_{r=1}^{p} \left(\sum_{s=1}^{n} a_{is} b_{sr} \right) c_{rj} = \sum_{r=1}^{p} \sum_{s=1}^{n} (a_{is} b_{sr}) c_{rj}$$

$$= \sum_{r=1}^{p} \sum_{s=1}^{n} a_{is} (b_{sr} . c_{rj}) \qquad \text{(associativity in R)}$$

$$= \sum_{s=1}^{n} a_{is} \sum_{r=1}^{p} b_{sr} c_{rj} = \sum_{s=1}^{n} a_{is} (BC)_{sj}$$

$$= [A . (BC)]_{ij}$$

$\therefore \quad [(A . B) . C]_{ij} = [A . (B . C)]_{ij} \quad \Rightarrow \quad (AB) C = A . (BC)$

DISTRIBUTIVITY :

$$\left.\begin{aligned} A(B + C) &= AB + AC \\ (A + B) C &= AC + BC \end{aligned}\right] \text{ Provided A, B \& C are conformable for respective products}$$

$A = m \times n$; $B = n \times p$; $c = n \times p$

$$[A . (B + C)]_{ij} = \sum_{r=1}^{n} a_{ir} (B + C)_{rj} = \sum_{r=1}^{n} a_{ir} (b_{rj} + c_{rj})$$

$$= \sum_{r=1}^{n} a_{ir} b_{rj} + \sum_{r=1}^{n} a_{ir} c_{rj}$$

$$= (AB)_{ij} + (AC)_{ij} = (AB + AC)_{ij}$$

POSITIVE INTEGRAL POWERS OF A SQUARE MATRIX :

For a square matrix A, $A^2 A = (AA) A = A (AA) = A^3$.

Note that for a unit matrix I of any order, $I^m = I$ for all $m \in N$.
It can be easily seen that $A^m . A^n = A^{m+n}$ and $(A^m)^n = A^{mn}$.
In particular we define, $A^0 = I_n$, n being the order of A.

MATRIX POLYNOMIAL :

If $f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + ... + a_nx^0$ then we define a matrix polynomial
$f(A) = a_0A^n + a_1A^{n-1} + a_2A^{n-2} + ... + a_nI_n$

where A is the given square matrix. If f (A) is the null matrix then A is called the zero or root of the matrix polynomial f (x).

Note that $(A)^0$ is not defined if A is a null matrix.

DEFINITIONS :

(A) Idempotent Matrix :

A square matrix is idempotent provide $A^2 = A$. For an idempotent matrix
$A,\ \mathbf{A^n = A \ \forall \ n \geq 2 , n \in N} \ \Rightarrow A^n = A,\ n \geq 2.$

For example if $A = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$ then $A^2 = A$ i.e. A is idempotent.

(B) Nilpotent Matrix:

A square matrix is said to be nilpotent matrix of index p, ($p \in N$), if $A^p = \mathbf{O}$, $A^{p-1} \neq \mathbf{O}$ i.e. if p is the least positive integer for which $A^p = \mathbf{O}$, then A is said to be nilpotent of index p.

e.g. (i) $A = \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & 3 \end{bmatrix}$ Note that $A^3 = 0$ but $A^2 \neq 0 \Rightarrow$ index of nilpotency = 3

(ii) $A = \begin{bmatrix} ab & b^2 \\ -a^2 & -ab \end{bmatrix}$ is a nilpotent matrix of index 2.

(iii) $A = \begin{bmatrix} a & -a^2 \\ 1 & -a \end{bmatrix} \begin{bmatrix} a & -a^2 \\ 1 & -a \end{bmatrix}$ nil potent

(C) Periodic Matrix :

A square matrix which satisfies the relation $A^{K+1} = A$, for some positive integer K then A is periodic with period K i.e. if K is the least positive integer for which $A^{K+1} = A$ then A is said to be periodic with period K. If K = 1 then A is called idempotent.

e.g. the matrix $\begin{bmatrix} 2 & -3 & -5 \\ -1 & 4 & 5 \\ 1 & -3 & -4 \end{bmatrix}$ has the period 1.

Note : (1) Period of a square null matrix is not defined.
(2) Period of an idempotent matrix is 1.

(D) Involutary Matrix :

If $A^2 = I$, the matrix is said to be an involutary matrix. An involutary matrix is its own inverse.

e.g. (i) $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$;

Illustration :

Prove that a square matrix A is involutary if and only if $(I - A)(I + A) = O$.

Sol. *L.H.S.* $= (I - A)(I + A) = I^2 + IA - AI - A^2 = I^2 - A^2$
above is null matrix if $A = I$.

Practice Problem

Q.1 If matrix $A = \begin{bmatrix} 0 & 1 & -1 \\ 4 & -3 & 4 \\ 3 & -3 & 4 \end{bmatrix} = B + C$, where B is symmetric matrix and C is skew-symmetric matrix.

Then find matrix B and C.

Q.2 Show that the matrix $A = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$ is idempotent.

Q.3 If $k \in R_0$, then det $\{adj\,(kI)_n\}$ is equal to
(A) k^{n-1} (B) $k^{n(n-1)}$ (C) k^n (D) k

Q.4 If $A_1, A_3, \ldots\ldots, A_{2n-1}$ are n skew symmetric matrices of same order, then $B = \sum_{r=1}^{n} (2r-1)(A_{2r} - 1)^{2r-1}$ will be
(A) symmetric (B) skew-symmetric
(C) neither symmetric nor skew-symmetric (D) data not adequate

Answer key

Q.1 $\frac{1}{2}\begin{bmatrix} 0 & -3 & -4 \\ 3 & 0 & 7 \\ 4 & -7 & 0 \end{bmatrix}$ Q.3 B Q.4 B

Illustration :

The matrix $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 1 \\ 2 & 2 & 1 \end{bmatrix}$ be a zero divisor of the polynomial $f(x) = x^2 - 4x - 5$. Find the trace of matrix A^3.

Sol. $A^3 = A \cdot A^2 = A(4A + 5I) = 4A^2 + 5A$

$= 4(4A + 5I) + 5A.$

$= 21A + 20I$

$= \begin{bmatrix} 21 & 42 & 42 \\ 42 & 21 & 21 \\ 42 & 42 & 21 \end{bmatrix}$

Trace $A^3 = 123$. **Ans.**

Illustration :

Define $a = \begin{bmatrix} 0 & 1 \\ 3 & 0 \end{bmatrix}$ find a vertical vector V such that $(A^8 + A^6 + A^4 + A^2 + I)\, V = \begin{bmatrix} 0 \\ 11 \end{bmatrix}$.

Sol. $A^2 = 3I : (8I + 27 + 9 + 3 + 1) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 9 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 11 \end{bmatrix}$.

$$\begin{bmatrix} |\,21\,| \\ |\,21\,| \end{bmatrix} = \begin{bmatrix} 0 \\ 11 \end{bmatrix}$$

$a = 0;\ b = \dfrac{1}{11}$. **Ans.**

Illustration :

If $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and $\det(A^n - I) = 1 - \lambda^n \quad n \in N$ then the value of λ is

Sol. $A^2 = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$; $A^2 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 2^2 & 2^2 \\ 2^2 & 2^2 \end{bmatrix}$; $A^n = \begin{bmatrix} 2^{n-1} & 2^{n-1} \\ 2^{n-1} & 2^{n-1} \end{bmatrix}$

$A^n - I = \begin{bmatrix} 2^{n-1} - 1 & 2^{n-1} \\ 2^{n-1} & 2^{n-1} - 1 \end{bmatrix}$

$\det(A^n - I) = (2^{n-1} - 1)^2 - (2^{n-1})^2$

$= 1 - 2^n \Rightarrow \lambda = 2$. **Ans.**

Illustration :

The product of n matrices $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \cdots\cdots \begin{bmatrix} 1 & n \\ 0 & n \end{bmatrix}$ is equal to matrix $\begin{bmatrix} 1 & 378 \\ 0 & 1 \end{bmatrix}$. Find n

Sol. $\begin{bmatrix} 1 & \dfrac{n(n+1)}{2} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 378 \\ 0 & 1 \end{bmatrix}$

$\dfrac{n(n+1)}{2} = 378$

$n = 27$ **Ans.**

Illustration :

If $A^3 = B^3$ and $A^2 B = A^2B$ then prove that at least one of let $(A^2 + B^2)$ or det $(A - B)$ must be zero.

Sol. $A^3 - A^2B = B^3 - B^2A$

$A^2(A - B) = B^2 (B - A)$

$(A^2 + B^2) (A - B) = 0$

Let $\left((A^2 + B^2) - (A - B)\right) = 0$

$\left(\det (A^2 + B^2) (\det A - B)\right)$. **Ans.**

Illustration :

Let the matrices $A = \begin{bmatrix} 2 & 1 \\ 4 & 1 \end{bmatrix}, B = \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix}, C = \begin{bmatrix} 3 & -4 \\ -2 & 3 \end{bmatrix}$

then find $tr + tr\left(\frac{ABC}{2}\right) + tr\left(\frac{A(BC)^2}{2}\right) + tr\left(\frac{A(BC)^3}{8}\right) .....\infty.$

Sol. $tr\,(A) + tr\left(\frac{A}{2}\right) + tr\left(\frac{A}{2^2}\right) + ...$

$$= tr\,A\left(1 + \frac{1}{2} + \frac{1}{2^2} .....\infty\right) = \frac{trA}{1 - \frac{1}{2}} = 2tr(A) = 2(2 + 1) = 6.$$

Illustration :

If the matrices $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ *comments find* $\frac{d-b}{a+c-b}$.

Sol. $AB = BA$

$$\begin{bmatrix} a+3b & 2a+4b \\ c+3d & 2c+4d \end{bmatrix} = \begin{bmatrix} a+2c & b+2d \\ 3a+4c & 3b+4d \end{bmatrix}$$

$3a + 4c = c + 3d$

$3(a + c) = 3d$

$a + 2c = a + 3b$

$b = \frac{2c}{3}$. **Ans.**

Illustration :

If A is involutary prove that $\frac{1}{2}(I+A)$ *and A is idempotent.*

Sol. $A^2 = I$, $A = \frac{I+A}{2}$; $A^2 = \frac{I^2 + A^2 + 2IA}{4} = \frac{1}{4}(I + 2A + I) = \frac{A+I}{2}$. ***Ans.***

Illustration :

If α *and* β *are roots of the equation*

$$[1 \quad 25]\begin{bmatrix} 0 & \frac{1}{2} \\ -1 & \frac{1}{2} \end{bmatrix}^5 \begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix}^{10} \begin{bmatrix} 0 & \frac{1}{2} \\ -1 & \frac{1}{2} \end{bmatrix}^5 \begin{bmatrix} x^2 - 5x + 20 \\ x + 2 \end{bmatrix} = [40]$$

then find the value of $(1 - \alpha)(1 - \beta)$.

Sol. $$[1 \quad 25]\begin{bmatrix} 0 & \frac{1}{2} \\ -1 & \frac{1}{2} \end{bmatrix}^5 \begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix}^{10} \begin{bmatrix} 0 & \frac{1}{2} \\ -1 & \frac{1}{2} \end{bmatrix}^5 \begin{bmatrix} x^2 - 5x + 20 \\ x + 2 \end{bmatrix} = [40].$$

Let $A = \begin{bmatrix} 0 & \frac{1}{2} \\ -1 & \frac{1}{2} \end{bmatrix}$ *and* $B = \begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix}$.

Here, $AB = BA = I$

$\therefore \quad A^5 B^{10} A^5 = I$

$$[1 \quad 25]\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x^2 - 5x + 20 \\ x + 2 \end{bmatrix} = [40] \Rightarrow [1 \quad 25]\begin{bmatrix} x^2 - 5x + 20 \\ x + 2 \end{bmatrix} = [40]$$

$$x^2 - 5x + 20 + 25x + 50 = 40 \Rightarrow x^2 + 20x + 30 = 0 \begin{matrix} \nearrow \alpha \\ \searrow \beta \end{matrix}$$

$(1 - \alpha)(1 - \beta) = 1 - (\alpha + \beta) + \alpha\beta = 1 - (-20) + 30 + 51$. ***Ans.***

Practice Problem

Q.1 Given the matrices A of order $m \times n$, B of order $n \times p$ and C of order $r \times q$. Under what conditions on p, q and r would matrices be conformable for finding the product and what is the order of each
(a) ABC, (b) ACB, (c) A(B+C)

Q.2 The restriction on n, k and p so that PY + WY will be defined are
(A) $k = 3, p = n$ (B) k is arbitrary, $p = 2$
(C) p is arbitrary, $k = 3$ (D) $k = 2, p = 3$

Q.3 If $n = p$, then the order of the matrix $7X - 5Z$ is
(A) $p \times 2$ (B) $2 \times n$ (C) $n \times 3$ (D) $p \times n$

Q.4 If A, B, C are the given matries such that $AB = O$ and $BC = I$ then prove that $(A + B)^2 (A + C)^2 = I$ where I is an identity matrix.

Q.5 Find all matrices which commute with the matrix $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

Q.6 Let $A = \begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix}$ and $B = \begin{pmatrix} a & 1 \\ b & -1 \end{pmatrix}$. If $(A + B)^2 = A^2 + B^2$, find a and b.

Answer key

Q.1 (a) $p = r$, $m \times q$, (b) $r = n = q$, $m \times p$, (c) $r = n$, $p = q$, $m \times q$ Q.2 A Q.3 B

Q.5 $B = \begin{pmatrix} x & y \\ 0 & x \end{pmatrix}$, where x, y are scalars; Let $B = \begin{pmatrix} x & y \\ a & b \end{pmatrix}$; now equate $AB = BA$ to get $a = 0$ and $b = x$.

Q.6 $a = 1$, $b = 4$

Illustration :

Paragraph for questions no. 1 & 2

Consider a square matrix A of order 2 which has it's element's as 0, 1, 2, 4. Let N denotes number of such matrices, all element's of which are distinct
Find

Q.1 *All possible matrices (distinct entries)*

Q.2 *Possible non negative values of det (A) and total number of such matrices.*

Sol.

(i) $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \Rightarrow 4 \cdot 3 \cdot 2 \cdot 1.$

(ii) $\begin{vmatrix} 1 & 0 \\ 4 & 2 \end{vmatrix} = 2 \Rightarrow$ *4 matrices*

$$\begin{vmatrix} 1 & 0 \\ 2 & 4 \end{vmatrix} = 4 \quad \Rightarrow \quad 4 \text{ matrices}$$

$$\begin{vmatrix} 2 & 0 \\ 1 & 4 \end{vmatrix} = 8 \quad \Rightarrow \quad 4 \text{ matrices.}$$

Also 12 more matrices are possible whose determinant value can be $\{-2, -4, -8\}$. **Ans. 12**

Illustration :

Let M be a 2×2 matric such that $M\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ and $M^2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. If x_1 and x_2 $(x_1 > x_2)$ are the two values of x for which $\det(M - xI) = 0$, where I is an identity matrix of order 2 then find the value of $(5x_1 + 2x_2)$.

Sol. Let $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

Now, $M\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix} \Rightarrow \begin{bmatrix} a & b \\ c & d \end{bmatrix} \times \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix} \Rightarrow \begin{bmatrix} a-b \\ c-d \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

So, $a - b = -1$ ....(1)

$c - d = 2$ ....(2)

Also, $M^2\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$$\Rightarrow \quad M\left(M\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow M\begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} -a+2b \\ -c+2d \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

So, $-a + 2b = 1$ ....(3)

$-c + 2d = 0$ ....(4)

$\therefore$ From (1), (2), (3), (4), we get

$a = -1, b = 0, c = 4, d = 2$

Hence, $M = \begin{bmatrix} -1 & 0 \\ 4 & 2 \end{bmatrix}$

Now, $\det(M - xI) = 0$ (Given)

$$\Rightarrow \quad \begin{bmatrix} -1 & 0 \\ 4 & 2 \end{bmatrix} - x\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 0$$

$$\begin{vmatrix} -1-x & 0 \\ 4 & 2-x \end{vmatrix} = 0$$

$\Rightarrow \quad -(1 + x)(2 - x) = 0$

$(x + 1) x - 2) = 0$

$x = 2, -1$

$x_1 = 2, x_2 = -1$

$5x_1 + 2x_2 = 10 - 2 = 8$. **Ans.**

Illustration :

Let $A = \begin{bmatrix} 2 & 0 & 7 \\ 0 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} -\alpha & 14\alpha & 7\alpha \\ 0 & 1 & 0 \\ \alpha & -4\alpha & -2\alpha \end{bmatrix}$. If $AB = I$, where I is an identity matrix of order 3 then trace B has value equal to

Sol. We have $AB = I$, so

$$\begin{bmatrix} 2 & 0 & 7 \\ 0 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} -\alpha & 14\alpha & 7\alpha \\ 0 & 1 & 0 \\ \alpha & -4\alpha & -2\alpha \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} 5\alpha & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 10\alpha - 2 & 5\alpha \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ (Given)}$$

$$\Rightarrow \quad 5a = 1 \Rightarrow \quad a = \frac{1}{5}.$$

Now, trace $(B) = (-\alpha) + 1 + (-2\alpha) = 1 - 3a = 1 - 3\left(\frac{1}{5}\right) = 1 - \frac{3}{5} = \frac{2}{5}$. **Ans.**

Illustration :

Let A and B be 3×3 matrices of real numbers, where A is symmetric, B is skew-symmetric, and $(A + B)(A - B) = (A - B)(A + B)$. If $(AB)^t = (-1)^k AB$, where $(AB)^t$ is the transpose of the matrix AB, then the possible value of k are

Sol. $(A + B)(A - B) = (A - B)(A + B) \Rightarrow AB = BA$

as A is symmetric and B is skew symmetric

$\Rightarrow \quad (AB)^t = -AB \Rightarrow \quad k = 1$ and $k = 3$.

$(AB)^T = (-1)^k AB$

$B^T \cdot A^T = (-1)^k AB$

$(-B)(A) = (-1)^k AB$

$(-1)^{k+1} AB = BA$

$(-1)^{k+1} = 1$

$k \Rightarrow$ odd.

Illustration :

Paragraph for questions no. 1 to 4

Let A be the 2 × 2 matrices given by $A = [a_{ij}]$ where $a_{ij} \in \{0, 1, 2, , 4\}$ such that $a_{11} + a_{12} + a_{21} + a_{22} = 4$.

Q.1 *The number of matrices A such that the trace of A is equal to 4 are*
(A) 3 (B) 4 (C) 5 (D) 6
[Note : The trace of a matrix is the sum of its diagonal entries.]

Q.2 *The number of matrices A such that A is invertible are*
(A) 20 (B) 18 (C) 15 (D) 12

Q.3 *The absolute value of the difference between maximum value and minimum value of det(A) is equal to*
(A) 0 (B) 4 (C) 6 (D) 8

Q.4 *The number of matrices A such that A is either symmetric or skew- symmetric or both and det (A) is divisible by 2 are*
(A) 5 (B) 3 (C) 7 (D) 9

Sol. *We have* $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ *where a, b, c, d ∈ {0, 1, 2, 3, 4} and a + b + c + d = 4*

S.No.	Category	$a+b+c+d=4$	Cases
1	Type – I	4, 0, 0, 0	$\frac{4!}{3!}=\frac{24}{6}=4$
2	Type – II	3, 1, 0, 0	$\frac{4!}{2!}=\frac{24}{2}=12$
3	Type – III	2, 1, 1, 0	$\frac{4!}{2!}=\frac{24}{2}=12$
4	Type – IV	2, 2, 0, 0	$\frac{4!}{2!\,2!}=\frac{24}{4}=6$
5	Type – V	1, 1, 1, 1	$\frac{4!}{4!}=\frac{24}{24}=1$

∴ *Total number of matrices A = 4 + 12 + 12 + 6 + 1 = 35.*

(i) $\begin{bmatrix} 4 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 4 \end{bmatrix}, \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ *are 5 matrices A where trace of A is equal to 4.*

(ii) *For matrix A to be invertible, det (A) ≠ 0*

Type-I has no determinant whose value is non-zero.

Type-II have 4 determinants whose value is non-zero.

Type-III have all its 12 determinants whose value is non-zero.

Type-IV have 2 determiants whose value is non-zero.

Type-V has no determinants whose value is non-zero.

∴ *Total number of matrices A such that A is invertible are = 0 + 4 + 12 + 2 + 0 = 18.* ***Ans.***

(iii) *Maximum value of* $\det(A) = \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} = 4$ *and minimum value of* $\det(A) = \begin{vmatrix} 0 & 2 \\ 2 & 0 \end{vmatrix} = -4$

∴ *Absolute value of difference* $= |4-(-4)| = 8.$

(iv) *There will not be any skew- symmetric matrix because no element is negative and sum of elements is 4. For symmetric matrix, pair of conjugate elements must be same.*

Type-I : $\begin{bmatrix} 4 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 4 \end{bmatrix}$; *Type-II* : $\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$

Type-III: $\begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}$; *Type-IV* $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}$

Clearly, there are 5 symmetric matrices A such that det(A) is divisible by 2.

ORTHOGONAL MATRICES (NOT IN NCERT):

A square matrix is said to be orthogonal matrix, if $AA^T = I = A^T A$

Note that :

$$A = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \text{ then } A^T = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$$

$$AA^T = \begin{bmatrix} a_1^2 + a_2^2 + a_3^2 & a_1b_1 + a_2b_2 + a_3b_3 & a_1c_1 + a_2c_2 + a_3c_3 \\ b_1a_1 + b_2a_2 + b_3a_3 & b_1^2 + b_2^2 + b_3^2 & b_1c_1 + b_2c_2 + b_3c_3 \\ c_1a_1 + c_2a_2 + c_3a_3 & c_1b_1 + c_2b_2 + c_3b_3 & c_1^2 + c_2^2 + c_3^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

comparing,

$$\sum_{i=1}^{3} a_i^2 = \sum b_i^2 = \sum c_i^2 = 1 \quad \text{and} \quad \sum_{i=1}^{3} a_i b_i = \sum b_i c_i = \sum c_i a_i = 0$$

Note : All the 3 rows or 3 columns of an orthogonal matrix are pair wise orthogonal triad of 3 unit vectors.

Illustration :

Find a, b, c, x and y if the matrix A given by $A = \begin{bmatrix} a & 2/3 & 2/3 \\ 2/3 & 1/3 & b \\ c & x & y \end{bmatrix}$ *is orthogonal.*

Sol. (I) $a = \frac{1}{3}, b = -\frac{2}{3}, c = \frac{2}{3}, x = -\frac{2}{3}$ and $y = \frac{1}{3}$

(II) $a = \frac{1}{3}, b = -\frac{2}{3}, c = -\frac{2}{3}, x = \frac{2}{3}$ and $y = -\frac{1}{3}$

Illustration :

If $P = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}, A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ *and* $Q = PAP^T$ *and* $x = P^TQ^{2005}P$, *then x is equal to*

(A) $\begin{bmatrix} 1 & 2005 \\ 0 & 1 \end{bmatrix}$

(B) $\begin{bmatrix} 4+2005\sqrt{3} & 6015 \\ 2005 & 4-2005\sqrt{3} \end{bmatrix}$

(C) $\frac{1}{4}\begin{bmatrix} 2+\sqrt{3} & 1 \\ -1 & 2-\sqrt{3} \end{bmatrix}$

(D) $\frac{1}{4}\begin{bmatrix} 2005 & 2-\sqrt{3} \\ 2+\sqrt{3} & 2005 \end{bmatrix}$

Sol. $P^TP = I$

$Q = PAP^T$ so that

$x = P^TQ^{2005}P = P^T(PAP^T)^{2005}P$

$= P^TPAP^T(PAP^T)^{2004}P$

$= A^{2005} = \begin{bmatrix} 1 & 2005 \\ 0 & 21 \end{bmatrix}.$

ADJOINT OF A SQUARE MATRIX :

Let $A = [a_{ij}] = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ be a square matrix and let the matrix formed by the cofactors of $[a_{ij}]$ in determinant $|A|$ is $= \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}$.

Then (adj A) $= \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}$

Hence the transpose of the matrix of cofactors of elements of A in det. A is called the adj A.

Note:

(i) If $A = \begin{bmatrix} p & q \\ r & s \end{bmatrix}$ then Adj A $= \begin{bmatrix} s & -q \\ -r & p \end{bmatrix}$ e.g. $A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$ the adj. $A = \begin{bmatrix} 4 & -3 \\ -1 & 2 \end{bmatrix}$

Hence adjoint of a square matrix of order 2 can be easily obtained by interchanging the diagonal elements and changing the signs of the off diagonal elements.

(ii) Adjoint of a scalar matrix is a also a scalar matrix, adjoint of a diagonal matrix and adjoint of a triangular matrix is a triangular matrix.

e.g., $A = \begin{bmatrix} 1 & 2 & 3 \\ 5 & 0 & 4 \\ 2 & 6 & 7 \end{bmatrix}$ adj $A = \begin{bmatrix} -24 & 4 & 8 \\ -27 & 1 & 11 \\ 30 & -2 & -10 \end{bmatrix}$.

Illustration :

Which of the following statement(s) is(are) correct ?

(A) If A is square matrix of order 3, then the value of $\det\left\{(A - A^T)^{2011}\right\}$ *is equal to 0.*

(B) If A is a skew-symmetric matrix of order 3, then maxtrix A^4 *is symmetric.*

(C) If $3A = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ x & 2 & y \end{pmatrix}$ *and* $AA^T = I$*, then* $(x + y)$ *is equal to* -3.

(D) If α, β, γ *are the roots of the cubic* $x^3 + px^2 + q = 0$*, then the value of the determinant*

$\begin{vmatrix} \alpha & \beta & \gamma \\ \beta & \gamma & \alpha \\ \gamma & \alpha & \beta \end{vmatrix}$ *is equal to* $-p^3$

Sol.

(A) *If A is square matrix of order 3, then* $A - A^T$ *is skew symmetric of order 3.*

$\therefore \quad \left|(A - A^T)\right| = 0 \Rightarrow \left|(A - A^T)^{2011}\right| = \left|A - A^T\right|^{2011} = 0.$ *Ans.*

(B) *Given* $A^T = -A$

Let $C = A^4$

$C^T = (A^T)^4 = (-A)^4 = A^4 = C.$

Hence C is symmetric matrix $\Rightarrow$ *B is true.*

(C) *We have* $AA^T = \begin{pmatrix} 9 & 0 & x+4+2y \\ 0 & 0 & 2x+2-2 \\ x+4+2y & 2x+2-2y & x^2+4+y^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ *(Given)*

$\Rightarrow \quad x = -2, y = -1$

Hence $(x + y) = (-2) + (-1) = -3.$ ***Ans.***

(D) *We have* $\alpha + \beta + \gamma = -p, \ \alpha\beta + \beta\gamma + \gamma\alpha = 0$

Now, $2\alpha\beta\gamma - \alpha^3 - \beta^3 - \gamma^3 = -(\alpha + \beta + \gamma)(\alpha + \beta + \gamma)^2 - 3(\alpha\beta + \beta\gamma + \gamma\alpha)$

$= p(p^2) = p^3.$ ***Ans.***

PROPERTIES OF ADJOINT :

Theorem-1 :

$A(\text{adj } A) = (\text{adj } A) = A = |A| I_n$
whose A is any square matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \ldots\ldots\ldots a_{1n} \\ a_{21} & a_{22} & \ldots\ldots\ldots a_{2n} \\ a_{n1} & a_{n2} & \ldots\ldots\ldots a_{nn} \end{bmatrix}$$

$$(\text{adj } A) = \begin{bmatrix} A_{11} & A_{21} & \ldots\ldots\ldots A_{n1} \\ A_{12} & A_{22} & \ldots\ldots\ldots A_{n2} \\ \vdots & \vdots & \vdots \\ A_{1n} & A_{n2} & A_{nn} \end{bmatrix}$$

$(\text{adj } A) = \Sigma$ element's of ith row of A multiplied by corresponding element's jth column of adj A.

$= a_{ij}A_{ji} + a_{a2}A_{j2} + \ldots\ldots\ldots + a_{in}A_{in} = 0$

if $\quad i \neq j$ and $|A|$ when $i = j$.

Thus in product A (adj A) only diagonal element's exist all of them equal to $|A|$ while all other element's are zero.

$$A(\text{adj } A) = \begin{bmatrix} |A| & 0 & 0\ldots\ldots\ldots \\ 0 & |A| & 0\ldots\ldots\ldots \\ 0 & 0 & |A|\ldots\ldots \\ \vdots & \vdots & \vdots \\ \vdots & & |A| \end{bmatrix}$$

also for $(\text{adj } A) \cdot A = |A| I_n$.

Theorem-2 :

Let A be a non singular matrix of order $n \times n$. Then

$$|(\text{adj } A)| = |A|^{n-1}$$

Take det

$$|A(\text{adj } A)| = \begin{vmatrix} |A| & 0 & \ldots\ldots 0 \\ 0 & |A| & \ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots & & |A| \end{vmatrix} = |A|^n$$

$|A||\text{adj } A| = |A|^n; \quad |\text{adj } A| = |A|^{n-1}$.

Also we can say that

$A(\text{adj } A) = |A| \cdot I_n$

$$A \cdot \frac{(\text{adj } A)}{|A|} = \frac{(\text{adj } A) \cdot A}{|A|} = I_n.$$

Theorem-3 :

If A and B are two $n \times n$ matrices, then adj (AB) = (adj B) · (adj A)
(A and B must be non-singular of same order)

Proof : *We know that*

$A \cdot (adj\ A) = |A|\ I$ mutliplying both sides by AB

L.H.S. $(AB) \cdot (adj\ AB) = |AB|\ I$ ....(1)

Now, $AB\ adj\ (AB) = AB\ (adj\ B) \cdot (adj\ A)$

R.H.S $(AB) \cdot (adj\ B) \cdot (adj\ A)$

$= A\ (B \cdot adj\ B) \cdot (adj\ A)$

$= A \cdot (|B|\,I) \cdot (adj\ A)$ $\quad \because B \cdot adj\ B = |B|\ I$

$= A \cdot |B|\ I \cdot (adj\ A)$

$= A \cdot |B|\ (adj\ A)$ $\quad \because I \cdot (adj\ A) = adj\ A$

$= |B|\ A \cdot (adj\ A)$

$= |B|\ |A|\ I$ $\quad \because A(adj\ A) = |A|\ I$

$= |A|\ |B|\ I$

$= |AB|\ I$ $\quad \because |A|\ |B| = |AB|$ ....(2)

In view of equation (1) and (2), we have

$(AB) \cdot (adj\ AB) = (AB) \cdot (adj\ B) \cdot (adj\ A) \Rightarrow adj\ (AB) = (adj\ B) \cdot (adj\ A)$

Generalisation : *The result can be generalised for square matrices A, B, C, D, ... each of order n as follows:*

$adj\ (ABCD) = ...(adj\ D) \cdot (adj\ C) \cdot (adj\ B) \cdot (adj\ A)$

Illustration :

If A is any square matrix of order $n \times n$, then adj (adj A) = $|A|^{n-2} A$.

Sol. *We know that*

$A \cdot (adj\ A) = |A|\ I$

$\therefore$ $adj\ \{A \cdot (adj\ A)\} = adj\ \{|A|\ I\}$

$\Rightarrow$ $\{adj\ (adj\ A)\} \cdot (adj\ A) = |A|^{n-1}$ $\quad | \because adj\ \{k\ I_n\} = k^{n-1}\ I_n$

$\Rightarrow$ $\{adj\ (adj\ A)\} \cdot (adj\ A)\ A = |A|^{n-1}\ I \cdot A$ $\quad$ | Post-multiplying both sides by A

$\Rightarrow$ $\{adj\ (adj\ A)\} \cdot |A|\ I = |A|^{n-1}\ A$

$\Rightarrow$ $\{adj\ (adj\ A)\}\ |A| = |A|^{n-1}\ A$

$\Rightarrow$ $adj\ (adj\ A) = |A|^{n-2}\ A$

Illustration :

If A is a square matrix of order n and $|A| \neq 0$, then $|adj\ (adj\ A)| = |A|^{(n-1)^2}$

Sol. *We know that*

$|adj\ A| = |A|^{n-1}\ A$

$= \{|A|^{n-1}\}^{n-1}$

$= |A|^{(n-1)^2}$

Illustration :

Let matrix $A = \begin{bmatrix} x & y & -z \\ 1 & 2 & 3 \\ 1 & 1 & 2 \end{bmatrix}$ *where x, y, z* $\in N$. *If* $|adj(adj(adj(adj\ A)))| = 2^{32}.3^{16}$ *then number of such matrix A is* ____.

Sol. $|A| = x + y + z$

$adj\ (adj\ A) = |A|^{n-2} A = |A|\ A$

$|adj\ (adj\ (adj\ (adj\ A)))| = \left||A|^5 A\right| = |A|^{16}$

$|A|^{16} = 2^{32} \cdot 3^{16} = 12^{16}$

$|A| = 12$

$\Rightarrow x + y + z = 12$

Number of matrix = 55.

INVERSE OF A MATRIX (RECIPROCAL MATRIX) :

Definition:

A square matrix A said to be invertible (non singular) if there exists a matrix B such that,

$$AB = I = BA$$

B is called the inverse (reciprocal) of A and is denoted by A^{-1}. Thus

$$A^{-1} = B \Leftrightarrow AB = I = BA.$$

Note that for an involutary matrix $A^2 = I \Rightarrow A = A^{-1}$.

PROPERTIES OF INVERSE:

For a non singular matrix

Property :

$$A^{-1} = \frac{(\text{adj } A)}{|A|}$$

We have, $A.(\text{adj } A) = |A|\ I_n$

$A^{-1} A\ (\text{adj } A) = A^{-1} I_n\ |A|$

$I_n\ (\text{adj } A) = A^{-1}\ |A|\ I_n$

$$\therefore \quad A^{-1} = \frac{(\text{adj } A)}{|A|}$$

Remark: (i) Note that A^{-1} exists if A is non singular.

(ii) If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $|A| = 1$ then $A^{-1} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Note :

(i) The necessary and sufficient condition for a square matrix A to be invertible is that $|A| \neq 0$.

(ii) Inverse of a non singular diagonal matrix dia $(k_1\ k_2\ k_3\ \ldots\ldots\ k_n)$ is the diagonal matrix diag $\left(k_1^{-1}, k_2^{-1}, k_3^{-1}\ldots\ldots, k_n^{-1}\right)$

Theorem-1 :

Every invertible matrix possesses a unique inverse.

Proof : Let A be an invertible matrix of order n. Let B and C be two inverse of A.

Then, $AB = BA = I_n$ ....(i)

and $AC = CA = I_n$ ....(ii)

Now, $AB = I_n$

$\Rightarrow C(AB) = C\,I_n$ [pre-multiplying by C]

$\Rightarrow (CA)B = C\,I_n$ [by associativity]

$\Rightarrow I_n B = C\,I_n$ [$\because$ $CA = I_n$ from (ii)]

$\Rightarrow B = C$ [$\because$ $I_n B = B,\ C\,I_n = C$]

Hence an invertible matrix possesses a unique inverse.

Theorem-2 :

If A is an invertible square matrix, then A^T is also invertible and $(A^T)^{-1} = (A^{-1})^T$

Proof : Since A is invertible matrix. Therefore,

$|A| \neq 0$

$\Rightarrow |A^T| \neq 0$ [$\because$ $|A^T| = |A|$]

$\Rightarrow$ A^T is also invertible.

Now, $AA^{-1} = I_n = A^{-1}A$

$\Rightarrow (AA^{-1})^T = (I_n)^T = (A^{-1}A)^T$

$\Rightarrow (A^{-1})^T (A^T) = I_n = A^T (A^{-1})^T$ (as good as $AB = I = BA \Rightarrow A^{-1} = B$)

$\Rightarrow (A^T)^{-1} = (A^{-1})^T$

Theorem-3 :

If A is a non-singular matrix, then prove that $|A^{-1}| = |A|^{-1}$ i.e. $|A^{-1}| = \dfrac{1}{|A|}$

Proof : Since $|A| \neq 0$, therefore A^{-1} exists such that $AA^{-1} = I = A^{-1}A$

$\Rightarrow |AA^{-1}| = |I|$

$\Rightarrow |A|\,|A^{-1}| = 1$ [$\because$ $|AB| = |A|\,|B|$ and $|I| = 1$]

$\Rightarrow |A^{-1}| = \dfrac{1}{|A|}$ [$\because$ $|A| \neq 0$]

Theorem-4 :

If A & B are invertible matrices of the same order, then $(AB)^{-1} = B^{-1}A^{-1}$.

Proof : A & B are invertible. (reversal law of inverse)

$\Rightarrow |A| \neq 0$ & $|B| \neq 0 \Rightarrow |A|\,|B| \neq 0 \Rightarrow |AB| \neq 0$

$\Rightarrow$ AB is invertible.

Now $(AB)(AB)^{-1} = I$

$A^{-1}(AB)(AB)^{-1} = A^{-1}$

$(IB)(AB)^{-1} = A^{-1}$

$B(AB)^{-1} = A^{-1}$

$B^{-1}B(AB)^{-1} = B^{-1}A^{-1}$

$(AB)^{-1} = B^{-1}A^{-1}$

Note :

(i) If A is invertible, **(a)** $(A^{-1})^{-1} = A$ **(b)** $(A^k)^{-1} = (A^{-1})^k = A^{-k}$, $k \in N$

(ii) A square matrix is said to be orthogonal if, $A^{-1} = A^T$.

(iii) $(AB)^{-1}$ may be equal to $A^{-1}B^{-1}$.

EXPLANATION :

(i) As A is invertible hence $AA^{-1} = I = A^{-1}A$

Hence (A^{-1}) and A are inverse of each other

$\therefore \quad (A^{-1})^{-1} = A$

again $A^k = \underbrace{A\ A\ A\ldots\ldots A}_{k \text{ times}}$

$\therefore \quad (A^k)^{-1} = (AA\ldots\ldots A)^{-1} = A^{-1}.A^{-1}.A^{-1}\ldots\ldots k \text{ times} = (A^{-1})^k$

hence $(A^k)^{-1} = (A^{-1})^k k \in N$

(ii) again if A is a square matrix and is orthogonal

then $AA^T = A^TA = I$

hence A and A^T are inverse of each other

$\therefore \quad A^{-1} = A^T$

alternatively if $A^{-1} = A^T$

$$\left.\begin{array}{l} \therefore\ AA^{-1} = I = AA^T \\ \text{or } A^{-1}A = I = A^TA \end{array}\right] \Rightarrow A \text{ is orthogonal}$$

(iii) Let $A = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$AB = I = BA \Rightarrow A = B^{-1}$ and $B = A^{-1}$

also $(AB)^{-1} = I^{-1} = I$

SYSTEM OF EQUATION & CRITERIAN FOR CONSISTENCY :

GAUSS - JORDAN METHOD :

Q1. $x + y + z = 6$
$x - y + z = 2$
$2x + y - z = 1$

Q2. $x + 2y + 3z = 2$
$2x + 4y + 5z = 3$
$3x + 5y + 6z = 4$

$$\begin{pmatrix} x+y+z \\ x-y+z \\ 2x+y-z \end{pmatrix} = \begin{pmatrix} 6 \\ 2 \\ 1 \end{pmatrix}$$

$$\underset{A}{\begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 2 & 1 & -1 \end{pmatrix}} \underset{X}{\begin{pmatrix} x \\ y \\ z \end{pmatrix}} = \underset{B}{\begin{pmatrix} 6 \\ 2 \\ 1 \end{pmatrix}}$$

$AX = B \quad \Rightarrow \quad A^{-1}AX = A^{-1}B$

$$X = A^{-1}B = \frac{(\text{Adj}A)B}{|A|}$$

Note:

(i) If $|A| \neq 0$, system is consistent having unique solution

(ii) If $|A| \neq 0$ & $(\text{adj } A).B \neq$ Null matrix,
system is consistent having unique non – trivial solution.

(iii) If $|A| \neq 0$ & $(\text{adj } A).B = 0$ (Null matrix),
system is consistent having trivial solution.

(iv) If $|A| = 0$, **matrix method fails**

If $(\text{adj } A).B =$ null matrix
$\downarrow$
Infinite solutions

If $(\text{adj } A).B \neq 0$
$\downarrow$
Inconsistent (no solution)

EQUIVALENT MATRICES :

If a matrix B is obtained from a matrix A by one or more elementary transformations, then A and B are equivalent matrices and we write $A \sim B$. Let.

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 3 & 1 & 2 & 4 \end{bmatrix}$$

Then $A \sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & -1 & 1 & -1 \\ 3 & 1 & 2 & 4 \end{bmatrix}$ [Applying $R_2 \to R_2 + (-1)R_1$]

$\sim \begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & -1 & 1 & -2 \\ 3 & 1 & 2 & 2 \end{bmatrix}$ [Applying $C_4 \to C_4 + (-1)C_3$]

An elementary transformation is called a row transformation or a column transformation accordingly as it is appled to rows or columns.

Theorem-1 :

Every elementary row (column) transformation of an $m \times n$ matrix (not identity matrix) can be obtained by pre-multiplication (post-multiplication) with the corresponding elementary matrix obtained from the identity matrix $I_m(I_n)$ by subjecting it to the same elementary row (column) transformation.

Theorem-2 :

Let $C = AB$ be a product of two matrices. Any elementary row (column) transformation of AB can be obtained by subjecting the pre-factor A (post factor B) to the same elementary row (column) transformation.

Method of finding the inverse of a matrix by Elementary transformation :

Let A be a non sigular matrix of order n. Then A can be reduced to the identity matrix I_n by a finite sequence of elementary transformation only. As we have discussed every elementary row transformation of a matrix is equivalent to pre-multiplication by the corresponding elementary matrix. Therefore there exist elementary matrices $E_1, E_2, \ldots\ldots, E_4$ such that

$$(E_k E_{k-1} \ldots\ldots E_2E_1) A = I_n$$

$$\Rightarrow \quad (E_k E_{k-1} \ldots\ldots E_2E_1)AA^{-1} = I_nA^{-1} \qquad \text{(post multiplying by } A^{-1}\text{)}$$

$$\Rightarrow \quad (E_k E_{k-1} \ldots\ldots E_2E_1)\, I_n = A^{-1} \;\; (\because\; I_nA^{-1} = A^{-1} \text{ and } AA^{-1} = I_n)$$

$$\Rightarrow \quad A^{-1} = (E_k E_{k-1} \ldots\ldots E_2E_1)\, I_n$$

Algorithm for finding the inverse of a non singular matrix by elementary row transformations :

Let A be non-singular matrix of order n

Step-I : Write $A = I_n A$

Step-II : Perform a sequence of elementary row operations successively on A on the LHS and the pre factor I_n on the RHS till we obtain the result $I_n = BA$

Step-III : Write $A^{-1} = B$

The following steps will be helpful to find the inverse of a square matrix of order 3 by using elementary row transformations.

Step-I : Introduce unity at the intersection of first row and first column either by interchanging two rows or by adding a constant multiple of elements of some other row to first row.

Step-II : After introducting unity at (1, 1) place introduce zeros at all other places in first column.

Step-III : Introduce unity at the intersection 2nd row and 2nd column with the help of 2nd and 3rd row.

Step-IV : Introduce zeros at all other places in the second column except at the intersection of 2nd and 2nd column

Step-V : Introduce unity at the intersection of 3rd row and third column

Step-VI : Finally introduce zeros at all other places in the third column except at the intersection of third row and third column.

Illustration :

Using elementary transformation, find the inverse of the matrix $A = \begin{bmatrix} a & b \\ c & \left(\dfrac{1+bc}{-a}\right) \end{bmatrix}$.

Sol. $A = \begin{bmatrix} a & b \\ c & \left(\dfrac{1+bc}{a}\right) \end{bmatrix}$

We write , $\begin{bmatrix} a & b \\ c & \left(\dfrac{1+bc}{a}\right) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} A \quad \Rightarrow \quad \begin{bmatrix} 1 & \dfrac{b}{a} \\ c & \left(\dfrac{1+bc}{a}\right) \end{bmatrix} = \begin{bmatrix} \dfrac{1}{a} & 0 \\ 0 & 1 \end{bmatrix} A \qquad \left(R_1 \to \dfrac{R_1}{a}\right)$

or $\begin{bmatrix} 1 & \dfrac{b}{a} \\ 0 & \dfrac{1}{a} \end{bmatrix} = \begin{bmatrix} \dfrac{1}{a} & 0 \\ \dfrac{-c}{a} & 1 \end{bmatrix} A \quad (R_2 \to R_2 - cR_1)$ *or* $\begin{bmatrix} 1 & \dfrac{b}{a} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \dfrac{1}{a} & 0 \\ -c & a \end{bmatrix} A \qquad (R_2 \to aR_2)$

or $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \dfrac{1+bc}{a} & -b \\ -c & a \end{bmatrix} A \quad \left(R_1 \to R_1 - \dfrac{b}{a} R_2\right) \quad \Rightarrow \quad A^{-1} = \begin{bmatrix} \dfrac{1+bc}{a} & -b \\ -c & a \end{bmatrix}$

Illustration :

Obtain the inverse of the following matrix using elementary operations $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$.

Sol. *We, have* $A = IA$

$$\Rightarrow \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \quad \Rightarrow \quad \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \quad (\text{Applying } R_1 \leftrightarrow R_2)$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & -5 & -8 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & -3 & 1 \end{bmatrix} A \qquad (\text{Applying } R_3 \to R_3 - 3R_1)$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & -5 & -8 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & -3 & 1 \end{bmatrix} A \qquad (\text{Applying } R_1 \to R_1 - 2R_2)$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 0 \\ 1 & 0 & 0 \\ 5 & -3 & 1 \end{bmatrix} A \qquad (Applying\ R_3 \to R_3 + 5R_2)$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 0 \\ 1 & 0 & 0 \\ \frac{5}{2} & \frac{-3}{2} & \frac{1}{2} \end{bmatrix} A \qquad (Applying\ R_3 \to \frac{1}{2} R_3)$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{-1}{2} & \frac{1}{2} \\ -4 & 3 & -1 \\ \frac{5}{2} & \frac{-3}{2} & \frac{1}{2} \end{bmatrix} A \qquad (R_1 \to R_1 + R_3\ and\ R_2 \to R_2 \to -2R_3)$$

$$Hence,\ A^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{-1}{2} & \frac{1}{2} \\ -4 & 3 & -1 \\ \frac{5}{2} & \frac{-3}{2} & \frac{1}{2} \end{bmatrix}$$

Practice Problem

Q.1 If $\begin{bmatrix} 1 & -2 & -3 \\ -4 & 2 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 3 \\ 4 & 5 \\ 2 & 1 \end{bmatrix}$ then find the product AB and BA.

Q.2 The matrix R(t) is defined by $R(t) = \begin{bmatrix} -\cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}$. Show that $R(s)\,R(t) = R(s+t)$.

Q.3 If $A = \begin{bmatrix} 1 & 2 & x \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -2 & y \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $AB = I_3$, then x + y equals

(A) 0 (B) –1 (C) 2 (D) none of these

Q.4 Let $A+2B=\begin{bmatrix}1 & 2 & 0\\ 6 & -3 & 3\\ -5 & 3 & 1\end{bmatrix}$ and $2A-B=\begin{bmatrix}2 & -1 & 5\\ 2 & -1 & 6\\ 0 & 1 & 2\end{bmatrix}$

then Tr (A) – Tr (B) has the value equal to

(A) 0 (B) 1 (C) 2 (D) none

Q.5 Consider a matrix $A=\begin{bmatrix}3 & 1\\ -6 & -2\end{bmatrix}$, then $(I+A)^{99}$ equals (where I is a unit matrix of order 2)

(A) $I+2^{98}A$ (B) $I+2^{99}A$ (C) $I+(2^{99}+1)A$ (D) $I+(2^{99}-1)A$

Q.6 Let the matrix A and B be defined as $A=\begin{bmatrix}3 & 2\\ 2 & \alpha\end{bmatrix}$ and $B=\begin{bmatrix}3 & 1\\ 7 & 3\end{bmatrix}$. If $\det(2A^9B^{-1})=-2$,

then the number of distinct possible real values of α equals

(A) 0 (B) 1 (C) 2 (D) 3

Answer key

Q.1 $\begin{bmatrix}-10 & 2 & 21\\ -16 & 2 & 37\\ -2 & -2 & 11\end{bmatrix}$ Q.3 A Q.4 C Q.5 D Q.6 B

SOLVED EXAMPLES

Q.1 If the system of equations

$2x + 3y - z = 0$

$3x + 2y + kz = 0$

$4x + y + z = 0$

have a set of non-zero integral solutions then, find the smallest positive value of z.

Sol. The system has a non-zero solution if $|A| = 0 \Rightarrow k = 0$.

Clearly, the solutions are $(2a, -3a, -5a)$.

So, the smallest positive integral value of $z = 5$. **Ans.**

Q.2 Given $a, b \in \{0, 1, 2, 3, 4, ..., 9, 10\}$. Consider the system of equations

$x + y + z = 4$

$2x + y + 3z = 6$

$x + 2y + az = b$

Let **L :** denotes number of ordered pairs (a, b) so that the system of equations has unique solution,

M : denotes number of ordered pairs (a, b) so that the system of equations has no solution and

N : denotes number of ordered pairs (a, b) so that the system of equations has infinite solutions.

Find $(L + M - N)$.

Sol. Clearly, $\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 1 & 3 \\ 1 & 2 & a \end{vmatrix} = 1(a-6) - 1(2a-3) + 1(4-1)$ (Expanding along R_1)

$\Rightarrow \quad \Delta = -a$

Case-I : If $a \neq 0$, then system of equations has unique solution.

Case-II: If $a = 0$, put $z = k$, we get $x + y = 4 - k$ and $2x + y = 6 - 3k$

$\therefore$ On solving, we get

$x = 2 - 2k, y = 2 + k$

Now, substituting these values of x, y and z in equation $x + 2y + a \cdot z = b$, we get

$(2 - 2k) + 2(2 + k) + a \cdot k = bn \Rightarrow 6 + 0k = b$ i.e, $b = 6$

Thus for $b \neq 6$, there is no solution and for $b = 6$, there are infinite solution.

Hence, for unique solution $a \neq 0,\ b \in R \quad \Rightarrow \quad L = 10 \times 11 = 110$

for no solution we must have $a = 0, b \neq 6 \Rightarrow \quad M = 1 \times 10 = 10$

for infinite solution $a = 0$ and $b = 6 \Rightarrow \quad N = 1 \times 1 = 1$

$\Rightarrow \quad L + M - N = 110 + 10 - 1 = 119$ **Ans.**

Alternatively:

$x+y+z=4$ .....(1)

$2x+y+3z=6$ .....(2)

$x+2y+az=b$ .....(3)

Solving (1) and (2) $\Rightarrow$ $x=2-2z$ and $y=2+z$

Put in equation (3), we get

$az=b-6$

Hence, for unique solution $a\neq 0,\ b\in R$ $\Rightarrow$ $L=10\times 11=110$

for no solution we must have $a=0, b\neq 6 \Rightarrow$ $M=1\times 10=10$

for infinite solution $a=0$ and $b=6$ $\Rightarrow$ $N=1\times 1=1$

$\Rightarrow$ $L+M-N=110+10-1=119$ **Ans.**

Q.3 If $f(x)=\begin{vmatrix} 1 & x & x+1 \\ 2x & x(x-1) & (x+1)x \\ 3x(x-1) & x(x-1)(x-2) & (x+1)x(x-1) \end{vmatrix}$, then $f(100)$ is equal to

(A) 0 (B) 1 (C) 100 (D) –100

Sol. We have

$$f(x)=x\,(x+1)\,(x-1)\begin{vmatrix} 1 & 1 & 1 \\ 2x & x-1 & x \\ 3x & x-2 & x \end{vmatrix}$$

$$=x\,(x+1)\,(x-1)\begin{vmatrix} 1 & 1 & 1 \\ 2x & x-1 & x \\ 3x & x-2 & x \end{vmatrix} \quad [C_1\to C_1-C_3 \text{ and } C_2\to C_2-C_3]\ =0$$

Hence, $f(100)=0$.

Q.4 If α,β,γ are the roots of $x^3-3x+2=0$, then the value of the determinant $\begin{vmatrix} \alpha & \beta & \gamma \\ \beta & \gamma & \alpha \\ \gamma & \alpha & \beta \end{vmatrix}$ is equal to

(A) –3 (B) 2 (C) 1 (D) None of these

Sol. We have

$$\begin{vmatrix} \alpha & \beta & \gamma \\ \beta & \gamma & \alpha \\ \gamma & \alpha & \beta \end{vmatrix}=\begin{vmatrix} \alpha+\beta+\gamma & \beta & \gamma \\ \alpha+\beta+\gamma & \gamma & \alpha \\ \alpha+\beta+\gamma & \alpha & \beta \end{vmatrix} \quad [C_1\to C_1+C_2+C_3]$$

$$=0 \quad [\because\ \alpha+\beta+\gamma=0 \text{ from the equation } x^3-3x+2=0]$$

Q.5 If $ax^4 + bx^3 + cx^2 + dx + e = \begin{vmatrix} 2x & x-1 & x+1 \\ x+1 & x^2-x & x-1 \\ x-1 & x+1 & 3x \end{vmatrix}$, then the value of e, is

(A) 0 (B) –2 (C) 3 (D) –2

Sol. Putting x = 0, we have

$$\begin{vmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 1 \\ 1 & -1 & -1 \\ -1 & 1 & 0 \end{vmatrix} \qquad [C_2 \to C_2 + C_3]$$

$= 1 - 1 = 0.$

Q.6 If the value of the determinant $\begin{vmatrix} x+1 & \alpha & \beta \\ \alpha & x+\beta & 1 \\ \beta & 1 & x+\alpha \end{vmatrix}$ is equal to -8, then the value of x, is

(A) ±2 (B) –2 (C) 0 (D) 1

Sol. We have

$\alpha = \omega$ and $\beta = \omega^2$

Thus, we have

$$\Delta = \begin{vmatrix} x & \alpha & \beta \\ x & x+\beta & 1 \\ x & 1 & x+\alpha \end{vmatrix} \qquad [C_1 \to C_1 + C_2 + C_3 \text{ and using } 1 + \omega + \omega^2 = 0]$$

$$= \begin{vmatrix} 1 & \alpha & \beta \\ 0 & x+\beta-\alpha & 1-\beta \\ 0 & 1-\alpha & x+\alpha-\beta \end{vmatrix} \qquad [R_2 \to R_2 - R_1 \text{ and } R_3 \to R_3 - R_1]$$

$= x\,[x^2 - (\alpha - \beta)^2 - (1 - \alpha)(1 - \beta)]$

$= x\,[x^2 - \alpha^2 - \beta^2 + 2\alpha\beta - 1 + \alpha + \beta - \alpha\beta]$

$= x\,[x^2 - \omega^2 - \omega + 2 - 1 + \omega + \omega^2 - 1]$ [using $\omega^3 = 1$]

$= x^3$

According to the given condition, we have

$x^3 = -8$

gives $x = -2.$

Q.7 If $\begin{vmatrix} \sin 2x & \cos^2 x & \cos 4x \\ \cos^2 x & \cos 2x & \sin^2 x \\ \cos 4x & \sin^2 x & \sin 2x \end{vmatrix} = a_0 x + a_1(\cos x) + a_2(\cos^2 x) + \ldots\ldots + a_n(\cos^n x)$, then the value of a_0, is

(A) –1 (B) 1 (C) 0 (D) 2

Sol. We can see that

$a_0 = \Delta'(0)$

Now, we have

$$\Delta'(x) = \begin{vmatrix} 2\cos 2x & -2\cos x \sin x & -4\sin 4x \\ \cos^2 x & \cos 2x & \sin^2 x \\ \cos 4x & \sin^2 x & \sin 2x \end{vmatrix} + \begin{vmatrix} \sin 2x & \cos^2 x & \cos 4x \\ -2\cos x \sin x & -2\sin 2x & 2\sin x \cos x \\ \cos 4x & \sin^2 x & \sin 2x \end{vmatrix}$$

$$+ \begin{vmatrix} \sin 2x & \cos^2 x & \cos 4x \\ \cos^2 x & \cos 2x & \sin^2 x \\ -4\sin 4x & 2\sin x \cos x & 2\cos 2x \end{vmatrix}$$

Hence, we have

$$a_0 = \Delta'(0) = \begin{vmatrix} 2 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= 0 + 0 + (-1) = -1.$$

Q.8 Let α, β be the roots of $ax^2 + bx + c = 0$. Let $S_n = \alpha^n + \beta^n, n \geq 1$ and $\Delta = \begin{vmatrix} 1+S_0 & 1+S_1 & 1+S_2 \\ 1+S_1 & 1+S_2 & 1+S_3 \\ 1+S_2 & 1+S_3 & 1+S_4 \end{vmatrix}$.

If α, β are distinct and real, then

(A) $\Delta \leq 0$ (B) $\Delta > 0$ (C) $\Delta < 0$ (D) $\Delta = 0$

Sol. We have

$$\Delta = \begin{vmatrix} 1+1+1 & 1+\alpha+\beta & 1+\alpha^2+\beta^2 \\ 1+\alpha+\beta & 1+\alpha+\beta^2 & 1+\alpha^3+\beta^3 \\ 1+\alpha^2+\beta^2 & 1+\alpha^3+\beta^3 & 1+\alpha^4+\beta^4 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 1 \\ 1 & \alpha & \beta \\ 1 & \alpha^2 & \beta^2 \end{vmatrix} \times \begin{vmatrix} 1 & 1 & 1 \\ 1 & \alpha & \beta \\ 1 & \alpha^2 & \beta^2 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 1 & \alpha & \beta \\ 1 & \alpha^2 & \beta^2 \end{vmatrix}^2$$

$$= \begin{vmatrix} 1 & 0 & 0 \\ 1 & \alpha-1 & \beta-1 \\ 1 & \alpha^2-1 & \beta^2-1 \end{vmatrix}^2 \qquad [C_2 \to C_2 - C_1 \text{ and } C_3 \to C_3 - C_1]$$

$$= [(\alpha - 1)(\beta^2 - 1) - (\alpha^2 - 1)(\beta - 1)]^2$$

$$= [\alpha\beta(\beta - \alpha) + (\beta - \alpha) + (\alpha^2 - \beta^2)]^2$$

$$= [\alpha\beta + 1 - (\alpha + \beta)]^2 [(\alpha + \beta)^2 - 4\alpha\beta]$$

$$= \left(\frac{c}{a} + \frac{b}{a} + 1\right)^2 \left(\frac{b^2}{a^2} - \frac{4c}{a}\right)$$

$$= \frac{(a+b+c)^2(b^2-4ac)}{a^4} > 0 \text{ if roots are real and distinct.}$$

Q.9 Let $\Delta(x) = \begin{vmatrix} 1+\sin^2 x & \cos^2 x & 4\sin 2x \\ \sin^2 x & 1+\cos^2 x & 4\sin 2x \\ \sin^2 x & \cos^2 x & 1+4\sin 2x \end{vmatrix}$. The value of $x\left(0 \le x \le \frac{\pi}{2}\right)$ for which $\Delta(x)$ is maximum, is equal to

(A) $\frac{\pi}{2}$ (B) $\frac{\pi}{6}$ (C) $\frac{\pi}{3}$ (D) $\frac{\pi}{4}$

Sol. We have

$$\Delta(x) = (1+\sin^2 x+\cos^2 x+4\sin 2x)\begin{vmatrix} 1 & \cos^2 x & 4\sin 2x \\ 1 & 1+\cos^2 x & 4\sin 2x \\ 1 & \cos^2 x & 1+4\sin 2x \end{vmatrix} \quad [C_1 \to C_1 + C_2 + C_3]$$

$$= (2+4\sin 2x)\begin{vmatrix} 1 & \cos^2 x & 4\sin 2x \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} \quad [R_2 \to R_2 - R_1 \text{ and } R_3 \to R_3 - R_1]$$

$$= (2+4\sin 2x)$$

which attains maxima at $x = \frac{\pi}{4}$.

Q.10 If $D_k = \begin{vmatrix} 3^k & \frac{1}{(k+1)(k+2)} & \cos(k+1)d\theta \\ \frac{3^n-1}{2} & \frac{n}{(n+1)} & \frac{\sin\frac{n\theta}{2}\cos\frac{(n-1)\theta}{2}}{\sin\frac{\theta}{2}} \\ a & b & c \end{vmatrix}$ then $\sum_{k=0}^{n-1} D_k$ is

(A) independent of n (B) independent of a, b, c

(C) a + b + c (D) none of these

Sol. We have

$$\sum_{k=0}^{n-1} 3^k = 1+3+3^2+\dots\dots+3^{n-1} = \frac{3^n-1}{3-1} = \frac{3^n-1}{2}$$

$$\sum_{k=0}^{n-1} \frac{1}{(k+1)(k+2)} = \sum_{k=0}^{n-1}\left(\frac{1}{k+1} - \frac{1}{k+2}\right)$$

$$= \left(\frac{1}{1}-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\dots\dots+\left(\frac{1}{n}-\frac{1}{n+1}\right) = 1-\frac{1}{n+1} = \frac{n}{n+1}$$

and $\sum_{k=0}^{n-1} \cos(k+1)\,\theta = 1 + \cos\theta + \cos 2\theta + \cos 3\theta + \ldots\ldots + \cos(n-1)\,\theta$

$= \text{Re}\,[1 + e^{i\theta} + e^{i\,2\theta} + \ldots\ldots + e^{i(n-1)\theta}]$

$$= \text{Re}\left(\frac{1-e^{in\theta}}{1-e^{i\theta}}\right) = \text{Re}\left(\frac{(1-e^{in\theta})(1-e^{-i\theta})}{(1-e^{i\theta})(1-e^{-i\theta})}\right)$$

$$= \text{Re}\left(\frac{1-e^{-i\theta}-e^{in\theta}+e^{i(n-1)\theta}}{2-(e^{i\theta}+e^{-i\theta})}\right) = \frac{1-\cos\theta-\cos n\theta+\cos(n-1)\theta}{2(1-\cos\theta)}$$

$$= \frac{2\sin^2\frac{\theta}{2}+2\sin\left(n-\frac{1}{2}\right)\theta\sin\frac{\theta}{2}}{4\sin^2\frac{\theta}{2}} = \frac{\sin\frac{\theta}{2}+\sin\left(n-\frac{1}{2}\right)\theta}{2\sin\frac{\theta}{2}} = \frac{\sin\frac{n\theta}{2}\cos\frac{(n-1)\theta}{2}}{\sin\frac{\theta}{2}}.$$

Q.11 The roots of the equation $\begin{vmatrix} \frac{1}{x}+x^2+\frac{1}{x} & \frac{1}{x}+ax+\frac{1}{a} & \frac{1}{x^2}+x+\frac{1}{b} \\ \frac{1}{a}+ax+\frac{1}{x} & \frac{1}{a}+a^2+\frac{1}{a} & \frac{1}{a}+ab+\frac{1}{b} \\ \frac{1}{b}+bx+\frac{1}{x} & \frac{1}{b}+ab+\frac{1}{a} & \frac{1}{b}+b^2+\frac{1}{b} \end{vmatrix} = 0$ are

(A) a, b (B) –a, –b (C) a + b, a – b (D) None of these

Sol. We have

$$\Delta = \begin{vmatrix} \frac{1}{x} & x & 1 \\ \frac{1}{a} & a & 1 \\ \frac{1}{b} & b & 1 \end{vmatrix} \times \begin{vmatrix} 1 & x & \frac{1}{x} \\ 1 & a & \frac{1}{a} \\ 1 & b & \frac{1}{b} \end{vmatrix} \text{ [row by row] } = (-1)\begin{vmatrix} \frac{1}{x} & x & 1 \\ \frac{1}{a} & a & 1 \\ \frac{1}{b} & b & 1 \end{vmatrix}^2$$

Thus, we have

$$\begin{vmatrix} \frac{1}{x} & x & 1 \\ \frac{1}{a} & a & 1 \\ \frac{1}{b} & b & 1 \end{vmatrix} = 0$$

i.e. $\frac{1}{abx}\begin{vmatrix} 1 & x^2 & x \\ 1 & a^2 & a \\ 1 & b^2 & b \end{vmatrix} = 0$ i.e. $\begin{vmatrix} 1 & x^2 & x \\ 1 & a^2-x^2 & a-x \\ 1 & b^2-x^2 & b-x \end{vmatrix} = 0$ $[R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1]$

i.e. $(a^2 - x^2)(b - x) - (b^2 - x^2)(x - a) = 0$ [expanding along C_1]

i.e. $(a - x)(b - x)[(a + x) - (b + x)] = 0$

gives $x = a, b$

Q.12 If α, β are the roots of the equation $ax^2 + bx + c = 0$, then the value of the determinant

$$\begin{vmatrix} 1 & \cos(\alpha-\beta) & \cos\alpha \\ \cos(\alpha-\beta) & 1 & \cos\beta \\ \cos\alpha & \cos\beta & 1 \end{vmatrix} \text{ is equal to}$$

(A) $a + b$ (B) 0 (C) $a - b$ (D) $a + b + c$

Sol. We have

$$\begin{vmatrix} 1 & \cos(\alpha-\beta) & \cos\alpha \\ \cos(\alpha-\beta) & 1 & \cos\beta \\ \cos\alpha & \cos\beta & 1 \end{vmatrix} \quad [\text{expanding along } R_1]$$

$= (1 - \cos^2\beta) + \cos(\alpha - \beta)[\cos\alpha\cos\beta - \cos(\alpha - \beta)] + \cos\alpha[\cos(\alpha - \beta)\cos\beta - \cos\alpha]$

$= \sin^2\beta + \cos(\alpha - \beta)[2\cos\alpha\cos\beta - \cos(\alpha - \beta)]$

$= \sin^2 + \cos(\alpha - \beta)\cos(\alpha + \beta) - \cos^2\alpha$

$= \sin^2\beta + (\cos^2\alpha - \sin^2\beta) - \cos^2\alpha = 0.$

Q.13 If $p + q + r = 0$, prove that $\begin{vmatrix} pa & qb & rc \\ qc & ra & pb \\ rb & pc & qa \end{vmatrix} = pqr\begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix}$

Sol. We have

L.H.S.$= pa(qra^2 - p^2bc) - qb(q^2ca - prb^2) + rc(pqc^2 - r^2ab)$

$= pqra^3 - abcp^3 - abcq^3 + pqrb^3 + pqrc^3 - abcr^3$

$= pqr(a^3 + b^3 + c^3) - abc(p^3 + q^3 + r^3)$

$= pqr(a^3 + b^3 + c^3 - 3abc) - abc(p^3 + q^3 + r^3 - 3pqr)$

$= pqr(a^3 + b^3 + c^3 - 3abc) - 0$

$[p^3 + q^3 + r^3 - 3pqr = (p + q + r)(p^2 + q^2 + r^2 - pq - qr - rp)$

$= 0 \quad \therefore\ p + q + r = 0$ (given)]

$= pqr(a^3 + b^3 + c^3 - 3abc)$

and R.H.S. $= pqr(a + b + c)\begin{vmatrix} 1 & b & c \\ 1 & a & b \\ 1 & c & a \end{vmatrix} \quad [C_1 \to C_1 + C_2 + C_3]$

$$= pqr(a + b + c)\begin{vmatrix} 0 & b-c & c-a \\ 0 & a-c & b-a \\ 1 & c & a \end{vmatrix} \quad \begin{bmatrix} R_1 \to R_1 - R_3 \\ R_2 \to R_2 - R_3 \end{bmatrix}$$

$= pqr(a + b + c)[(b - c)(b - a) + (c - a)^2]$

$= pqr(a + b + c)[a^2 + b^2 + c^2 - ab - bc - ca]$

$= pqr(a^2 + b^2 + c^3 - 3abc) = $ L.H.S.

Q.14 If A is non-singular, prove that the eigen values of A^{-1} are the reciprocals of the eigen value of A.

Sol. Let A be an eigen value of A and X be a corresponding eigenvector. Then

$$AX = \lambda X$$

$$\Rightarrow \quad X = A^{-1}(\lambda X) = \lambda(A^{-1}X)$$

$$\Rightarrow \quad \frac{1}{\lambda}X = A^{-1}X \quad [\because \text{ A is non-singular} \Rightarrow \lambda = 0]$$

$$\Rightarrow \quad A^{-1}X = \frac{1}{\lambda}X$$

Therefore, $\frac{1}{\lambda}$ is an eigenvalue of A^{-1} and X is a corresponding eigenvector.

Q.15 If α is a characteristic root of a non-singular matrix, then prove that $\left[\frac{A}{\alpha}\right]$ is a characteristic root of adj A.

Sol. Since a is a characteristic root of a non-singular matrix, therefore $a \neq 0$. Also a is a characteristic root of A implies that there exists a non-zero vector X such that

$$AX = \alpha X$$

$$\Rightarrow \quad (\text{adj } A)(AX) = (\text{adj } A)(\alpha X) \quad \Rightarrow \quad [(\text{adj } A)A]X = \alpha(\text{adj } A)X$$

$$\Rightarrow \quad |A|\ IX = \alpha(\text{adj } A)X \quad [\because (\text{adj})A]A = |A|\,I]$$

$$\Rightarrow \quad |A|X = \alpha(\text{adj } A)X \quad \Rightarrow \quad \frac{|A|}{\alpha}X = (\text{adj } A)X$$

$$\Rightarrow \quad (\text{adj } A)X = \frac{|A|}{\alpha}X$$

Since X is a non-zero vector, therefore $\left[\frac{A}{\alpha}\right]$ is a characteristic root of the matrix adj A.

Q.16 Solve the following system of equations, using matrix method : $x+2y+z=7,\ x+3z=11,\ 2x-3y=1$.

Sol. The given system of equation is

$$x+2y+z=7, \quad x+0y+3z=11, \quad 2x-3y+0z=1$$

or $\begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & 3 \\ 2 & -3 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7 \\ 11 \\ 1 \end{bmatrix}$ or $AX = B$, where $A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & 3 \\ 2 & -3 & 0 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} 7 \\ 11 \\ 1 \end{bmatrix}$

Now, $|A| = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & 3 \\ 2 & -3 & 0 \end{bmatrix} = 18$

So, the given system of equation has a unique solution given by $X = A^{-1}B$

$$\therefore \quad \text{adj } A = \begin{bmatrix} 9 & 6 & -3 \\ -3 & -2 & 7 \\ 6 & -2 & 2 \end{bmatrix}^T = \begin{bmatrix} 9 & -3 & 6 \\ 6 & -2 & -2 \\ -3 & 7 & -2 \end{bmatrix} \quad \Rightarrow \quad A^{-1} = \frac{1}{|A|}\text{adj } A = \frac{1}{8}\begin{bmatrix} 9 & -3 & 6 \\ 6 & -2 & -2 \\ -3 & 7 & -2 \end{bmatrix}$$

Now $X = A^{-1}B$

$$\Rightarrow X = \frac{1}{18}\begin{bmatrix} 9 & -3 & 6 \\ 6 & -2 & -2 \\ -3 & 7 & -2 \end{bmatrix}\begin{bmatrix} 7 \\ 11 \\ 1 \end{bmatrix} = \frac{1}{18}\begin{bmatrix} 63-33+6 \\ 42-22-2 \\ -21+77-2 \end{bmatrix} \quad \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{18}\begin{bmatrix} 36 \\ 18 \\ 54 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$$

$\Rightarrow x = 2, y = 1, z = 3$

Q.17 If $A = \begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix}$, then find the value of $|A|\ |\text{adj } A|$.

Sol. $|A|\ |\text{adj } A| = |A \text{ adj } A| = |\ |A|\ I| = \begin{vmatrix} |A| & 0 & 0 \\ 0 & |A| & 0 \\ 0 & 0 & |A| \end{vmatrix} = |A|^3 = (a^3)^3 = a^9$

Q.18 For the matrix $A = \begin{bmatrix} 3 & 1 \\ 7 & 5 \end{bmatrix}$, then x and y so that $A^2 + xI = yA$. Hence, find A^{-1}.

Sol. We have

$$A = \begin{bmatrix} 3 & 1 \\ 7 & 5 \end{bmatrix}$$

$$A^2 = AA = \begin{bmatrix} 3 & 1 \\ 7 & 5 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 7 & 5 \end{bmatrix} = \begin{bmatrix} 9+7 & 3+5 \\ 21+35 & 7+25 \end{bmatrix} = \begin{bmatrix} 16 & 8 \\ 56 & 32 \end{bmatrix}$$

Now, $A^2 + xI = yA$

$$\Rightarrow \begin{bmatrix} 16 & 8 \\ 56 & 32 \end{bmatrix} + x\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = y\begin{bmatrix} 3 & 1 \\ 7 & 5 \end{bmatrix} \quad \Rightarrow \begin{bmatrix} 16+x & 8+0 \\ 56+0 & 32+x \end{bmatrix} = \begin{bmatrix} 3y & y \\ 7y & 5y \end{bmatrix}$$

$\Rightarrow \quad 16 + x = 3y, \ \ y = 8, \ \ 7y = 56, \ \ 5y = 32 + x$

Putting $y = 8$ in $16 + x = 3y$, we get $x = 24 - 16 = 8$. Clearly, $x = 8$ and $y = 8$ also satisfy $7y = 56$ and $5y = 32 + x$. Hence, $x = 8$ and $y = 8$. We have

$$|A| = \begin{vmatrix} 3 & 1 \\ 7 & 5 \end{vmatrix} = 8 \neq 0$$

So, A is invertible.

Putting $x = 8, y = 8$ in $A^2 = xI = yA$, we get

$A^2 + 8I = 8A$

$\Rightarrow \quad A^{-1}(A^2 + 8I) = 8A^{-1}A$ [re-multiplying througout by A^{-1}]

$\Rightarrow \quad A^{-1}A^2 + 8A^{-1}I = 8A^{-1}A$

$\Rightarrow \quad A + 8A^{-1} = 8I$ [$\because A^{-1}A^2 = (A^{-1}A)A = IA = A, \ A^{-1}I = A^{-1}$ and $A^{-1}A = I$]

$\Rightarrow \quad 8A^{-1} = 8I - A$

$$\Rightarrow \quad A^{-1} = \frac{1}{8}(8I - A) = \frac{1}{8}\left\{\begin{bmatrix} 8 & 0 \\ 0 & 8 \end{bmatrix} - \begin{bmatrix} 3 & 1 \\ 7 & 5 \end{bmatrix}\right\}$$

$$\Rightarrow \quad A^{-1} = \frac{1}{8}\begin{bmatrix} 8-3 & 0-1 \\ 0-7 & 8-5 \end{bmatrix} = \frac{1}{8}\begin{bmatrix} 5 & -1 \\ -7 & 3 \end{bmatrix} = \begin{bmatrix} 5/8 & -1/8 \\ -7/8 & 3/8 \end{bmatrix}$$

Q.19 By the method of matrix inversion, solve the system.

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 5 & 7 \\ 2 & 1 & -1 \end{bmatrix}\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \end{bmatrix} = \begin{bmatrix} 9 & 2 \\ 52 & 15 \\ 0 & 1 \end{bmatrix}$$

Sol. $$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 5 & 7 \\ 2 & 1 & -1 \end{bmatrix}\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \end{bmatrix} = \begin{bmatrix} 9 & 2 \\ 52 & 15 \\ 0 & 1 \end{bmatrix}$$

$\Rightarrow \quad AX = B \qquad \ldots(i)$

Clearly $|A| = -4 \neq 0$. Therefore

$$\text{adj } A = \begin{bmatrix} -12 & 16 & -8 \\ 2 & -3 & 1 \\ 2 & 1 & 3 \end{bmatrix}^T = \begin{bmatrix} -12 & 2 & 2 \\ 16 & -3 & -5 \\ -8 & 1 & 3 \end{bmatrix} \qquad \therefore \; A^{-1} = \frac{\text{adj. } A}{|A|} = \frac{-1}{4}\begin{bmatrix} -12 & 2 & 2 \\ 16 & -3 & -5 \\ -8 & 1 & 3 \end{bmatrix}$$

Now, $$A^{-1}B = \frac{-1}{4}\begin{bmatrix} -12 & 2 & 2 \\ 16 & -3 & -5 \\ -8 & 1 & 3 \end{bmatrix}\begin{bmatrix} 9 & 2 \\ 52 & 15 \\ 0 & -1 \end{bmatrix} = \frac{-1}{4}\begin{bmatrix} -4 & 4 \\ -12 & -8 \\ -20 & -4 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 3 & 2 \\ 5 & 1 \end{bmatrix}$$

From equation (i),

$$X = A^{-1}B \;\Rightarrow\; \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 3 & 2 \\ 5 & 1 \end{bmatrix}$$

$\Rightarrow \quad x_1 = 1,\; x_2 = 3,\; x_3 = 5 \quad \text{or} \quad y_1 = -1,\; y_2 = 2,\; y_3 = 1$

Q.20 If A, B and C are $n \times n$ matrix and $\det(A) = 2$, $\det(B) = 3$ and $\det(C) = 5$, then find the value of the $\det(A^2BC^{-1})$.

Sol. Given that $|A| = 2,\; |B| = 3,\; |C| = 5$. Now

$$\det(A^2BC^{-1}) = |A^2BC^{-1}| = \frac{|A|^2|B|}{|C|} = \frac{4\times 3}{5} = \frac{12}{5}$$

Q.21 Matrices A and B satisfy $AB = B^{-1}$, where $B = \begin{bmatrix} 2 & -1 \\ 2 & 0 \end{bmatrix}$. Find

(i) without finding B^{-1}, the value of K for which $KA = 2B^{-1} + I = 0$

(ii) without finding A^{-1}, the matrix X satsifying $A^{-1}XA = B$.

Sol.

(i) $AB = B^{-1} \;\Rightarrow\; AB^2 = I$

Now,

$KA - 2B^{-1} + I = O \;\Rightarrow\; KAB - 2B^{-1}B + IB = O \;\Rightarrow\; KAB - 2I + B = O$

$\Rightarrow \; KAB^2 - 2B + B^2 = O \;\Rightarrow\; KI - 2B + B^2 = O$

$$\Rightarrow \quad K\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - 2\begin{bmatrix} 2 & -1 \\ 2 & 0 \end{bmatrix} + \begin{bmatrix} 2 & -1 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} K & 0 \\ 0 & K \end{bmatrix} - \begin{bmatrix} 4 & -2 \\ 4 & 0 \end{bmatrix} + \begin{bmatrix} 2 & -2 \\ 4 & -2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} K-2 & 0 \\ 0 & K-2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$\Rightarrow \quad K = 2$

(ii) $A^{-1}XA = B$

$\Rightarrow \quad AA^{-1}XA = AB \quad \Rightarrow \quad IXA = AB \quad \Rightarrow \quad XAB = AB^2$

$\Rightarrow \quad XAB = I \quad \Rightarrow \quad XAB^2 = B \quad \Rightarrow \quad XI = B$

$\Rightarrow \quad X = B$

Paragraph for question nos. 22 to 24

For $x > 0$, let $A = \begin{bmatrix} x+\frac{1}{x} & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & 16 \end{bmatrix}$ and $B = \begin{bmatrix} \frac{5x}{x^2+1} & 0 & 0 \\ 0 & \frac{3}{x} & 0 \\ 0 & 0 & \frac{1}{4} \end{bmatrix}$ be two matrices.

Three other matrices X, Y and Z are defined as

$X = (AB)^{-1} + (AB)^{-2} + (AB)^{-3} + \ldots\ldots\ldots\ldots\ldots\ (AB)^{-n}$, $Y = \lim\limits_{n\to\infty} X$ and $Z = Y^{-1} - 2I$,

where I is identity matrix of order 3.

[**Note:** tr (P) denotes the trace of matrix P.]

Q.22 The value of $\det.\left(\text{adj}\,(\sqrt{5}\,Y^{-1})\right)$ is equal to

(A) $(5!)^2$ (B) $5^3\,(5!)^2$ (C) $5\,(5!)^2$ (D) $5^2\,(5!)^2$

Q.23 Least positive integral value of tr (AY) is equal to

(A) 8 (B) 7 (C) 6 (D) 5

Q.24 If $\text{tr}\,(Z + Z^2 + Z^3 + \ldots\ldots + Z^{10}) = 2^a + b$ where $a, b \in N$, then least value of $(a + b)$ is equal to

(A) 11 (B) 12 (C) 18 (D) 19

Sol. $AB = \begin{bmatrix} \frac{x^2+1}{x} & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & 16 \end{bmatrix} \begin{bmatrix} \frac{5x}{x^2+1} & 0 & 0 \\ 0 & \frac{3}{x} & 0 \\ 0 & 0 & \frac{1}{4} \end{bmatrix} = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$

$(AB)^{-1} = \begin{bmatrix} \frac{1}{5} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{4} \end{bmatrix}, (AB)^{-2} = \begin{bmatrix} \frac{1}{5^2} & 0 & 0 \\ 0 & \frac{1}{3^2} & 0 \\ 0 & 0 & \frac{1}{4^2} \end{bmatrix}$ and so on

$$\therefore \quad X = \begin{bmatrix} \frac{1}{5}+\frac{1}{5^2}+\ldots\ldots\frac{1}{5^n} & 0 & 0 \\ 0 & \frac{1}{3}+\frac{1}{3^2}+\ldots\ldots\frac{1}{3^n} & 0 \\ 0 & 0 & \frac{1}{4}+\frac{1}{4^2}+\frac{1}{4^3}+\ldots\ldots+\frac{1}{4^n} \end{bmatrix}$$

$$Y = \lim_{n\to\infty} X = \begin{bmatrix} \frac{\frac{1}{5}}{1-\frac{1}{5}} & 0 & 0 \\ 0 & \frac{\frac{1}{3}}{1-\frac{1}{3}} & 0 \\ 0 & 0 & \frac{\frac{1}{4}}{1-\frac{1}{4}} \end{bmatrix} \Rightarrow Y = \begin{bmatrix} \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix}$$

$$\therefore \quad Y^{-1} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

(i) $$\left|\operatorname{adj}\left(\sqrt{5}\,Y^{-1}\right)\right| = \left|\left(\sqrt{5}\right)^2 \operatorname{adj}\left(Y^{-1}\right)\right| = 5^3 \left|\operatorname{adj}\left(Y^{-1}\right)\right| = 5^3 \cdot \left|Y^{-1}\right|^2 = 5^3 \cdot (4\cdot 2\cdot 3)^2 = 5\cdot(5\,!)^2$$

(ii) $$AY = \begin{bmatrix} x+\frac{1}{x} & 0 & 0 \\ 0 & \frac{1}{x} & 0 \\ 0 & 0 & 16 \end{bmatrix}\begin{bmatrix} \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix} = \begin{bmatrix} \frac{1}{4}\left(x+\frac{1}{x}\right) & 0 & 0 \\ 0 & \frac{1}{2x} & 0 \\ 0 & 0 & \frac{16}{3} \end{bmatrix}$$

$$\therefore \quad t_r(AY) = \frac{1}{4}x + \frac{1}{4x} + \frac{1}{2x} + \frac{16}{3}$$

$$= \frac{1}{4}x + \frac{3}{4x} + \frac{16}{3} \geq 2\cdot\sqrt{\frac{1}{4}\cdot x\cdot\frac{3}{4x}} + \frac{16}{3} \geq \frac{\sqrt{3}}{2}+\frac{16}{3} \geq 0.866 + 5.333 \geq 6.199$$

$\therefore$ Least integral value of $t_r(AY) = 7$.

(iii) $$Z = Y^{-1} - 2I = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} - 2\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\therefore \quad t_r(Z + Z^2 + Z^3 + \ldots\ldots\ldots\ldots Z^{10}) = 2 + 2^2 + 2^3 + \ldots\ldots\ldots\ldots + 2^{10} + 10$$

$$= 2\cdot\left(\frac{2^{10}-1}{2-1}\right) + 10 = 2^{11} + 8 = 2^a + b$$

$\therefore$ a + b = 11 + 8 = 19 Ans.

Paragraph for question nos. 25 to 27

Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ satisfy $A^n = A^{n-2} + A^2 - I \qquad \forall\, n \ge 3$

Further consider a matrix $\underset{3\times 3}{U}$ with its column as U_1, U_2, U_3 such that

$$A^{50}U_1 = \begin{bmatrix} 1 \\ 25 \\ 25 \end{bmatrix}; \qquad A^{50}\,U_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}; A^{50}\,U_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Then answer the following

Q.25 Trace of A^{50} equals

(A) 0 (B) 1 (C) 2 (D) 3

Q.26 $|\text{Adj Adj Adj Adj Adj } A^{50}| =$

(A) 2^n (B) 2^5 (C) 2^{25} (D) 2^{50}

Q.27 Find sum of all entries of $UA^{50}U^{-1}$

(A) 50 (B) 51 (C) 52 (D) 53

Sol. $A^n - A^{n-2} = A^2 - I$

$A^{50} - A^{48} = A^2 - I$

$A^{48} - A^{46} = A^2 - I$

.. .. ..

.. .. ..

$A^4 - A^2 = A^2 - I$

$A^{50} - A^2 = 24A^2 - 24I$

$$A^{50} = 25A^2 - 24I \Rightarrow A^{50} = \begin{bmatrix} 1 & 0 & 0 \\ 25 & 1 & 0 \\ 25 & 0 & 1 \end{bmatrix}$$

$|A^{50}| = 1 \qquad ; \qquad \text{tr } A^{50} = 3$

$$A^{50}U_1 = \begin{bmatrix} 1 \\ 25 \\ 25 \end{bmatrix}; \begin{bmatrix} 1 & 0 & 0 \\ 25 & 1 & 0 \\ 25 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 25 \\ 25 \end{bmatrix}$$

$$\Rightarrow \quad U_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \text{ similarly } \quad U_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}; U_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \text{ So, } \underset{3\times 3}{U} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$ **Ans.**

Q.28 Consider $I_{n,m} = \int_0^1 \frac{x^n}{x^m - 1} dx$ and $J_{n,m} = \int_0^1 \frac{x^n}{x^m + 1} dx$ $\forall$ $n > m$ and $n, m \in N$.

(a) Consider a matrix $A = [a_{ij}]_{3 \times 3}$, where $a_{ij} = \begin{cases} I_{6+i,3} - I_{i+3,3}, & i = j \\ 0, & i \neq j \end{cases}$. Then find trace (A^{-1}).

[**Note:** Trace of a square matrix is sum of the diagonal elements.]

(b) Let $A = \begin{bmatrix} J_{6,5} & 72 & J_{11,5} \\ J_{7,5} & 63 & J_{12,5} \\ J_{8,5} & 56 & J_{13,5} \end{bmatrix}$ and $B = \begin{bmatrix} I_{6,5} & 72 & I_{11,5} \\ I_{7,5} & 63 & I_{12,5} \\ I_{8,5} & 56 & I_{13,5} \end{bmatrix}$,

then find the value of $\det(A) - \det(B)$.

[Ans. (a) 18; (b) 0]

Sol.

(a) Given, $I_{n,m} = \int_0^1 \frac{x^n}{x^m - 1} dx$ and $A = [a_{ij}]_{3 \times 3}$, where $a_{ij} = \begin{cases} I_{6+i,3} - I_{i+3,3}, & i = j \\ 0, & i \neq j \end{cases}$.

Hence, $A = \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix}$

Now, $a_{11} = I_{7,3} - I_{4,3} = \int_0^1 x^4 \frac{(x^3 - 1)}{(x^3 - 1)} dx = \frac{1}{5}$

$|||^{ly}$, $a_{22} = I_{8,3} - I_{5,3} = \int_0^1 x^5 \frac{(x^3 - 1)}{(x^3 - 1)} dx = \frac{1}{6}$

and $a_{33} = I_{9,3} - I_{6,3} = \int_0^1 x^6 \frac{(x^3 - 1)}{(x^3 - 1)} dx = \frac{1}{7}$

Hence, $A = \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix} = \begin{bmatrix} \frac{1}{5} & 0 & 0 \\ 0 & \frac{1}{6} & 0 \\ 0 & 0 & \frac{1}{7} \end{bmatrix}$

$\therefore$ $A^{-1} = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 7 \end{bmatrix} \Rightarrow$ trace $(A^{-1}) = 5 + 6 + 7 = 18$. **Ans.**

(b) Clearly, $\det.(A) = \begin{vmatrix} J_{6,5} & 72 & J_{11,5} \\ J_{7,5} & 63 & J_{12,5} \\ J_{8,5} & 56 & J_{13,5} \end{vmatrix} = \begin{vmatrix} \frac{1}{7} & 72 & J_{11,5} \\ \frac{1}{8} & 63 & J_{12,5} \\ \frac{1}{9} & 56 & J_{13,5} \end{vmatrix}$

Applying $C_1 \to C_1 + C_3$ and using equation (2), we get

$$= \frac{1}{7\times8\times9}\begin{vmatrix} 72 & 72 & J_{11,5} \\ 63 & 63 & J_{12,5} \\ 56 & 56 & J_{13,5} \end{vmatrix} = 0. \qquad \text{(As } C_1 \text{ and } C_2 \text{ are identical.)}$$

Similarly, $\det.(B) = \begin{vmatrix} I_{6,5} & 72 & I_{11,5} \\ I_{7,5} & 63 & I_{12,5} \\ I_{8,5} & 56 & I_{13,5} \end{vmatrix} = \begin{vmatrix} I_{6,5} & 72 & \frac{1}{7} \\ I_{7,5} & 63 & \frac{1}{8} \\ I_{8,5} & 56 & \frac{1}{9} \end{vmatrix}$

Applying $C_3 \to C_3 - C_1$ and using equation (1), we get

$$= \frac{1}{7\times8\times9}\begin{vmatrix} I_{6,5} & 72 & 72 \\ I_{7,5} & 63 & 63 \\ I_{8,5} & 56 & 56 \end{vmatrix} = 0 \qquad \text{(As, } C_2 \text{ and } C_3 \text{ are identical.)}$$

LIMIT

1. CONCEPT OF LIMITS :

Suppose f(x) is a real-valued function c is a real number. The expression $\lim_{x \to c} f(x) = L$ means that f(x) can be as close to L as desired by making x sufficiently close to c. In such a case, we say that limit of f, as x approaches c, is L. Note that this statement is true even if $f(c) \neq L$. Indeed, the function f(x) need not even be defined at c. Two examples help illustrate this.

Consider $f(x) = x - 1$ as x approaches 2. In this case, f(x) is defined at 2, and it equals its limiting value 1.

f(1.9)	f(1.99)	f(1.999)	f(2)	f(2.001)	f(2.01)	f(2.1)
0.9	0.99	0.999	$\rightarrow 1 \leftarrow$	1.001	1.01	1.1

As x approaches 2, f(x) approaches 1 and hence we have $\lim_{x \to 2} f(x) = 1$.

$f(x) = \dfrac{x^2 - 4}{x - 2}$ in this case x approaches 2 the limiting value of f(x) is equal to 4 even if f(x) is not defined at $x = 2$.

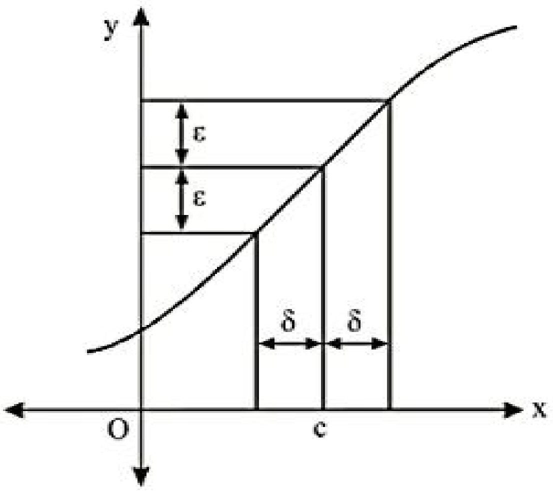
f(1.9)	f(1.99)	f(1.999)	f(2.0)	f(2.001)	f(2.01)	f(2.1)
3.9	3.99	3.999	$\Rightarrow$ undefined $\Leftarrow$	4.001	4.01	4.10

Thus, f(x) can be made arbitrarily close to the limit of 4 just by making x sufficiently close to 2.

Formal Definition of Limit :

Karl Weierstrass formally defined limit as follows :

Let f be a function defined on an open interval containing c (except possibly at c) and let L be a real number.



$\lim_{x \to c} f(x) = L$ means that for each real $\varepsilon > 0$ there exists a real $\delta > 0$ such that for all x with $0 < |x - c| < \delta$, we have $|f(x) - L| < \varepsilon$ or, symbolically.

$\forall \varepsilon > 0, \exists \delta > 0, \forall x (0 < |x - c| < \delta \Rightarrow |f(x) - L| < \varepsilon)$

Neighbourhood (NBD) of a point :

Let 'a' be a real number and let be a positive real number. Then the set of all real numbers lying between $a-\delta$ and $a+\delta$ is called the neighbourhood of 'a' of radius 'δ' and is denoted by $N_\delta(a)$.
Thus, $N_\delta(a) = (a-\delta, a+\delta) = \{x \in R | a-\delta < x < a+\delta\}$
The set $(a-\delta, a)$ is called the left NBD of 'a' and the set $(a, a+\delta)$ is known as the right NBD of 'a'.

Left-and Right-Hand Limits :

Let $f(x)$ be a function with domain D and let 'a' be a point such that every NBD of 'a' contains infinitely many points of D. A real number l is called left limit of $f(x)$ at $x = a$ iff for every $\varepsilon > 0$ then exists a $\delta > 0$ such that $a-\delta < x < a \Rightarrow |f(x)-l| < \varepsilon$

In such a case, we write $\lim\limits_{x\to a^-} f(x) = l$.

Thus, $\lim\limits_{x\to a^-} f(x) = l$, if $f(x)$ tends to l as x tends to 'a' from the left-hand side.

Similarly, a real number l' is a right limit of f(x) at $x = a$ iff for every $\varepsilon > 0$, there exists $\delta > 0$ such that $a < x < a+\delta \Rightarrow |f(x)-l| < \varepsilon$ and we write $\lim\limits_{x\to a^+} f(x) = l'$.

In order words, l' is a right limit of $f(x)$ at $x = a$ iff f(x) tends to l' as x tends to 'a' from the right-hand side.

Existence of Limit :

If follows from the discussions made in the previous two sections that $\lim\limits_{x\to a} f(x)$ exists if $\lim\limits_{x\to a^-} f(x)$ and $\lim\limits_{x\to a^+} f(x)$ exists and both are equal.

Thus, $\lim\limits_{x\to a} f(x)$ exists $\Leftrightarrow \lim\limits_{x\to a^-} f(x) = \lim\limits_{x\to a^+} f(x)$.

$$f(a^-) = \lim_{h\to 0} f(a-h)$$

$$f(a^+) = \lim_{h\to 0} f(a+h)$$

$$\lim_{x\to a} f(x) \text{ exists } \Leftrightarrow f(a^-) = f(a^+)$$

One sided limit :

Let the function $f(x)$ is defined in $x \in [a, b]$. Sometime we need to calculate $\lim\limits_{x\to b} f(x)$ or $\lim\limits_{x\to a} f(x)$. In such $\lim\limits_{x\to a} f(x) = \lim\limits_{x\to a^+} f(x) =$ RHL at $x = a$, as their is no left neighbourhood of $x = a$.

Similarly $\lim\limits_{x\to b} f(x) = \lim\limits_{x\to b^-} f(x) =$ LHL at $x = b$. As their is no right neighbourhood of $x = b$.

For example, $f(x) = \cos^{-1} x$

$$\lim_{x\to 1} f(x) = \lim_{x\to 1^-} \cos^{-1} x = 0$$

$$\lim_{x\to -1} f(x) = \lim_{x\to -1^+} \cos^{-1} x = \pi$$

Difference between limit of function at x = a and f (a) :

Case	y = f(x)	Explanation
$\lim\limits_{x\to a} f(x)$ exists but f(a) does not exist	$f(x)=\dfrac{x^2-a^2}{x-a}$	The value of function at x = a is of the form $\frac{0}{0}$ which is indeterminate, i.e., f (a) does not exist. But $\lim\limits_{x\to a^+} f(x)=\lim\limits_{x\to a^-} f(x)=2a$. Hence, $\lim\limits_{x\to a} f(x)$ exists.
$\lim\limits_{x\to a} f(x)$ does not exist but f (a) exists	f (x) = [x] where [·] denotes greatest integer function.	The value of function at x = n (n ∈ I) is n i.e. f(n) = n. But $\lim\limits_{x\to n^-}[x]=n-1$ and $\lim\limits_{x\to n^+}[x]=n$ Hence $\lim\limits_{x\to n}[x]=\text{DNE}$
$\lim\limits_{x\to a} f(x)$ and f(a) both exist and are equal	$f(x)=\begin{cases}\sin x, & x<0\\ x, & x\ge 0\end{cases}$	The value of function at x = 0 is 0, i.e., 0. Also $\lim\limits_{x\to 0^-} f(x)=\lim\limits_{x\to 0}\sin x=0$ and $\lim\limits_{x\to 0^+} f(x)=\lim\limits_{x\to 0^+} x=0$ $\Rightarrow \lim\limits_{x\to 0} f(x)$ exists
$\lim\limits_{x\to a} f(x)$ and f(a) both exist but are unequal	$f(x)=\begin{cases}\dfrac{x^2-9}{x-3}, & x\ne 3\\ 3, & x=3\end{cases}$	The value of function at x = 3 is 3. Also i.e. f(3) = 3. Also $\lim\limits_{x\to 3^-} f(x)=\lim\limits_{x\to 3^+} f(x)=\lim\limits_{x\to 3}\dfrac{x^2-9}{x-3}$ $=\lim\limits_{x\to 3}(x+3)=6$, i.e., $\lim\limits_{x\to 3} f(x)$ exists. But $\lim\limits_{x\to 3} f(x)\ne f(3)$

Thus, for limit to exist at x = a, it is not necessary that function is defined at that point.

Illustration :

Evaluate the following limits ([·] , {·} denotes greatest integer function and fractional part respectively)

(i) $\lim\limits_{x\to n}[x]$, $\lim\limits_{x\to n}\{x\}$, $n\in I$ (ii) $\lim\limits_{x\to 0}\dfrac{|x|}{x}$ (iii) $\lim\limits_{x\to 0}\tan^{-1}\dfrac{1}{x}$

(iv) $\lim\limits_{x\to 0}\dfrac{1}{\ln|x|}$ (v) $\lim\limits_{x\to 0}\cot^{-1}x^2$ (vi) $\lim\limits_{x\to 0}[x]+\sqrt{\{x\}}$ (vii) $\lim\limits_{x\to 1} x\,\text{sgn}(x-1)$

(viii) $\lim\limits_{x\to 1}\dfrac{x}{[x]}$ (ix) $\lim\limits_{x\to 1}\dfrac{[x]}{x}$ (x) $\lim\limits_{x\to 0}\sin^{-1}[\sec x]$

Sol.

(i) $\lim\limits_{x\to n^+} [x] = n,\ \lim\limits_{x\to n^-} [x] = n-1 \quad \Rightarrow \quad \lim\limits_{x\to n} [x] = DNE$ (Does not exists)

Similarly $\lim\limits_{x\to n^+} \{x\} = 0,\ \lim\limits_{x\to n^-} \{x\} = 1 \quad \Rightarrow \quad \lim\limits_{x\to n} \{x\} = DNE$ (Does not exists)

$\Rightarrow$ $f(x) = [x]$ and $\{x\}$ has no limit at all integers

(ii) $f(x) = \dfrac{|x|}{x}$ has no limit at $x = 0$ $<$ $\begin{matrix} f(0^+) = 1 \\ f(0^-) = -1 \end{matrix}$

(iii) $\lim\limits_{x\to 0} \tan^{-1}\dfrac{1}{x}$ does not exist at $x = 0$ $<$ $\begin{matrix} f(0^-) = \dfrac{-\pi}{2} \\ f(0^+) = \dfrac{\pi}{2} \end{matrix}$ even if $f(0)$ is not defined.

(iv) $\lim\limits_{x\to 0} \dfrac{1}{\ln|x|}$ exists at $x = 0$ $f(0^-) = f(0^+) = 0$ even if $f(0)$ is not defined.

(v) $\lim\limits_{x\to 0} \cot^{-1} x^2 = \dfrac{\pi}{2}$.

(vi) $f(x) = [x] + \sqrt{\{x\}}$ limit exists at $x = 0$ as $\lim\limits_{x\to 0^+} f(x) = 0+0 = 0$, $\lim\limits_{x\to 0^-} f(x) = -1+\sqrt{1} = 0$.

(vii) $\lim\limits_{x\to 1} x\ \mathrm{sgn}(x-1)$ does not exist $<$ $\begin{matrix} f(1^+) \to 1 \\ f(1^-) \to -1 \end{matrix}$ at $x = 1$.

(viii) $\lim\limits_{x\to 1} \dfrac{x}{[x]} = \lim\limits_{x\to 1^+} \dfrac{x}{[x]} = \lim\limits_{x\to 1^+} \dfrac{x}{1} = 1,\ \lim\limits_{x\to 1^-} \dfrac{x}{[x]}$ is undefined

(ix) $\lim\limits_{x\to 1} \dfrac{[x]}{x} = D.N.E.$ as $\lim\limits_{x\to 1^+} \dfrac{[x]}{x} = 1$, $\lim\limits_{x\to 1^-} \dfrac{[x]}{x} = 0$

(x) $\lim\limits_{x\to 0} \sin^{-1}[\sec x]$ where [ ] denotes greatest integer function, exists and is equal to $\pi/2$.

Illustration :

Consider the function: $f(x) = \begin{cases} \dfrac{1}{x^2} & \text{if } x < -1 \\ 2, & \text{if } -1 \le x < 1 \\ 3, & \text{if } x = 1 \\ x+1, & \text{if } 1 < x \le 2 \\ \dfrac{-1}{(x-2)^2}, & \text{if } x > 2 \end{cases}$

(i) Sketch the graph of f.

(ii) Determine the following limits.

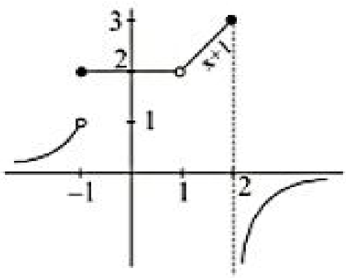
(a) $\lim\limits_{x \to -1^+} f(x)$ (b) $\lim\limits_{x \to -1^-} f(x)$ (c) $\lim\limits_{x \to -1} f(x)$ (d) $\lim\limits_{x \to 1^+} f(x)$

(e) $\lim\limits_{x \to 1^-} f(x)$ (f) $\lim\limits_{x \to 1} f(x)$ (g) $\lim\limits_{x \to 2^+} f(x)$ (h) $\lim\limits_{x \to 2^-} f(x)$

(i) $\lim\limits_{x \to 2} f(x)$ (j) $\lim\limits_{x \to -3} f(x)$ (k) $\lim\limits_{x \to 5} f(x)$ (l) $\lim\limits_{x \to 1.5} f(x)$

Sol.

(i)



(ii) (a) $\lim\limits_{x \to -1^+} f(x) = \lim\limits_{x \to -1^+} 2 = 2$

(b) $\lim\limits_{x \to -1^-} f(x) = \lim\limits_{x \to -1^-} \dfrac{1}{x^2} = \lim\limits_{x \to -1^-} \dfrac{1}{(-1)^2} = 1$

(c) $\lim\limits_{x \to -1} f(x) = DNE$

(d) $\lim\limits_{x \to 1^+} f(x) = \lim\limits_{x \to 1^+} (x+1) = 2$

(e) $\lim\limits_{x \to 1^-} f(x) = \lim\limits_{x \to 1^+} 2 = 2$

(f) $\lim\limits_{x \to 1} f(x) = DNE$

(g) $\lim\limits_{x \to 2^+} f(x) = \lim\limits_{x \to 2^+} \dfrac{-2}{(x-2)^2} = DNE.$

(h) $\lim\limits_{x \to 2^-} f(x) = \lim\limits_{x \to 2^-} (x+1) = 3$

(i) $\lim\limits_{x \to 2} f(x) = DNE$

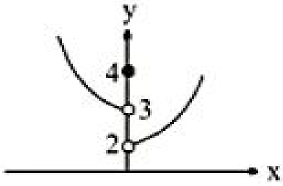
(j) $\lim\limits_{x \to -3} f(x) = \lim\limits_{x \to -3} \dfrac{1}{x^2} = \dfrac{1}{9}$

(k) $\lim\limits_{x \to 5} f(x) = \lim\limits_{x \to 5} \dfrac{-1}{(x-2)^2} = \dfrac{-1}{9}$

(l) $\lim\limits_{x \to 1.5} f(x) = \lim\limits_{x \to 1.5} x+1 = 2.5.$

Illustration :

Refer the figure,



the value of λ *for which* $2\left(\lim_{x\to 0} f(x^3-x^2)\right) = \lambda\left(\lim_{x\to 0} f(2x^4-x^5)\right)$ *is*

(A) $\frac{4}{3}$ (B) 2 (C) 3 (D) 5

Sol. $\lim_{x\to 0} f\left(x^3-x^2\right) = \lim_{x\to 0^+} f\left(x^3-x^2\right) = \lim_{x\to 0^-} f\left(x^3-x^2\right) = f(0^-) = 3$

$\lim_{x\to 0} f\left(x^4-x^5\right) = \lim_{x\to 0^+} f\left(x^4-x^5\right) = \lim_{x\to 0^-} f\left(x^4-x^5\right) = f(0^+) = 2$

$2\lim_{x\to 0} f\left(x^3-x^2\right) = x\lim_{x\to 0} f\left(2x^4-x^5\right) \Rightarrow 2(3) = \lambda\,(2) \Rightarrow \lambda = 3$. **Ans.**

Illustration :

Evaluate the left and right-hand limits of the function $f(x) = \begin{cases} \dfrac{|x-4|}{x-4}, & x \neq 4 \\ 0, & x = 4 \end{cases}$.

Sol. *L.H.L of f(x) at x = 4*

$$= \lim_{x\to 4^-} f(x) = \lim_{h\to 0} f(4-h) = \lim_{h\to 0} \frac{|4-h-4|}{4-h-4} = \lim_{h\to 0} \frac{|-h|}{-h} = \lim_{h\to 0} \frac{h}{-h} = \lim_{h\to 0} -1 = -1$$

R.H.L. of f(x) at x = 4

$$= \lim_{x\to 4^+} f(x) = \lim_{h\to 0} f(4+h) = \lim_{h\to 0} \frac{|4+h-4|}{4+h-4} = \lim_{h\to 0} \frac{|h|}{h} = \lim_{h\to 0} \frac{h}{h} = \lim_{h\to 0} 1 = 1$$

Illustration :

Evaluate the left and the right-hand limits of the function defined by $f(x) = \begin{cases} 1+x^2, & if\ 0 \le x < 1 \\ 2-x, & if\ x > 1 \end{cases}$

at x = 1. Also, show that $\lim_{x\to 1} f(x)$ *does not exist.*

Sol. *L.H.L. of f (x) at x = 1*

$$= \lim_{x\to 1^-} f(x) = \lim_{x\to 0} f(1-h) = \lim_{x\to 0} 1+(1-h)^2 = \lim_{h\to 0} 2-2h+h^2 = 2$$

R.H.L. of f (x) at x = 1.

$$= \lim_{x\to 1^+} f(x) = \lim_{h\to 0} f(1+h) = \lim_{x\to 0} [2-(1+h)] = \lim_{h\to 0} (1-h) = 1$$

Clearly, $\lim_{x\to 1^-} f(x) \neq \lim_{x\to 1^+} f(x)$

So, $\lim_{x\to 1} f(x)$ *does not exist.*

Illustration :

Let $f(x) = \begin{cases} \cos x, \text{if } x \geq 0 \\ x + k, \text{if } x < 0 \end{cases}$. *Find the value of constant k, given that* $\lim_{x \to 0} f(x)$ *exists.*

Sol. $\lim_{x \to 0} f(x)$ *exists*

$$\Rightarrow \lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) \Rightarrow \lim_{x \to 0} x + k = \lim_{x \to 0} \cos x \Rightarrow 0 + k = \cos 0 \Rightarrow k = 1.$$

Practice Problem

Q.1 If $f(x) = \begin{cases} \dfrac{x - |x|}{2}, & x \neq 0 \\ 2, & x = 0 \end{cases}$ show that $\lim_{x \to 0} f(x)$ does not exist.

Q.2 Show that $\lim_{x \to 0} \dfrac{e^{1/x} - 1}{e^{1/x} + 1}$ does not exist.

Q.3 Evaluate $\lim_{x \to 0} \dfrac{3x + |x|}{7x - 5|x|}$.

Q.4 If $f(x) = \begin{cases} x, & x < 0 \\ 1, & x = 0 \\ x^2, & x > 0 \end{cases}$, then find $\lim_{x \to 0} f(x)$ if exists.

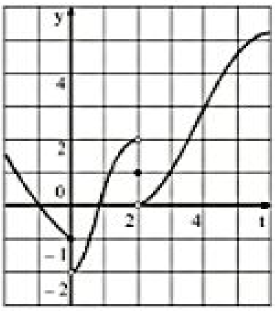
Q.5 For the function g whose graph is given, state the value of each quantity, if it exists.

(a) $\lim_{t \to 0^-} g(t)$ (b) $\lim_{t \to 0^+} g(t)$

(c) $\lim_{t \to 0} g(t)$ (d) $\lim_{t \to 2^-} g(t)$

(e) $\lim_{t \to 2^+} g(t)$ (f) $\lim_{t \to 2} g(t)$

(g) g(2) (h) $\lim_{t \to 4} g(t)$



Q.6 For the function R whose graph is shown, state the following

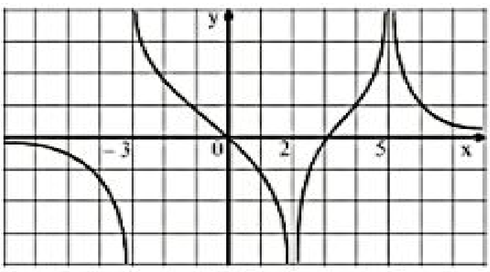
(a) $\lim_{x \to 2} R(x)$

(b) $\lim_{x \to 5} R(x)$

(c) $\lim_{x \to 3^-} R(x)$

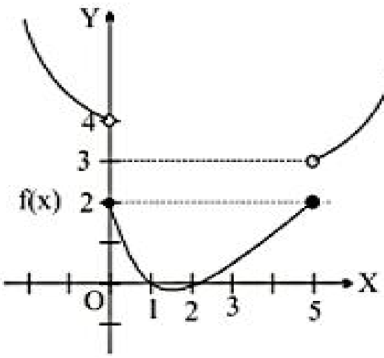
(d) $\lim_{x \to 3^+} R(x)$

(e) The equations of the vertical asymptotes.



Q.7 Refer to the graph of $y = f(x)$
and $g(x) = (x-2)^2, x < 2$
$= 7 - x, x \geq 2$
then which of the following limits are non existent.

(a) $\lim\limits_{x \to 2} f(g(x))$ (b) $\lim\limits_{x \to 0} g(f(x))$ (c) $\lim\limits_{x \to 5} g(f(x))$



Answer key

Q.3 Does not exists **Q.4** **0**
Q.5 (a) – 1 ; (b) – 2 ; (c) does not exist ; (d) 2 ; (e) 0 ; (f) does not exist ; (g) 1 ; (h) 3
Q.6 (a) $-\infty$; (b) ∞ ; (c) 0 ; (d) 0 ; (e) $x = 5$; $x = 2$; $x + 3 = 0$ **Q.7** b, c

THE ALGEBRA OF LIMITS :

Let $\lim\limits_{x \to a} f(x) = l$ and $\lim\limits_{x \to a} g(x) = m$. If l and m exist, then

1. $\lim\limits_{x \to a} (f \pm g)(x) = \lim\limits_{x \to a} f(x) \pm \lim\limits_{x \to a} g(x) = l \pm m$

2. $\lim\limits_{x \to a} (fg)(x) = \lim\limits_{x \to a} f(x) \lim\limits_{x \to a} g(x) = lm$

3. $\lim\limits_{x \to a} \left(\frac{f}{g}\right)(x) = \frac{\lim\limits_{x \to a} f(x)}{\lim\limits_{x \to a} g(x)} = \frac{l}{m}$, provided $m \neq 0$

4. $\lim\limits_{x \to a} k\, f(x) = k. \lim\limits_{x \to a} f(x)$, where k is a constant

5. $\lim\limits_{x \to a} |f(x)| = \left|\lim\limits_{x \to a} f(x)\right| = |l|$

6. $\lim\limits_{x \to a} (f(x))^{g(x)} = \lim\limits_{x \to a} f(x)^{\lim\limits_{x \to a} g(x)} = l^m$

7. $\lim\limits_{x \to a} fog(x) = f\left(\lim\limits_{x \to a} g(x)\right) = f(m)$, only if f is continous at $g(x) = m$
In particular,

(a) $\lim\limits_{x \to a} \log f(x) = \log\left(\lim\limits_{x \to a} f(x)\right) = \log l$ (b) $\lim\limits_{x \to a} e^{f(x)} = e^{\lim\limits_{x \to a} f(x)} = e^l$

8. If $\lim\limits_{x \to a} f(x) = +\infty$ or $-\infty$, then $\lim\limits_{x \to a} \frac{1}{f(x)} = 0$.

9. If $f(x) \leq g(x)$ for every x in the NBD of a, then $\lim\limits_{x \to a} f(x) \leq \lim\limits_{x \to a} g(x)$.

Points to Remember :

1. If $\lim\limits_{x\to c} f(x)\,g(x)$ exists, then we can have the following cases :

(a) Both $\lim\limits_{x\to c} f(x)$ and $\lim\limits_{x\to c} g(x)$ exist. Obviously, then $\lim\limits_{x\to c} f(x)\,g(x)$ exists.

(b) $\lim\limits_{x\to c} f(x)$ exists and $\lim\limits_{x\to c} g(x)$ does not exist.

Consider $f(x) = x;\ g(x) = \dfrac{1}{x}$, now $\lim\limits_{x\to 0} f(x)\cdot g(x)$ exists $= 1$. Also $\lim\limits_{x\to 0} f(x) = 0$ exists but $\lim\limits_{x\to 0} g(x)$ does not exist.

(c) Both $\lim\limits_{x\to c} f(x)$ and $\lim\limits_{x\to c} g(x)$ do not exist.

Let f be defined as $f(x) = \begin{cases} 1 & \text{if } x \le 0 \\ 2 & \text{if } x > 0 \end{cases}$. Let $g(x) = \begin{cases} 2 & \text{if } x \le 0 \\ 1 & \text{if } x > 0 \end{cases}$.

Then $f(x)\,g(x) = 2$, and so $\lim\limits_{x\to 0} f(x)\times g(x)$ exists, while $\lim\limits_{x\to 0} f(x)$ and $\lim\limits_{x\to 0} g(x)$ do not exist.

2. If $\lim\limits_{x\to c}[f(x)+g(x)]$ exists then we can have the following cases :

(a) If $\lim\limits_{x\to c} f(x)$ exists, then $\lim\limits_{x\to c} g(x)$ must exist.

Proof : This is true as $g = (f+g) - f$.

Therefore, by the limit theorem, $\lim\limits_{x\to c} g(x) = \lim\limits_{x\to c}(f(x)+g(x)) - \lim\limits_{x\to 0} f(x)$ which exists.

(b) Both $\lim\limits_{x\to c} f(x)$ and $\lim\limits_{x\to c} g(x)$ do not exist.

Consider $\lim\limits_{x\to 1}[x]$ and $\lim\limits_{x\to 1} g[x]$, where $[\cdot]$ and $\{\cdot\}$ represent greatest integer and fractional part function, respectively. Here both the limits do not exist but $\operatorname{Lim}\limits_{x\to 1}[x] + \{x\} = \lim\limits_{x\to 1} x = 1$ exists.

For Example :

(a) $f(x) = \dfrac{1}{\sin x}$ and $g(x) = \dfrac{1}{\tan x}$ at $x = 0$ $\operatorname{Lim}\limits_{x\to 0} f(x) = \text{DNE}$, $\operatorname{Lim}\limits_{x\to 0} g(x) = \text{DNE}$

But $\operatorname{Lim}\limits_{x\to 0}\left(\dfrac{1}{\sin x} - \dfrac{1}{\tan x}\right) = 0 = \operatorname{Lim}\limits_{x\to 0}\dfrac{1-\cos x}{\sin x}$

$$= \operatorname{Lim}_{x\to 0}\frac{2\sin^2\frac{x}{2}}{2\sin\frac{x}{2}\cos\frac{x}{2}} = \operatorname{Lim}_{x\to 0}\tan\frac{x}{2} = 0 \text{ (exist).}$$

(b) $f(x) = \text{sgn } x$ and $g(x) = [x]$ then $\lim\limits_{x\to 0}(\text{sgn } x + [x])$ does not exist

as $\lim\limits_{x\to 0^+} \sin x + [x] = 1$, $\lim\limits_{x\to 0^-} \sin x + [x] = -2$

while $\lim\limits_{x\to 0} f(x) = \text{DNE}$, $\lim\limits_{x\to 0} g(x) = \text{DNE}$.

(c) $f(x) = [x]$ and $g(x) = \{x\}$; $F(x) = [x]\{x\}$

$\lim\limits_{x\to 0}[x]\cdot\{x\}$ does not exist but $\lim\limits_{x\to 1}[x]\{x\}$ exist and is equal to zero.

(d) If $f(x) = e^{[x]}$; $g(x) = e^{\{x\}}$ then $\lim\limits_{x\to 0} e^{[x]}\cdot e^{\{x\}} = \lim\limits_{x\to 0} e^{x}$ which exist.

Illustration :

Let $f(x) = \begin{cases} x+1, & x>0 \\ 2-x, & x\le 0 \end{cases}$ *and* $g(x) = \begin{cases} x+3, & x<1 \\ x^2-2x-2, & 1\le x<2 \\ x-5, & x\ge 2 \end{cases}$. *Find L.H.L. and R.H.L. of g(f(x)) at x = 0 and hence find* $\lim\limits_{x\to 0} g(f(x))$

Sol. *As* $x \to 0^-$ $\Rightarrow f(x) \to f(0^-) = 2^+ \Rightarrow \lim\limits_{x\to 0^-} g(f(x)) = g(2^+) = -3$

Also as $x \to 0^+$ $\Rightarrow$ $f(x) \to f(0^+) = 1^+$ $\Rightarrow$ $\lim\limits_{x\to 0^+} g(f(x)) = g(1^+) = -3$

Hence, $\lim\limits_{x\to 0} g(f(x))$ *exists and is equal to* $-3 \Rightarrow \lim\limits_{x\to 0} g(f(x)) = -3$

Indeterminate forms :

Indeterminant forms are $\frac{0}{0}$, $\frac{\infty}{\infty}$, $\infty-\infty$, $0\times\infty$, 1^∞, 0^0 and ∞^0

EVALUATION OF ALGEBRAIC LIMITS :

(i) Direct Substitution Method :

Consider the following limits : (i) $\lim\limits_{x\to a} f(x)$ (ii) $\lim\limits_{x\to a}\frac{\Phi(x)}{\Psi(x)}$

If f(a) and $\frac{\Phi(a)}{\Psi(a)}$ exist and are fixed real numbers and $\Psi(a) \ne 0$ then we say that $\lim\limits_{x\to a} f(x) = f(a)$ and

$\lim\limits_{x\to a}\frac{\Phi(x)}{\Psi(x)} = \frac{\Phi(a)}{\Psi(a)}$.

Illustration :

Evaluate

(i) $\lim\limits_{x \to 1} 3x^2 + 4x + 5$ (ii) $\lim\limits_{x \to 2} \dfrac{x^2 - 4}{x + 3}$

Sol. (i) $\lim\limits_{x \to 1} 3x^2 + 4x + 5 = 3(1)^2 + 4(1) + 5 = 12$

(ii) $\lim\limits_{x \to 2} \dfrac{x^2 - 4}{x + 3} = \dfrac{4 - 4}{2 + 3} = \dfrac{0}{5} = 0$

(ii) Fractorization Method :

Consider $\lim\limits_{x \to a} \dfrac{f(x)}{g(x)}$

If by substituting $x = a$, $\dfrac{f(x)}{g(x)}$ reduces to the form $\dfrac{0}{0}$, then $(x - a)$ is a factor of both f(x) and g(x). So, we first factorize f(x) and g(x) and then cancel out the common factor to evaluate the limit.

Illustration :

Evaluate $\lim\limits_{x \to 2} \dfrac{x^2 - 5x + 6}{x^2 - 4}$.

Sol. $\lim\limits_{x \to 2} \dfrac{x^2 - 5x + 6}{x^2 - 4} = \lim\limits_{x \to 2} \dfrac{(x-2)(x-3)}{(x+2)(x-2)} = \lim\limits_{x \to 2} \dfrac{x-3}{x+2} = \dfrac{2-3}{2+2} = -\dfrac{1}{4}$

Illustration :

Evalute $\lim\limits_{x \to 1} \dfrac{x^2 + x\log_e x - \log_e x - 1}{(x^2 - 1)}$

Sol. $\lim\limits_{x \to 1} \dfrac{x^2 + x\log_e x - \log_e x - 1}{(x^2 - 1)} \left(\dfrac{0}{0} \text{ form}\right) = \lim\limits_{x \to 1} \dfrac{(x-1)(\log_e x + x + 1)}{(x+1)(x-1)} \left(\dfrac{0}{0} \text{ form}\right)$

$= \lim\limits_{x \to 1} \dfrac{\log_e x + x + 1}{(x+1)} = \dfrac{\log_e 1 + 1 + 1}{1 + 1} = \dfrac{0 + 2}{2} = 1$

Illustration :

Evaluate $\lim\limits_{x \to \frac{\pi}{4}} \dfrac{1 - \sin 2x}{1 + \cos 4x}$

Sol. $\lim\limits_{x \to \frac{\pi}{4}} \dfrac{1 - \sin 2x}{1 + \cos 4x} \left(\dfrac{0}{0} \text{ form}\right) = \lim\limits_{x \to \frac{\pi}{4}} \dfrac{(\sin x - \cos x)^2}{2\cos^2 2x} \left(\dfrac{0}{0} \text{ form}\right)$

$= \lim\limits_{x \to \frac{\pi}{4}} \dfrac{(\sin x - \cos x)^2}{2(\cos^2 x - \sin^2 x)^2} \left(\dfrac{0}{0} \text{ form}\right) = \lim\limits_{x \to \frac{\pi}{4}} \dfrac{1}{2(\cos x + \sin x)^2} = \dfrac{1}{4}$

(iii) Rationalization Method :

This is particularly used when either the numerator or the denominator or both involve expression consists of squares roots and on substituting the value of x the rational expression takes the form $\frac{0}{0}, \frac{\infty}{\infty}$.

Following examples illustrate the procedure.

Illustration :

Evaluate $\lim\limits_{x\to 0} \frac{\sqrt{2+x}-\sqrt{2}}{x}$

Sol. *When* $x = 0$, *the expression* $\frac{\sqrt{2+x}-\sqrt{2}}{x}$ *takes the form* $\frac{0}{0}$, *Rationalizing the numerator, we have*

$$\lim_{x\to 0} \frac{\sqrt{2+x}-\sqrt{2}}{x}$$

$$= \lim_{x\to 0} \frac{(\sqrt{2+x}-\sqrt{2})(\sqrt{2+x}+\sqrt{2})}{x(\sqrt{2+x}+\sqrt{2})} = \lim_{x\to 0} \frac{2+x-2}{(\sqrt{2+x}+\sqrt{2})x} = \lim_{x\to 0} \frac{1}{\sqrt{2+x}+\sqrt{2}} = \frac{1}{2\sqrt{2}}$$

Illustration :

Evalute $\lim\limits_{x\to a} \frac{\sqrt{a+2x}-\sqrt{3x}}{\sqrt{3a+x}-2\sqrt{x}}$

Sol. $\lim\limits_{x\to a} \frac{\sqrt{a+2x}-\sqrt{3x}}{\sqrt{3a+x}-2\sqrt{x}} \left(\text{form } \frac{0}{0}\right)$

$$= \lim_{x\to a} \frac{(\sqrt{a+2x}-\sqrt{3x})(\sqrt{a+2x}+\sqrt{3x})}{(\sqrt{3a+x}-2\sqrt{x})(\sqrt{3a+x}+2\sqrt{2})} \frac{(\sqrt{3a+x}+2\sqrt{x})}{(\sqrt{a+2x}+\sqrt{3x})} \left(\text{form } \frac{0}{0}\right)$$

$$= \lim_{x\to a} \frac{(a+2x-3x)}{(3a+x-4x)} \frac{(\sqrt{3a+x}+2\sqrt{x})}{(\sqrt{a+2x}+\sqrt{3x})} = \lim_{x\to a} \frac{\sqrt{3a+x}+2\sqrt{x}}{3(\sqrt{a+2x}+\sqrt{3x})}$$

$$= \frac{\sqrt{3a+a}+2\sqrt{a}}{3(\sqrt{a+2a}+\sqrt{3a})} = \frac{1}{3} \times \frac{4\sqrt{a}}{2\sqrt{3a}} = \frac{2}{3\sqrt{3}}$$

(iv) Evalutation of Algebraic Limit Using Some Standard Limits :

Recall the binomial expansion for any rational power

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 \ldots\ldots$$

where $|x| < 1$

When x is infinitely small (approaching to zero) such that we can ignore higher powers of x, then we have $(1+x)^n = 1 + nx$ (approximately).

Following theorem will be used to evaluate some algebraic limits :

Theorem : If $n \in Q$, then $\lim\limits_{x \to a} \dfrac{x^n - a^n}{x - a} = na^{n-1}$

Proof : We have $\lim\limits_{x \to a} \dfrac{x^n - a^n}{x - a}$

$$= \lim_{x \to a^+} \frac{x^n - a^n}{x - a} = \lim_{h \to 0} \frac{(a+h)^n - a^n}{a+h-a} = \lim_{h \to 0} \frac{a^n\left\{\left(1+\frac{h}{a}\right)^n - 1\right\}}{h} \text{ [when } x \to 0, (1+x)^n \to 1 + nx]$$

$$= a^n \lim_{h \to 0} \frac{\left\{1 + n\frac{h}{a}\right\} - 1}{h} = a^n \frac{n}{a} = na^{n-1}$$

Illustration :

Evaluate $\lim\limits_{x \to 2} \dfrac{x^{10} - 1024}{x^5 - 32}$

Sol. $$\lim_{x \to 2} \frac{x^{10} - 1024}{x^5 - 32} = \lim_{x \to 2} \frac{x^{10} - 2^{10}}{x^5 - 2^5} = \lim_{x \to 2} \frac{\frac{x^{10} - 2^{10}}{x-2}}{\frac{x^5 - 2^5}{x-2}} = \frac{\lim\limits_{x \to 2} \frac{x^{10} - 2^{10}}{x-2}}{\lim\limits_{x \to 2} \frac{x^5 - 2^5}{x-2}} = \frac{10 \times 2^{10-1}}{5 \times 2^{5-1}} = 64$$

Illustration :

Evaluate $\lim\limits_{x \to 2} \dfrac{\sqrt{(x+7)} - 3\sqrt{(2x-3)}}{\sqrt[3]{(x+6)} - 2\sqrt[3]{(3x-5)}}$

Sol. *We have* $L = \lim\limits_{x \to 2} \dfrac{\sqrt{(x+7)} - 3\sqrt{(2x-3)}}{\sqrt[3]{(x+6)} - 2\sqrt[3]{(3x-5)}} \left(\dfrac{0}{0} \text{ form}\right)$

Let $x - 2 = t$ *such that when* $x \to 2$, $t \to 0$

Then $L = \lim\limits_{t \to 0} \dfrac{(t+9)^{\frac{1}{2}} - 3(2t+1)^{\frac{1}{2}}}{(t+8)^{\frac{1}{3}} - 2(3t+1)^{\frac{1}{3}}} \left(\dfrac{0}{0} \text{ form}\right)$

$$= \frac{3}{2} \lim_{t \to 0} \frac{\left(1+\frac{t}{9}\right)^{\frac{1}{2}} - (2t+1)^{\frac{1}{2}}}{\left(1+\frac{t}{8}\right)^{\frac{1}{3}} - (3t+1)^{\frac{1}{3}}} \left(\frac{0}{0} \text{ form}\right) = \frac{3}{2} \lim_{t \to 0} \frac{\frac{1}{2}\frac{t}{9} - (2t)\frac{1}{2}}{\frac{t}{8}\frac{1}{3} - (3t)\frac{1}{3}} = \frac{3}{2} \frac{\left(\frac{1}{18} - 1\right)}{\left(\frac{1}{24} - 1\right)} = \frac{34}{23}$$

(v) Evaluation of Algebraic Limits at Infinity :

We know that $\lim\limits_{x\to+\infty}\frac{1}{x}=0$ and $\lim\limits_{x\to+\infty}\frac{1}{x^2}=0$

$\therefore \quad \lim\limits_{x\to\infty} f(x)=\lim\limits_{y\to 0} f\left(\frac{1}{y}\right)$

Illustration :

Evaluate $\lim\limits_{x\to\infty}\dfrac{ax^2+bx+c}{dx^2+ex+f}$.

Sol. *Herethe expression assumes the form* $\frac{\infty}{\infty}$. *We notice that the highest power of x in both the numberator and the denominator is 2. So we divide each term in both the numerator and denominator by* x^2.

$$\therefore \quad \lim_{x\to\infty}\frac{ax^2+bx+c}{dx^2+ex+f}=\lim_{x\to\infty}\frac{a+\frac{b}{x}+\frac{c}{x^2}}{d+\frac{e}{x}+\frac{f}{x^2}}=\frac{a+0+0}{d+0+0}=\frac{a}{d}$$

Illustration :

Eavalue $\lim\limits_{x\to\infty}\dfrac{\sqrt{3x^2-1}-\sqrt{2x^2-1}}{4x+3}$

Sol. $$\lim_{x\to\infty}\frac{\sqrt{3x^2-1}-\sqrt{2x^2-1}}{4x+3}=\operatorname*{Lim}_{x\to\infty}\frac{|x|\sqrt{3-\frac{1}{x^2}}-|x|\sqrt{2-\frac{1}{x^2}}}{4x+3}$$

$$=\lim_{x\to\infty}\frac{\sqrt{3-1/x^2}-\sqrt{2-1/x^2}}{4+3/x}=\frac{\sqrt{3}-\sqrt{2}}{4}$$

Illustration :

Evaluate $\lim\limits_{x\to\infty}\dfrac{\sqrt{x^2+1}-\sqrt[3]{x^3+1}}{\sqrt[4]{x^4+1}-\sqrt[5]{x^4+1}}$

Sol. *We have* $\lim\limits_{x\to\infty}\dfrac{\sqrt{x^2+1}-\sqrt[3]{x^3+1}}{\sqrt[4]{x^4+1}-\sqrt[5]{x^4+1}}$

$$\lim_{x\to\infty}\frac{\sqrt{1+\frac{1}{x^2}}-\sqrt[3]{1+\frac{1}{x^3}}}{\sqrt[4]{1+\frac{1}{x^4}}-\sqrt[5]{\frac{1}{x}+\frac{1}{x^5}}}=\frac{1-1}{1-0}=0$$

Illustration :

Evaluate $\lim\limits_{x\to-\infty}\left(\sqrt{25x^2-3x}+5x\right)$

Sol. We have $\lim\limits_{x\to-\infty}\left(\sqrt{25x^2-3x}+5x\right)$ $\quad(\infty-\infty\,form)$

$$=\lim_{y\to\infty}\left(\sqrt{25y^2+3y}-5y\right),\ where\ y=-x$$

$$=\lim_{y\to\infty}\frac{25y^2+3y-25y^2}{\sqrt{25y^2+3y}+5y}=\lim_{y\to\infty}\frac{3y}{\sqrt{25y^2+3y}+5y}$$

$$=\lim_{y\to\infty}\frac{3}{\sqrt{25+\frac{3}{y}}+5}=\frac{3}{5+5}=\frac{3}{10}$$

Illustration :

If $\lim\limits_{x\to0}\dfrac{\sqrt{ax+b}-2}{x}=1$ find a and b.

Sol. $\lim\limits_{x\to0}\dfrac{\sqrt{ax+b}-2}{x}=1$

For limit to exist, $\sqrt{0+b}-2=0\ \Rightarrow b=4$

$$\lim_{x\to0}\frac{\sqrt{ax+4}-2}{x}=1\ \Rightarrow\ 2\lim_{x\to0}\frac{\sqrt{1+\frac{ax}{4}}-1}{x}=1$$

$$\Rightarrow 2\lim_{x\to0}\frac{1+\frac{ax}{2\cdot4}-1}{x}=1\ \Rightarrow\ 2\cdot\frac{a}{8}=1\ \Rightarrow a=4.$$ ***Ans.***

Illustration :

(i) $\lim\limits_{x\to-2}\left(\dfrac{1}{x+2}-\dfrac{12}{x^3+8}\right)$ $(\infty-\infty)$ form $\quad$ (ii) $\lim\limits_{x\to\infty}\left(\sqrt{4x^2+x}-\sqrt{\dfrac{4x^3}{x+2}}\right)$ $(\infty-\infty)$ form

Sol.

(i) $$\lim_{x\to-2}\left(\frac{1}{x+2}-\frac{12}{x^3+8}\right)=\lim_{x\to-2}\frac{x^2-2x+4-12}{x^3+8}$$

$$=\lim_{x\to-2}\frac{x^2-2x-8}{x^3+8}=\lim_{x\to-2}\frac{(x+2)(x-4)}{(x+2)(x^2-2x+4)}$$

$$=\lim_{x\to-2}\frac{x-4}{x^2-2x+4}=\frac{-2-4}{4+4+4}=\frac{-6}{12}=\frac{-1}{2}.$$ ***Ans.***

(ii) $\lim\limits_{x\to\infty} \sqrt{4x^2+x} - \sqrt{\dfrac{4x^3}{x+2}} = \lim\limits_{y\to 0} \sqrt{\dfrac{4}{y^2}+\dfrac{1}{y}} - \sqrt{\dfrac{\dfrac{4}{y^3}}{\left(\dfrac{1}{y}+2\right)}} = \lim\limits_{y\to 0} \dfrac{\sqrt{4+y} - 2\sqrt{\dfrac{1}{1+2y}}}{y}$

$$= 2\lim_{y\to 0} \frac{\sqrt{1+\frac{y}{4}} - (1+2y)^{\frac{-1}{2}}}{y} = 2\lim_{y\to 0} \frac{1+\frac{y}{8}-(1-y)}{y} = 2\left(\frac{9}{8}\right) = \frac{9}{4}.\ \textbf{Ans.}$$

Illustration :

$$\lim_{n\to\infty} \frac{1^2+2^2+3^2+....+n^2}{n^3}$$

Sol. $\lim\limits_{n\to\infty} \dfrac{1^2+2^2+3^2+......+n^2}{n^3} = \lim\limits_{n\to\infty} \dfrac{\frac{n}{6}(n+1)(2n+1)}{n^3}$

$$= \lim_{n\to\infty} \frac{\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)}{6} = \frac{(1+0)(2+0)}{6} = \frac{1}{3}.\ \textbf{Ans.}$$

Illustration :

$\lim\limits_{x\to 0} \dfrac{(\cos x)^{1/3}-(\cos x)^{1/2}}{\sin^2 x}$ equals

(A) 1/12 (B) 1/6 (C) 1/3 (D) 1/2

Sol. Let $y = \cos x$

$$\lim_{x\to 0} \frac{(\cos x)^{\frac{1}{3}}-(\cos x)^{\frac{1}{2}}}{1-\cos^2 x} = \lim_{y\to 1} \frac{y^{\frac{1}{3}}-y^{\frac{1}{2}}}{1-y^2}$$

$$= \lim_{y\to 1} \frac{\left(\dfrac{y^{\frac{1}{3}}-1}{y-1}\right)-\left(\dfrac{y^{\frac{1}{2}}-1}{y-1}\right)}{\left(\dfrac{1-y^2}{1-y}\right)} = \frac{\frac{1}{3}(1)^{2/3}-\frac{1}{2}(1)^{-1/2}}{-2} = \frac{\frac{1}{3}-\frac{1}{2}}{-2} = \frac{1}{12}.\ \textbf{Ans.}$$

Illustration :

$$\lim_{x \to \infty} \left((x+a)(x+b)(x+c)\right)^{1/3} - x$$

Sol. $$\lim_{x \to \infty} \left((x+a)(x+b)(x+c)^{\frac{1}{3}} - x \right) \Rightarrow \lim_{y \to 0} \left(\frac{(1+ay)(1+by)(1+cy)^{\frac{1}{3}} - 1}{y} \right)$$

$$\lim_{y \to 0} \frac{\left(\left(1+(a+b+c)+abcy^3\right)^{\frac{1}{3}} - 1 \right)}{y}$$

$$\lim_{y \to 0} \frac{1 + \frac{1}{3}(a+b+c)y + \text{Higher terms} - 1}{y} = \frac{(a+b+c)}{3}. \ \textbf{Ans.}$$

Illustration :

$$\lim_{x \to 0} \frac{\ln(\sin 2x)}{\ln(\sin x)} \quad \text{is equal to}$$

(A) 0 (B) 1 (C) 2 (D) non existent

Sol. $$\lim_{x \to 0} \frac{\ln(\sin 2x)}{\ln(\sin x)} = \lim_{x \to 0} \frac{\ln 2 + \ln \sin x + \ln \cos x}{\ln \sin x}$$

$$= \lim_{x \to 0} \left(1 + \frac{\ln 2}{\ln \sin x} + \frac{\ln \cos x}{\ln \sin x} \right) = 1 + 0 + 0 = 1. \ \textbf{Ans.}$$

Illustration :

$$\lim_{x \to 0} \left(1^{\operatorname{cosec}^2 x} + 2^{\operatorname{cosec}^2 x} + 3^{\operatorname{cosec}^2 x} + \ldots\ldots + 100^{\operatorname{cosec}^2 x} \right)^{\sin^2 x} \qquad [\infty^0]$$

Sol. $$\lim_{x \to 0} \left(1^{\operatorname{cosec}^2} + 2^{\operatorname{cosec}^2 x} + 3^{\operatorname{cosec}^2 x} + \ldots\ldots + 100^{\operatorname{cosec}^2 x} \right)^{\sin^2 x}$$

$$100 \lim_{x \to 0} \left(\left(\frac{1}{100}\right)^{\operatorname{cosec}^2 x} + \left(\frac{2}{100}\right)^{\operatorname{cosec}^2 x} + \ldots\ldots + \left(\frac{99}{100}\right)^{\operatorname{cosec}^2 x} + 1 \right)^{\sin^2 x}$$

$$= 100\,(0 + 0 + \ldots\ldots\ldots + 0 + 1)^0 = 100. \ \textbf{Ans.}$$

Practice Problem

Q.1 Evaluate the following limits

(i) $\lim\limits_{x \to \frac{\pi}{4}} \dfrac{-1+\cot^3 x}{-2+\cot x+\cot^3 x}$ (ii) $\lim\limits_{x \to 2} \dfrac{2^x+2^{3-x}-6}{\sqrt{2^{-x}}-2^{1-x}}$ (iii) $\lim\limits_{x \to 9} \dfrac{3-\sqrt{x}}{4-\sqrt{2x-2}}$

Q.2 Evaluate the following limits

(i) $\lim\limits_{n \to \infty} \dfrac{\sqrt{n^3-2n^2+1}+\sqrt[3]{n^4+1}}{\sqrt[4]{n^6+6n^5+2}-\sqrt[5]{n^7+3n^3+1}}$ (ii) $\lim\limits_{x \to \pm\infty} \left(\sqrt{x^2-2x-1}-\sqrt{x^2-7x-3}\right)$

(iii) $\lim\limits_{x \to \frac{\pi}{2}} \tan^2 x\left(\sqrt{2\sin^2 x+3\sin x+4}-\sqrt{\sin^2 x+6\sin x+2}\right)$

Q.3 Evaluate the following limits

(i) $\lim\limits_{x \to \infty} \sqrt[3]{x^3+3x^2}-\sqrt{x^2-2x}$ (ii) $\lim\limits_{x \to \infty} ((x+1)(x+2)(x+3))^{\frac{1}{3}}-x$

(iii) $\lim\limits_{x \to \infty} 100\left[[(x+1)(x+2)(x+3).......(x+100)]^{\frac{1}{100}}-x\right]$ (iv) $\lim\limits_{x \to 1} \dfrac{x^n-1}{x^m-1}$ $(m, n \in N)$

Q.4 If $\lim\limits_{x \to 2}\left(\dfrac{x^{n+1}-2^{n+1}}{x-2}\right)=80$ and $n \in N$, find n.

Q.5 Evaluate the following limits

(i) $\lim\limits_{x \to 1} \dfrac{(2x-3)(\sqrt{x}-1)}{2x^2+x-3}$ (ii) $\lim\limits_{x \to 1} \dfrac{\left[\sum\limits_{k=1}^{100} x^k\right]-100}{x-1}$

(iii) $\lim\limits_{x \to \infty}\left[\sqrt{a^2x^2+ax+1}-\sqrt{a^2x^2+1}\right]$ (iv) $\lim\limits_{x \to a} \dfrac{\sqrt{3x-a}-\sqrt{x+a}}{x-a}$

(v) $\lim\limits_{n \to \infty} \dfrac{(1^2-2^2+3^2-4^2+5^2+......n \text{ terms})}{n^2}$ (vi) $\lim\limits_{h \to 0}\left[\dfrac{1}{h\sqrt[3]{8+h}}-\dfrac{1}{2h}\right]$

Answer key

Q.1 (i) $\frac{3}{4}$, (ii) 8, (iii) $\frac{2}{3}$ **Q.2** (i) 1, (ii) $\pm\frac{5}{2}$, (iii) $\frac{1}{12}$

Q.3 (i) 2, (ii) 2, (iii) 5050, (iv) $\frac{n}{m}$ **Q.4** $n=4$

Q.5 (i) $\frac{-1}{10}$, (ii) 5050, (iii) $\frac{1}{2}$, (iv) $\frac{1}{\sqrt{2a}}$, (v) when n is even, limit $=\frac{-1}{2}$, when n is odd limit $=\frac{1}{2}$, (vi) $\frac{-1}{48}$

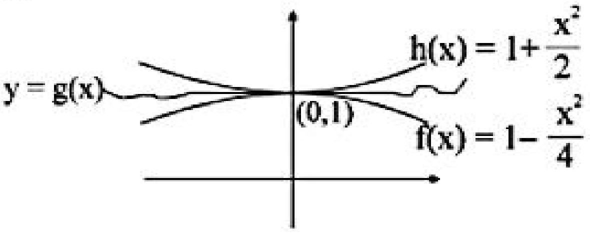
SANDWICH THEOREM OR SQUEEZE PLAY THEOREM FOR EVALUTATING LIMITS :

General: The squeeze principle is used on limit problems where the usual algebraic methods (factorisation or algebraic manipulation etc.) are not effective. However it requires to "squeeze" our problem in between two other simpler function whose limits can be easily computed and equal. Use of Squeeze principle requires accurate analysis, indepth algebra skills and careful use of inequalities.

Statement: If f, g and h are 3 functions such that $f(x) \le g(x) \le h(x)$ for all x in some interval containing the point x = c, and if

$$\underset{x\to c}{\text{Lim}}\ f(x) = \underset{x\to c}{\text{Lim}}\ h(x) = L \ \Rightarrow \underset{x\to c}{\text{Lim}}\ g(x) = L$$

From the figure note that $\underset{x\to 0}{\text{Lim}}\ g(x) = 1$.



Note: **(i)** the quantity c may be a finite number, $+\infty$ or $-\infty$.
Similarly L may be finite number, $+\infty$ or $-\infty$.

Illustration :

Evaluate $\underset{x\to\infty}{Lim}\dfrac{x+7\sin x}{-2x+13}$ *using Sandwich theorem.*

Sol. *We know that* $-1 \le \sin x \le 1$ *for all x.*

$\Rightarrow \quad -7 \le 7\sin x \le 7$

$\Rightarrow \quad x-7 \le x + 7\sin x \le x + 7$

Dividing throughout by $-2x + 13$, *we get*

$$\frac{x-7}{-2x+13} \ge \frac{x+7\sin x}{-2x+13} \ge \frac{x+7}{-2x+13} \ \textit{for all x that are large.}$$

[Why did we switch the inequality signs?]

Now, $$\lim_{x\to\infty}\frac{x-7}{-2x+13} = \lim_{x\to\infty}\frac{1-\frac{7}{x}}{-2+\frac{13}{x}} = \frac{1-0}{-2+0} = -\frac{1}{2}$$

and $$\lim_{x\to\infty}\frac{x+7}{-2x+13} = \lim_{x\to\infty}\frac{1+\frac{7}{x}}{-2+\frac{13}{x}} = \frac{1+0}{-2+0} = -\frac{1}{2}$$

$$\therefore \quad \underset{x\to\infty}{Lim}\frac{x+7\sin x}{-2x+13} = \frac{-1}{2}.$$

Illustration :

If [·] denotes the greatest integer function, then find the value of $\lim\limits_{x\to\infty} \dfrac{[x]+[2x]+......+[nx]}{n^2}$.

Sol. $nx - 1 < [nx] \le nx$. Putting $n = 1, 2, 3, \ldots\ldots, n$ and adding them, $x\Sigma n - n < \Sigma[nx] \le x\Sigma n$

$$\therefore \quad x\frac{\Sigma n}{n^2} - \frac{1}{n} < \frac{\Sigma[nx]}{n^2} \le x.\frac{\Sigma n}{n^2} \quad \ldots(i)$$

Now, $\lim\limits_{n\to\infty}\left\{x.\dfrac{\Sigma n}{n^2} - \dfrac{1}{n}\right\} = x.\lim\limits_{n\to\infty}\dfrac{\Sigma n}{n^2} - \lim\limits_{n\to\infty}\dfrac{1}{n} = \dfrac{x}{2}$

$$\lim_{n\to\infty}\left\{x.\frac{\Sigma n}{n^2}\right\} = x\lim_{n\to\infty}\frac{\Sigma n}{n^2} = \frac{x}{2}$$

As the two limits are equal by equation (i) $\lim\limits_{n\to\infty}\dfrac{\Sigma[nx]}{n^2} = \dfrac{x}{2}$.

Illustration :

Evaluate $\lim\limits_{n\to\infty}\dfrac{1}{1+n^2} + \dfrac{2}{2+n^2} + ... + \dfrac{n}{n+n^2}$.

Sol. $P_n = \dfrac{1}{1+n^2} + \dfrac{2}{2+n^2} + ... + \dfrac{n}{n+n^2}$

Now, $P_n < \dfrac{1}{1+n^2} + \dfrac{2}{1+n^2} + ... + \dfrac{n}{1+n^2} = \dfrac{1+2+...+n}{(1+n^2)} = \dfrac{n(n+1)}{2(1+n^2)}$

Also, $P_n > \dfrac{1}{n+n^2} + \dfrac{2}{n+n^2} + \dfrac{3}{n+n^2} + ... + \dfrac{n}{n+n^2} = \dfrac{1+2+.....+4}{(n+n^2)} = \dfrac{n(n+1)}{2(n+n^2)}$

Thus, $\dfrac{n(n+1)}{2(n+n^2)} < P_n < \dfrac{n(n+1)}{2(1+n^2)}$

$$\Rightarrow \quad \lim_{n\to\infty}\frac{n(n+1)}{2(n+n^2)} < \lim_{n\to\infty}P_n < \lim_{n\to\infty}\frac{n(n+1)}{2(1+n^2)}$$

$$\Rightarrow \quad \lim_{n\to\infty}\frac{1\left(1+\frac{1}{n}\right)}{2\left(\frac{1}{n}+1\right)} < \lim_{n\to\infty}P_n < \lim_{n\to\infty}\frac{1\left(1+\frac{1}{n}\right)}{2\left(\frac{1}{n^2}+1\right)}$$

$$\Rightarrow \quad \frac{1}{2} < \lim_{n\to\infty}P_n < \frac{1}{2} \quad \Rightarrow \quad \lim_{n\to\infty}P_n = \frac{1}{2}$$

Practice Problem

Evaluate following limits based on Sandwich theorem :

Q.1 If $4x - 9 \le f(x) \le x^2 - 4x + 7 \ \forall \ x \ge 0$ find $\lim\limits_{x \to 4} f(x)$

Q.2 $2x \le g(x) \le x^4 - x^2 + 2$ for all x then $\lim\limits_{x \to 1} g(x)$

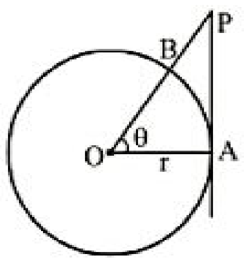
Answer key

Q.1 7 **Q.2** 2

EVALUATION OF TRIGONOMETRIC LIMITS :

(i) $\lim\limits_{\theta \to 0} \dfrac{\sin\theta}{\theta} = 1$ (where θ is in radians)

Proof : Consider a circle of radius r. Let O be the centre of the circle such that $\angle AOB = \theta$ where θ is measured in radians and it is very small. Suppose the tangent at A meets OB produced at P. From figure, we have



Area of ΔOAB < Area of sector OAB < Area of ΔOAP

$\Rightarrow \quad \frac{1}{2} OA \times OB \sin\theta < \frac{1}{2}(OA)^2\theta < \frac{1}{2} OA \times AP$

$\Rightarrow \quad \frac{1}{2} r^2 \sin\theta < \frac{1}{2} r^2 \theta < \frac{1}{2} r^2 \tan\theta$ [In ΔOAP, $AP = OA \tan\theta$]

$\Rightarrow \quad \sin\theta < \theta < \tan\theta \quad \Rightarrow \quad 1 > \dfrac{\sin\theta}{\theta} > \cos\theta$

$\Rightarrow \quad 1 > \lim\limits_{\theta \to 0} \dfrac{\sin\theta}{\theta} > \lim\limits_{\theta \to 0} \cos\theta$ or, $\lim\limits_{\theta \to 0} \cos\theta < \lim\limits_{\theta \to 0} \dfrac{\sin\theta}{\theta} < 1$

$\Rightarrow \quad 1 < \lim\limits_{\theta \to 0} \dfrac{\sin\theta}{\theta} < 1 \quad \Rightarrow \lim\limits_{\theta \to 0} \dfrac{\sin\theta}{\theta} = 1$ (By Sandwich Theorem)

(ii) $\lim\limits_{\theta \to 0} \dfrac{\sin\theta}{\theta} = 1$

We have $\lim\limits_{\theta \to 0} \dfrac{\tan\theta}{\theta} = \lim\limits_{\theta \to 0} \dfrac{\sin\theta}{\theta} \dfrac{1}{\cos\theta} = \lim\limits_{\theta \to 0} \dfrac{\sin\theta}{\theta} \cdot \lim\limits_{\theta \to 0} \dfrac{1}{\cos\theta} = 1$

(iii) $\lim\limits_{\theta \to a} \frac{\sin(\theta - a)}{\theta - a} = 1$

We have $\lim\limits_{\theta \to a} \frac{\sin(\theta - a)}{\theta - a} = \lim\limits_{h \to 0} \frac{\sin(a + h - a)}{(a + h - a)} = \lim\limits_{h \to 0} \frac{\sin h}{h} = 1$

(iv) $\lim\limits_{\theta \to a} \frac{\tan(\theta - a)}{\theta - a} = 1$

(v) $\lim\limits_{\theta \to 0} \frac{\sin^{-1} x}{x} = 1$

(vi) $\lim\limits_{x \to 0} \frac{\tan^{-1} x}{x} = 1$

(vii) $\operatorname{Lim}\limits_{x \to \infty} \frac{1 - \cos x}{x^2} = \frac{1}{2}$ (remember).

Note : Let [·] denotes greatest integer function

(i) $\operatorname{Lim}\limits_{x \to 0} \left[\frac{\sin x}{x} \right] = 0$

(ii) $\left[\operatorname{Lim}\limits_{x \to 0} \frac{\sin x}{x} \right] = 1$

(iii) $\operatorname{Lim}\limits_{x \to 0} \left[\frac{\tan x}{x} \right] = 1$

(iv) $\operatorname{Lim}\limits_{x \to 0} \left[\frac{\sin^{-1} x}{x} \right] = 1$

(v) $\left[\operatorname{Lim}\limits_{x \to 0} \frac{\sin^{-1} x}{x} \right] = 0$

Illustration :

Evaluate the following limits

(i) $\lim\limits_{x \to 0} \frac{\sin 3x}{x}$ *(ii)* $\lim\limits_{x \to 0} \frac{\sin ax}{\sin bx}$ *(iii)* $\lim\limits_{x \to 1} \frac{\sin(\log x)}{\log x}$

Sol. *(i)* *We have* $\lim\limits_{x \to 0} \frac{\sin 3x}{x} = \lim\limits_{x \to 0} \left(3 \frac{\sin 3x}{3x} \right) = 3 \lim\limits_{x \to 0} \frac{\sin 3x}{3x} = 3(1) = 3$ $\left[\because \lim\limits_{x \to 0} \frac{\sin 3x}{3x} = 1 \right]$

(ii) *We have* $\lim\limits_{x \to 0} \frac{\sin ax}{\sin bx} = \lim\limits_{x \to 0} \frac{\left(\frac{\sin ax}{ax} \right) ax}{\left(\frac{\sin bx}{bx} \right) ax} = \frac{a}{b}$

(iii) *Given* $L = \lim\limits_{x \to 1} \frac{\sin(\log x)}{\log x}$

Let $\log x = t$ *then*

$\Rightarrow \quad L = \lim\limits_{t \to 0} \frac{\sin t}{t} = 1$

Illustration :

Evaluate $\lim\limits_{x \to 0} \frac{1}{x} \sin^{-1}\left(\frac{2x}{1 + x^2} \right)$.

Sol. *We know that* $\sin^{-1}\left(\frac{2x}{1 + x^2} \right) = 2 \tan^{-1} x$, *for* $-1 \le x \le 1$

$\Rightarrow \quad \lim\limits_{x \to 0} \frac{1}{x} \sin^{-1}\left(\frac{2x}{1 + x^2} \right) = \lim\limits_{x \to 0} \frac{2 \tan^{-1} x}{x} = 2$

Illustration :

Evaluate $\lim_{x\to 0}\frac{1-\cos 2x}{x^2}$

Sol. We know $\lim_{x\to 0}\frac{1-\cos 2x}{x^2}\left(\frac{0}{0}\text{ form}\right) = \lim_{x\to 2}\frac{2\sin^2 x}{x^2} = 2\left(\lim_{x\to 0}\frac{\sin x}{x}\right)^2 = 2$

Illustration :

Evaluate $\lim_{x\to 0}\frac{\tan x-\sin x}{x^3}$

Sol. We have $\lim_{x\to 0}\frac{\tan x-\sin x}{x^3}\left(\frac{0}{0}\text{ form}\right)$

$$= \lim_{x\to 0}\left(\frac{\sin x-\sin x\cos x}{x^3\cos x}\right)$$

$$= \lim_{x\to 0}\left\{\frac{\sin x(1-\cos x)}{x^3\cos x}\right\} = \lim_{x\to 0}\left\{\frac{\sin x}{x}\frac{1-\cos x}{x^2}\frac{1}{\cos x}\right\}$$

$$= \left\{\lim_{x\to 0}\frac{\sin x}{x}\right\}\left\{\lim_{x\to 0}\frac{2\sin^2\frac{x}{2}}{\left(\frac{x}{2}\right)^2\times 4}\right\}\lim_{x\to 0}\frac{1}{\cos x} = \left\{\lim_{x\to 0}\frac{\sin x}{x}\right\}\frac{1}{2}\left\{\lim_{x\to 0}\left(\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right)^2\right\}\lim_{x\to 0}\frac{1}{\cos x}$$

$$= 1\times\frac{1}{2}(1)^2\times\frac{1}{1} = \frac{1}{2}$$

Illustration :

Evaluate $\lim_{x\to\frac{\pi}{2}}\frac{1+\cos 2x}{(\pi-2x)^2}$

Sol. We have, $\lim_{x\to\frac{\pi}{2}}\frac{1+\cos 2x}{(\pi-2x)^2}\left(\frac{0}{0}\text{ form}\right)$

$$= \lim_{h\to 0}\frac{1+\cos 2\left(\frac{\pi}{2}+h\right)}{\left[\pi-2\left(\frac{\pi}{2}+h\right)\right]^2} = \lim_{h\to 0}\frac{1+\cos(\pi+2h)}{4h^2} = \lim_{h\to 0}\frac{1-\cos 2h}{4h^2}$$

$$= \lim_{h\to 0}\frac{2\sin^2 h}{4h^2} = \frac{2}{4}\left(\lim_{h\to 0}\frac{\sin h}{h}\right)^2 = \frac{1}{2}$$

Illustration :

Evaluate $\lim\limits_{x\to\infty} 2^{x-1}\tan\left(\frac{a}{2^x}\right)$.

Sol. We have $\lim\limits_{x\to\infty} 2^{x-1}\tan\left(\frac{a}{2^x}\right) = \lim\limits_{x\to\infty}\frac{a}{2}\frac{\tan\left(\frac{a}{2^x}\right)}{\left(\frac{a}{2^x}\right)}\left(\frac{0}{0}\text{ form}\right)$

$$=\frac{a}{2}\lim_{y\to 0}\frac{\tan y}{y},\quad \text{where } y=\frac{a}{2^x}\quad =\left(\frac{a}{2}\right)$$

Illustration :

Evaluate $\lim\limits_{x\to 2}\frac{x^2-x-2}{x^2-2x-\sin(x-2)}$

Sol. $\lim\limits_{x\to 2}\frac{x^2-x-2}{x^2-2x-\sin(x-2)}\left(\frac{0}{0}\text{ form}\right)$

$$=\lim_{x\to 2}\frac{(x-2)(x+1)}{x(x-2)-\sin(x-2)}\quad =\lim_{x\to 2}\frac{(x+1)}{x-\frac{\sin(x-2)}{x-2}}=\frac{2+1}{2-1}=3$$

Illustration :

Evaluate $\lim\limits_{x\to\infty} x\left(\tan^{-1}\frac{x+1}{x+4}-\frac{\pi}{4}\right)$.

Sol. We have $\lim\limits_{x\to\infty} x\left(\tan^{-1}\frac{x+1}{x+4}-\frac{\pi}{4}\right)$

$$=\lim_{x\to\infty} x\left(\tan^{-1}\frac{x+1}{x+4}-\tan^{-1}1\right)\quad =\lim_{x\to\infty} x\tan^{-1}\left(\frac{\frac{x+1}{x+4}-1}{1+\frac{x+1}{x+4}}\right)$$

$$=\lim_{x\to\infty} x\tan^{-1}\left(\frac{-3}{2x+5}\right)=\lim_{x\to\infty}\left(\frac{\tan^{-1}\left(\frac{-3}{2x+5}\right)}{\frac{-3}{2x+5}}\right)\left(\frac{-3x}{2x+5}\right)$$

$$=\lim_{x\to\infty}\left(\frac{\tan^{-1}\left(\frac{-3}{2x+5}\right)}{\frac{-3}{2x+5}}\right)\lim_{x\to\infty}\left(\frac{-3x}{2x+5}\right)=1\times\lim_{x\to\infty}\left(\frac{-3}{2+\frac{5}{x}}\right)=1\times\frac{-3}{2}=-\frac{3}{2}$$

Practice Problem

Q.1 Evaluate the following limits

(i) $\lim_{x\to 0} \sin 8x \cot 3x$ (ii) $\lim_{x\to 0} \frac{1-\cos x}{x^2}$ (iii) $\lim_{x\to 0} \frac{1-\cos 5x}{3x^2}$

(iv) $\lim_{x\to 0} \frac{\sqrt{1+\tan x}-\sqrt{1+\sin x}}{x^3}$ (v) $\lim_{x\to 1} \frac{x^3+x^2-2}{\sin(x-1)}$

Q.2 Evaluate the following limits

(i) $\lim_{x\to 1} (1-x)\tan\frac{\pi x}{2}$ (ii) $\lim_{x\to 0} \frac{x\tan 2x - 2x\tan x}{(1-\cos 2x)^2}$ (iii) $\lim_{x\to 0} \frac{\cos 7x - \cos 9x}{\cos x - \cos 5x}$

(iv) $\lim_{x\to 0} \frac{1-\cos x\sqrt{\cos 2x}}{\tan^2 x}$ (v) $\lim_{x\to 0} \frac{1-\cos(1-\cos x)}{\sin^4 x}$

Q.3 Evaluate the following limits

(i) $\lim_{x\to 0} \frac{1-\cos mx}{1-\cos nx}$ (ii) $\lim_{x\to \frac{\pi}{4}} \frac{\sqrt{2}\cos x - 1}{\cot x - 1}$ (iii) $\lim_{x\to 0} \frac{\cot 2x - \operatorname{cosec} 2x}{x}$

(iv) $\lim_{x\to 0} \frac{\tan 2x - x}{3x - \sin x}$ (v) $\lim_{n\to\infty} n\cos\left(\frac{\pi}{4n}\right)\sin\left(\frac{\pi}{4n}\right)$

Answer key

Q.1 (i) $\frac{8}{3}$, (ii) $\frac{1}{2}$, (iii) $\frac{25}{6}$, (iv) $\frac{1}{4}$, (v) 5

Q.2 (i) $\frac{2}{\pi}$, (ii) $\frac{1}{2}$, (iii) $\frac{4}{3}$, (iv) $\frac{3}{2}$, (v) $\frac{1}{8}$

Q.3 (i) $\frac{m^2}{n^2}$, (ii) $-\frac{1}{2}$, (iii) -1, (iv) $\frac{1}{2}$, (v) $\frac{\pi}{4}$

EVALUATION OF EXPONENTIAL AND LOGARITHMIC LIMITS :

In order to evaluate these type of limit, we use the following standard results.

1. $\lim_{x\to 0} \frac{a^x - 1}{x} = \log_e a$

Proof : $\lim_{x\to 0} \frac{a^x - 1}{x}$

$$= \lim_{x\to 0} \frac{\left(1+\frac{x(\log a)}{1!}+\frac{x^2(\log a)^2}{2!}+\ldots\ldots\right)-1}{x} = \lim_{x\to 0}\left(\frac{\log a}{1!}+\frac{x(\log a)^2}{2!}+\ldots\ldots\right) = \log_e a$$

2. $\lim\limits_{x\to 0}\frac{e^x-1}{x}=1$ (replace a by e in the above proof)

3. $\lim\limits_{x\to 0}\frac{\log(1+x)}{x}=1$

Proof : $\lim\limits_{x\to 0}\frac{\log(1+x)}{x}=\lim\limits_{x\to 0}\frac{x-\frac{x^2}{2}+\frac{x^3}{3}-......}{x}=\lim\limits_{x\to 0}\left(1-\frac{x}{2}+\frac{x^2}{3}......\right)=1$

Illustration :

Evaluate $\lim\limits_{x\to 0}\frac{2^x-1}{\sqrt{1+x}-1}$

Sol. *We have* $\lim\limits_{x\to 0}\frac{2^x-1}{\sqrt{1-x}-1}\left(\frac{0}{0}\text{ form}\right)$

$=\lim\limits_{x\to 0}\frac{2^x-1}{\sqrt{1+x}-1}\frac{(\sqrt{1+x}+1)}{(\sqrt{1+x}+1)}=\lim\limits_{x\to 0}\frac{2^x-1}{x}\lim\limits_{x\to 0}\left(\sqrt{1+x}+1\right)$

$=(\log 2)\left(\sqrt{1+0}+1\right)=2\log 2.$

Illustration :

Evaluate $\lim\limits_{x\to 1}\frac{a^{x-1}-1}{\sin\pi x}.$

Sol. *We have* $\lim\limits_{x\to 1}\frac{a^{x-1}-1}{\sin\pi x}\left(\frac{0}{0}\text{ form}\right)$

$=\lim\limits_{h\to 0}\frac{a^{1+h-1}-1}{\sin\pi(1+h)}=\lim\limits_{h\to 0}\frac{a^h-1}{-\sin\pi h}=\frac{-1}{\pi}\lim\limits_{h\to 0}\left(\frac{a^h-1}{h}\right)\frac{\pi h}{\sin\pi h}=-\frac{1}{\pi}\log a$

Illustration :

Evaluate $\lim\limits_{x\to 0}\frac{10^x-2^x-5^x+1}{x\tan x}$

Sol. *We have* $\lim\limits_{x\to 0}\frac{10^x-2^x-5^x+1}{x\tan x}\left(\frac{0}{0}\text{ form}\right)$

$=\lim\limits_{x\to 0}\frac{5^x.2^x-2^x-5^x+1}{x\tan x}\quad=\lim\limits_{x\to 0}\frac{5^x-1}{x}\frac{2^x-1}{x}\frac{x}{\tan x}$

$=\lim\limits_{x\to 0}\frac{5^x-1}{x}\lim\limits_{x\to 0}\frac{2^x-1}{x}\lim\limits_{x\to 0}\frac{x}{\tan x}$

$=(\log 5)(\log 2)(1)=(\log 5)(\log 2)$

Illustration :

Evaluate $\lim\limits_{x\to a}\dfrac{\log x-\log a}{x-a}$

Sol. Let $x-a=h$, then if $x\to a,\ h\to 0 \Rightarrow \lim\limits_{x\to a}\dfrac{\log x-\log a}{x-a}\left(\dfrac{0}{0}\text{ form}\right)$

$$=\lim_{h\to 0}\frac{\log(a+h)-\log a}{h}\quad=\lim_{x\to 0}\frac{\log\left(1+\frac{h}{a}\right)}{\frac{h}{a}a}=\frac{1}{a}$$

Illustration :

Evaluate $\lim\limits_{x\to 0}\dfrac{\log(5+x)-\log(5-x)}{c}$

Sol. We have $\lim\limits_{x\to 0}\dfrac{\log(5+x)-\log(5-x)}{c}\left(\dfrac{0}{0}\text{ form}\right)$

$$=\lim_{x\to 0}\frac{\log\left\{5\left(1+\frac{x}{5}\right)\right\}-\log\left\{5\left(1-\frac{x}{5}\right)\right\}}{x}=\lim_{x\to 0}\frac{\left\{\log 5\log\left(1+\frac{x}{5}\right)\right\}-\left\{\log 5+\log\left(1-\frac{x}{5}\right)\right\}}{x}$$

$$=\lim_{x\to 0}\frac{\log\left(1+\frac{x}{5}\right)-\log\left(1-\frac{x}{5}\right)}{x}$$

$$=\lim_{x\to 0}\frac{1}{5}\frac{\log\left(1+\frac{x}{5}\right)}{\frac{x}{5}}-\lim_{x\to 0}\frac{\log\left(1-\frac{x}{5}\right)}{-\frac{x}{5}}\frac{1}{(-1)}=\frac{1}{5}+\frac{1}{5}=\frac{2}{5}$$

Illustration :

Let $P_n=a^{P_n-1}-1,\ \forall\, n=2,3\ldots\ldots$ and let $P_1=a^x-1$ where $a\in R^+$, then evaluate $\lim\limits_{x\to 0}\dfrac{P_n}{x}$.

Sol. Clearly, if $P_k\to 0\Rightarrow P_{k+1}\to 0$

Now, as $x\to 0\Rightarrow P_1\to 0\Rightarrow P_2P_3P_4\ldots\ldots P_n\to 0$

$$\Rightarrow\quad \lim_{x\to 0}\frac{P_n}{x}=\lim_{x\to 0}\frac{P_n}{P_{n-1}}\frac{P_{n-1}}{P_{n-2}}\ldots\ldots\frac{P_1}{x}=\mathrm{Lim}_{x\to 0}\left(\frac{P_n}{P_{n-1}}\right)\cdot\mathrm{Lim}_{x\to 0}\left(\frac{P_{n-1}}{P_{n-2}}\right)\ldots\ldots\mathrm{Lim}_{x\to 0}\left(\frac{P_1}{x}\right)$$

Now, $\lim\limits_{x\to 0}\dfrac{P_k}{P_{k-1}}=\lim\limits_{x\to 0}\dfrac{a^{P_{k-1}-1}}{P_{k-1}}=\ln\ a$

$\Rightarrow$ Required limit $=(\ln a)^n$

Practice Problem

Q.1 Evaluate the following limits

(i) $\lim\limits_{x\to 0} \dfrac{e^{\tan x}-e^{x}}{\tan x - x}$ (ii) $\lim\limits_{x\to 0} \dfrac{a^{\tan x}-a^{\sin x}}{\tan x-\sin x}, a>0$ (iii) $\lim\limits_{x\to 0} \dfrac{e^{x^2}-\cos x}{x^2}$

(iv) $\lim\limits_{x\to 0} \dfrac{e^{\sin 2x}-e^{\sin x}}{x}$ (v) $\lim\limits_{x\to 0} \dfrac{\cos(xe^{x})-\cos(xe^{-x})}{x^3}$ (vi) $\lim\limits_{x\to \infty} \dfrac{e^{1/x^2}-1}{2\arctan x^2-\pi}$

Q.2 If $\lim\limits_{x\to 1} \dfrac{\frac{\pi}{4}-\tan^{-1}x}{e^{\sin(\ln x)}-x^{n}}$ exists and has the value equal to $\dfrac{1}{8}$, then find n.

Q.3 Evaluate the following limits :

(i) $\lim\limits_{x\to\infty} [x(a^{1/x}-1)], a>1$ (ii) $\lim\limits_{x\to 0} \dfrac{x2^{x}-x}{1-\cos x}$ (iii) $\lim\limits_{x\to 2} \dfrac{\sin(e^{x-2}-1)}{\log(x-1)}$

(iv) $\lim\limits_{x\to 0} \dfrac{e^{x^2}-\cos x}{x^2}$ (v) $\lim\limits_{x\to 0} \dfrac{e^{x}+e^{-x}-2}{x^2}$ (vi) $\lim\limits_{x\to a} \dfrac{\log(x-a)}{\log(e^{x}-e^{a})}$

(vii) $\lim\limits_{x\to 0} \dfrac{a^{\tan x}-a^{\sin x}}{\tan x-\sin x}, a>0$ (viii) $\lim\limits_{x\to 0} \dfrac{(1-3^{x}-4^{x}+12^{x})}{\sqrt{(2\cos x+7)}-3}$

(ix) $\lim\limits_{x\to 0} \dfrac{(729)^{x}-(243)^{x}-(81)^{x}+9^{x}+3^{x}-1}{x^3}$

Answer key

Q.1 (i) 1, (ii) ln a, (iii) $\dfrac{3}{2}$, (iv) 1, (v) 2, (vi) $-\dfrac{1}{2}$ **Q.2** 5

Q.3 (i) $\ln a$, (ii) $\ln 4$, (iii) 1, (iv) $\dfrac{3}{2}$, (v) 1, (vi) 1, (vii) $\ln a$, (viii) $-12\ln 2\times \ln 3$, (ix) $6(\ln 3)^3$

LIMITS OF THE FORM $\lim\limits_{x \to a} (f(x))^{g(x)}$:

Form : $0^0, \infty^0$

Let $L = \lim\limits_{x \to a} (f(x))^{g(x)} \Rightarrow \log_e L = \log_e \left[\lim\limits_{x \to a} (f(x))^{g(x)}\right]$

$= \lim\limits_{x \to a} g(x) \log_e [f(x)]$

Form : 1^∞

1. $\lim\limits_{x \to 0} (1+x)^{\frac{1}{x}} = e$ or $\lim\limits_{x \to \infty} \left(1 + \frac{1}{x}\right)^x = e$

Proof: $\lim\limits_{x \to \infty} (1+x)^{\frac{1}{x}}$

$$= \lim_{x \to 0} \left(1 + \frac{1}{x} x + \frac{\frac{1}{x}\left(\frac{1}{x} - 1\right)}{2!} x^2 + \frac{\frac{1}{x}\left(\frac{1}{x} - 1\right)\left(\frac{1}{x} - 2\right)}{3!} x^3 + \ldots\ldots \right)$$

$$= \lim_{x \to 0} \left(1 + 1 + \frac{1(1-x)}{2!} + \frac{1(1-x)(1-2x)}{3!} + \ldots\ldots \right)$$

$$= \left(1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \ldots\ldots \right) = e$$

2. $L = \lim\limits_{x \to a} f(x)^{g(x)}$ if $\lim\limits_{x \to a} f(x) = 1$ and $\lim\limits_{x \to a} g(x) = \infty$

Then $L = \lim\limits_{x \to a} f(x)^{g(x)}$

$$= \lim_{x \to a} (1 + (f(x) - 1))^{\frac{1}{f(x)-1}(f(x)-1) \times g(x)}$$

$$= \left[\lim_{x \to a} \left((1 + (f(x) - 1))^{\frac{1}{f(x)-1}} \right) \right]^{\lim\limits_{x \to a} (f(x)-1) \times g(x)} \qquad = e^{\lim\limits_{x \to a} (f(x)-1) \times g(x)}$$

Illustration :

Evaluate $\lim\limits_{x \to 0} (1+x)^{\operatorname{cosec} x}$.

Sol. $\lim\limits_{x \to 0} (1+x)^{\operatorname{cosec} x} = \lim\limits_{x \to 0} \left[(1+x)^{\frac{1}{x}} \right]^{\frac{x}{\sin x}} = \left[\lim\limits_{x \to 0} (1+x)^{\frac{1}{x}} \right]^{\lim\limits_{x \to 0} \frac{x}{\sin x}} = e^1 = e$

Illustration :

Evaluate $\lim_{x\to 0}(\cos x)^{\cos 2x}$.

Sol. $\lim_{x\to 0}(\cos x)^{\cos 2x}$

$$= \lim_{x\to 0}\left[(1+(\cos x-1))^{\frac{1}{\cos x-1}}\right]^{\frac{\cos x-1}{\tan x}}$$

$$= \left[\lim_{x\to 0}(1+(\cos x-1))^{\frac{1}{\cos x-1}}\right]^{\lim_{x\to 0}\frac{\cos x-1}{\tan x}} = e^{\lim_{x\to 0}\frac{\cos x-1}{\tan x}\cos x}$$

$$= e^{\lim_{x\to 0}\frac{\cos x-1}{\sin^2 x}\cos x\sin x} = e^{\lim_{x\to 0}\left(\frac{\cos x-1}{1-\cos^2 x}\cdot\cos x\cdot\sin x\right)}$$

$$= e^{-\lim_{x\to 0}\frac{\sin x\cos x}{1+\cos x}} = e^{0} = 1$$

Illustration :

Evaluate $\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^{\left(\frac{\sin x}{x-\sin x}\right)}$

Sol. Since $\lim_{x\to 0}\frac{\sin x}{x} = 1$ and $\lim_{x\to 0}\frac{\sin x}{x} = \lim_{x\to 0}\frac{\sin x}{x-\sin x} = \lim_{x\to 0}\frac{1}{\left(\frac{x}{\sin x}-1\right)} = \frac{1}{1-1} = \infty$

$$\Rightarrow \quad \lim_{x\to 0}\left(\frac{\sin x}{x}\right)^{\left(\frac{\sin x}{x-\sin x}\right)} = e^{\lim_{x\to 0}\left(\frac{\sin x}{x}-1\right)\left(\frac{\sin x}{x-\sin x}\right)} = e^{-\lim_{x\to 0}\frac{\sin x}{x}} = e^{-1} = \frac{1}{e}$$

Illustration :

Evaluate $\lim_{x\to 0}\left(\frac{a^x+b^x+c^x}{3}\right)^{2/x}$; (a, b, c > 0).

Sol. We have $\lim_{x\to 0}\left(\frac{a^x+b^x+c^x}{3}\right)^{2/x}$

$$= e^{\lim_{x\to 0}\left(\frac{a^x+b^x+c^x}{3}-1\right)\frac{2}{x}} = e^{\frac{2}{3}\lim_{x\to 0}\left(\frac{a^x+b^x+c^x-3}{x}\right)}$$

$$= e^{\frac{2}{3}\lim_{x\to 0}\left(\frac{a^x-1}{x}+\frac{b^x-1}{x}+\frac{c^x-1}{x}\right)} = e^{\frac{2}{3}\left\{\lim_{x\to 0}\frac{a^x-1}{x}+\lim_{x\to 0}\frac{b^x-1}{x}+\lim_{x\to 0}\frac{c^x-1}{x}\right\}}$$

$$= e^{\left(\frac{2}{3}\right)[\ln a+\ln b+\ln c]} = e^{\left(\frac{2}{3}\right)\ln(abc)} = e^{\ln(abc)^{\frac{2}{3}}} = (abc)^{\frac{2}{3}}$$

Practice Problem

Q.1 Evaluate the following limits

(i) $\lim\limits_{x\to\infty}\left(\dfrac{1^{1/x}+2^{1/x}+3^{1/x}+.........+n^{1/x}}{n}\right)^{nx}$ $n \in N$

(ii) $\lim\limits_{x\to\infty}\left(\dfrac{x+2}{x+1}\right)^{x+3}$

(iii) $\operatorname*{Lim}\limits_{x\to\infty}\left(\dfrac{x+6}{x+1}\right)^{x+4}$

(iv) $\lim\limits_{x\to\infty}\left(1+\dfrac{1}{a+bx}\right)^{c+dx}$ where a, b, c and d are positive.

(v) $\operatorname*{Lim}\limits_{x\to 0}\left(\sin^2\dfrac{\pi}{2-ax}\right)^{\sec^2\frac{\pi}{2-bx}}$

Q.2 **Column-I** **Column-II**

(A) Let $f(x)=\begin{cases}3-x & 2\le x\le 3\\ x-1 & 1\le x<2\end{cases}$, then $\operatorname*{Lim}\limits_{x\to 2}\,(f(x))^{\operatorname{cosec}\frac{\pi x}{2}}=$ (P) 1

(B) $\operatorname*{Lim}\limits_{x\to 1}\,(2-x)^{\tan\frac{\pi x}{2}}$ (Q) e

(C) $\operatorname*{Lim}\limits_{x\to\infty}\left(\sin\dfrac{1}{x}+\cos\dfrac{1}{x}\right)^{x}$ (R) e^2

(D) $\operatorname*{Lim}\limits_{x\to 0}\left[\tan\left(\dfrac{\pi}{4}+x\right)\right]^{\frac{1}{x}}$ (S) $e^{\frac{2}{\pi}}$

(E) $\operatorname*{Lim}\limits_{x\to 0}\left(\dfrac{1+\tan x}{1+\sin x}\right)^{\operatorname{cosec} x}$ (T) $e^{\frac{-2}{\pi}}$

(F) $\operatorname*{Limit}\limits_{x\to 0}\,(\cos x)^{\cot^2 x}$ (U) non existent

(G) $\operatorname*{Lim}\limits_{x\to\infty}\left(\dfrac{x}{1+x}\right)^{x}$ (V) $e^{-1/2}$

(H) $\operatorname*{Lim}\limits_{x\to\infty}\left(\dfrac{2\tan^{-1}x}{\pi}\right)^{x}$ (W) e^{-1}

(X) $e^{1/2}$

Answer key

Q.1 (i) n !, (ii) e, (iii) e^5, (iv) $e^{d/b}$, (v) $e^{-\frac{a^2}{b^2}}$

Q.2 (A) U; (B) S; (C) Q; (D) R; (E) P ; (F) V; (G) W; (H) T

L' HOSPITAL'S RULE FOR EVALUATING LIMITS :

Rule : If $\lim\limits_{x\to a}\dfrac{f(x)}{g(x)}$ takes $\dfrac{0}{0}$ or $\dfrac{\infty}{\infty}$ form, then, $\lim\limits_{x\to a}\dfrac{f(x)}{g(x)} = \lim\limits_{x\to a}\dfrac{f'(x)}{g'(x)}$

Illustration :

Evaluate $\lim\limits_{x\to 0} \log_{\tan^2 x}(\tan^2 2x)$.

Sol. $L = \lim\limits_{x\to 0}\dfrac{\log(\tan^2 2x)}{\log(\tan^2 x)}\left(\dfrac{\infty}{\infty}\text{ form}\right)$

Using L' hospital Rule

$$\text{We have } L = \lim_{x\to 0}\frac{\left(\dfrac{1}{\tan^2 2x}2\tan 2x\sec^2 2x\right)\times 2}{\dfrac{1}{\tan^2 x}2\tan x\sec^2 x}$$

$$= \lim_{x\to 0}\frac{2\left(\dfrac{1}{\sin 2x\cos 2x}\right)}{\dfrac{1}{\sin x\cos x}} = \lim_{x\to 0}\frac{\left(\dfrac{1}{\sin 2x\cos 2x}\right)}{\left(\dfrac{1}{\sin 2x}\right)} = 1\ \lim_{x\to 0}\frac{1}{\cos 2x} = 1$$

Illustration :

Evaluate $\lim\limits_{x\to 0^+} x^m(\log x)^n,\ m, n \in N$.

Sol. $\lim\limits_{x\to 0^+} x^m(\log x^n) = \lim\limits_{x\to 0^+}\dfrac{(\log x)^n}{x^{-m}}\left(\dfrac{\infty}{\infty}\text{ form}\right)$

$$= \lim_{x\to 0^+}\frac{n(\log x)^{(n-1)}\dfrac{1}{x}}{-mx^{-m-1}} \qquad \textit{(using L' hospital Rule)}$$

$$= \lim_{x\to 0^+}\frac{n(\log x)^{n-1}}{-mx^{-m}}\left(\frac{\infty}{\infty}\text{ form}\right)$$

$$= \lim_{x\to 0^+}\frac{n(n-1)(\log x)^{(n-2)}\dfrac{1}{x}}{(-m)^2x^{-m-1}} \qquad \textit{(using L' hospital Rule)}$$

$$= \lim_{x\to 0^+}\frac{n(n-1)(\log x)^{(n-2)}}{m^2x^{-m}}\left(\frac{\infty}{\infty}\text{ form}\right)$$

$$= \lim_{x\to 0^+}\frac{n!}{(-m)^n x^{-m}} = 0 \quad \textit{(differentiating } N^r \textit{ and } D^r\ n \textit{ times)}$$

Illustration :

Evaluate $\lim\limits_{x\to 0}\dfrac{\sin^{-1}x-\tan^{-1}x}{x^3}$.

Sol. $\lim\limits_{x\to 0}\dfrac{\sin^{-1}x-\tan^{-1}x}{x^3}$

$$=\lim_{x\to 0}\frac{(1-x^2)-\sqrt{1-x^2}}{3x^2\sqrt{1-x^2}(1+x^2)} \qquad \textit{(Using L'hospital's Rule)}$$

$$=\lim_{x\to 0}\frac{(1+x^2)-\sqrt{1-x^2}}{3x^2\sqrt{1-x^2}(1+x^2)}$$

$$=\lim_{x\to 0}\frac{(1+x^2)^2-(1-x^2)}{3x^2\sqrt{1-x^2}(1+x^2)}\times\frac{1}{(1+x^2)+\sqrt{1-x^2}} \qquad \textit{(Rationalizing)}$$

$$=\lim_{x\to 0}\frac{x^4+3x^2}{3x^2\sqrt{1-x^2}(1+x^2)}\times\frac{1}{(1+x^2)+\sqrt{1-x^2}}$$

$$=\lim_{x\to 0}\frac{x^2+3}{3\sqrt{1-x^2}(1+x^2)}\times\frac{1}{(1+x^2)+\sqrt{1-x^2}}=\frac{1}{2}$$

LIMITS OF FUNCTIONS HAVING BUILT IN LIMIT WITH THEM :

EXAMPLES :

$$\lim_{n\to\infty} a^n=\begin{cases}0, & 0<a<1\\ 1, & a=1\\ \infty, & a>1\end{cases}, \quad \lim_{n\to\infty} a^{-n}=\begin{cases}\infty, & 0<a<1\\ 1, & a=1\\ 0, & a>1\end{cases}.$$

Illustration :

$f(x)=\lim\limits_{n\to\infty}\dfrac{\tan\pi x^2+(x+1)^n\sin x}{x^2+(x+1)^n}$, find $\lim\limits_{x\to 0}f(x)$.

Sol. $f(x)=\begin{cases}\sin x, & x>0\\ \dfrac{\tan\pi x^2}{x^2}, & x<0\end{cases}$

$$\lim_{x\to 0^+}f(x)=\lim_{x\to 0^+}\sin x=0$$

$$\lim_{x\to 0^-}f(x)=\lim_{x\to 0^-}\frac{\tan\pi x^2}{x^2}=\pi$$

$\Rightarrow$ $LHL\neq RHL\Rightarrow\lim\limits_{x\to 0}f(x)=DNE$.

Illustration :

$$f(x) = \lim_{n\to\infty} = \frac{\cos\pi x - x^{2n}\sin(x-1)}{1+x^{2n+1}-x^{2n}},\ \text{find} \lim_{x\to 1} f(x)$$

Sol. $$f(x) = \lim_{n\to\infty} \frac{\cos\pi x - x^{2n}\sin(x-1)}{1+x\cdot x^{2n}-x^{2n}}$$

when $0 < x^2 < 1$, $\quad \lim_{n\to\infty} x^{2n} \to 0$

$\therefore \quad f(x) = \cos\pi x$

when $x^2 = 1 \quad \lim_{n\to\infty} x^2 \to 1$

$$\therefore \quad f(x) = \frac{\cos\pi x - \sin(x-1)}{1+x-1} = \frac{\cos\pi x - \sin(x-1)}{x}$$

when $x^2 > 1$

$\lim_{n\to\infty} x^{2n} \to \infty$

$$\therefore \quad \lim_{n\to\infty} \frac{1}{x^{2n}} \to 0$$

$$f(x) = \lim_{n\to\infty} \frac{\frac{\cos\pi x}{x^{2n}} - \sin(x-1)}{\frac{1}{x^{2n}} + x - 1} = \frac{0-\sin(x-1)}{0+x-1} = \frac{-\sin(x-1)}{(x-1)}$$

$$f(x) = \begin{cases} \cos\pi x\ , & 0 < x^2 < 1 \\ \frac{\cos\pi x - \sin(x-1)}{x}, & x^2 = 1 \\ \frac{-\sin(x-1)}{(x-1)}, & x^2 > 1 \end{cases}$$

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} \cos\pi x = -1, \qquad \lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} \frac{-\sin(x-1)}{(x-1)} = -1$$

$\therefore \quad \lim_{x\to 1} f(x) = 1.$

Illustration :

Let $f(x) = \min\left(1, x^{2n}, x^{2n+1}\right), n \in N$. The value of $\lim_{x\to 0} \frac{e^{\tan(f(x))} - e^{\sin(f(x))}}{\tan(f(x)) - \sin(f(x))}$, is equal to

(A) 0 (B) 1 (C) 2 (D) does not exist.

Sol. $f(x) = \min.\ (1, x^{2n}, x^{2n+1}),\ n \in N$

$$= \begin{cases} x^{2n+1}, & x < 0 \\ x^{2n+1}, & 0 \le x \le 1 \\ 1, & x \ge 1 \end{cases} = \begin{cases} x^{2n+1}, & x \le 1 \\ 1, & x \ge 1 \end{cases}$$

$$\lim_{x\to 0}\frac{e^{\tan(f(x))}-e^{\sin(f(x))}}{\tan(f(x))-\sin(f(x))}$$

$$=\lim_{x\to 0}\frac{e^{\sin(f(x))}\left(e^{\tan(f(x))-\sin(f(x))}-1\right)}{\tan(f(x))-\sin(f(x))}$$

$$=\lim_{x\to 0}\frac{\left(e^{\tan\left(x^{2n+1}\right)-\sin\left(x^{2n+1}\right)}-1\right)}{\tan\left(x^{2n+1}\right)-\sin\left(x^{2n+1}\right)}=\lim_{y\to 0}\frac{e^{y}-1}{y}=1$$

where $y=\tan(x^{2n+1})-\sin(x^{2n+1})$ **Ans.**

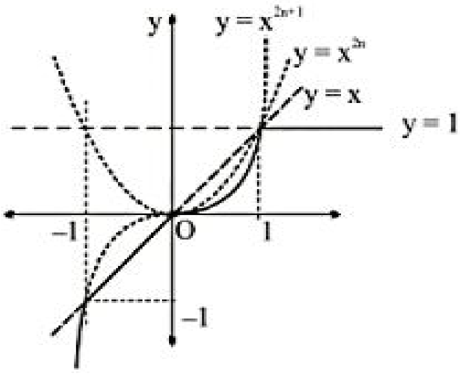


Illustration :

Let $f(x)=\lim_{n\to\infty}\dfrac{\sin(\pi x^{4})+(x+2)^{n}\cdot\dfrac{\tan\pi x}{x+1}}{1+(x+2)^{n}-x^{4}}$, then $\lim_{x\to -1}f(x)$ is equal to

(A) π (B) 22/7 (C) 1 (D) non existent

Sol. Let $f(x)=\lim_{n\to\infty}\dfrac{\sin(\pi x^{4})+(x+2)^{n}\dfrac{\tan\pi x}{(x+1)}}{1+(x+2)^{n}-x^{4}}$

$$f(x)=\begin{cases}\dfrac{\sin(\pi x^{4})}{1-x^{4}}, & x<-1\\[2mm] \dfrac{\tan\pi x}{(x+1)}, & x>-1\end{cases}$$

$$\lim_{x\to -1}f(x)=\lim_{x\to -1^{+}}\frac{\tan\pi x}{(x+1)}=\lim_{x\to -1^{+}}\frac{(\tan\pi(1+x))}{(1+x)}=\pi$$

$$\lim_{x\to -1^{-}}f(x)=\lim_{x\to -1}\frac{\sin(\pi x^{4})}{(1-x^{4})}=\lim_{x\to -1^{-}}\frac{\sin(\pi(1-x^{4}))}{(1-x^{4})}=\pi$$

$$\lim_{x\to -1^{-}}f(x)=\pi.$$

ONE SIDED LIMITS:

Illustration :

Evaluate $\lim_{x\to 0}\left(1+\tan^{2}\sqrt{x}\right)^{1/x}$

Sol. Let $\left(1+\tan^{2}\sqrt{x}\right)^{\frac{1}{x}}$, $l=\lim_{x\to 0^{+}}\dfrac{1}{x}\ln\left(1+\tan^{2}\sqrt{x}\right)$

$$l=e^{\lim_{x\to 0^{+}}\frac{1}{x}\left(\tan^{2}\sqrt{x}-\frac{\left(\tan^{2}\sqrt{x}\right)^{2}}{2}+\ldots\right)}=e^{1}=e$$

$$l = e^{\lim_{x\to 0^-} \frac{1}{x} \ln\left(1+\tan^2\sqrt{x}\right)}.$$

So here left hand limit has so significance as $\sqrt{x}$ is not defined for $x < 0$.

Illustration :

$$\lim_{x\to \tan^{-1}3} \frac{[\tan^2 x] - 2[\tan x] - 3}{[\tan^2 x] - 4[\tan x] + 3}$$ *(where [x] is the greatest integer function of x)*

(A) is 1/3 *(B) is 2* *(C) is 3* *(D) does not exist*

Sol. $$\lim_{x\to \tan^{-1}3} \frac{[\tan^2 x] - 2[\tan x] - 3}{[\tan^2 x] - 4[anx] + 3} = \lim_{x\to \tan^{-1}3} \frac{8-4-3}{8-8+3} = \frac{1}{3}.$$ ***Ans.***

EXPANSION OF FUNCTION:

Expansion of function like Binomial expansion, exponential & logarithmic expansion, expansion of sinx, cosx, tanx should be remembered by heart & are given below :

(i) $a^x = 1 + \frac{x\ln a}{1!} + \frac{x^2\ln^2 a}{2!} + \frac{x^3\ln^3 a}{3!} + \ldots\ldots\ldots a > 0$ **(ii)** $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\ldots\ldots\ldots \ \forall\, x \in R$

(iii) $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\ldots\ldots \text{for } -1 < x \le 1$

(iv) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots\ldots\ldots \left(-\frac{\pi}{2} < x < \frac{\pi}{2}\right)$ **(v)** $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots\ldots \ x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

(vi) $\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \ldots\ldots\ldots \ x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ **(vii)** $\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots\ldots\ldots$

(viii) $\sin^{-1}x = x + \frac{1^2}{3!}x^3 + \frac{1^2.3^2}{5!}x^5 + \frac{1^2.3^2.5^2}{7!}x^7 + \ldots\ldots\ldots$ **(ix)** $\sec^{-1}x = 1 + \frac{x^2}{2!} + \frac{5x^4}{4!} + \frac{61x^6}{6!} + \ldots.$

Illustration :

Evaluate the following limit :

(i) $\lim_{x\to 0} \frac{e^x - 1 - x}{x^2}$ *(ii)* $\lim_{x\to 0} \frac{x - \sin x}{x^3}$ *(iii)* $\lim_{x\to 0} \frac{e^x - e^{-x} - 2x}{x^3}$ *(iv)* $\lim_{x\to 0} \frac{x - \tan x}{x^3}$

Sol.

(i) $$\lim_{x\to 0} \frac{e^x - 1 - x}{x^2} = \lim_{x\to 0} \frac{\left(1 + x + \frac{x^2}{2!} + \ldots.\right) - 1 - x}{x^2} = \frac{1}{2!} = \frac{1}{2}.$$

(ii) $$\lim_{x\to 0} \frac{x - \sin x}{x^3} = \lim_{x\to 0} \frac{x - \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots..\right)}{x^3} = \frac{1}{3!} = \frac{1}{6}.$$

(iii) $$\lim_{x\to 0}\frac{e^x-e^{-x}-2x}{x^3} = \lim_{x\to 0}\frac{\left(1+x+\frac{x^2}{2!}+....\right)-\left(1-x+\frac{x^2}{2!}...\right)-2x}{x^3}$$

$$= \lim_{x\to 0}\frac{\left(2x+\frac{2x^3}{3!}+....\right)-2x}{x^3} = \frac{2}{3!}=\frac{1}{3}.\ \textbf{Ans.}$$

(iv) $$\lim_{x\to 0}\frac{x-\tan x}{x^3} = \lim_{x\to 0}\frac{x-\left(x+\frac{x^3}{3}+\frac{2x^5}{15}+......\right)}{x^3} = \frac{-1}{3}.\ \textbf{Ans.}$$

Illustration :

Evaluate $\lim_{x\to 0}\frac{e^{x^3}-1-x^3}{\sin^6 2x}$

Sol. $$\lim_{x\to 0}\frac{e^{x^3}-1-x^3}{\sin^6 2x} = \lim_{x\to 0}\frac{\left(1+x^3+\frac{x^6}{2!}+.....\right)-1-x^3}{\frac{\sin^6 2x}{(2x)^6}\cdot(2x)^6}$$

$$= \lim_{x\to 0}\frac{\frac{1}{2!}+\frac{x^9}{3!}+.....}{2^6}\ \frac{\frac{1}{2}}{2^6} = \frac{1}{128}.\ \textbf{Ans.}$$

Illustration :

Evaluate $\lim_{x\to\infty} x-x^2\ \ell n\left(1+\frac{1}{x}\right)$

Sol. $\lim_{x\to\infty} x-x^2\ \ln\left(1+\frac{1}{x}\right)$ put $x = \frac{1}{y}$

$$= \lim_{y\to 0}\frac{1}{y}-\frac{\ln(1+y)}{y^2}=\frac{y-\ln(1+y)}{y^2} = \lim_{y\to 0}\frac{y-\left(y-\frac{y^2}{2}+\frac{y^3}{3!}-...\right)}{y^2} = \frac{1}{2}.\ \textbf{Ans.}$$

Don't do it: $\lim_{y\to 0}\left(\frac{1}{y}-\frac{\ln(1+y)}{y}\cdot\frac{1}{y}\right)=\frac{1}{y}-\frac{1}{y}=0$ as $\lim_{y\to 0}\frac{\ln(1+y)}{y}=1$, is not correct.

Illustration :

$$\text{Evaluate } \lim_{x\to 0}\frac{(1+x)^{1/x}-e}{x}.$$

Sol. $$\lim_{x\to 0}\frac{(1+x)^{\frac{1}{x}}-e}{x} = \lim_{x\to 0}\frac{e^{\frac{1}{x}\ln(1+x)}-e}{x} = \lim_{x\to 0}\frac{e\left(e^{\frac{\ln(1+x)-x}{x}}-1\right)}{\left(\frac{\ln(1+x)-x}{x}\right)}\cdot\left(\frac{\ln(1+x)-x}{x^2}\right)$$

$$= e\lim_{x\to 0}\frac{e^{\frac{\ln(1+x)}{x}-1}}{\left(\frac{\ln(1+x)}{x}-1\right)}\cdot\lim_{x\to 0}\left(\frac{\ln(1+x)-x}{x^2}\right) = e\,(1)\cdot\left(\frac{-1}{2}\right) = \frac{-e}{2}.\ \textbf{Ans.}$$

Don't do it

$$\lim_{x\to 0}\frac{(1+x)^{1/x}-e}{x} = l$$

Let $x = y/2$

$$\therefore l = \lim_{y\to 0}\frac{\left(1+\frac{y}{2}\right)^{2/y}-e}{y/2} = 2.\lim_{y\to 0}\frac{\left(1+\frac{y^2}{4}+y\right)^{1/y}-e}{y} = 2.\lim_{y\to 0}\frac{e^{\frac{1}{y}\left(1+\frac{y^2}{4}+y-1\right)}-e}{y}\ *$$

$$= 2.\lim_{y\to 0}\frac{e^{\left(\frac{y^2}{4}+y\right).\frac{1}{y}}-e}{y} = 2.\lim_{y\to 0}\frac{e^{\frac{y}{4}+1}-e}{y} = 2e.\lim_{y\to 0}\frac{e^{\frac{y}{4}}-1}{y/4}.\frac{1}{4} = \frac{2e}{4} = \frac{e}{2}$$

Note that mistake occured at *.

Illustration :

$$\text{Evaluate } \lim_{x\to 0}\frac{1}{(\sin^{-1}x)^2}-\frac{1}{x^2}.$$

Sol. Put $x = \sin\theta \Rightarrow \lim_{\theta\to 0}\frac{1}{\theta^2}-\frac{1}{\sin^2\theta} = \lim_{\theta\to 0}\frac{\sin^2\theta-\theta^2}{\theta^2\sin^2\theta}$

$$= \lim_{\theta\to 0}\frac{(\sin\theta-\theta)(\sin\theta+\theta)}{\theta^4} = 2.\lim_{\theta\to 0}\frac{\sin\theta-\theta}{\theta^3}\qquad \textbf{Ans.}\ -\frac{1}{3}$$

Don't do it

$$\lim_{x\to 0}\frac{1}{(\sin^{-1}x)^2}-\frac{1}{x^2} = \lim_{x\to 0}\frac{1}{x^2}.\frac{x^2}{(\sin^{-1}x)^2}-.\frac{1}{x^2} = \lim_{x\to 0}\frac{1}{x^2}-\frac{1}{x^2} = 0, \text{ is wrong.}$$

Illustration :

If $\displaystyle\lim_{x\to 0}\frac{A\cos x+Bx\sin x-5}{x^4}$ exists & finite. Find A & B and also the limit.

Sol. Let $L=\displaystyle\lim_{x\to 0}\frac{A\cos x+Bx\sin x-5}{x^4}$

$$=\lim_{x\to 0}\frac{A\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}.....\right)+B\,x\left(x-\frac{x^3}{3!}+....\right)-5}{x^4}$$

$$L=\lim_{x\to 0}\frac{(A-5)+\left(B-\frac{A}{2}\right)x^2+\left(\frac{A}{24}-\frac{B}{6}\right)x^4}{x^4}=\text{finite value}$$

$$\Rightarrow\quad A=5,\ B=\frac{A}{2}=\frac{5}{2},\ L=\frac{A}{24}-\frac{B}{6}=\frac{-5}{24}.\ \textbf{Ans.}$$

Illustration :

Let $f(x)=\dfrac{4+\sin 2x+A\sin x+B\cos x}{x^2}$. If $\displaystyle\lim_{x\to 0}f(x)$ exists and finite find A and B and the limit.

Sol. Let $L=\displaystyle\lim_{x\to 0}\frac{4+\sin 2x+A\sin x+B\cos x}{x^2}$

$$L=\lim_{x\to 0}\frac{4+2x-\frac{(2x)^3}{3!}+.......+A\left(x-\frac{x^3}{3!}+.....\right)+B\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}+....\right)}{x^2}$$

$$L=\lim_{x\to 0}\frac{(4+B)+(A+2)\,x-\frac{B}{2}x^2+....}{x^2}$$

$$\Rightarrow\quad B=-4,\ A=-2,\ L=2.\ \textbf{Ans.}$$

Illustration :

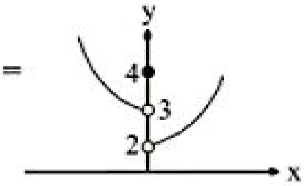
Evaluate $\displaystyle\lim_{x\to 0}\frac{\tan^2x-x^2}{x^2\tan^2x}$

Sol. $\displaystyle\lim_{x\to 0}\frac{\tan^2x-x^2}{x^2\tan^2x}=\lim_{x\to 0}\frac{(\tan x-x)(\tan x+x)}{x^2\tan^2x}$

$$=\lim_{x\to 0}\frac{\left(x+\frac{1}{3}x^3+......-x\right)(\tan x+x)}{x^4\left(\frac{\tan x}{x}\right)^2}=\frac{1}{3}\frac{\left(\frac{\tan x}{x}+1\right)}{\left(\frac{\tan x}{x}\right)^2}=\frac{1}{3}\left(\frac{1+1}{1}\right)=\frac{2}{3}.\ \textbf{Ans.}$$

Illustration :

Refer the figure, the value of $\displaystyle\lim_{x\to 0^-}\left(\left[3f\left(\frac{x^3-\sin^3 x}{x^4}\right)\right]-f\left(\left[\frac{\sin x^3}{x}\right]\right)\right)=$



where [·] denote greatest integer function.

(A) 3 (B) 5 (C) 7 (D) 9

Sol. *Let* $\displaystyle L=\lim_{x\to 0^-}\left(\left[3f\left(\frac{x^3-\sin^3 x}{x^4}\right)\right]\right)-f\left(\left[\frac{\sin x^3}{x}\right]\right)$

when $\displaystyle x\to 0^-,\ \frac{x^3-\sin^3 x}{x^4}=\frac{(x-\sin x)\left(x^2+\sin^2 x+x\sin x\right)}{x^4}$

$$=\frac{x-\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}.......\right)\left(1+\frac{\sin^2 x}{x^2}+\frac{\sin x}{x}\right)}{x^2}$$

$$=x\left(\frac{1}{3!}-\frac{x^2}{5!}+.......\right)\left(1+\frac{\sin^2 x}{x^2}+\frac{\sin x}{x}\right)\to 0^-$$

$f(0^-)=3^+$

$\left[3f\left(0^-\right)\right]=\left[9^+\right]=9$

$f\left(\left[\frac{\sin x^3}{x^3}\cdot x^2\right]\right)=f(0)=4$

$L=9-4=5.$ ***Ans.***

Illustration :

Evaluate the following limits

(a) $\displaystyle\lim_{x\to 0}\frac{e^{x^2}-\cos x}{x^2}$ (b) $\displaystyle\lim_{x\to 0}\frac{e^{\sin 2x}-e^{\sin x}}{x}$

Sol. (a) $$\lim_{x\to 0}\frac{e^{x^2}-\cos x}{x^2}=\lim_{x\to 0}\frac{1+x^2+\frac{(x^2)^2}{2!}+\frac{(x^2)^3}{3!}-\left(1-\frac{x^2}{2!}+.....\right)}{x^2}=\frac{3}{2}$$

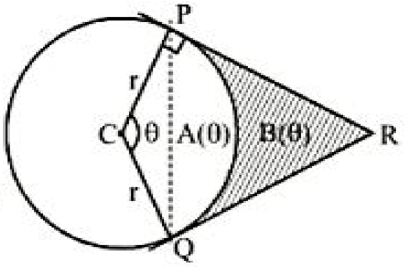
(b) $$\lim_{x\to 0}\frac{e^{\sin 2x}-e^{\sin x}}{x}=\lim_{x\to 0}\frac{\left(1+\sin 2x+\frac{(\sin 2x)^2}{2!}+....\right)-(1+\sin x+....)}{x}$$

$$=\lim_{x\to 0}\frac{\sin 2x-\sin x}{x}=2\ \ 1=1.$$ ***Ans.***

Illustration :

An arc PQ of a circle subtends a central angle θ as shown. Let $A(\theta)$ be the area between the chord PQ and the arc PQ. Let $B(\theta)$ be the area between the tangent lines PR and QR and the arc PQ.

Find $\lim\limits_{\theta \to 0} \dfrac{A(\theta)}{B(\theta)}$



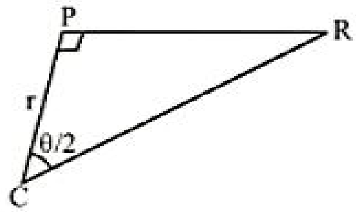
Sol. $A(\theta)$ = *Area of sector PCQ – Area of* ΔPCR

$$= \frac{1}{2} r^2\theta - \frac{1}{2} r^2 \sin\theta = \frac{1}{2} r^2 (\theta - \sin\theta)$$

$B(\theta)$ = *Area of quadrilateral PCQR – Area of sector PCQR*

= 2 (*Area of* ΔCPR) – *Area of sector PCQP*

$$= 2\left(\frac{1}{2} \cdot r \cdot \tan\frac{\theta}{2}\right) - \frac{1}{2} r^2\theta = \frac{r^2}{2}\left(2\tan\frac{\theta}{2} - \theta\right)$$

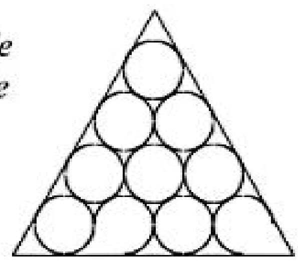


$$\lim_{\theta \to 0} \frac{A(\theta)}{B(\theta)} = -\lim_{\theta \to 0} \frac{\theta - \sin\theta}{2\tan\left(\frac{\theta}{2}\right) - \theta}$$

$$= \lim_{\theta \to 0} \frac{\theta - \left(\theta - \frac{\theta^3}{3!} + \frac{\theta^4}{4!} \ldots\ldots\right)}{2\left(\frac{\theta}{2} + \frac{\left(\frac{\theta}{2}\right)^3}{3} + \frac{2\left(\frac{\theta}{2}\right)^5}{5!} \ldots\ldots\right) - \theta} = \frac{\frac{1}{3!}}{2 \cdot \frac{1}{3 \cdot 2^5}} = \frac{3 \cdot 8}{2 \cdot 6} = 2. \ \textbf{Ans.}$$

Illustration :

Suppose that circle of equal diameter are packed tightly in n rows inside an equilateral triangle. (The figure illustrates the case n=4.) If A is the area of the triangle and A_n is the total area occupied by the circles in n rows then $\lim\limits_{n \to \infty} \dfrac{A_n}{A}$ *equals*



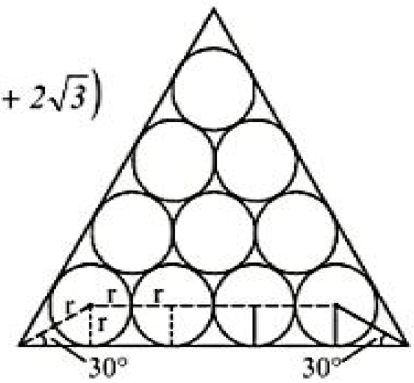
(A) $\dfrac{\pi}{\sqrt{3}}$ (B) $\dfrac{\pi\sqrt{3}}{6}$ (C) $\dfrac{\pi}{2\sqrt{3}}$ (D) $\dfrac{\pi}{6}$

Sol. *Let radius of each circle = r and side of triangle = a*

$a = (n-2)\,2r + 2(r + 30°) = r\left(2n - 4 + 2 + 2\sqrt{3}\right) = r\left(2n - 2 + 2\sqrt{3}\right)$

$$A = \frac{\sqrt{3}}{4} a^2 = \frac{\sqrt{3}}{4} r^2 \ 4\left(n - 1 + \sqrt{3}\right)^2$$

$$A = \sqrt{3}\, r^2 \left(n + \sqrt{3} - 1\right)^2$$



$$A_n = \pi r^2 \quad (1 + 2 + \ldots\ldots + n) = \frac{n(n+1)}{2}\pi r^2$$

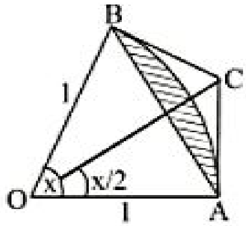
$$\lim_{n\to\infty} \frac{A_n}{A} = \lim_{n\to\infty} \frac{\frac{n(n+1)}{2}\pi r^2}{\sqrt{3}r^2\left(n+\sqrt{3}-1\right)^2}$$

$$= \lim_{n\to\infty} \frac{\pi}{2\sqrt{3}} \frac{\left(1+\frac{1}{n}\right)}{\left(1+\frac{\sqrt{3}-1}{n}\right)^2} = \frac{\pi}{2\sqrt{3}} \frac{(1+0)}{(1+0)^2} = \frac{\pi}{2\sqrt{3}}. \ \textit{Ans.}$$

Illustration :

A circular arc of radius 1 subtends an angle of x radians, $0 < x < \frac{\pi}{2}$ *as shown in the figure. The point C is the intersection of the two tangent lines at A & B. Let T(x) be the area of triangle ABC & let S(x) be the area of the shaded region. Compute:*

(a) T(x) *(b) S(x)* & *(c) the limit of* $\frac{T(x)}{S(x)}$ *as* $x \to 0$.



Sol. $\tan\frac{x}{2} = \frac{CA}{1}$ so $CA = CB = \tan\frac{x}{2}$

$$\angle ABC = \pi - x \ ; \ \Delta ABC = \frac{1}{2}\left(\tan^2\frac{x}{2}\right)\sin x$$

(a) $$\text{so } T(x) = \frac{1}{2}\tan^2\frac{x}{2}\cdot\sin x = \frac{1}{2}\left(\sec^2\frac{x}{2} - 1\right)\sin x$$

$$= \frac{1}{2}\sec^2\frac{x}{2}\sin x \quad \frac{\sin x}{2} - \frac{\sin x}{2} = \tan\left(\frac{x}{2}\right) - \frac{\sin x}{2}$$

$$\Delta OAB = \frac{1}{2}\sin x\ ; \ \text{sector } OAB = \frac{1}{2}x.$$

(b) $$S(x) = \frac{1}{2}x - \frac{1}{2}\sin x.$$

(c) $$\lim_{x\to 0}\frac{T(x)}{S(x)} = \lim_{x\to 0}\frac{\tan\left(\frac{x}{2}\right) - \frac{\sin x}{2}}{\frac{x}{2} - \frac{\sin x}{2}} = \frac{\left(\frac{x}{2} + \frac{\left(\frac{x}{2}\right)^3}{3} + \ldots\ldots\right) - \left(\frac{x}{2} - \frac{x^3}{12} + \ldots\ldots\right)}{\frac{1}{2}\left(x - x + \frac{x^3}{3!} + \ldots\ldots\right)}$$

$$= \lim_{x\to 0}\frac{\frac{1}{24} + \frac{2}{24}}{\frac{1}{12}} = 12\left(\frac{3}{24}\right) = \frac{3}{2}.$$

CONTINUITY

1. GENERAL INTRODUCTION :

After conceiving the notion of limits the next element which is taken into consideration is the continuity of function. Qualitatively the graph of a function is said to be continuous at $x = a$ if while travelling along the graph of the function and in crossing over the point at $x = a$ either from Left to Right or from Right to Left one does not have to lift his pen. In case one has to lift his pen the graph of the function is said to have a break or discontinuous at $x = a$. Different type of situations which may come up at $x = a$ along the graph can be :

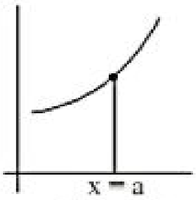


figure (1)

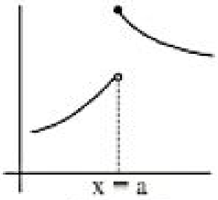


figure (2)

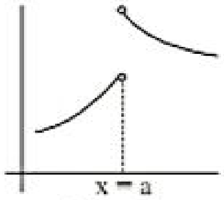


figure (3)

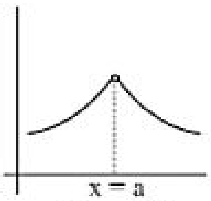


figure (4)

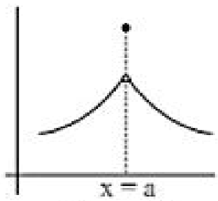


figure (5)

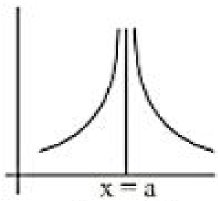


figure (6)

Figure (2) – (6) is discontinuous at $x = a$ and in figure (1) f is continuous at $x = a$

2. DEFINITION OF CONTINUITY OF A FUNCTION :

A function $f(x)$ is said to be continuous at $x = a$,

if $\quad \lim_{x \to a} f(x) = f(a)$.

$\Rightarrow \quad \lim_{h \to 0} f(a - h) = \lim_{h \to 0} f(a + h) = f(a) =$ a finite quantity.

i.e. LHL at $x = a$ = RHL at $x = a$ = value of $f(x)$ at $x = a$ = a finite quantity.

Note:

(i) Continuity at $x = a$ $\Rightarrow$ existence of limit at $x = a$, but not the converse

(ii) Continuity at $x = a$ $\Rightarrow$ f is well defined at $x = a$, but not the converse

(iii) Discontinuity at $x = a$ is meaningful to talk if in the immediate neighbourhood of $x = a$, i.e. the function has a graph in the immidiate neighbourhood of $x = a$, not necessarily at $x = a$.

(iv) Continuity is always talk in the domain of function and hence $f(x) = \frac{1}{x-1}, \frac{1}{x}$, $\tan x$ are all continuous functions but if you want to talk of discontinuity then we can say $\frac{1}{x-1}$ is discontinuous at $x = 1$, $\frac{1}{x}$ is discontinuous at $x = 0$.

Note that all rational functions are continuous. Because continuity is always talk in the domain of $f(x)$.

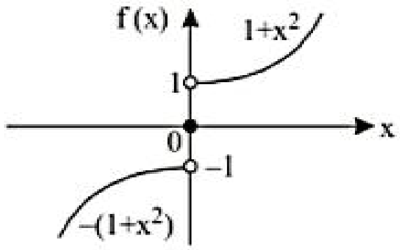
(v) Point function are continuous.

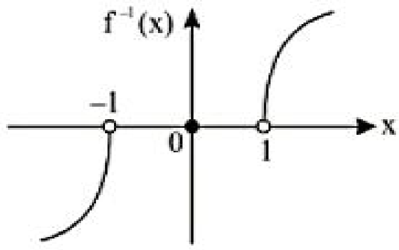
e.g. $\quad \sqrt{1-x} + \sqrt{x-1}, \quad \sqrt{x} + \sqrt{-x}$

(vi) Inverse of a discontinuous function can be continuous.

e.g. $f(x) = \begin{bmatrix} 1+x^2 & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \\ -(1+x^2) & \text{if } x \neq 0 \end{bmatrix}$ is discontinuous at $x = 0$ but its inverse function

$f^{-1}(x) = \begin{bmatrix} \sqrt{x-1} & \text{if } x > 1 \\ 0 & \text{if } x = 0 \\ \sqrt{-(1+x)} & \text{if } x < -1 \end{bmatrix}$ which is a continuous function and its graph is as shown.





3. CONTINUITY IN AN INTERVAL :

(a) A function f is said to be continuous in (a, b) if f is continuous at each & every point $\in (a, b)$.

(b) A function f is said to be continuous in a closed interval $[a, b]$ if :

(i) f is continuous in the open interval (a, b) &

(ii) f is right continuous at '**a**' i.e. $\lim\limits_{x \to a^+} f(x) = f(a) =$ a finite quantity.

(iii) f is left continuous at '**b**' i.e. $\lim\limits_{x \to b^-} f(x) = f(b) =$ a finite quantity.

4. REASONS OF DISCONTINUITY :

A function can be discontinuous due to the following reasons.

(i) $\lim\limits_{x \to a} f(x)$ does not exist (f(a) may or may not be defined)

i.e. $\lim\limits_{h \to 0} f(a+h) \neq \lim\limits_{h \to 0} f(a-h)$

e.g. $f(x) = [x]$ discontinuous at all integer points $f(x) = \operatorname{sgn} x$ discontinuous at $x = 0$

$f(x) = \dfrac{x}{x-1}$ discontinuous at $x = 1$.

(ii) $\lim\limits_{x \to a} f(x)$ exist but is not equal to f(a) i.e. $\lim\limits_{h \to 0} f(a+h) = \lim\limits_{h \to 0} f(a-h) \neq f(a)$

$$f(x) = \begin{bmatrix} (1-x)\tan\dfrac{\pi x}{2} & \text{if } x \neq 1 \\ \dfrac{\pi}{2} & \text{if } x = 1 \end{bmatrix}$$

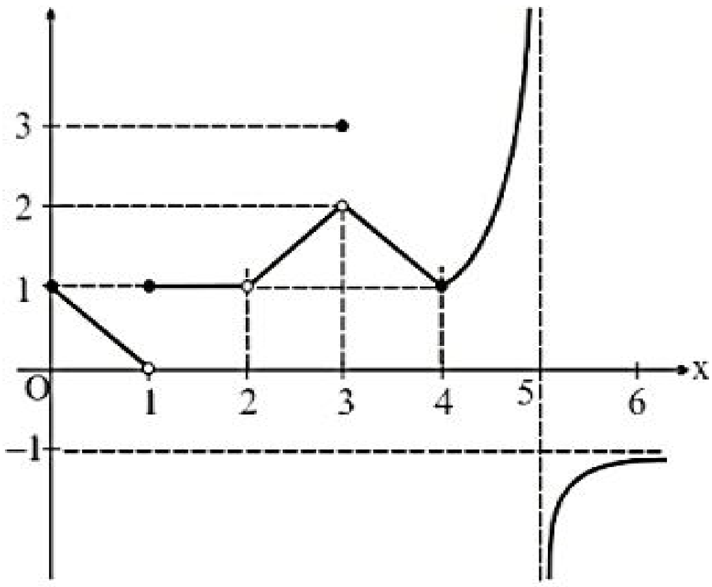
$$\lim_{x \to 1} f(x) = \lim_{x \to 1} (1-x) \tan\left(\frac{\pi x}{2}\right) = \lim_{h \to 1} \frac{-h}{\tan\left(\dfrac{\pi h}{2}\right)} = \frac{2}{\pi}$$

$\Rightarrow$ $\lim f(x) \neq f(1) \Rightarrow f(x)$ is discontinuous at $x = 1$.

(iii) f(a) is not defined

eg. $f(x)=\dfrac{1}{x-1}$

(iv) To understand explicitly the reasons of discontinuity. Consider the following graph of a function.



(a) f is continuous at $x=0$ and $x=4$

(b) f is discontinuous at $x=1$ as limit does not exist

(c) f is discontinuous at $x=2$ as $f(2)$ is not defined although limit exist.

(d) f is discontinuous at $x=3$ as $\lim\limits_{x\to 3} f(x)\neq f(3)$

(e) f is discontinuous at $x=5$ as neither the limit exist nor f is defined at $x=5$

Note:

(i) Every polynomial function is continous at every point of the real line.

$$f(x)=a_0x^n+a_1x^{n-1}+a_2x^{n-2}+\ldots\ldots+a_n\ \forall\ x\in R$$

(ii) Every rational function is continuous at every point where its denominator is different from zero.

(iii) Logarithmic functions, exponential functions, trigonometric functions, inverse circular functions, and modulus functions are continuous in their domain.

Illustration :

Find the points of discontinuity of the following functions.

(i) $f(x)=\dfrac{1}{2\sin x-1}$; *(ii)* $f(x)=\dfrac{1}{x^2-3|x|+2}$; *(iii)* $f(x)=\dfrac{1}{x^4+x^2+1}$; *(iv)* $f(x)=\dfrac{1}{1-e^{\frac{x-1}{x-2}}}$

(v) $f(x)=[[x]]-[x-1]$, *where [·] represents the greatest integer function.*

Sol.

(i) $f(x)=\dfrac{1}{2\sin x-1}$

f (x) is discontinuous when $2\sin x-1=0$

$$\Rightarrow\ \sin x=\frac{1}{2}\ \Rightarrow\ x=2n\pi+\frac{\pi}{6}\quad or\quad x=2n\pi+\frac{5\pi}{6},\ n\in Z$$

(ii) $f(x)=\dfrac{1}{x^2-3|x|+2}$

f (x) is discontinuous when $x^2-3|x|+2=0$

$$\Rightarrow\quad |x|^2-3|x|+2=0\Rightarrow\quad (|x|-1)(|x|-2)=0\Rightarrow|x|=1,2\Rightarrow x=\pm 1,\pm 2$$

(iii) $f(x)=\dfrac{1}{x^4+x^2+1}=\dfrac{1}{\left(x^2+\dfrac{1}{2}\right)^2+\dfrac{3}{4}}$

Now, $x^4+x^2+1=\left(x^2+\dfrac{1}{2}\right)^2+\dfrac{3}{4}\geq 1\ \forall x\in R$

$\Rightarrow$ $f(x)$ is continuous $\forall x \in R$

(iv) $f(x)=\dfrac{1}{1-e^{\frac{x+1}{x-2}}}$

$f(x)$ is discontinuous when $x-2=0$ also

when $1-e^{\frac{x+1}{x-2}}=0$

$\Rightarrow$ $x=2$ and $e^{\frac{x+1}{x-2}}=1$

$\Rightarrow$ $x=2$ and $\dfrac{x-1}{x-2}=0$

$\Rightarrow$ $x=2$ and $x=1$

(v) $f(x)=[[x]]-[x-1]=[x]-([x]-1)=1$

$\Rightarrow f(x)$ is continuous $\forall x\in R$.

Illustration :

(a) $f(x)=\begin{cases}(\cos x)^{\cot^2 x} & x\neq 0\\ e^{-1/2} \quad if & x=0\end{cases}$ find whether the $f(x)$ is continuous at $x=0$ or not.

(b) If $f(x)=\begin{cases}\dfrac{(e^x-1)^3\,cosec(ax)}{\ln(1+x^2)} & x\neq 0\\ b & x=0\end{cases}$ is continuous, find b.

Sol.

(a) $\lim\limits_{x\to 0} f(x)=\lim\limits_{x\to 0}(\cos x)^{\cot^2 x}$

$$= e^{\lim\limits_{x\to 0}(\cos x-1)\cot^2 x}=e^{\lim\limits_{x\to 0}\frac{-(1-\cos x)}{x^2}\cdot\frac{x^2}{\tan^2 x}}=e^{\frac{-1}{2}}=f(0)$$ $\Rightarrow f(x)$ is continuous at $x=0$.

(b) $$\lim_{x\to 0}\frac{(e^x-1)^3\,cosec\,(ax)}{\ln(1+x^2)}=\lim_{x\to 0}\frac{\left(\dfrac{e^x-1}{x}\right)^3}{\dfrac{\ln(1+x^2)}{x^2}}\cdot\left(\frac{x}{\sin ax}\right)$$

$b=\dfrac{1}{-}.$

Illustration :

Find the values of 'a' and 'b' so that the function $f(x)=\begin{cases} x+a\sqrt{2}\sin x & 0\le x<\frac{\pi}{4} \\ 2x\cot x+b & \frac{\pi}{4}\le x\le\frac{\pi}{2} \\ a\cos 2x-b\sin x & \frac{\pi}{2}<x\le\pi \end{cases}$

is continuous in [0, π]

Sol.

f(x) is continuous in the interval $0\le x<\frac{\pi}{4},\ \frac{\pi}{4}<x<\frac{\pi}{2},\ \frac{\pi}{2}<x\le\pi$.

We need to make the function continuous at $x=\frac{\pi}{4},\frac{\pi}{2}$

For continuity at $x=\frac{\pi}{4}$, $\lim\limits_{x\to\frac{\pi}{4}^-} f(x)=\lim\limits_{x\to\frac{\pi}{4}^+} f(x)=f\left(\frac{\pi}{4}\right)$

$$\lim_{x\to\left(\frac{\pi}{4}\right)^-}\left(x+a\sqrt{2}\sin x\right)=\lim_{x\to\left(\frac{\pi}{4}\right)^+}\left(2x\cot x+b\right)=f\left(\frac{\pi}{4}\right)$$

$$\Rightarrow \frac{\pi}{4}+a\sqrt{2}\sin\left(\frac{\pi}{4}\right)=2\cdot\frac{\pi}{4}\cdot\cot\left(\frac{\pi}{4}\right)+b=2\frac{\pi}{4}\cdot\cot\left(\frac{\pi}{4}\right)+b$$

$$\Rightarrow \quad \frac{\pi}{4}+a=\frac{\pi}{2}+b \Rightarrow a-b=\frac{\pi}{4} \qquad ....(1)$$

For continuity at $x=\frac{\pi}{2}$, $\lim\limits_{x\to\frac{\pi}{2}^-} f(x)=\lim\limits_{x\to\frac{\pi}{2}^+} f(x)=f\left(\frac{\pi}{2}\right)$

$$\Rightarrow \quad \lim_{x\to\frac{\pi}{2}^-}\left(2x\cot x+b\right)=\lim_{x\to\left(\frac{\pi}{2}\right)^+}\left(a\cos 2x-b\sin x\right)=a\cos\pi-b\sin\pi$$

$$\Rightarrow \quad 0+b=-a-b \Rightarrow a+2b=0 \qquad ....(2)$$

From equation (1) and (2)

$$a=\frac{\pi}{6},\ b=\frac{-\pi}{12}.$$

Illustration :

$$f(x)=\begin{cases} \dfrac{1-\cos 4x}{x^2} & \text{if } x<0 \\ a & \text{if } x=0 \\ \dfrac{\sqrt{x}}{\sqrt{16+\sqrt{x}}-4} & \text{if } x>0 \end{cases}$$

Determine 'a' if possible so that the function is continuous at x = 0.

Sol. $f(0^-) = \lim\limits_{x\to0^-} f(x) = \lim\limits_{x\to0^-} \dfrac{1-\cos 4x}{x^2} = 8$

$$f(0^+) = \lim_{x\to0^+} f(x) = \lim_{x\to0^+} \frac{\sqrt{x}}{\sqrt{16+\sqrt{x}}\ -4} = \lim_{x\to0^+} \frac{\sqrt{x}\cdot\left(\sqrt{16+\sqrt{x}}+4\right)}{16+\sqrt{x}\ -16}$$

$$= \lim_{x\to0^+}\left(\sqrt{16+\sqrt{x}}+4\right) = 8$$

$f(0^-) = f(0^+) = 8 = f(0) \Rightarrow a = 8.$

Illustration :

$$\text{Let } f(x) = \begin{cases} (1+|\sin x|)^{\frac{a}{|\sin x|}} & \text{for } -\dfrac{\pi}{6}<x<0 \\ b & \text{for } x=0 \\ e^{\frac{\tan 2x}{\tan 3x}} & \text{for } 0<x<\dfrac{\pi}{6} \end{cases}$$

Find 'a' and 'b' if f is continuous at x =0.

Sol. $\lim\limits_{x\to0^-} f(x) = \lim\limits_{x\to0} (1+|\sin x|)^{\frac{a}{|\sin x|}} = e^a$

$$\lim_{x\to0^+} f(x) = \lim_{x\to0^+} e^{\frac{\tan 2x}{\tan 3x}} = e^{\lim\limits_{x\to0^+}\frac{\tan 2x}{2x}\cdot\frac{3x}{\tan 3x}\cdot\frac{3}{2}} = e^{\frac{3}{2}}$$

$\therefore \quad e^a = b = e^{\frac{3}{2}} \Rightarrow a = \dfrac{3}{2},\ b = \ln\left(\dfrac{3}{2}\right).$

Illustration :

$$\text{Let } f(x) = \begin{cases} \dfrac{(e^{2x}+1)-(x+1)(e^x+e^{-x})}{x(e^x-1)} & \text{if } x\neq 0 \\ k & \text{if } x=0 \end{cases}$$

if f (x) is continuous at x = 0 then k is equal to

(A) 1/2 *(B) 1* *(C) 3/2* *(D) 2*

Sol. $k = \lim\limits_{x\to0} \dfrac{\left(e^{2x}+1\right)-(x+1)(e^x+e^{-x})}{x(e^x-1)} = \lim\limits_{x\to0} \dfrac{\left(e^{2x}+1\right)-(x+1)(e^x+e^{-x})}{x^2}$

$$= \lim_{x \to 0} \frac{2e^{2x} - (x+1)(e^x - e^{-x}) - (e^x + e^{-x})}{2x}$$

By L Hospital Rule

$$= \lim_{x \to 0} \frac{4e^{2x} - (x+1)\left(e^x + e^{-x}\right) - \left(e^x - e^{-x}\right) - \left(e^x - e^{-x}\right)}{2} = \frac{4-2-0-0}{2} = 1.$$

Illustration :

Let $f(x) = \dfrac{\sqrt{x^2 + kx + 1}}{x^2 - k}$. *The interval(s) of all possible values of k for which f is continuous for every* $x \in R$, *is*

(A) $(-\infty, -2]$ (B) $[-2, 0)$ (C) $R - (-2, 2)$ (D) $(-2, 2)$

Sol. $x^2 - k \neq 0 \ \forall x \in R$

$\Rightarrow \quad k < 0$ ....(1)

$x^2 + x + 1 \geq 0 \ \forall x \in R$

$\Rightarrow \quad k^2 - 4 \leq 0 \Rightarrow -2 \leq k \leq 2$ ....(2)

$\therefore$ From (1) and (2)

$k \in [-2, 0)$

Practice Problem

Q.1 What value must be assigned to k so that the function f(x) is continuous at $x = 4$?

$$f(x) = \begin{cases} \dfrac{x^4 - 256}{x - 4}, & x \neq 4 \\ k, & x = 4 \end{cases}$$

Q.2 Let $f(x) = \begin{cases} \dfrac{\sin ax^2}{x^2}, & x \neq 0 \\ \dfrac{3}{4} + \dfrac{1}{4a}, & x = 0 \end{cases}$. For what values of a, f(x) is continuous at $x = 0$.

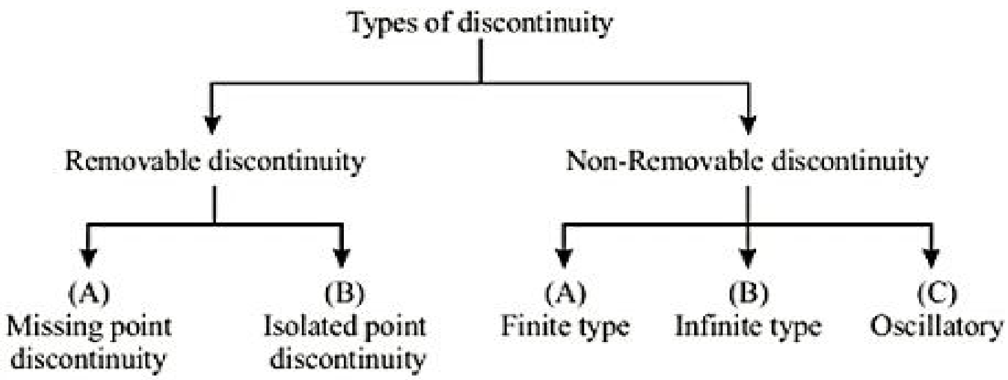
Q.3 Let $f(x) = \begin{cases} \dfrac{a + 3\cos x}{x^2}, & x \neq 0 \\ b\tan\left(\dfrac{\pi}{[x+3]}\right), & x = 0 \end{cases}$

If f(x) is continuous at $x = 0$, then find a and b, where $[\cdot]$ denotes the greatest integer function.

Answer key

Q.1 $k = 256$ **Q.2** $a = -\dfrac{1}{4}, 1$ **Q.3** $a = -3, b = -\dfrac{\sqrt{3}}{2}$

5. TYPES OF DISCONTINUITY :



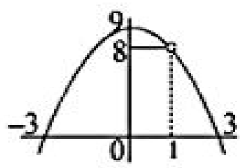
5.1 REMOVABLE DISCONTINUITY :

Here $\lim\limits_{x \to a} f(x)$ necessarily exists, but is either not equal to $f(a)$ or $f(a)$ is not defined. In this case, therefore it is possible to redefine the function in such a manner that $\operatorname{Lim}\limits_{x \to a} f(x) = f(a)$ and thus making the function continuous. These discontinuities can be further classified as

(A) Missing point discontinuity :

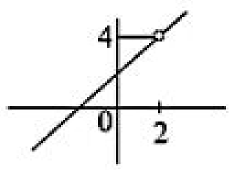
Here $\operatorname{Lim}\limits_{x \to a} f(x)$ exists. But f(a) is not defined.

(a) $f(x) = \dfrac{(x-1)(9-x^2)}{x-1} \quad x \neq 1$



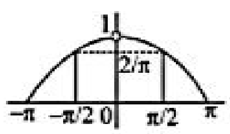
at $x = 1$, $f(1)$ is not defined. Hence $f(x)$ has missing point of discontinuity at $x = 1$.

(b) $f(x) = \dfrac{x^2-4}{x-2} \quad x \neq 2$



$f(2)$ is not defined. Hence, $f(x)$ has missing point of discontinuity at $x = 2$.

(c) $f(x) = \dfrac{\sin x}{x}, x \neq 0$

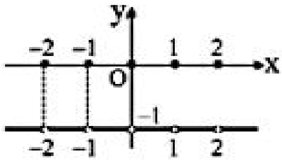


$f(0)$ is not defined. $f(x)$ has missing point of discontinuity at $x = 0$.

(B) Isolated point discontinuity :

Here $\lim_{x \to a} f(x)$ exists, also f(a) is defined but $\lim_{x \to a} f(x) \neq f(a)$

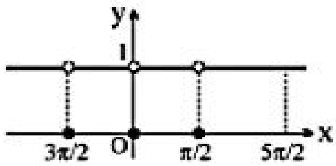
(a) $f(x) = [x] + [-x] = \begin{cases} 0 & \text{if } x \in I \\ -1 & \text{if } x \notin I \end{cases}$



has isolated point of discontinuity at all integeral points.

(b) $f(x) = \text{sgn}(\cos 2x - 2\sin x + 3) = \text{sgn}\left(2(2 + \sin x)(1 - \sin x)\right) = \begin{cases} 0 & \text{if } x = 2n\pi + \dfrac{\pi}{2} \\ +1 & \text{if } x \neq 2n\pi + \dfrac{\pi}{2} \end{cases}$

has an isolated point at x = 0 discontinuity as $x = 2n\pi + \dfrac{\pi}{2}$



5.2 NON-REMOVABLE DISCONTINUITY :

Here $\lim_{x \to a} f(x)$ does not exists and therefore it is not possible to redefine the function in any manner to make it continuous. Such discontinuities can be further classified into 3 fold.

(a) Finite type (both limits finite and unequal)

(i) $\lim_{x \to 0} \tan^{-1}\left(\dfrac{1}{x}\right) < \begin{matrix} f(0^+) = \dfrac{\pi}{2} \\ f(0^-) = -\dfrac{\pi}{2} \end{matrix}$; jump = π

(ii) $\lim_{x \to 0} \dfrac{|\sin x|}{x} < \begin{matrix} f(0^+) = 1 \\ f(0^-) = -1 \end{matrix}$; jump = 2

(iii) $\lim_{x \to 2} \dfrac{[x]}{x} < \begin{matrix} f(2^+) = 1 \\ f(2^-) = \dfrac{1}{2} \end{matrix}$; jump = $\dfrac{1}{2}$

In this case non negative difference between the two limits is called the Jump of discontinuity. A function having a finite number of jumps in a given interval I is called a Piece Wise Continuous or Sectionally Continuous function in this interval.

(b) Infinite type (at least one of the two limit are infinity)

(i) $f(x)=\dfrac{x}{1-x}$ at $x=1$ $\Big\langle$ $f(1^+)=-\infty$, $f(1^-)=+\infty$

(ii) $f(x)=2^{\tan x}$ at $x=\dfrac{\pi}{2}$ $\Big\langle$ $f\left(\dfrac{\pi}{2}^+\right)=0$, $f\left(\dfrac{\pi}{2}^-\right)=\infty$

(iii) $f(x)=\dfrac{1}{x^2}$ at $x=0$ $\Big\langle$ $f(0^+)=\infty$, $f(0^-)=\infty$

(c) Oscillatory (limits oscillate between two finite quantities)

(i) $\left.\begin{array}{l} f(x)=\sin\dfrac{1}{x} \\ \text{or} \\ f(x)=\cos\dfrac{1}{x}\end{array}\right]$ at $x=0$ oscillates between -1 & 1

(ii) $f(x)=\left[1+\dfrac{1}{3}\sin(\ln|x|)\right]$ at $x=0$ oscillates between 0 & 1.

Illustration :

State the number of point of discontinuities and discuss the nature of discontinuity for the function $f(x)=\dfrac{1}{\ln|x|}$ and also sketch its graph.

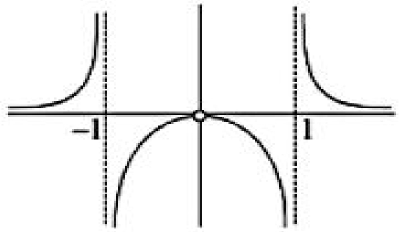
Sol. $f(x)=\left[\begin{array}{ll}\dfrac{1}{\ln x} & \text{if } x>0,\ x\neq 1 \\ \dfrac{1}{\ln(-x)} & \text{if } x<0,\ x\neq -1\end{array}\right.$ *function is obviously discontinuous at $x = 0, 1, -1$. as it is not defined.*

$\left.\begin{array}{l}\displaystyle\lim_{x\to 0^+} f(x)=0 \\ \displaystyle\lim_{x\to 0^-} f(x)=0\end{array}\right\}$ *Limit exists at $x = 0$. Hence removable discontinuity at $x = 0$. (Missing point discontinuity)*

$\left.\begin{array}{l}\displaystyle\lim_{x\to 1^+} f(x)=\infty \\ \displaystyle\lim_{x\to 1^-} f(x)=-\infty\end{array}\right\}$ *Limit DNE. Hence non removable discontinuity (infinite type) at $x = 1$*

$$\left.\begin{array}{l}\lim\limits_{x\to -1^+} f(x) = -\infty \\ \lim\limits_{x\to -1^-} f(x) = \infty\end{array}\right\}$$ *Limit DNE. Hence non removable discontinuity (infinite type) at x = 0*

Note that f (x) is even ⇒ symmetric about y axis.
The graph of f (x) is as follows.



Practice Problem

Q.1 The function $f: R - \{0\} \to R$ given by $f(x) = \dfrac{1}{x} - \dfrac{2}{e^{2x}-1}$, is continuous at $x = 0$, then find the value of $f(0)$.

Q.2 Let $f(x) = \begin{cases}(1+3x)^{1/x}, & x \neq 0 \\ e^3, & x = 0\end{cases}$. Discuss the continuity of $f(x)$ at (i) $x = 0$, (ii) $x = 1$.

Q.3 Which of the following functions is not continuous $\forall\, x \in R$?

(A) $\sqrt{2\sin x + 3}$ (B) $\dfrac{e^x+1}{e^x+3}$ (C) $\left(\dfrac{2^{2x}+1}{2^{3x}+5}\right)^{5/7}$ (D) $\sqrt{\operatorname{sgn} x + 1}$

Q.4 Discuss the continuity of $f(x) = \begin{cases}\dfrac{x^4-5x^2+4}{|(x-1)(x+2)|}, & x \neq 1, 2 \\ 6, & x = 1 \\ 12, & x = 2\end{cases}$

Q.5

	Column-I		Column-II
(A)	$f(x) = \dfrac{1}{x-1}$	(P)	Removable discontinuity
(B)	$f(x) = \dfrac{x^3-x}{x^2-1}$	(Q)	Non-removable discontinuity
(C)	$f(x) = \dfrac{\lvert x-1\rvert}{x-1}$	(R)	Jump of discontinuity
(D)	$f(x) = \sin\left(\dfrac{1}{x-1}\right)$	(S)	Discontinuity due to vertical asymptote
		(T)	Missing point discontinuity
		(U)	Oscillating discontinuity

Answer key

Q.1	1	**Q.2**	Continuous at $x = 0$
Q.3	D	**Q.4**	$f(x)$ is discontinuous at $x = 1, 2$

Q.5 (A) $\rightarrow$ (S), (Q); (B) $\rightarrow$ (T), (P); (C) $\rightarrow$ (R), (Q); (D) $\rightarrow$ (U), (Q)

6. CONTINUITY OF FUNCTIONS DEFINED BY SOME FUNCTIONAL RULE :

Illustration :

If $f(x+y) = f(x) \cdot f(y)$ for all x & y & $f(x) = 1 + g(x)$. $G(x)$ where $\lim\limits_{x \to 0} g(x) = 0$ & $\lim\limits_{x \to 0} G(x)$ exist. Prove that $f(x)$ is continuous for all x.

Sol. $\lim\limits_{x \to a^-} f(x) = \lim\limits_{h \to 0} f(a-h) = \lim\limits_{h \to 0} f(a)\, f(-h) = \lim\limits_{h \to 0} f(-h)$

$$= f(a)\left(1 + \lim_{h \to 0} g(-h)\, g(-h)\right) = f(a)$$

Similarly $\lim\limits_{x \to a^+} f(x) = f(a) = \lim\limits_{x \to a^-} f(x)$ *so continuous at $x = a$.*

7. THEOREMS ON CONTINUITY :

T-1 : Sum, difference, product and quotient of two continuous functions is always a continuous function. However $h(x) = \dfrac{f(x)}{g(x)}$ is continuous at $x = a$ only if $g(a) \neq 0$.

8. FOLLOWING IMPORTANT NOTES SHOULD BE REMEMBERED :

(a) If $f(x)$ is continuous and $g(x)$ is discontinuous then prove that $f(x) + g(x)$ is a discontinuous function.

Proof : Let $f(x) + g(x)$ is a continuous function.

so, $\lim\limits_{x \to a} \left(f(x) + g(x)\right) = f(a) + g(a)$ ....(1)

Also, $f(x)$ is a continuous function $\lim\limits_{x \to a} f(x) + g(a)$ ....(2)

From (1) and (2)

$\lim\limits_{x \to a} g(x) = g(a) \Rightarrow g(x)$ is continuous at $x = a$.

But given $g(x)$ is discontinuous at $x = a$.

(b) If f(x) is continuous & g(x) is discontinuous at $x = a$ then the product function $\phi(x) = f(x) \cdot g(x)$ is not necessarily be discontinuous at $x = a$. e.g.

(i) $f(x) = x$ & $g(x) = \begin{bmatrix} \sin\frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{bmatrix}$.

Then $f(x) \cdot g(x) = \begin{cases} x \sin\left(\dfrac{1}{x}\right), & x \neq 0 \\ 0, & x = 0 \end{cases}$ is continuous at $x = 0$.

(ii) $f(x) = \cos\left(\dfrac{2x-1}{2}\right)\pi$ is continuous at $x = 1$ and $g(x) = [x]$ and $[\cdot]$ denotes the greatest integer functions is discontinuous at $x = 1$ but $f(x) \cdot g(x)$ is continuous at $x = 1$.

$$\lim_{x \to 1^+} \cos\left(\frac{2x-1}{2}\right)\pi \cdot [x] = \cos\frac{\pi}{2} \cdot (1) = 0$$

$$\lim_{x \to 1^-} \cos\left(\frac{2x-1}{2}\right)\pi \cdot [x] = \cos\left(\frac{\pi}{2}\right) \cdot (0) = 0,\ f(1) = 0 \Rightarrow \text{continuous at } x = 1.$$

(c) If f(x) and g(x) both are discontinuous at $x = a$ then the product function $\phi(x) = f(x) \cdot g(x)$ is not necessarily be discontinuous at $x = a$. e.g.

$f(x) = -g(x) = \begin{bmatrix} 1 & x \geq 0 \\ -1 & x < 0 \end{bmatrix}$

$\therefore$ $f(x)\, g(x) = 1\ \forall\, x \in R$ which is continuous function.

Illustration :

If $f(x) = \begin{cases} |x+1|; & x \leq 0 \\ x; & x > 0 \end{cases}$ *and* $g(x) = \begin{cases} |x+1|; & x \leq 0 \\ -|x+2|; & x > 0 \end{cases}$

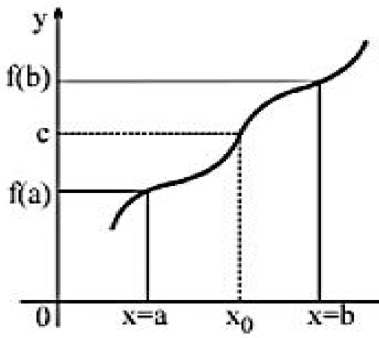
Draw its graph and discuss continuity of $f(x) + g(x)$.

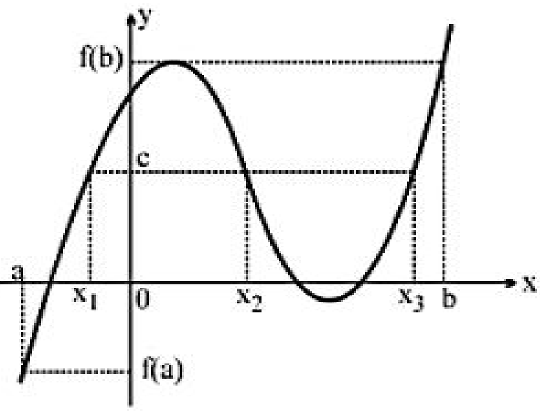
Sol. *Since* $f(x)$ *is discontinuous at* $x = 0$ *and* $g(x)$ *is continuous at* $x = 0$, *then* $f(x) + g(x)$ *is discontinuous at* $x = 0$.

Since $f(x)$ *is continuous at* $x = 1$ *and* $g(x)$ *is discontinuous at* $x = 1$, *then* $f(x) + g(x)$ *discontinuous at* $x = 1$.

T-2 : Intermediate value theorem :

If f is continuous on [a, b] and $f(a) \neq f(b)$ then for any value $c \in (f(a), f(b))$, there is at least one number x_0 in (a, b) for which $f(x_0) = c$





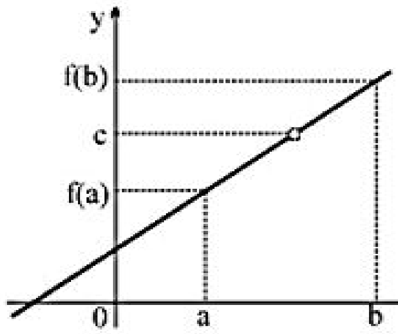


Figure-2

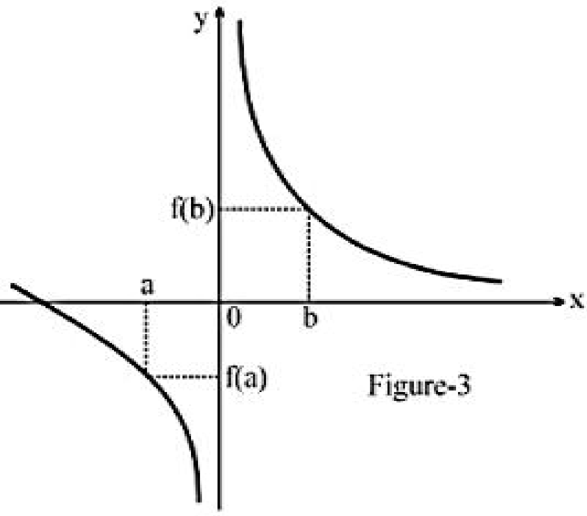


Figure-3

NOTE:

(1) Continuity through the interval [a, b] is essential for the validity of this theorem.

(2) in figure-3, f (a) and f (b) are of opposite sign but f(x) has no root in (a, b) as f is continuous.

Illustration :

Show that the function $f(x) = (x-a)^2 (x-b)^2 + x$ *takes the value* $\frac{a+b}{2}$ *for some value of* $x \in [a, b]$

Sol. $f(x) = (x-a)^2 (x-b)^2 + x$; *as* $f(x)$ *is continuous on* $[a, b]$ *and* $f(a) = a$ *and* $f(b) = b$, *then for any value* $c \in (a, b)$, *there is at least one number* x_0 *in* (a, b) *for which* $f(x_0) = c = \frac{a+b}{2}$.

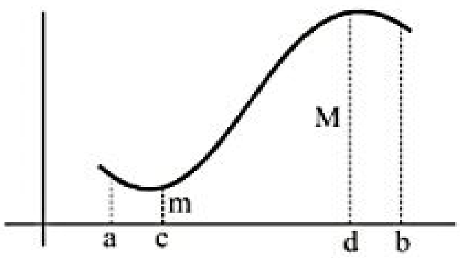
Illustration :

Suppose that $f(x)$ is continuous in $[0, 1]$ and $f(0) = 0, f(1) = 0$. Prove that $f(c) = 1 - 2c^2$ for some $c \in (0, 1)$.

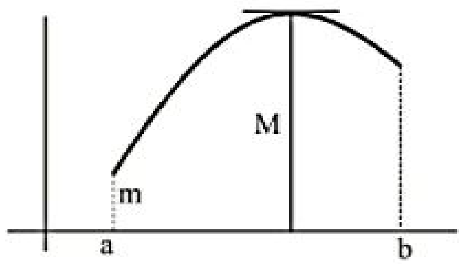
Sol. *Let $f(x) = f(x) + 2x^2 - 1$ is a continuous function in $(0, 1)$.*
$f(0) = f(0) - 1 = -1$ and $f(1) = F(1) + 1 = 1$ then there exists same $c \in (0, 1)$ such that $F(x) = 0$
$f(x) = 1 - 2c^2$.

T-3: Extreme Value Theorem :

If f is continuous on [a, b] then f takes on, a least value of m and a greatest value M on this interval.



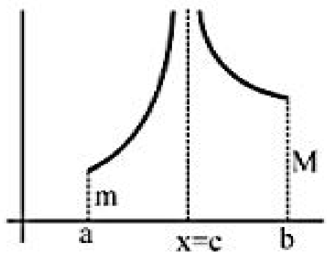
Minimum value 'm' occurs at $x = c$ and maximum value M occurs at $x = d$. $c, d \in (a, b)$



Minimum value 'm' occurs at the end point $x = a$ and the maximum value M occurs inside the interval

☞ **Note :** To see that continuity is necessary for the extreme value theorem to be true refer the graph shown.

There is a discontinuity at $x = c$ interval. The function has a minimum value at the left end point $x = a$ and f has no maximum value.



9. PROPERTIES OF FUNCTION CONTINUOUS IN [a, b] :

(i) If a function f is continuous on a closed interval [a, b] then it is bounded.

(ii) A continuous function whose domain is some closed interval must have its range also in closed interval.

(iii) If f is continuous and onto on [a, b] and is onto then f^{-1}(from the range of f) is also continuous.

(iv) If f(a) and f(b) possess opposite signs then $\exists$ at least one solution of the equation $f(x) = 0$ in the open interval (a, b) provided f is continous in [a, b].

Illustration :

Let f be a continuous function defined onto on [0, 1] with range [0, 1]. Show that there is some c in [0,1] such that $f(c) = 1 - c$.

Sol. Consider $g(x) = f(x) - 1 + x$

$g(0) = f(0) - 1 \le 0$ [as $f(0) \le 1$]

$g(1) = f(1) \ge 0$ [as $f(1) \ge 0$]

Hence, g(0) and g(1) have valuse of opposite signs.

Hence, there exists at least one $c \in (0, 1)$ such that $g(c) = 0$.

$\therefore \quad g(c) = f(c) - 1 + c = 0;\ f(c) = 1 - c.$

Illustration :

Let f be continuous on the inteval [0, 1] to R such that f (0) = f (1). Prove that there exists a point c in $\left[0, \frac{1}{2}\right]$ *such that* $f(c) = f\left(c + \frac{1}{2}\right)$.

Sol. Consider a continuous function $g(x) = f\left(x + \frac{1}{2}\right) - f(x)$ $\left(g \text{ is continous } \forall\, x \in \left[0, \frac{1}{2}\right]\right)$

$\Rightarrow \quad g(0) = f\left(\frac{1}{2}\right) - f(0) = f\left(\frac{1}{2}\right) - f(1)$ [as $f(0) = f(1)$]

and $\quad g\left(\frac{1}{2}\right) = f(1) - f\left(\frac{1}{2}\right) = -\left[f\left(\frac{1}{2}\right) - f(1)\right]$

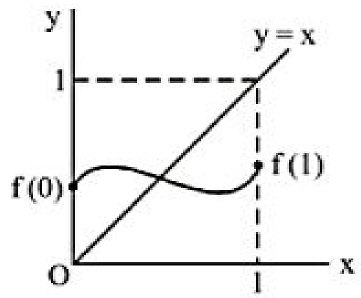
Since g is continuous and g(0) and $g\left(\frac{1}{2}\right)$ have opposite signs, hence the equation g(x) = 0 must have at one root in $\left[0, \frac{1}{2}\right]$.

Hence, for some $c \in \left[0, \frac{1}{2}\right]$, $g(c) = 0 \Rightarrow f\left(c + \frac{1}{2}\right) = f(c)$.

Illustration :

Let $f : [0, 1] \to [0, 1]$ *be a continuous function. Then prove* $f(x) = x$ *for at least one* $0 \le x \le 1$.

Sol. Clearly, $0 \le f(0) \le 1$ and $0 \le f(1) \le 1$. As $f(x)$ is continuous, $f(x)$ attains all the values between f (0) and f (1) and the graph will have no breaks. So, the graph will cut the line $y = x$ at one point x at least where $0 \le x \le 1$. So, $f(x) = x$ at that point.

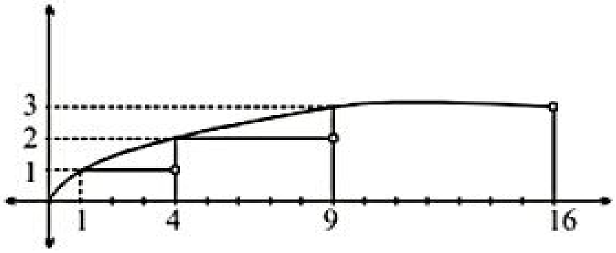


10. CONTINUITY OF SPECIAL TYPES OF FUNCTIONS :

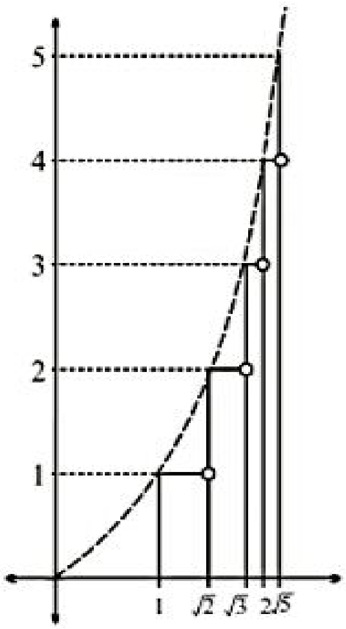
10.1 Continuity of functions in which greatest integer function is involved:

$f(x) = [x]$ is discontinuous when x is an integer.
Similarly, $f(x) = [g(x)]$ is discontinuous when g(x) is an integer, but this is true only when g(x) is monotonic (g(x) is strictly increasing or strictly decreasing).

For example, $f(x) = [\sqrt{x}]$ is discountinuous when $\sqrt{x}$ is an integer, as $\sqrt{x}$ is strictly increasing (monotonic function).



$f(x) = [x^2], x \geq 0$ is discountinuous when x^2 is an integer, as x^2 is strictly inreasing for $x \geq 0$.



Now consider, $f(x) = [\sin x], x \in [0, 2\pi]$. $g(x) = \sin x$ is not monotonic in $[0, 2\pi]$.

For this type of function, points of discontinuity can determined easily by graphical methods. We can note that at $x = \frac{3\pi}{2}$, sin x takes integral value –1, but at $x = \frac{3\pi}{2}$, $f(x) = [\sin x]$ is countinuous.

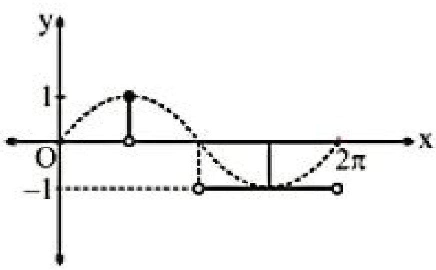


Illustration :

Discuss the continuity of following functions ([·] represents the greatest integer function.)

(a) $f(x) = [\log_e x]$ (b) $f(x) = [\sin^{-1} x]$ (c) $f(x) = \left[\dfrac{2}{1+x^2}\right], x \geq 0$

Sol.

(a) $\log_e x$ function is a montonically increasing function.

Hence $f(x) = [\log_e x]$ is discontinuous, where $\log_e x = k$ or $x = e^k$, $k \in Z$.

Thus $f(x)$ is discontinuous at $x = \ldots\ldots e^{-2}, e^{-1}, e^0, e^1, e^2, \ldots\ldots$

(b) $\sin^{-1} x$, is a monotonically increasing function.

Hence, $f(x) = [\sin^{-1} x]$ is discontinuous where $\sin^{-1} x$ is discontinuous where $\sin^{-1} x$ is an integer.

$\Rightarrow \quad \sin^{-1} x = -1, 0, 1$ or $x = -\sin 1, 0, \sin 1$

(c) $\dfrac{2}{1+x^2}, x \geq 0$, is a monotonically decreasing function.

Hence, $f(x) = \left[\dfrac{2}{1+x^2}\right], x \geq 0$ is discontinuous, when $\dfrac{2}{1+x^2}$ is an integer.

$$\Rightarrow \quad \frac{2}{1+x^2} = 1,2$$

$$\Rightarrow \quad x = 1, 0$$

Illustration :

Draw the graph and find the points of discontinuity for $f(x) = [2 \cos x]$, $x \in [0, 2\pi]$, ([·] represents the gratest integer function).

Sol. $f(x) = [2\cos x]$

Clearly from the graph given in figure

$f(x)$ is discontinuous at $x = 0$

and when $2 \cos x = \pm 1$

or $\quad x = 0$ and when $2 \cos x = \pm 1$

or $\quad x = 0$ and $\cos x = \pm\dfrac{1}{2}$

or $\quad x = 0$ and $x = \dfrac{\pi}{3}, \dfrac{2\pi}{3}, \dfrac{4\pi}{3}, \dfrac{5\pi}{3}$

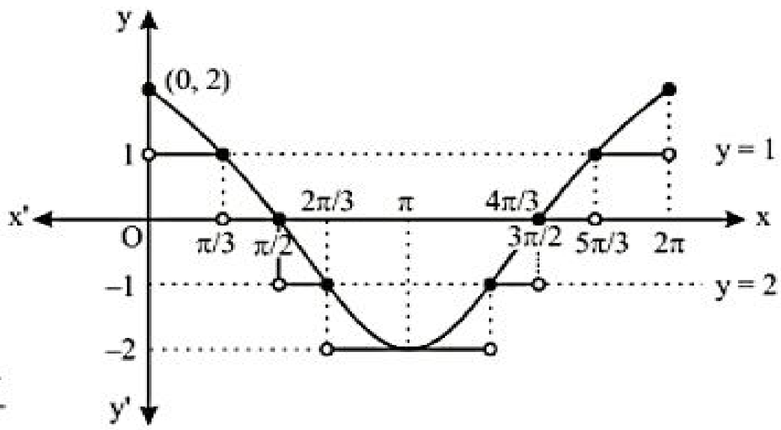


Illustration :

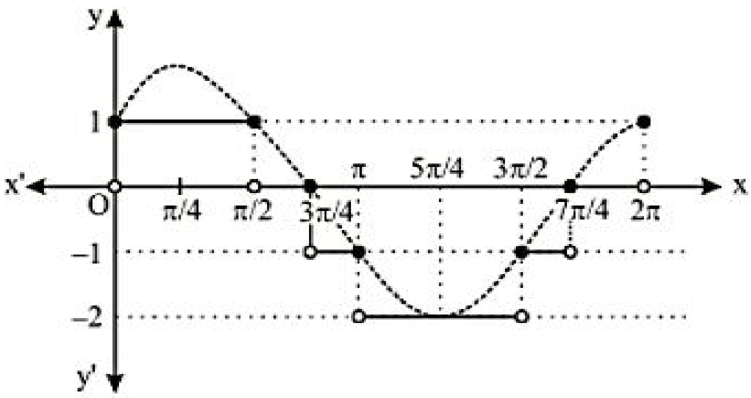
Draw the graph and discuss the continuity of $f(x) = [\sin x + \cos x]$, $x \in [0, 2\pi]$, where $[\cdot]$ represents the greatest integer function.

Sol. $f(x) = [\sin x + \cos x] = [g(x)]$ where $g(x) = \sin x + \cos x$

$g(0) = 1,\ g\left(\frac{\pi}{4}\right) = \sqrt{2},\ g\left(\frac{\pi}{2}\right) = 1$

$g\left(\frac{3\pi}{4}\right) - 0$ or $g(\pi) = -1$ $g\left(\frac{5\pi}{2}\right) = \sqrt{2}$

$g\left(\frac{3\pi}{2}\right) = -1,\ g\left(\frac{7\pi}{2}\right) = 0,\ g(2\pi) = 1$



Clearly from the graph given in figure f (x) is discontinuous at $x = 0, \frac{\pi}{2}, \frac{3\pi}{4}, \pi\ \frac{3\pi}{2}, \frac{7\pi}{2}, 2\pi$.

Illustration :

If the function $f(x) = \left[\frac{(x-2)^3}{a}\right] \sin(x-2) + a\cos(x-2)$. $[\cdot]$ *denotes the greatest integer function which is continuous in [4, 6], then find the value of a.*

Sol. *sin (x – 2) and cos (x – 2) are continuous for all x.*

Since [x] is not continuous at integral point.

So, $f(x)$ *is continuous in [4, 6] if* $\left[\frac{(x-2)^3}{a}\right] = 0\ \forall x \in [4, 6]$.

Now $(x-2)^3 \in [8, 64]$ *for* $x \in [4, 6]$.

$\Rightarrow \quad a > 64$ for $\left[\frac{(x-2)^3}{a}\right] = 0$

10.2 Continuity of functions is which signum function is involved :

We know that $f(x) = \text{sgn}(x)$ is discontinuous at $x = 0$.

In general, $f(x) = \text{sgn}(g(x))$ is discontinuous at $x = a$ if $g(a) = 0$.

Illustration :

Discuss the continuity of

(a) $f(x) = \text{sgn}(x^3 - x)$, (b) $f(x) = \text{sgn}(2\cos x - 1)$, (c) $f(x) = \text{sgn}(x^2 - 2x + 3)$.

Sol.

(a) $f(x) = \text{sgn}(x^3 - x)$

Here $x^3 - x = 0 \Rightarrow x = 0, -1, 1$

Here $f(x)$ is discontinuous at $x = 0, -1, 1$

(b) $f(x) = \text{sgn}(2\cos x - 1)$

Here, $2\cos x - 1 = 0 \Rightarrow \cos x = \frac{1}{2} \Rightarrow x = 2n\pi + \left(\frac{\pi}{3}\right)$

$n \in Z$, where $f(x)$ is discontinuous.

(c) $f(x) = \text{sgn}(x^2 - 2x + 3)$

Here, $x^2 - 2x + 3 > 0$ for all x.

Thus, $f(x) = 1$ for all x, hence continuous for all x.

Illustration :

If $f(x) = \text{sgn}(2\sin x + a)$ is continuous for all x, then find the possible values of a.

Sol. $f(x) = \text{sgn}(2\sin x + a)$ is continuous for all x.

Then $2\sin x + a \neq 0$ for any real x.

$$\Rightarrow \quad \sin x \neq -\frac{a}{2}$$

$$\Rightarrow \quad \left|\frac{a}{2}\right| > 1 \quad \Rightarrow \quad a < -2 \text{ or } a > 2$$

Illustration :

If $f(x) = \begin{cases} \text{sgn}(x-2) \times [\log_e x], & 1 \le x \le 3 \\ \{x^2\}, & 3 < x \le 3.5 \end{cases}$ where $[\cdot]$ denotes the greatest integer function and $\{\cdot\}$ represents the fractional part function. Find the point where the continuity of f(x) should be checked. Hence, find the points of discontinity.

Sol.

(a) Continuity should be checked at the endpoints of intervals of each definition, i.e., $x = 1, 3, 3.5$.

For $\{x^2\}$, continuity should be checked when $x^2 = 10$.

(b) 11, 12 or $x = \sqrt{10}, \sqrt{11}, \sqrt{12}$. $[x^2]$ discontinuous for those value of x where x^2 is an integer (note, here x^2 is monotonic for given domain).

(c) For sgn $(x - 2)$, continuity should be checked when $x - 2 = 0$ or $x = 2$.

(d) For $[\log_e x]$, continuity should be checked when $\log_e x = 1$ or $x = e\ (\in [1, 3])$.

Hence, the overall continuity must be checked at $x = 1, 2, e, 3, \sqrt{10}, \sqrt{11}, \sqrt{12}, 3.5$.

Checking continuity at $x = 1$

$$f(1) = 0 \text{ and } \lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} \text{sgn}(x-2) \times [\log_e x] = 0.$$

Hence $f(x)$ is continuous at $x = 1$.

Checking continuity at $x = 2$

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} \operatorname{sgn}(x-2) \times [\log_e x] = (-1) \times 0 = 0$$

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} \operatorname{sgn}(x-2) \times [\log_e x] = (1) \times 0 = 0$$

Hence, $f(x)$ is continuous at $x = 2$.

Checking continuity at $x = 3$

$$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} \{x^2\} = 0$$

Hence, $f(x)$ is discontinuous at $x = 3$.

Also $\{x^2\}$ and hence $f(x)$ is discontinuous at $x = \sqrt{10}, \sqrt{11}, \sqrt{12}$.

Checking continuity at $x = 3.5$

$$\lim_{x\to 3.5^-} f(x) = \lim_{x\to 3.5^-} \{x^2\} = 0.25 = f(3.5)$$

Hence, $f(x)$ is discontinuous at $x = 3, \sqrt{10}, \sqrt{11}, \sqrt{12}$.

10.3 Continuity of functions involving limit $\lim\limits_{n\to\infty} a^n$:

We know that $\lim\limits_{n\to\infty} a^n = \begin{cases} 0, & 0 \le a < 1 \\ 1, & a = 1 \\ \infty, & a > 1 \end{cases}$

Illustration :

Discuss the continuity of $f(x) = \lim\limits_{n\to\infty} \dfrac{x^{2n}-1}{x^{2n}+1}$.

Sol. $$f(x) = \lim_{n\to\infty} \frac{(x^2)^n - 1}{(x^2)^n + 1} = \lim_{x\to\infty} \frac{1 - \frac{1}{(x^2)^n}}{1 + \frac{1}{(x^2)^n}} = \begin{cases} -1, & 0 \le x^2 < 1 \\ 0, & x^2 = 1 \\ 1, & x^2 > 1 \end{cases} = \begin{cases} 1, & x < -1 \\ 0, & x = -1 \\ -1, & -1 < x < 1 \\ 0, & x = 1 \\ 1, & x > 1 \end{cases}$$

Thus, $f(x)$ is discontinuous at $x = \pm 1$.

Illustration :

Discuss the continuity of $f(x) = \lim\limits_{n\to\infty} \cos^{2n} x$.

Sol. $$f(x) = \lim_{n\to\infty} (\cos^2 x)^n$$

$$= \begin{cases} 0, & 0 \le \cos^2 x < 1 \\ 1, & \cos^2 x = 1 \end{cases} = \begin{cases} 0, & x \ne n\pi, \ n \in I \\ 1, & x = n\pi, \ n \in I \end{cases}$$

Hence, $f(x)$ is discontinuous when $x = n\pi$, $n \in I$.

10.4 Continuity of functions in which f (x) is defined differently for rational and irrational values of x :

Illustration :

Discuss the continuity of the following function : $f(x) = \begin{cases} 1, & \text{if } x \text{ is rational} \\ 0, & \text{if } x \text{ is irrational} \end{cases}$.

Sol. *For any* $x = a$,

$L.H.L. = \lim_{x \to a^-} f(x) = \lim_{x \to a^-} f(a-h) = 0$ or 1 [as $\lim_{h \to 0} (a-h)$ can be rational or irrational]

Similarly, $R.H.L. = \lim_{x \to a^+} f(x) = \lim_{h \to 0} f(a+h) = 0$ or 1.

Hence, $f(x)$ *oscillates between 0 and 1 as for all values of a.*

$\therefore$ *L.H.L. and R.H.L. do not exist.*

$\Rightarrow f(x)$ *is discontinuous at a point* $x = a$ *for all values of a.*

Illustration :

Find the value of x where $f(x) = \begin{cases} x, & \text{if } x \text{ is rational} \\ 1-x, & \text{if } x \text{ is irrational} \end{cases}$.

Sol. $f(x)$ *is continuous at some* $x = a$, *where* $x = 1 - x$ *or* $x = \frac{1}{2}$.

Hence, $f(x)$ *is continuous at* $x = \frac{1}{2}$

If $x \to \frac{1}{2^+}$ *then x may be rational or irrational*

$$\Rightarrow \quad f\left(\frac{1}{2^+}\right) = \frac{1}{2} \text{ or } 1 - \frac{1}{2} = \frac{1}{2}$$

If $x \to \frac{1}{2^-}$ *then x may be rational or irrational*

$$\Rightarrow \quad f\left(\frac{1}{2^-}\right) = \frac{1}{2} \text{ or } 1 - \frac{1}{2} = \frac{1}{2}$$

Hence $f(x)$ *is continuous at* $x = \frac{1}{2}$

For some other point, say, $x = 1$

$\Rightarrow \quad f(1) = 1$

If $x \to 1^+$ *then x may be rational or irrational.*

$\Rightarrow \quad f(1^+) = 1$ or $1 - 1 = 0$

Hence, $f(1^+)$ *oscillates between 1 and 0, which causes discontinuity at* $x = 1$.

Similarly, $f(x)$ *osciallates between 0 and 1 for all* $x \in R - \left(\frac{1}{2}\right)$.

11. CONTINUITY OF COMPOSITE FUNCTIONS :

If f is continuous at x = a & g is continuous at x = f(a) then the composite g[f(x)] is continuous at x = a. e.g. $f(x) = \frac{x\sin x}{x^2+2}$ & $g(x) = |x|$ are continuous at x = 0, hence the composite $(gof)(x) = \left|\frac{x\sin x}{x^2+2}\right|$ will also be continuous at x = 0.

Illustration :

If $f(x) = \frac{x+1}{x-1}$ *and* $g(x) = \frac{1}{x-2}$, *then discuss the continuity of f (x), g(x) and fog (x).*

Sol.

(a) $f(x) = \frac{x+1}{x-1}$

$\therefore$ *f is not defined at x = 1.* $\therefore$ *f is discontinuous at x = 1.*

(b) $g(x) = \frac{1}{x-2}$

g(x) is not defined at x = 2. $\therefore$ *g is discontinuous at x = 2.*

(c) *Now, fog will be discontinuous at*

x = 2 [point of discontinuity of g(x)]

g(x) = 1 [when g(x) = point of discontinuity of f (0)]

if $g(x) = 1 \Rightarrow \frac{1}{x-2} = 1 \Rightarrow x = 3$

$\therefore$ *fog (x) is discontinuous at x = 2 and x = 3.*

Also, $fog(x) = \dfrac{\frac{1}{x-2}+1}{\frac{1}{x-2}-1}$

Here fog (2) is not defined.

$$\lim_{x\to 2} fog(x) = \lim_{x\to 2}\dfrac{\frac{1}{x-2}+1}{\frac{1}{x-2}-1} = \lim_{x\to 2}\frac{1+x-2}{1-x+2} = 1$$

$\therefore$ *fog (x) is discontinuous at x = 2 and it has a removable discontinuity at x = 2.*

For continuity at x = 3.

$$\lim_{x\to 3^+} fog(x) = \lim_{x\to 3}\dfrac{\frac{1}{x-2}+1}{\frac{1}{x-2}-1} = \infty$$

$$\lim_{x\to 3^-} fog(x) = \lim_{x\to 3}\dfrac{\frac{1}{x-2}+1}{\frac{1}{x-2}-1} = -\infty$$

$\therefore$ *fog (x) is discontinuous at x = 3 and it is a non-removable discontinuity at x = 3.*

Practice Problem

Q.1 Find the number of points in $[1, 3]$ where the function is $[x^2 + 1]$ is discontinuous. ($[\cdot]$ represents the greatest integer function).

Q.2 Find the number of points of discontinuity for $f(x) = [6 \sin x], 0 \le x \le \pi$, ($[\cdot]$ represents the greatest integer function).

Q.3 Discuss the continuity of $f(x) = [\tan^{-1} x]$ ($[\cdot]$ represents the greatest integer function).

Q.4 Discuss the continuity of $f(x) = \{\cot^{-1} x\}$ ($\{\cdot\}$ represent the fraction part function).

Q.5 Discuss the continuity of f(x) in $[0, 2]$, where $f(x) = \lim\limits_{n\to\infty}\left(\sin\dfrac{\pi x}{2}\right)^{2n}$.

Q.6 Discuss the continuity of $f(x) = \begin{cases} x^2, & x \text{ is rational} \\ -x^2, & x \text{ is irrational} \end{cases}$.

Q.7 If $y = \dfrac{1}{t^2+t-2}$ where $t = \dfrac{1}{x-1}$, then find the number of points where $f(x)$ is discontinuous.

Q.8 Prove that :

(i) $f(x) = \left[\begin{array}{l} x \text{ if } x \in Q \\ 0 \text{ if } x \notin Q \end{array}\right.$ is continuous only at $x = 0$.

(ii) $f(x) = \left[\begin{array}{ll} x & \text{if } x \in Q \\ -x & \text{if } x \notin Q \end{array}\right.$ is continuous only at $x = 0$.

(iii) $f(x) = \left[\begin{array}{ll} x & \text{if } x \in Q \\ 1-x & \text{if } x \notin Q \end{array}\right.$ is continuous only at $x = 1/2$.

(iv) $f(x) = \left[\begin{array}{ll} x^2 & \text{if } x \in Q \\ 1 & \text{if } x \notin Q \end{array}\right.$ is continuous only at $x = 1$ or -1.

Answer key

Q.1 9

Q.2 11

Q.3 Not continuous at $x = -\tan 1, 0, \tan 1$

Q.4 Not continuous at $x = \cot 1, \cot 2, \cot 3$

Q.5 Discontinuous at $x = 1$

Q.6 Continuous at $x = 0$

Q.7 Discontinuous at $x = 1, \dfrac{1}{2}, 2$

DIFFERENTIABILITY

1.0 DIFFERENTIABILITY / DERIVABILITY:

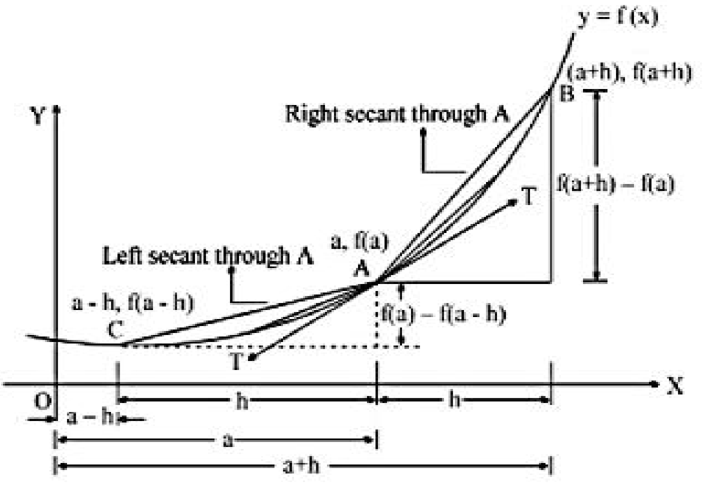
(Two fold meaning of derivability)

Geometrical meaning of derivative
Slope of the tangent drawn to the curve at $x = a$ if it exists

Physical meaning of derivative
(functions which are differentiable)
Instantaneous rate of change of function

Note : "Tangent at a point 'A' is the the limiting case of secant through A."

1.1 RIGHT AND LEFT HAND DERIVATIVES :



1.2 EXISTENCE OF DERIVATIVE :

(I) Right hand & Left hand Derivatives ; By definition : $f'(a) = \lim\limits_{h \to 0} \dfrac{f(a+h) - f(a)}{h}$ if it exist

(a) The right hand derivative of f at $x = a$ denoted by $f'(a^+)$ is defined by : $f'(a^+) = \lim\limits_{h \to 0^+} \dfrac{f(a+h) - f(a)}{h}$,

provided the limit exists & is finite.
when $h \to 0$, the point B moving along the curve tends to A, i.e., $B \to A$ then the chord AB approaches the tangent line AT at the point A.

$$\Rightarrow \qquad f'(a^+) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \to 0} \tan\phi = \tan\psi$$

(b) The left hand derivative of f at x = a denoted by $f'(a^-)$ is defined by : $f'(a^-) = \lim\limits_{h\to 0^+} \dfrac{f(a-h)-f(a)}{-h}$,

Provided the limit exists & is finite.

When $h \to 0$, the point C moving along the curve tends to A, i.e., $C \to A$ then the chord CA approaches the tangent line AT at the point A then

$$\Rightarrow \quad f'(a^-) = \lim_{h\to 0} \frac{f(a-h)-f(a)}{-h}$$

f is said to be derivable at x = a if $f'(a^+) = f'(a^-) =$ a finite quantity.

This geomtrically means that a unique tangent with finite slope can be drawn at x = a as shown in the figure.

(II) Theorem :

If a function f is derivable at x = a then f is continuous at x = a.

Proof :

Let f is derivable at x = a. Hence

For : $f'(a) = \lim\limits_{h\to 0} \dfrac{f(a+h)-f(a)}{h}$ exists .

Also $\quad f(a+h)-f(a) = \dfrac{f(a+h)-f(a)}{h}\cdot h \qquad (h \neq 0)$

Therefore : $\quad \lim\limits_{h\to 0} [(f(a+h)-f(a)] = \lim\limits_{h\to 0} \dfrac{f(a+h)-f(a)}{h}.h = f'(a).0 = 0$

Therefore $\lim\limits_{h\to 0} [(f(a+h)-f(a)] = 0 \Rightarrow \lim\limits_{h\to 0} f(a+h) = f(a) \Rightarrow$ f is continuous at x.

Note: If f(x) is derivable for every point of its domain of definition, then it is continuous in that domain.

The Converse of the above result is not true :

For a function f :

Differentiability $\Rightarrow$ Continuity ; Continuity $\not\Rightarrow$ derivability ;

Non derivibality $\not\Rightarrow$ discontinuous But discontinuity $\Rightarrow$ Non derivability

Note

(a) Let $f'(a^+) = p$ & $f'(a^-) = q$ where p & q are finite then :

(i) $p = q \Rightarrow$ f is derivable at x = a $\quad \Rightarrow$ f is continuous at x = a .

(ii) $p \neq q \Rightarrow$ f is not derivable at x = a.

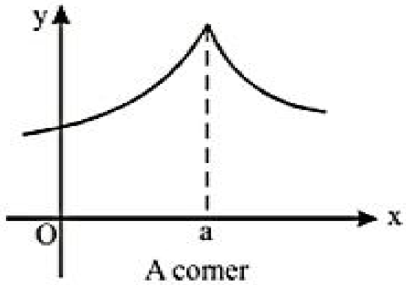
It is very important to note that f may be still continuous at x = a.

(b) If a function f is not differentiable but is continuous at x = a it geometrically implies a sharp corner at x = a.

(III) How can a function fail to be differntiable :

The function $f(x)$ is said to non-differentiable at $x = a$ if

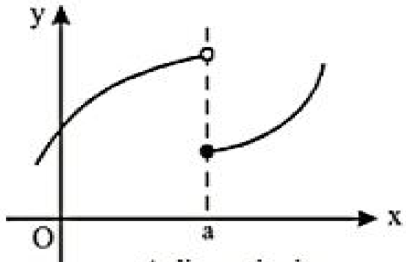
(a) Both left and right and hand derivative exists but are not equal



A corner

The function $y = |x|$ is not differentiable at 0 as its graph change direction abrupty when $x = 0$. In general, if the graph of a function has a 'corner' or 'kink' in it, then the graph of f has no tangent at this point and f is not differentiable there. (To compute $f'(a)$, we find that the left and right limits are different.)

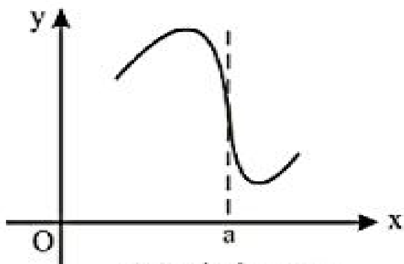
(b) Function is discontinuous at $x = a$



A discontinuity

If f is not continuous at a then f is not differentiable at a. So at any discontinuity (for instance, a jump of discontinuity) f fails to be differentiable.

(c) Either or both left and right hand derivative are not finite.



A vertical tangent

A third possibility is that the curve has a vertical tangent line when $x = a$, that is f is continuous at a and $\lim\limits_{x \to a} |f'(x)| = \infty$.

This means that the tangent lines becomes steeper and sleeper as $x \to a$.

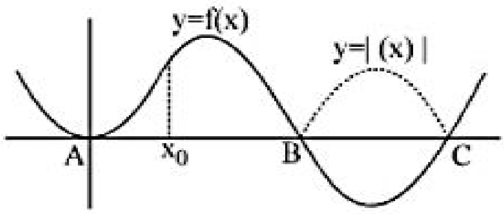
(IV) Derivability Over An Interval :

$f(x)$ is said to be derivable over an open interval (a, b) if it is derivable at each & every point of the interval $f(x)$ is said to be derivable over the closed interval $[a, b]$ if :

(i) for the points a and b, $f'(a+)$ & $f'(b-)$ exist &

(ii) for any point c such that $a < c < b$, $f'(c+)$ & $f'(c-)$ exist & are equal.

Note : Consider the graph of a differentiable function



(1) If $f(x)$ is derivable at $x = x_0$ then $|f(x)|$ must be derivable at $x = x_0$ provided $f(x_0) \neq 0$. However if $f(x_0) = 0$ then $|f(x)|$ may or may not be derivable at $x = 0$

e.g. $f(x) = x^3$ is derivable at $x = 0$ and $|f(x)|$ is also derivable at $x = 0$.

$f(x) = x - 1$ is derivable at $x = 1$ but $|f(x)|$ is not derivable at $x = 1$

i.e. if $f'(x_0) = 0$ and $f(x_0) = 0$ then $|f(x)|$ will also derivable at $x = x_0$ and if $f'(x_0)$ is non zero finite then $|f(x)|$ is non derivable at $x = x_0$.

In figure $f(x) = 0$ at A, B, C and $f(x)$ is derivable at A, B and C but $|f(x)|$ is non derivable at $x =$B and C but derivable at $x = A$.

Consider $f(x) = x^3$ at $x = 0$ when $f'(0) = 0$ and $f(0) = 0$ but from $|f(x)| = |x^3|$ is derivable but if $f(x) = x$, where $f'(x) \neq 0$,

hence $|f(x)|$ is not derivable at $x = 0$

Illustration :

Find the left and right hand derivative of the following function at given point
(a) $f(x) = |\ln x|$ at $x = 1$ (b) $f(x) = \ln^2 x$ at $x = 1$ (c) $f(x) = e^{-|x|}$ at $x = 0$

Sol.

(a) $f(x) = \begin{cases} \ln x, & x \ge 1 \\ -\ln x, & 0 < x \le 1 \end{cases}$

$\Rightarrow$ $f(x)$ is continuous at $x = 1$

$$f'(1^+) = \lim_{h\to 0} \frac{f(1+h)-f(1)}{h} = \lim_{h\to 0} \frac{\ln(1+h)-\ln 1}{h} = \lim_{h\to 0} \frac{\ln(1+h)}{h} = 1$$

$$f'(1^-) = \lim_{h\to 0} \frac{f(1-h)-f(1)}{-h} \lim_{h\to 0} \frac{-\ln(1-h)-0}{-h} = \lim_{h\to 0} \frac{\ln(1-h)}{h} = -1$$

(b) $f(x) = \ln^2 x$ is continuous and differentiable function.

$$f'(x) = 2\ln x\left(\frac{1}{x}\right)$$

$$f'(1) = 2\ln 1\left(\frac{1}{1}\right) = 0$$

(c) $f(x) = e^{-|x|} = \begin{cases} e^{-x}, & x \ge 0 \\ e^{x}, & x < 0 \end{cases}$

$\Rightarrow$ $f(x)$ is continuous at $x = 0$.

$$f'(0^+) = \lim_{h\to 0} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0} \frac{e^{-h}-e^{0}}{h} = -1$$

$$f'(0^-) = \lim_{h\to 0} \frac{f(0-h)-f(0)}{-h} = \lim_{h\to 0} \frac{e^{-h}-e^{0}}{-h} = 1$$

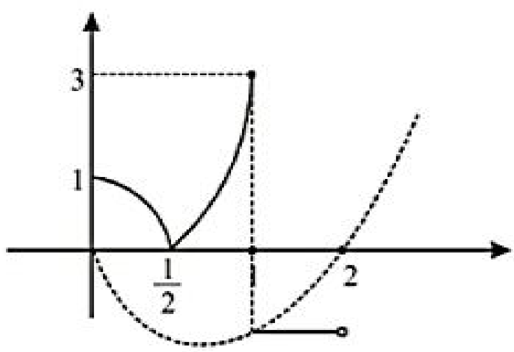
$\Rightarrow$ $f(x)$ is not differential at $x = 0$.

Illustration :

$f(x) = \begin{cases} |1-4x^2|, & 0 \le x < 1 \\ [x^2-2x], & 1 \le x \le 2 \end{cases}$, *check differentiability in (0, 2), where [ ] denotes greatest integer function.*

Sol. $f(x) = \begin{cases} |1-4x^2|, & 0 \le x < 1 \\ [x^2-2x], & 1 \le x \le 2 \end{cases}$

$$f(x) = \begin{cases} 1-4x^2, & 0 \le x \le \frac{1}{2} \\ 4x^2-1, & \frac{1}{2} \le x < 1 \\ 1, & 1 \le x < 2 \\ 0, & x = 2 \end{cases}$$



$f(x)$ is not differentiable at $x = \frac{1}{2}$, 1 in $x \in (0, 2)$

Illustration :

$$\text{If } f(x) = \begin{cases} ax+b, & x \le -1 \\ ax^3+x+2b, & x > -1 \end{cases} \text{ is differentiable for all } x \in R \text{ find 'a' \& 'b'.}$$

Sol. $f(x) = \begin{cases} ax+b, & x \le -1 \\ ax^3+x+2b, & x > -1 \end{cases}$

For continuity at $x = -1$, $-a + b = a(-1)^3 + (-1) + 2b$

$\Rightarrow \quad b = 1$

For differentiability at $x = -1$

$$\frac{d}{dx}(ax+b)\bigg|_{x=-1} = \frac{d}{dx}(ax^3+x+2b)\bigg|_{x=-1}$$

$\Rightarrow \quad a = 3a(-1)^2 + 1 \quad \Rightarrow a = \dfrac{-1}{2}.$

Illustration :

Find derivative of

(i) $\quad f(x) = \cos x + |\cos x|$ *at* $x = \dfrac{\pi}{2}$

(ii) $\quad f(x) = \max.\{(1-x), (1+x), 2\}$ *number of points where f is not differentiable.*

(iii) *Find the number of points at which the function* $f(x) = \begin{bmatrix} \max.(|x|, x^2) \text{ if } -\infty < x < 1 \\ \min.(2x-1, x^2) \text{ if } x \ge 1 \end{bmatrix}$ *is not derivable.*

(iv) *Let f be differentiable at x = a and let* $f(a) \neq 0$. *Evaluate* $\lim\limits_{n\to\infty} \left\{\dfrac{f(a+1/n)}{f(a)}\right\}^n$.

Sol.

(i) $\quad f(x) = \cos x + |\cos x| = \begin{cases} 2\cos x, & 0 \le x \le \dfrac{\pi}{2} \\ 0, & \dfrac{\pi}{2} < x < \pi \end{cases}$

$f(x)$ *is continuous at* $x = \dfrac{\pi}{2}$

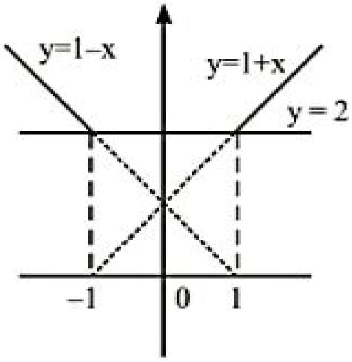
$$f'\left(\frac{\pi^-}{2}\right) = \lim_{h\to 0} \frac{f\left(\frac{\pi}{2}-h\right) - f\left(\frac{\pi}{2}\right)}{-h} = \lim_{h\to 0} \frac{2\cos\left(\frac{\pi}{2}-h\right) - \cos\frac{\pi}{2}}{-h} = \lim_{h\to 0} \frac{2\sin h}{-h} = -2$$

$$f'\left(\frac{\pi^+}{2}\right) = \lim_{h\to 0} \frac{f\left(\frac{\pi}{2}+h\right) - f\left(\frac{\pi}{2}\right)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0$$

$\Rightarrow f(x)$ *is not differentiable at* $x = \dfrac{\pi}{}$

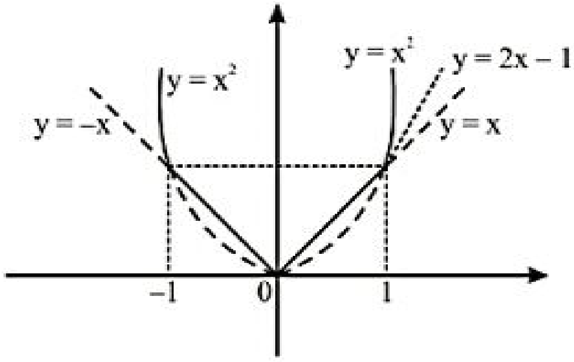
(ii) $f(x) = \max\{(1-x), (1+x), 2\}$

$\Rightarrow$ *f(x) is not differentiable at* $x = \pm 1$.



(iii) $f(x) = \begin{cases} \max\{|x|, x^2\}, & x < 1 \\ \min(2x-1, x^2) & x \geq 1 \end{cases}$

$\Rightarrow$ *f(x) is not differentiable at* $x = \pm 1$.



(iv) $$\lim_{n\to\infty}\left(\frac{f\left(a+\frac{1}{n}\right)}{f(a)}\right)^n = e^{\lim_{n\to\infty}\left(\frac{f\left(a+\frac{1}{n}\right)}{f(a)}-1\right)n} = e^{\lim_{n\to\infty}\left(\frac{f\left(a+\frac{1}{n}\right)-f(a)}{\left(\frac{1}{n}\right)}\right)\cdot\frac{1}{f(a)}} = e^{\frac{f'(a)}{f(a)}}$$

Illustration :

If $f(x) = \begin{cases} x^m.\sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$ *is continuous but not differentiable at x = 0 then find m.*

Sol. $f(x) = \begin{cases} x^m \sin\left(\frac{1}{x}\right), & x \neq 0 \\ 0, & x = 0 \end{cases}$

For continuity at $x = 0$.

$$\lim_{h\to 0} f(0+h) = f(0) \Rightarrow \lim_{h\to 0} h^m \sin\left(\frac{1}{n}\right) = 0 \Rightarrow m > 0$$

For function to be not differentiable at $x = 0$

$$\lim_{h\to 0}\frac{f(0+h)-f(0)}{h} = DNE$$

$$\lim_{h\to 0}\frac{h^m \sin\left(\frac{1}{h}\right)-0}{h} = \lim_{h\to 0} h^{m-1}\sin\left(\frac{1}{h}\right) = DNE$$

$\Rightarrow$ $m - 1 \leq 0 \Rightarrow m \leq 1$.

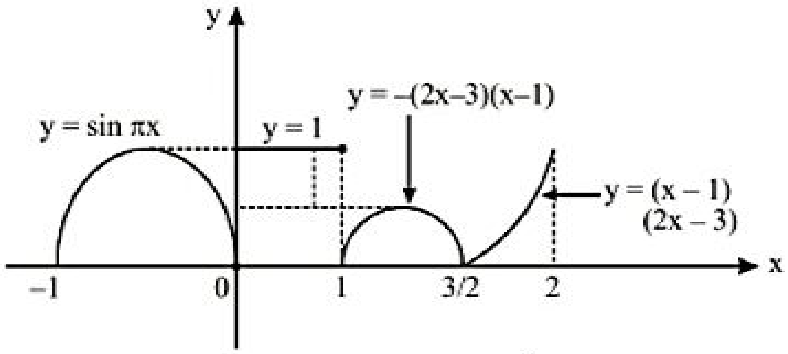
$\therefore$ $m \in (0, 1]$ *for function f(x) to be continuous and not differentiable at x = 0.*

Illustration :

$$f(x) = \begin{cases} \sqrt{4x^2-12x+9}.\{x\}; & x \ge 1 \\ \cos\left(\frac{\pi}{2}(|x|-\{x\})\right); & x<1 \end{cases}$$ *check the differentiability in* $x \in [-1, 2]$

Sol. $f(x) = \begin{cases} \sqrt{4x^2-12x+9}\cdot\{x\}, & x \ge 1 \\ \cos\left(\frac{\pi}{2}(|x|)-\{x\}\right), & x<1 \end{cases} = \begin{cases} |2x-3|\cdot\{x\}, & x \ge 1 \\ \cos\frac{\pi}{2}(|x|-\{x\}), & x<1 \end{cases}$

$$= \begin{cases} -(2x-3)(x-1), & 1 \le x \le \frac{3}{2} \\ (2x-3)(x-1), & \frac{3}{2} \le x \le 2 \\ \cos\frac{\pi}{2}(-x-(x+1)), & -1 \le x < 0 \\ \cos\frac{\pi}{2}(x-(x)), & 0 \le x < 1 \end{cases} = \begin{cases} -\sin\pi x, & -1 \le x < 0 \\ 1, & 0 \le x < 1 \\ -(2x-3)(x-1), & 1 \le x < \frac{3}{2} \\ (2x-3)(x-1), & \frac{3}{2} \le x \le 2 \end{cases}$$



$\Rightarrow$ *f(x) is not differentiate at* $x = 0, 1, \frac{3}{2}$.

Illustration :

Let f (x) = sgn x and g (x) = x (1 – x²). Investigate the composite functions $f(g(x))$ *and* $g(f(x))$ *for continuity and differentiability.*

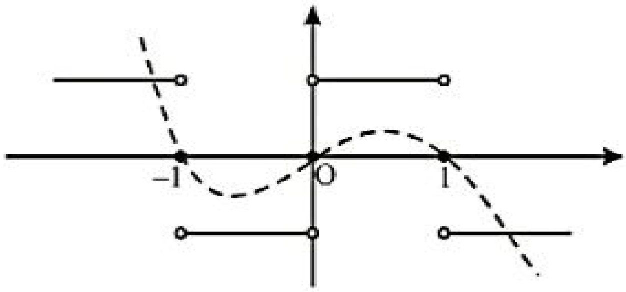
Sol. $f(x) = \text{sgn}\,(x)$

$g(x) = x(1-x^2)$

$f(g(x)) = \text{sgn}\,(x(1-x^2))$

$\Rightarrow$ *fog(x) is not differentiable at* $x = 0, \pm 1$.

$gof\,(x) = g(f(x)) = (\text{sgn}\,x)(1-(\text{sgn}\,x)^2)$



as $\text{sgn}\,(x) = \begin{cases} 1, & x>0 \\ 0, & x=0 \\ -1, & x<0 \end{cases}$

$\therefore$ $gof\,(x) = 0\ \forall x \in R$

$\Rightarrow$ *gof (x) is always differentiable.*

f (g (x)) is discontinuous & non derivable at – 1, 0, 1

Illustration :

$$f(x) = \begin{cases} \left(\ell n\left(e\left|[x]+[-x]\right|\right)^{x}\right)\cdot\left(\dfrac{2e^{\left(\frac{\{x\}+\{-x\}}{|x|}\right)}-5}{3+e^{\frac{1}{|x|}}}\right) & \text{for } x<0 \\ 0 & \text{for } x=0 \\ x\cdot\dfrac{1-e^{|x|+\{x\}}}{|x|+\{x\}} & \text{for } x>0 \end{cases}$$

Where [ ], { } represents integral and fractional part functions respectively. Compute the Right hand derivative and Left hand derivative at x = 0 and comment on the continuity and derivability at x = 0.

Sol. As $\{x\} + \{-x\} = x - [x] - x - [-x] = -\left([x]+[-x]\right) = -(-1+0)$ where $x < 0$.

$[x] + [-x] = -1 + 0 = -1$ when $x < 0$.

$$f(x) = \ln\left(e\left|(-1)\right|\right)^{x}\cdot\frac{\left(2e^{\frac{1}{|x|}}-5\right)}{\left(3+e^{\frac{1}{|x|}}\right)} \text{ for } x<0$$

$$= \frac{x\left(1-e^{x+x}\right)}{(x+x)} \text{ for } x>0 \qquad [\because |x| = x\ \forall x>0 \text{ and } \{x\} = x\ \forall\, 0<x<1]$$

$f(x) = 0$ at $x = 0$.

$$f(x) = x\,\frac{\left(2e^{\frac{-1}{x}}-5\right)}{\left(3+e^{\frac{-1}{x}}\right)} \text{ for } x<0$$

$$= x\,\frac{\left(1-e^{2x}\right)}{2x} \text{ for } x>0$$

$$\text{L.H.D.} = \lim_{h\to 0}\frac{f(0-h)-f(0)}{-h} = \lim_{h\to 0}\frac{(-h)}{(-h)}\frac{\left(2e^{\frac{1}{h}}-5\right)}{\left(3-e^{\frac{1}{h}}\right)} = \lim_{h\to 0}\frac{\left(2-5e^{\frac{-1}{h}}\right)}{\left(3e^{-\frac{1}{h}}+1\right)} = 2$$

$$\text{R.H.D.} = \lim_{h\to 0}\frac{f(0+h)-f(0)}{h} = \lim_{h\to 0}\frac{(1-e^{2h})}{2h} = -1$$

f (x) is continuous at x = 0 but not differentiable at x = 0.

Illustration :

$$f(x) = \begin{cases} x+a & \text{if } x<0 \\ |x-1| & \text{if } x\ge 0 \end{cases} \quad \text{and} \quad g(x) = \begin{cases} x+1 & \text{if } x<0 \\ (x-1)^2+b & \text{if } x\ge 0 \end{cases}$$

Where a and b are non negative real numbers. Determine the composite function gof. If (gof) (x) is continuous for all real x, determine the values of a and b. Further, for these values of a and b, is gof differentiable at $x = 0$? Justify your answer.

Sol. $f(x) = \begin{cases} x+a & \text{if} & x<0 \\ |x-1| & \text{if} & x\ge 0 \end{cases} \Rightarrow f(x) = \begin{cases} x+a & \text{if} & x<0 \\ (x-1) & \text{if} & 0\le x<1 \\ (x-1) & \text{if} & 1\le x<\infty \end{cases}$

$$g(x) = \begin{cases} x+1 & \text{if} & x<0 \\ (x-1)^2+b & \text{if} & x\ge 0 \end{cases}$$

$g(f(x)) = f(x) + 1$ if $f(x) < 0$

$= (f(x)-1)^2 + b$ if $f(x) \ge 0.$

So $\quad g(f(x)) = x + a + 1$ if $\quad x + a < 0$ and $x < 0 \Rightarrow x < -a$

$= -x + 1 + 1$ if $-x + 1 < 0$ and $0 \le x < 1 \Rightarrow$ Null set

$= x$ if $x - 1 < 0$ and $1 \le x < \infty \Rightarrow$ Null set

$= (x + a - 1)^2 + b$ if $x + a \ge 0$ and $x < 0 \Rightarrow -a \le x < 0$

$= (-x)^2 + b$ if $-x + 1 \ge 0$ and $0 \le x < 1 \Rightarrow 0 \le x < 1$

$= (x - 2)^2 + b$ if $x - 1 \ge 0$ and $1 \le x < \infty \Rightarrow 1 \le x \le \infty$

$g(f(x)) = x + a + 1 \quad$ if $\quad -\infty < x < -a \quad$ ...(1)

$= (x + a - 1)^2 + b \quad$ if $\quad -a \le x < 0 \quad$ ...(2)

$= x^2 + b \quad$ if $\quad 0 \le x < 1 \quad$ ...(3)

$= (x - 2)^2 + b \quad$ if $\quad 1 \le x \le \infty \quad$ ...(4)

$g(f(x))$ *is continuous every where so it must be continuous at* $x = -a, 0, 1.$

from (1) and (2), checking continuity at $x = -a$

$1 = 1 + b \Rightarrow b = 0$

Similarly checking continuity at $x = 0.$

$(0 + a - 1)^2 + b = 0 + b \Rightarrow a = 1$

$g(f(x)) = x + 2 : \forall \ -\infty < x < -1$

$= x^2$ if $-1 \le x < 0$

$= x^2$ if $0 \le x < 1$

$= (x - 2)^2$ if $1 \le x < \infty$

so $\quad (gof)'(x) = 2x$ at $x = 0$ $(gof)'(0) = 0$

(V) Theorem :

If $f(x)$ and $g(x)$ both are derivable at $x = a$, then

(i) $f(x) \pm g(x)$ will be differentiable at $x = a$.

(ii) $f(x) \cdot g(x)$ will be differentiable at $x = a$.

(iii) $\dfrac{f(x)}{g(x)}$ will be differentiable at $x = a$ if $g(a) \neq 0$.

Note that :

(1) If $f(x)$ and $g(x)$ are both derivable at $x = a$, $f(x) \pm g(x)$; $g(x) \cdot f(x)$ and $\dfrac{f(x)}{g(x)}$ will also be derivable at $x = a$. (only if $g(a) \neq 0$)

(2) If $f(x)$ is derivable at $x = a$ and $g(x)$ is not derivable at $x = a$ then the $f(x) + g(x)$ or $f(x) - g(x)$ will not be derivable at $x = a$.

e.g. $f(x) = \cos|x|$ is derivable at $x = 0$ and $g(x) = |x|$ is not derivable at $x = 0$,
then $\cos|x| + |x|$ or $\cos|x| - |x|$ will not be derivable at $x = 0$.
However nothing can be said about the product function in this case.

$f(x) = x$ derivable at $x = 0$

$g(x) = |x|$ not derivable at $x = 0$

$$x|x| = \begin{cases} x^2 & \text{if } x \geq 0 \\ -x^2 & \text{if } x < 0 \end{cases}$$

(3) If both $f(x)$ and $g(x)$ are non derivable then nothing definite can be said about the sum / difference / product function.

e.g. $f(x) = \sin|x|$ not derivable at $x = 0$

$g(x) = |x|$ not derivable at $x = 0$

then the function

$F(x) = \sin|x| - |x|$ is derivable at $x = 0$

$G(x) = \sin|x| + |x|$ is not derivable at $x = 0$.

(4) If $f(x)$ is derivable at $x = a$ and $f(a) = 0$ and $g(x)$ is continuous at $x = a$ then the product function
$F(x) = f(x) \cdot g(x)$ will be derivable at $x = a$

$$F'(a^+) = \lim_{h \to 0} \frac{f(a+h)g(a+h) - 0}{h} = f'(a) \cdot g(a)$$

$$F'(a^-) = \lim_{h \to 0} \frac{f(a-h)\,g(a-h) - 0}{-h} = f'(a) \cdot g(a)$$

(5) Derivative of a continuous function need not be a continuous function

$$f(x)=\begin{cases} x^2\sin\frac{1}{x} & \text{if } x\neq 0 \\ 0 & \text{if } x=0 \end{cases};\quad \begin{matrix} f'(0^+)=0 \\ f'(0^-)=0 \end{matrix}$$

$$f'(x)=\begin{cases} \sin\frac{1}{x}\cdot 2x - x^2\cos\frac{1}{x}\left(\frac{-1}{x^2}\right) & x\neq 0 \\ 0 & \text{if } x=0 \end{cases}$$

$f'(x)$ is not continuous at $x=0$.

Illustration :

Let f be defined as follows :

$$f(x)=\begin{cases} \sin x & \text{if } x<\pi \\ mx+n & \text{if } x\geq \pi \end{cases}$$

where m and n are constants. Determine m and n such that f is derivable on set of real numbers.

Sol. $f(x)=\begin{cases} \sin x, & \text{if } x<\pi \\ mx+n, & \text{if } x\geq\pi \end{cases}.$

Since f(x) is continuous at $x=\pi$

$0=m\pi+n$ ....(1)

Now f(x) is differentiable at $x=\pi$

$\cos x\big|_{x=\pi}=m \Rightarrow m=-1 \Rightarrow n=\pi$

Illustration :

Check the differentiability of function at x = e.

$$f(x)=\begin{cases} (x-e)2^{-2(1/(e-x))}, & x\neq e \\ 0, & x=e \end{cases}$$

Sol. $f(x)=\begin{cases} (x-e)\,2^{-2\frac{1}{(e-x)}}, & x\neq e \\ 0, & x=e \end{cases}$

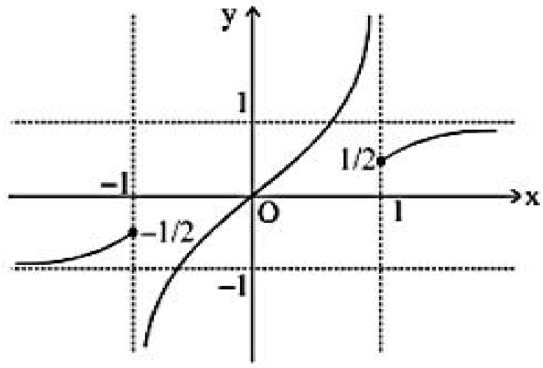
$$R.H.D. = \lim_{h\to 0}\frac{f(e+h)-f(e)}{h} = \lim_{h\to 0}\frac{(-h)\,2^{\frac{-2}{h}}}{h}=\infty$$

$$L.H.D. = \lim_{h\to 0}\frac{f(e-h)-f(e)}{-h} = \lim_{h\to 0}\frac{(-h)\,2^{\frac{-2}{h}}-0}{(-h)}=0$$

$\Rightarrow$ *f(x) is not differentiable at x = 0.*

Illustration :

Find the domain of $f'(x)$ *if* $f(x) = \begin{cases} \dfrac{x}{1+|x|} & \text{if } |x| \geq 1 \\ \dfrac{x}{1-|x|} & \text{if } |x| < 1 \end{cases}$.



Sol. $f(x) = \begin{cases} \dfrac{x}{1-x}; & -\infty < x < -1 \\ \dfrac{x}{1+x}; & -1 < x < 0 \\ \dfrac{x}{1-x}; & 0 < x < 1 \\ \dfrac{x}{1+x}; & 1 < x < \infty \end{cases}$

so domain of $f'(x)$ *will be* $R \sim \{-1, 1\}$

Illustration :

If $f(x)$ *is differentiable at* $x = a$ *and* $f'(a) = \dfrac{1}{4}$. *Find* $\lim\limits_{h\to 0} \dfrac{f(a+2h^2)-f(a-2h^2)}{h^2}$

Sol. Limit $= \lim\limits_{h\to 0} \dfrac{f(a+2h^2)-f(a-2h^2)}{h^2}$ *put* $t = h^2$

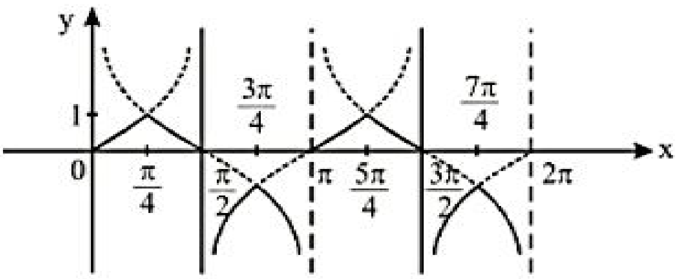
$= \lim\limits_{h\to 0} \dfrac{f(a+2t)-f(a-2t)}{t}$ *differentiating numerator and denominator*

$= \lim\limits_{t\to 0} \dfrac{2f'(a+2t)-2f'(a-2t)}{1} = 2f'(a) + 2f'(a) = 4f'(a) = 1.$

Illustration :

Consider the function $f(x) = \text{Min}(\tan x, \cot x)$ *in* $(0, 2\pi)$. *Number of points where f is either fails to be derivable is 'm' and number of points where it is discontinuous in 'n'. Find (m, n)*

Sol. *Number of points where* f *is non derivable* $= 7$.



Number of points where $f(x)$ *is discontinuous* $= 3$
$(m, n) = (7, 3)$.

(VI) Determination of function which are differentiable and satisfying the given functional rule :

BASIC STEPS :

(i) Write down the expression for $f'(x)$ as $f'(x) = \dfrac{f(x+h)-f(x)}{h}$

(ii) Manipulate $f(x+h)-f(x)$ in such a way that the given functional rule is applicable. Now apply the functional rule and simplify the RHS to get $f'(x)$ as a function of x along with constants if any.

(iii) Integrate $f'(x)$ get $f(x)$ as a function of x and a constant of integration. In some cases a Differential Equation in formed which can be solved to get $f(x)$.

(iv) Apply the boundary value conditions to determine the value of this constant.

Illustration :

Let f be a differentiable function satisfying $f\left(\dfrac{x}{y}\right) = f(x) - f(y)$ *for all* $x, y > 0$. *If* $f'(1) = 1$ *then find* $f(x)$.

Sol. $f'(x) = \lim\limits_{h\to 0} \dfrac{f(x+h)-f(x)}{h} = \lim\limits_{h\to 0} \dfrac{1}{h} f\left(\dfrac{x+h}{x}\right) = \lim\limits_{h\to 0} \dfrac{1}{h} f\left(1+\dfrac{h}{x}\right) = \lim\limits_{h\to 0} \dfrac{f\left(1+\dfrac{h}{x}\right)-f(1)}{h}$

as $f(1) = 0$

So, $f'(x) = \dfrac{f'(1)}{x} = \dfrac{1}{x}$

integrating w.r.t. to x

$f(x) = \ln x + k$ *as* $f(1) = 0$ *so* $k = 0$

$f(x) = \ln x$

Illustration :

Suppose f is a derivable function that satisfies the equation

$$f(x+y) = f(x) + f(y) + x^2y + xy^2$$

for all real numbers x and y. Suppose that $\lim\limits_{x\to 0} \dfrac{f(x)}{x} = 1$, *find*

(a) $f(0)$ *(b)* $f'(0)$ *(c)* $f'(x)$ *(d)* $f(3)$

Sol. $f'(x) = \lim\limits_{h\to 0} \dfrac{f(x+h)-f(x)}{h} = \lim\limits_{h\to 0} \dfrac{f(h)+x^2h+xh^2}{h}$

$f'(x) = \lim\limits_{h\to 0} \dfrac{f(h)}{h} + x^2 = 1 + x^2$

$f(x) = x + \dfrac{x^3}{3} + c$ *as* $f(0) = 0$ *so* $c = 0$

$f(x) = x + \dfrac{x^3}{3}$

Illustration :

A differentiable function satisfies the relation

$$f(x + y) = f(x) + f(y) + 2xy - 1 \quad \forall x, y \in R$$

If $f'(0) = \sqrt{3+a-a^2}$ *find* $f(x)$ *and prove that* $f(x) > 0 \quad \forall x \in R$

Sol. $f'(x) = \lim\limits_{h\to 0} \dfrac{f(x+h)-f(x)}{h} = \lim\limits_{h\to 0} \dfrac{f(h)+2xh-1}{h}$

$f(0) = 2f(0) - 1 \Rightarrow f(0) = 1$

$$f'(x) = \lim_{h\to 0} \frac{f(h)-f(0)}{h} + 2x = f'(0) + 2x$$

$$f'(x) = 2x + \sqrt{(3+a-a^2)}$$

$f(x) = x^2 + \sqrt{(3+a-a^2)}\,x + c \qquad$ as $f(0) = 1$ so $c = 1$

$$f(x) = x^2 + \sqrt{(3+a-a^2)}\,x + 1$$

$D = 3 + a - a^2 - 4 = a - a^2 - 1 = -(a^2 - a + 1) < 0$

as $a^2 - a + 1 > 0 \ \forall a$

So, $f(x) > 0 \quad \forall x \in R$

Illustration :

(i) *If* $f(x + y) = f(x) \cdot f(y)$, $\forall\ x,\ y \in R$ *and* $f(x)$ *is a differentiable function everywhere. Find* $f(x)$.

(ii) *If* $f(x + y) = f(x) + f(y)$, $\forall\ x,\ y \in R$ *then prove that* $f(kx) = k\,f(x)$ *for* $\forall\, k, x \in R$.

Sol.

(i) $f(x + y) = f(x)\,f(y) \Rightarrow f(0) = f(0)^2 \Rightarrow f(0) = 0$ or $f(0) = 1$

$f'(x) = \lim\limits_{h\to 0} \dfrac{f(x+h)-f(x)}{h} = \lim\limits_{h\to 0} \dfrac{f(x)\,f(h)-f(x)}{h} = f(x)\,f'(0)$ assume $f'(0) = k_2$

$$\int \frac{f'(x)}{f(x)} = \int k_2 dx \quad \Rightarrow \quad f(x) = k_1 e^{k_2 x}$$

(ii) $f'(x) = \lim\limits_{h\to 0} \dfrac{f(x+h)-f(x)}{h} = \lim\limits_{h\to 0} \dfrac{f(x)+f(h)-f(x)}{h} = f'(0) = k$

$\dfrac{d}{dx}(f(x)) = k \Rightarrow f(x) = kx + c$ as $f(0) = 0$

$f(x) = kx \Rightarrow f(f(x)) = k\,f(x).$

Illustration :

Discuss the differentiability of $f(x)=\begin{cases} \dfrac{\sin x^2}{x}, & x \neq 0 \\ 0, & x=0 \end{cases}$ at $x=0$

Sol. For continuity, $\lim\limits_{x\to 0} f(x) = \lim\limits_{h\to 0} \dfrac{\sin h^2}{h}$

$$= \lim_{h\to 0} h \frac{\sin h^2}{h^2} = 0$$

Hence, $f(x)$ is continuous at $x = 0$.

Also, $f'(0^+) = \lim\limits_{h\to 0} \dfrac{f(h)-f(0)}{h} = \lim\limits_{h\to 0} \dfrac{\sin h^2}{h^2} = 1$

and $f'(0) = \lim\limits_{h\to 0} \dfrac{f(-h)-f(0)}{-h} = \lim\limits_{h\to 0} \dfrac{\sin h^2}{h^2} = 1$

Thus, $f(x)$ is differentiable at $x = 0$.

DIFFERENTIABILITY USING THEOREMS ON DIFFERENTIABILITY :

Illustration :

Discuss the differentiability of $f(x) = |x| + |x - 1|$.

Sol. $f(x) = |x| + |x - 1|$

$f(x)$ is continuous everywhere at $|x|$ and $|x - 1|$ are continuous for all x.

Also $|x|$ and $|x - 1|$ are non-differentiable at $x = 0$ and $x = 1$, respectively.

Hence, $f(x)$ is non-differentiable at $x = 0$ and $x = 1$.

Illustration :

Discuss the differentiability of $f(x) = \text{maximum}\ \{2 \sin x, 1 - \cos x\}\ \forall x \in (0, \pi)$.

Sol. $f(x) = \max.\ \{2 \sin x, 1 - \cos x\}$ can be plotted as

Thus, $f(x) = \text{maximum}\ \{2 \sin x, 1 - \cos x\}$ is not differentiable, when $2 \sin x = 1 - \cos x$

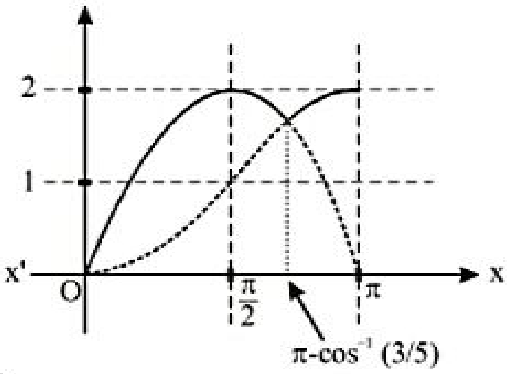
$\Rightarrow \quad 4 \sin^2 x = (1 - \cos x^2)$

$\Rightarrow \quad 4 (1 + \cos x) = (1 - \cos x)$

$\Rightarrow \quad 4 + 4 \cos x = 1 - \cos x$

$\Rightarrow \quad \cos x = -\dfrac{3}{5}$

$\Rightarrow \quad x = \cos^{-1}\left(-\dfrac{3}{5}\right)$



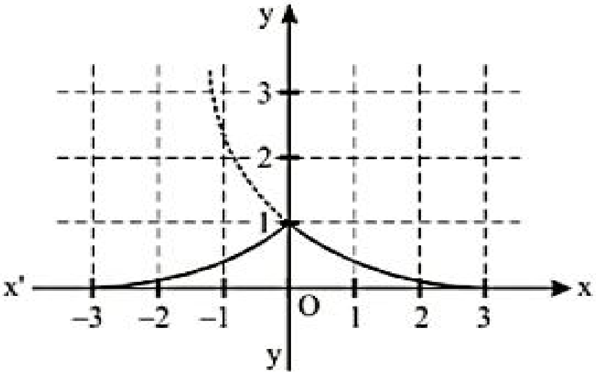
$\Rightarrow \quad f(x)$ is not differentiable at $x = \pi - \cos^{-1}\left(\dfrac{3}{5}\right),\ \forall x \in (0, \pi)$.

Illustration :

Discuss the differentiability of $f(x) = e^{-|x|}$.

Sol. *We have* $f(x) = \begin{cases} e^{-x}, & x \ge 0 \\ e^{x}, & x < 0 \end{cases}$

Clearly from the graph,
$f(x)$ *is non-differentiable at* $x = 0$.



DIFFERENTIABILITY BY DIFFERENTIATION :

Illustration :

If $f(x) = \begin{cases} x, & x \le 1 \\ x^2 + bx + c, & x > 1 \end{cases}$, *then find the values of b and c if* $f(x)$ *differentiable at* $x = 1$.

Sol. $f(x) = \begin{cases} x, & x \le 1 \\ x^2 + bx + c, & x > 1 \end{cases} \Rightarrow f'(x) = \begin{cases} 1, & x < 1 \\ 2x + b, & x > 1 \end{cases}$

$f(x)$ *is differentiable at* $x = 1$
Then, it must be continuous at $x = 1$

for which $\lim\limits_{x \to 1^+} f(x) = \lim\limits_{x \to 1^-} f(x)$

$\Rightarrow \quad 1 + b + c = 1$

$\Rightarrow \quad b + c = 0 \qquad \ldots(i)$

Also $f'(0) = f'(0^-)$

$\Rightarrow \quad \lim\limits_{x \to 1^+} f'(x) = \lim\limits_{x \to 1^-} f'(x) \quad \Rightarrow \quad 2 + b = 1 \quad \Rightarrow \quad b = -1$

$\Rightarrow \quad c = 1 \qquad$ *[from equation (i)]*

Illustration :

Find the values of a and b if $f(x) = \begin{cases} a + \sin^{-1}(x + b), & x \ge 1 \\ x, & x < 1 \end{cases}$ *is differentiable at* $x = 1$.

Sol. $f(x) = \begin{cases} a + \sin^{-1}(x + b), & x \ge 1 \\ x, & x < 1 \end{cases} \Rightarrow f'(x) = \begin{cases} \dfrac{1}{\sqrt{1 - (x + b)^2}}, & x > 1 \\ 1, & x < 1 \end{cases}$

For $f(x)$ *to be continuous at* $x = 1$.

$f(1^+) = f(1^-) \Rightarrow \quad a + \sin^{-1}(1 + b) = 1 \qquad \ldots(i)$

Also $\quad f'(1^+) = f'(1) \Rightarrow \quad \dfrac{1}{\sqrt{1 - (1 + b)^2}} = 1 \quad \Rightarrow \quad b = -1.$

Practice Problem

Q.1 Discuss the continuity and differentiability of $f(x) = |x+1| + |x| + |x-1|, \forall x \in R$; also draw the graph of $f(x)$.

Q.2 Find x where $f(x) = \text{maximum}\left\{\sqrt{x(2-x)}, 2-x\right\}$ is non-differentiable.

Q.3 Discuss the differentiability of function $f(x) = |[x]\,x|$ in $-1 < x \le 2$, where $[\cdot]$ represents gratest integer function.

Q.4 Discuss the differentiability of $f(x) = \max\{\tan^{-1} x, \cot^{-1} x\}$.

Q.5 Find the value of a and b if $f(x) = \begin{cases} ax^2 + 1, & x \le 1 \\ x^2 + ax + b, & x > 1 \end{cases}$ is differentiable at $x = 1$.

Q.6 Discuss the differentiability of $f(x) = \cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right)$.

Answer key

Q.1 Continuous $\forall x \in R$, differentiable for $x \in R - \{-1, 0, 1\}$

Q.2 $x = 1$

Q.3 $x = 0, 1, 2$

Q.4 $x = 0$

Q.5 $a = 2, b = 0$

Q.6 $a = 2, b = 0$

Solved Examples

Q.1 The value of $\lim\limits_{x\to 0}\dfrac{\sqrt{2}-\sqrt{1+\cos x}}{\sin^2 x}$ is

(A) $\dfrac{1}{2\sqrt{2}}$ (B) $\dfrac{1}{8\sqrt{2}}$ (C) $\dfrac{1}{4\sqrt{2}}$ (D) $-\dfrac{1}{4\sqrt{2}}$

Sol. We have $\lim\limits_{x\to 0}\dfrac{\sqrt{2}-\sqrt{1+\cos x}}{\sin^2 x}$

$$= \lim_{x\to 0}\frac{2-(1+\cos x)}{\sin^2 x}\times\frac{1}{\sqrt{2}+\sqrt{1+\cos x}}$$

$$= \lim_{x\to 0}\frac{1-\cos x}{(1+\cos x)(1-\cos x)}\times\lim_{x\to 0}\frac{1}{\sqrt{2}+\sqrt{1+\cos x}}$$

$$= \lim_{x\to 0}\frac{1}{(1+\cos x)}\times\frac{1}{2\sqrt{2}}=\frac{1}{4\sqrt{2}}$$

Q.2 If $a_1 = 1$ and $a_{n+1}=\dfrac{4+3a_n}{3+2a_n}$, $n \ge 1$ and if $\lim\limits_{n\to\infty} a_n = n$ then the value of a is

(A) $\sqrt{2}$ (B) $-\sqrt{2}$ (C) 2 (D) None of these

Sol. We have $a_{n+1}=\dfrac{4+3a_n}{3+2a_n}$

$$\Rightarrow \quad \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty}\frac{4+3a_n}{3+2a_n}$$

$$\Rightarrow \quad a=\frac{4+3a}{3+2a} \quad\Rightarrow\quad 2a^2=4 \quad\Rightarrow a=\sqrt{2} \quad (\text{where } \lim_{n\to\infty} a_n = 0)$$

$(a \ne -\sqrt{2}$ because each $a_n > 0$. therefore $\lim a_n = a > 0)$

Q.3 Evaluate $\lim\limits_{n\to\infty}\cos\left(\pi\sqrt{n^2+n}\right)$. when n is an integer.

(A) 1 (B) – 1 (C) 0 (D) None

Sol. $L=\lim\limits_{n\to\infty}\cos\left(\pi\sqrt{n^2+n}\right) = \lim\limits_{x\to\infty}\pm\cos\left(n\pi-\pi\sqrt{n^2+n}\right) = \lim\limits_{x\to\infty}\pm\cos\left(\pi\left(n-\sqrt{n^2+n}\right)\right)$

$$=\pm\lim_{x\to\infty}\cos\left(\frac{-n\pi}{n+\sqrt{n^2+n}}\right) = \pm\lim_{x\to\infty}\cos\left(\frac{n\pi}{n+n\sqrt{1+\frac{1}{n}}}\right) = \pm\lim_{x\to\infty}\cos\left(\frac{\pi}{1+\sqrt{1+\frac{1}{n}}}\right) = \pm\cos\frac{\pi}{2}\to 0$$

Q.4 Evaluate $\lim\limits_{n\to\infty}\prod\limits_{r=3}^{n}\frac{r^3-8}{r^3+8}$, where $\prod$ represents product of function.

(A) $\frac{2}{3}$ (B) $\frac{2}{5}$ (C) $\frac{1}{3}$ (D) $\frac{2}{7}$

Sol. Let $P=\lim\limits_{n\to\infty}\prod\limits_{r=3}^{n}\frac{r^3-8}{r^3+8}=\lim\limits_{n\to\infty}\prod\limits_{r=3}^{n}\left(\frac{r-2}{r+2}\right)\left(\frac{r^2+2r+4}{r^2-2r+4}\right)=\lim\limits_{n\to\infty}\prod\limits_{r=3}^{n}\left(\frac{r-2}{r+2}\right)\prod\limits_{r=3}^{n}\left(\frac{r^2+2r+4}{r^2-2r+4}\right)$

$$=\lim_{n\to\infty}\left\{\frac{1}{5}\frac{2}{6}\frac{3}{7}\frac{4}{8}\frac{5}{9}\ldots\ldots\frac{(n-5)}{(n-1)}\frac{(n-4)}{(n)}\frac{(n-3)}{(n+1)}\frac{(n-2)}{(n+2)}\right\}$$

$$\times\left\{\frac{19}{7}\times\frac{28}{12}\times\frac{39}{19}\ldots\ldots\frac{(n^2-2n+4)}{(n^2-6n+12)}\times\frac{(n^2+3)}{(n^2-4n+7)}\times\frac{(n^2+2n+4)}{(n^2-2n+4)}\right\}$$

$$=\lim_{n\to\infty}\left\{\frac{1\cdot2\cdot3\cdot4}{(n-1)n(n+1)(n+2)}\times\frac{(n^2+3)(n^2+2n+4)}{7\times12}\right\}=\frac{2}{7}\lim_{n\to\infty}\left\{\frac{(n^2+3)(n^2+2n+4)}{(n-1)n(n+1)(n-2)}\right\}$$

$$=\frac{2}{7}\lim_{n\to\infty}\left\{\frac{\left(1+\frac{3}{n^2}\right)\left(1+\frac{2}{n}+\frac{4}{n^2}\right)}{\left(1-\frac{1}{n}\right)1\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)}\right\}=\frac{2}{7}\frac{(1+0)(1+0+0)}{(1-0)1(1+0)(1+0)}=\frac{2}{7}$$

Hence $P=\frac{2}{7}$

Q.5 If $[x]$ denotes the greatest integer $\le x$, then evaluate $\lim\limits_{x\to\infty}\frac{1}{n^3}\{[1^2x]+[2^2x]+[3^2x]+\ldots\ldots+[n^2x]\}$

(A) $\frac{x}{2}$ (B) $\frac{x}{3}$ (C) $\frac{x}{4}$ (D) x

Sol. $\lim\limits_{x\to\infty}\frac{1}{n^3}\{[1^2x]+[2^2x]+[3^2x]+\ldots\ldots+[n^2x]\}$

$$=\lim_{n\to\infty}\left\{\frac{\sum\limits_{r=1}^{n}[r^2x]}{n^3}\right\}=\lim_{n\to\infty}\left(\frac{\sum\limits_{r=1}^{n}r^2x-\{r^2x\}}{n^3}\right)=\lim_{n\to\infty}\left(\frac{x\,\frac{n(n+1)(2n+1)}{6}}{n^3}-\sum_{r=1}^{n}\frac{\{r^2x\}}{n^3}\right)$$

$$=x\frac{(1)(1)(2)}{6}-0=\frac{x}{3}$$

Q.6 Let $f(x) = \begin{cases} \dfrac{(x^2+3x-1)\tan x}{x^2+2x} & \text{if } x \neq 0 \\ k & \text{if } x = 0 \end{cases}$, then find the value of k.

Sol. $f(x) = \begin{cases} \dfrac{(x^2+3x-1)\tan x}{x^2+2x} & \text{if } x \neq 0 \\ k & \text{if } x = 0 \end{cases}$; $k = \lim\limits_{x \to 0} \dfrac{(x^2+3x-1)}{(x+2)x} \tan x$

so $k = \dfrac{-1}{2}$.

Q.7 $f(x) = \begin{cases} \dfrac{\sin x + \sin 5x}{\cos x + \cos 5x} & \text{if } x \neq -\dfrac{\pi}{4} \\ k & \text{if } x = -\dfrac{\pi}{4} \end{cases}$ Find k if f is continuous at $x = -\dfrac{\pi}{4}$

(A) 1 (B) – 1 (C) 0 (D) $-1/\sqrt{2}$

Sol. $f(x) = \begin{cases} \dfrac{\sin x + \sin 5x}{\cos x + \cos 5x} & \text{if } x \neq -\dfrac{\pi}{4} \\ k & \text{if } x = -\dfrac{\pi}{4} \end{cases}$

$$k = \lim_{x \to \frac{-\pi}{4}} \frac{\sin x + \sin 5x}{\cos x + \cos 5x}$$

$$k = \lim_{x \to \frac{-\pi}{4}} \frac{2\sin 3x \cos 2x}{2\cos 3x \cos 2x} = 1.$$

Q.8 If $f(x) = \begin{cases} x+1 & x \leq 1 \\ 3-ax^2 & x > 1 \end{cases}$ is continuous at $x = 1$, then find the value of a.

Sol. $f(x) = \begin{cases} x+1, & x \leq 1 \\ 3-ax^2, & x > 1 \end{cases}$; $\lim\limits_{x \to 1^-} f(x) = \lim\limits_{x \to 1^+} f(x) = \lim\limits_{x \to 1^+} f(x)$

$2 = 3 - a \Rightarrow a = 1.$

Q.9 What kind of discontinuity the function $\dfrac{\cos x}{x}$ has at $x = 0$.

Sol. $f(x) = \dfrac{\cos x}{x}$ $\lim\limits_{x \to 0^+} f(x) = \lim\limits_{x \to 0^+} \dfrac{\cos x}{x} = \infty$

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \frac{\cos x}{x} = -\infty$$

Non removable infinite type

Q.10 If $f(x) = \begin{cases} \dfrac{8^x - 4^x - 2^x + 1^2}{x^2}, & x > 0 \\ e^x \sin x + 4x + k\,ln4, & x \le 0 \end{cases}$ is continuous at $x = 0$, then find the value of k.

Sol. $f(x) = \begin{cases} \dfrac{8^x - 4^x - 2^x + 1^2}{x^2}, & x > 0 \\ e^x \sin x + 4x + k\,ln4, & x \le 0 \end{cases}$

$$\lim_{x \to 0^+} \frac{8^x - 4^x - 2^x + 1}{x^2} = \lim_{x \to 0^-} \frac{e^{x\,ln_e 8} - e^{x\,ln_e 4} - e^{x\,ln_e 2} + 1}{x^2}$$

$$= \lim_{x \to 0^+} \frac{\left(1 + x\,ln_e 8 + \frac{(x\,ln_e 8)^2}{2!} + ...\right) - \left(1 + x\,ln_e 4 + \frac{(x\,ln_e 4)^2}{2!} + ...\right) - \left(1 + x\,ln_e 2 + \frac{(x\,ln_e 2)^2}{2!} + ...\right) + 1}{x^2}$$

$$= \lim_{x \to 0^+} \frac{x\left(ln_e 8 - ln_e 4 - ln_e 2\right) + x^2\left(\frac{(ln_e 8)^2}{2} - \frac{(ln_e 4)^2}{2} - \frac{(ln_e 2)^2}{2}\right)}{x^2}$$

$$= \lim_{x \to 0^+} \frac{1}{2}\left((ln_e 8)^2 - (ln_e 4)^{2x^2} - (ln_e 2)^2\right) = \frac{1}{2}\left((ln_e 32)(ln_e 2) - (ln_e 2)^2\right)^2$$

$$= \frac{1}{2}(ln_e 2)(ln_e 16) - (ln_e 2)(ln_e 4)$$

$$\lim_{x \to 0^-} f(x) = 0 + 0 + k\,ln\,4 = (ln_e 2)\,(ln_e 4) \Rightarrow k = ln_e 2.$$

METHOD OF DIFFERENTIATION

1.0 INTRODUCTION :

The essence of calculus is the derivative. The derivative is the instantaneous rate of change of a function with respect to one of its variables. This is equivalent to finding the slope of the tangent line to the function at a point. Let us use the view of derivatives as tangents to motivate a geometric definition of the derivative.

1.1 GEOMETRICAL MEANING OF A DERIVATIVE :

Let P $(x_0, f(x))$ and $Q(x_0 + h, f(x_0 + h))$ be two points very close to each other on the curve $y = f(x)$. Draw PM and QN perpendiculars from P and Q on x-axis, and draw PL as perpendicular from P on QN. Let the chord PQ produced meet the x-axis at R and $\angle QRN = \angle QPL = \phi$.
Now in right-angled triangle QPL

$$\tan\phi = \frac{QL}{PL} = \frac{NQ - NL}{MN} = \frac{NQ - MP}{ON - OM} = \frac{f(x_0 + h) - f(x_0)}{(x_0 + h) - x_0}$$

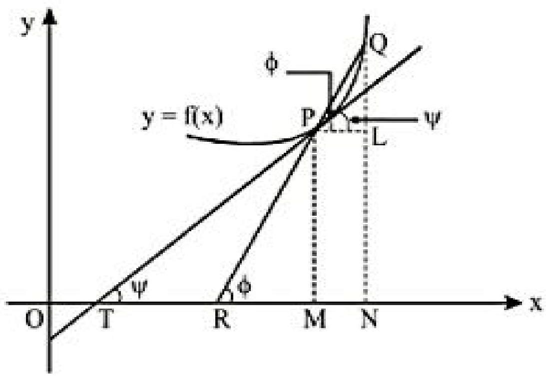
$$= \frac{f(x_0 + h) - f(x_0)}{h} \qquad \text{.........(1)}$$

when $h \to 0$, the point Q moving along the curve tends to P, i.e., $Q \to P$. The chord PQ approaches the tangent line PT at the point P and then $\phi \to \psi$. Now applying $\lim\limits_{h \to 0}$ in equation (1), we get

$$\lim_{h \to 0} \tan\phi = \frac{f(x_0 + h) - f(x_0)}{h}$$

$$\tan\psi = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

$$\Rightarrow \qquad f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$



This definition of derivative is also called the first principle of derivative. Clearly, the domain of definition of f'(x) is wherever the above limit exits.
Note that if $y = f(x)$ then the symbols

$$\frac{dy}{dx} = Dy = f'(x) = y_1 \text{ or } y' \text{ have the same meaning.}$$

However a dot, denotes the time derivative.

e.g. $\dot{S} = \frac{dS}{dt}$; $\dot{\theta} = \frac{d\theta}{dt}$ etc.

Illustration :

Find the derivative of $e^{\sqrt{x}}$ *w.r.t. x using first principle.*

Sol. *Let* $f(x) = e^{\sqrt{x}}$, *then* $f(x+h) = e^{\sqrt{x+h}}$

$$\therefore \quad \frac{d}{dx}f(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{e^{\sqrt{x+h}} - e^{\sqrt{x}}}{h} = e^{\sqrt{x}} \lim_{h \to 0} \left(\frac{e^{\sqrt{x+h} - \sqrt{x}} - 1}{h} \right)$$

$$= e^{\sqrt{x}} \lim_{h \to 0} \left(\frac{e^{\sqrt{x+h} - \sqrt{x}} - 1}{\sqrt{x+h} - \sqrt{x}} \right) \left(\frac{\sqrt{x+h} - \sqrt{x}}{h} \right)$$

$$= e^{\sqrt{x}} \lim_{h \to 0} \left(\frac{e^{\sqrt{x+h} - \sqrt{x}} - 1}{\sqrt{x+h} - \sqrt{x}} \right) \times \frac{\left(\sqrt{x+h} - \sqrt{x}\right)\left(\sqrt{x+h} + \sqrt{x}\right)}{h\left(\sqrt{x+h} + \sqrt{x}\right)}$$

$$= e^{\sqrt{x}} \lim_{h \to 0} \left(\frac{e^{y} - 1}{y} \right) \lim_{h \to 0} \frac{x+h-x}{h\left(\sqrt{x+h} + \sqrt{x}\right)},$$

where $y = \sqrt{x+h} - \sqrt{x}$ $\quad (\because$ when $h \to 0, y \to 0)$

$$= e^{\sqrt{x}} \times 1 \times \left(\frac{1}{\sqrt{x} + \sqrt{x}} \right) = \frac{e^{\sqrt{x}}}{2\sqrt{x}}.$$

Illustration :

Find the derivative of the following functions with respect to x using first principle.

(i) $y = \dfrac{x}{x^2+1}$; (ii) $y = \cos^2 x$; (iii) $y = \sin 3x$; (iv) $y = x^3 - 3^x$; (v) $y = \ln^2 x$.

Sol.

(i) $\quad y = \dfrac{x}{x^2+1} = f(x)$

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\dfrac{x+h}{(x+h)^2+1} - \dfrac{x}{x^2+1}}{h}$$

$$= \lim_{h \to 0} \frac{(x+h)(x^2+1) - x\left[(x^2+1) + 2hx + h^2\right]}{h(x^2+1)\left[(x+h)^2+1\right]} = \lim_{h \to 0} \frac{(x^2+1) - x(2x+h)}{(x^2+1)\left[(x+h)^2+1\right]}$$

$$= \lim_{h \to 0} \frac{(x^2+1) - 2x^2}{(x^2+1)^2} = \lim_{h \to 0} \frac{1-x^2}{(x^2+1)^2}.$$

(ii) $y = \cos^2 x = f(x)$

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\cos^2(x+h) - \cos^2 x}{h}$$

$$= \lim_{h \to 0} \frac{2\sin\left(x + \frac{h}{2}\right)\sin\left(\frac{-h}{2}\right)}{h} (\cos(x+h) + \cos x))$$

$$= 2\sin x \left(\frac{-1}{2}\right) (2\cos x) = -2\sin x \cos x = -\sin 2x.$$

(iii) $y = \sin 3x = f(x)$

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\sin(3x+3h) - \sin 3x}{h}$$

$$= \lim_{h \to 0} \frac{2\sin\left(\frac{3h}{2}\right)\cos\left(3x + \frac{3h}{2}\right)}{h} = 2 \cdot \left(\frac{3}{2}\right) \cdot \cos(3x+0) = 3\cos 3x.$$

(iv) $y = x^3 - 3^x = f(x)$

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{(x+h)^3 - 3^{x+h} - (x^3 - 3^x)}{h}$$

$$= \lim_{h \to 0} \frac{(x+h)^3 - x^3}{h} - \lim_{h \to 0} \frac{(3^{x+h} - 3^x)}{h} = \lim_{h \to 0} \frac{3xh(x+h) + h^3}{h} - 3^x \lim_{h \to 0} \frac{(3^h - 1)}{h}$$

$$= 3x(x+0) - 3^x \ln 3 = 3x^2 - 3^x \ln 3.$$

(v) $y = f(x) = \ln^2 x$

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\ln^2(x+h) - \ln^2 x}{h}$$

$$= \lim_{h \to 0} \left(\frac{\ln(x+h) - \ln x}{h}\right)(\ln(x+h) + \ln x)$$

$$= \lim_{h \to 0} \frac{\ln\left(1 + \frac{h}{x}\right)}{h} (\ln(x+h) + \ln x) = \frac{2\ln x}{x}.$$

1.2 STANDARD DERIVATIVES :

(i) $\frac{d}{dx}x^n = nx^{n-1}, x \in R, n \in R, x > 0$

(ii) $\frac{d}{dx}(e^x) = e^x$

(iii) $\frac{d}{dx}(a^x) = a^x \ln a$

(iv) $\frac{d}{dx}(\ln|x|) = \frac{1}{x}$

(v) $\frac{d}{dx}(\log_a x) = \frac{1}{x}\log_a e$

(vi) $\frac{d}{dx}(\sin x) = \cos x$

(vii) $\frac{d}{dx}(\cos x) = -\sin x$

(viii) $\frac{d}{dx}(\tan x) = \sec^2 x$

(ix) $\frac{d}{dx}(\cot x) = -\text{cosec}^2 x$

(x) $\frac{d}{dx}(\sec x) = \sec x \tan x$

(xi) $\frac{d}{dx}(\text{cosec}\, x) = -\text{cosec}\, x \cot x$

Illustration :

If $y = \left(1+x^{\frac{1}{4}}\right)\left(1+x^{\frac{1}{2}}\right)\left(1-x^{\frac{1}{4}}\right)$, *then find* $\frac{dy}{dx}$.

Sol. $y = \left(1+x^{\frac{1}{4}}\right)\left(1+x^{\frac{1}{2}}\right)\left(1-x^{\frac{1}{4}}\right) = \left(1+x^{\frac{1}{4}}\right)\left(1-x^{\frac{1}{4}}\right)\left(1+x^{\frac{1}{2}}\right) = \left(1-x^{\frac{1}{2}}\right)\left(1+x^{\frac{1}{2}}\right)$

$= 1 - x \Rightarrow \frac{dy}{dx} = -1.$

Illustration :

If $f(x) = x\,|x|$, *then prove that* $f'(x) = 2\,|x|$.

Sol. $f(x) = \begin{cases} -x^2, & x < 0 \\ x^2, & x > 0 \end{cases} \Rightarrow f'(x) = \begin{cases} -2x, & x < 0 \\ 2x, & x > 0 \end{cases}$

$\therefore \quad f'(x) = 2\,|x|.$

Illustration :

If $y = \sqrt{\frac{1-\cos 2x}{1+\cos 2x}}, x \in \left(0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, 0\right)$, *then find* $\frac{dy}{dx}$.

Sol. *We have*

$y = \sqrt{\frac{1-\cos 2x}{1+\cos 2x}} = \sqrt{\frac{2\sin^2 x}{2\cos^2 x}} = \sqrt{\tan^2 x} \Rightarrow y = |\tan x|$, *where* $x \in \left(0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right)$

$\Rightarrow y = \begin{cases} \tan x, & x \in \left(0, \frac{\pi}{2}\right) \\ -\tan x, & x \in \left(\frac{\pi}{2}, \pi\right) \end{cases} \Rightarrow \frac{dy}{dx} = \begin{cases} \sec^2 x, \text{ if } x \in \left(0, \frac{\pi}{2}\right) \\ -\sec^2 x, \text{ if } x \in \left(\frac{\pi}{2}, \pi\right) \end{cases}$

Illustration :

(i) $D\left(\frac{1}{\sin x}\right)$; (ii) $D(\tan(\tan^{-1}x))$; (iii) $D\left(\cos^4\frac{x}{2}-\sin^4\frac{x}{2}\right) =$;

(iv) $D(\cos^{-1}x + \sin^{-1}x)^n =$; (v) $D(\cos x \operatorname{cosec} x) =$ ____;

(vi) $D\left(\frac{1}{\log_x e}\right)$ when $x = \frac{1}{e}$ = ____ ; (vii) $D\left(\frac{1-\cos 2x}{\sin 2x}\right) =$ ____

Sol.

(i) $D\left(\frac{1}{\sin x}\right) = -\operatorname{cosec} x \cot x$

(ii) $D(\tan(\tan^{-1}x)) = 1$

(iii) $D\left(\cos^4\frac{x}{2}-\sin^4\frac{x}{2}\right) = \frac{d}{dx}\left(\cos^2\frac{x}{2}-\sin^2\frac{x}{2}\right) = \frac{d}{dx}(\cos x) = -\sin x$

(iv) $D(\cos^{-1}x + \sin^{-1}x)^n = \frac{d}{dx}\left(\frac{\pi}{2}\right)^n = 0$

(v) $D(\cos x \operatorname{cosec} x) = \frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$

(vi) $D\left(\frac{1}{\log_x e}\right) = \frac{1}{x}$ $\therefore$ $D\left(\frac{1}{\log_x e}\right)_{at\ x=\frac{1}{e}} = e$

(vii) $D\left(\frac{1-\cos 2x}{\sin 2x}\right) = D\left(\frac{2\sin^2 x}{2\sin x\cos x}\right) = D(\tan x) = \sec^2 x$

2.0 THEOREM ON DERIVATIVES :

T-1: $\frac{d}{dx}(f_1(x) \pm f_2(x)) = \frac{d}{dx}f_1(x) \pm \frac{d}{dx}f_2(x)$.

T-2: $\frac{d}{dx}(kf(x)) = k\frac{d}{dx}f(x)$, where k is any constant.

T-3 PRODUCT RULE :

$$\frac{d}{dx}\{f_1(x)\,f_2(x)\} = f_1(x)\frac{d}{dx}f_2(x) + f_2(x)\frac{d}{dx}f_1(x).$$

Illustration :

Find the derivative of the following functions

(i) $x \sin x$; (ii) $e^x \cdot \tan x$

Sol.

(i) $y = x \sin x$

$$y = x\sin x \text{ so } \frac{dy}{dx} = \sin x\frac{d}{dx}(x) + x\frac{d}{dx}(\sin x)$$

$$\frac{dy}{dx} = \sin x + x\cos x$$

(ii) $y = e^x \tan x$

$$\text{so } \frac{dy}{dx} = \tan x\frac{d}{dx}(e^x) + e^x\frac{d}{dx}(\tan x) = e^x(\tan x + \sec^2 x)$$

Note : If 3 functions are involved then remember

$$D\left(f(x)\cdot g(x)\cdot h(x)\right) = f(x)\cdot g(x)\cdot h'(x) + g(x)\cdot h(x)\cdot f'(x) + h(x)\cdot f(x)\cdot g'(x)$$

$$= \frac{(fg)'(h) + (gh)'(f) + (hf)'(g)}{2}$$

This result can be generalised to the product of n functions

Illustration :

Let $F(x) = f(x)\cdot g(x)\cdot h(x)$. *If for some* $x = x_0$, $F'(x_0) = 21F(x_0)$; $f'(x_0) = 4f(x_0)$; $g'(x_0) = -7g(x_0)$ *and* $h'(x_0) = k\,h(x_0)$ *then find k.*

Sol. $F(x) = f(x)\,g(x)\,h(x)$

Given that $F'(x_0) = 21f(x_0)$, $f'(x_0) = 4f(x_0)$; $g'(x_0) = -7g(x_0)$ and $h'(x_0) = kh(x_0)$

$$F'(x_0) = f'(x_0)\,g(x_0)\,h(x_0) + f(x_0)\,g'(x_0)\,h(x_0) + f(x_0)\,g(x_0)\,h'(x_0)$$

$$21F(x_0) = (4 - 7 + k)\,f(x_0)\,g(x_0)\,h(x_0) \quad \Rightarrow 21 = -3 + k \Rightarrow k = 24.$$

Illustration :

Let f, g and h are differentiable functions. If $f(0) = 1$; $g(0) = 2$; $h(0) = 3$ *and the derivatives of their pair wise products at* $x = 0$ *are*

$(fg)'(0) = 6$; $(gh)'(0) = 4$ and $(hf)'(0) = 5$

then compute the value of $(fgh)'(0)$.

Sol. Let $wx = f(x)\,g(x)\,h(x)$

$$\frac{d}{dx}(w(x)) = f'(x)\,g(x)\,h(x) + f(x)\,g'(x)\,h(x) + f(x)\,g(x)\,h'(x)$$

$$2\frac{d}{dx}(w(x)) = 2f'(x)\,g(x)\,h(x) + 2f(x)\,g'(x)\,h(x) + 2f(x)\,g(x)\,h'(x)$$

$$= h(x)\frac{d}{dx}(f(x)g(x)) + g(x)\frac{d}{dx}(f(x)h(x)) + f(x)\frac{d}{dx}(g(x)h(x))$$

$$2\frac{d}{dx}(w(0)) = (3)(6) + (2)(5) + (1)(4) = 32$$

$$\frac{d}{dx}(w(0)) = 16.$$

Illustration :

If $f(x) = 1 + x + x^2 + \ldots\ldots + x^{100}$ *then* $f'(1) =$ ____.

Sol. $f(x) = 1 + x + x^2 + \ldots\ldots + x^{100}$

$f'(x) = 0 + 1 + 2x + 3x^2 + \ldots\ldots + 100\,x^{99}$

$$\therefore\ f'(1) = 1 + 2 + 3 + \ldots\ldots + 100 = \frac{100(100+1)}{2} = 5050$$

T-4 QUOTIENT RULE :

$$y = \frac{f(x)}{g(x)}$$

$$D\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)\cdot f'(x) - f(x)\cdot g'(x)}{g^2(x)}$$, to be remembered as

$$D\left(\frac{N^r}{D^r}\right) = \frac{D^r\frac{d}{dx}(N^r) - N^r\frac{d}{dx}(D^r)}{(D^r)^2};$$

Note: If $y = \dfrac{1}{f(x)}$ then $D(y) = -\dfrac{f'(x)}{f^2(x)}$

Illustration :

Find the derivative of the following functions

(i) $y = \dfrac{1-\ln x}{1+\ln x}$; *(ii)* $y = \dfrac{x^3+2^x}{e^x}$; *(iii)* $y = \dfrac{x\sin x}{1+\tan x}$

Sol.

(i) $y = \dfrac{1-\ln x}{1+\ln x}$

$$\frac{dy}{dx} = \frac{(1+\ln x)\left(\frac{-1}{x}\right) - \frac{(1-\ln x)}{x}}{(1+\ln x)^2} = \frac{-(1+\ln x+1-\ln x)}{x(1+\ln x)^2} = \frac{-2}{x(1+\ln x)^2}$$

(ii) $y = \dfrac{x^3+2^x}{e^x} = x^3 e^{-x} + 2^x e^{-x}$

$$\frac{dy}{dx} = 3x^2e^{-x} - x^3e^{-x} - 2^xe^{-x} + 2^xe^{-x}(\ln 2) = (3x^2 - x^3 - 2^x + 2^x\ln 2)\,e^{-x}$$

(iii) $y = \dfrac{x \sin x}{(1+\tan x)}$

$$\frac{dy}{dx} = \frac{(1+\tan x)(\sin x + x\cos x) - (x\sin x)\sec^2 x}{(1+\tan x)^2}$$

$$= \frac{\sin x + x\cos x + \sin x \tan x + x\sin x - x\tan x \sec x}{(1+\tan x)^2}$$

$$= \frac{(1+x)\sin x + x\cos x + (\sin x - x\sec x)\tan x}{(1+\tan x)^2}$$

Illustration :

(i) If $y = \dfrac{x^4+x^2+1}{x^2+x+1}$ then $\dfrac{dy}{dx} = ax + b$ find a and b.

(ii) If $y = \dfrac{\sec x + \tan x - 1}{\tan x - \sec x + 1}$, find $\left.\dfrac{dy}{dx}\right]_{x=\pi/4}$

(iii) If $y = \dfrac{x^3+x^2+x}{1+x^2}$, find $\dfrac{dy}{dx}$.

Sol.

(i) $y = \dfrac{x^4+x^2+1}{x^2+x+1} = \dfrac{(x^4+2x^2+1)-x^2}{x^2+x+1} = x^2 - x + 1$

$$\frac{dy}{dx} = 2x - 1 = ax + b \Rightarrow a = 2,\ b = -1$$

(ii) $y = \dfrac{\sec x + \tan x - 1}{\tan x - \sec x + 1} = \dfrac{\sec x + \tan x - (\sec^2 x - \tan^2 x)}{\tan x - \sec x + 1}$

$$y = \frac{(\sec x + \tan x)(1 - \sec x + \tan x)}{(\tan x - \sec x + 1)}$$

$$y = \sec x + \tan x$$

$$\therefore \frac{dy}{dx} = \sec x \tan x + \sec^2 x = \sec x (\tan x + \sec x)$$

(iii) $y = \dfrac{x^3+x^2+x}{1+x^2} = (x+1) - \dfrac{1}{x^2+1}$

$$\frac{dy}{dx} = 1 - \frac{(x^2+1)\frac{d}{dx}(1) - 1\cdot\frac{dy}{dx}(x^2+1)}{(x^2+1)^2} = 1 - \frac{(0-2x)}{(x^2+1)^2} = 1 + \frac{2x}{(x^2+1)^2}.$$

T-5 DIFFERENTIATION OF COMPOSITE FUNCTION (CHAIN RULE) :

If f(x) and g(x) are differentiable functions, then fog is also differentiable and $(fog)'(x) = f'(g(x)).g'(x)$.

or, $$\frac{d}{dx}\{fog(x)\} = \frac{d}{d\,g(x)}\{(fog)(x)\}\frac{d}{dx}(g(x))$$

or

If y is a function of t and t is a function of x, then

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$

Thus, if $y = f(t)$ and $t = \phi(x)$

$$\frac{dt}{dx} = \phi'(x)$$

$\therefore$ $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = f'(t)\,\phi'(x)$. This rule is called Chain Rule.

This chain rule can be extended as follows

Let $y = f(t),\ t = \phi(z),\ z = \psi(x)$

then $$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dz} \times \frac{dt}{dx} = f'(t)\,\phi'(z)\,\psi'(x)$$

Let $y = \log \sin x^3 = \log t$

Putting $t = \sin x^3 = \sin z,\ z = x^3$

$\Rightarrow$ $$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dz} \times \frac{dt}{dx} = (1/t)\cos z\, 3x^2 = \left(\frac{1}{\sin x^3}\right)(\cos x^3) \times 3x^2 = 3x^2 \cot x^3$$

Note : If $y = f(u)$ and $u = g(x)$ then $\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}$.

It can be extended for any number of chain. In general if $y = (E)^C$ where E = function of x and C = constant then

$$\frac{dy}{dx} = C(E)^{C-1}\cdot\frac{d}{dx}(E)$$

Illustration :

Find derivative of following functions ?

(i)	$y = \sin^3\sqrt{x}$	*(ii)*	$y = \ln(\sec x)$	*(iii)*	$y = \cos(\ln x)$
(iv)	$y = e^{ax}\sin bx.$	*(v)*	$y = e^{ax}\cos bx$		

Sol.

(i) $y = \sin^3\sqrt{x}$

Let $v = \sqrt{x}$; $u = \sin\sqrt{x} = \sin v$

$y = u^3$

$$\frac{dy}{dx} = \frac{d}{dx}(u^3) = \frac{d}{dx}(u^3)\cdot\left(\frac{du}{dx}\right) = 3u^2\cdot\frac{d}{dv}(\sin v)\left(\frac{dv}{dx}\right) = 3u^2\cos v\,\frac{d}{dx}\left(\sqrt{x}\right)$$

$$= 3\sin^2 v\cos v\left(\frac{1}{2\sqrt{x}}\right) = \frac{3\sin^2\sqrt{x}\cdot\cos\sqrt{x}}{2\sqrt{x}}$$

(ii) $y = \ln(\sec x)$

Let $u = \sec x$

$\therefore y = \ln u$

$$\frac{dy}{dx} = \frac{d}{dx}(\ln u) = \frac{d}{du}(\ln u)\frac{du}{dx} = \frac{1}{u}\cdot\frac{d}{dx}(\sec x) = \frac{1}{\sec x}(\sec x \tan x) = \tan x$$

(iii) $y = \cos(\ln x)$

Let $u = \ln x$

$\therefore y = \cos u$

$$\frac{dy}{dx} = \frac{d}{dx}(\cos u) = \frac{d}{du}(\cos u)\frac{du}{dx} = -\sin u \cdot \frac{d}{dx}(\ln x) = \frac{-\sin(\ln x)}{x}$$

(iv) $y = e^{ax}\sin bx$

$$\frac{dy}{dx} = e^{ax}\frac{d}{dx}(\sin bx) + \sin bx\frac{d}{dx}e^{ax}$$

$$= e^{ax}\cos bx\frac{d}{dx}(bx) + \sin bx \cdot e^{ax}\frac{d}{dx}(ax)$$

$= be^{ax}\cos bx + a\sin bx \cdot e^{ax}$

$= (b\cos bx + a\sin bx)\,e^{ax}$

(v) $y = e^{ax}\cos bx$

$$\frac{dy}{dx} = e^{ax}\frac{d}{dx}(\cos bx) + \cos bx\frac{d}{dx}(e^{ax}) = e^{ax}(-\sin bx)\frac{d}{dx}(bx) + \cos bx\, e^{ax}\cdot a$$

$= (a\cos bx - b\sin bx)\,e^{ax}$

Illustration :

Find derivative of following functions ?

(i)	$y = (fog)(x)$	(ii)	$y = (gof)(x)$	(iii)	$y = \dfrac{1}{(f(x))^n}$
(iv)	$y = \sec^2\left(f^3(x)\right)$	(v)	$y = \sqrt{f(x)}$	(vi)	$y = f(1/x)$

Sol.

(i) $y = fog(x)\ \ f(g(x))$

$$\frac{dy}{dx} = \frac{d}{dx}f(g(x)) = f'(g(x))\cdot\frac{d}{dx}g(x) \qquad = f'(g(x)).\,g'(x)$$

(ii) $y = gof(x) = g(f(x))$

$$\frac{dy}{dx} = \frac{d}{dx}g(f(x)) = g'(f(x))\frac{d}{dx}f(x) = g'(f(x))\cdot f'(x)$$

(iii) $y = \dfrac{1}{(f(x))^n} = (f(x))^{-n}$

$$\frac{dy}{dx} = -n\,(f(x))^{-n-1}\frac{d}{dx}f(x) = \frac{-n\,f'(x)}{(f(x))^{n+1}}$$

(iv) $y = sec^2 (f^3(x))$

$$\frac{dy}{dx} = 2\,sec\left(f^3(x)\right)\,\frac{d}{dx}sec\left(f^3(x)\right)$$

$$= 2\,sec\,(f^3(x)) \cdot sec\,(f^3(x))\,tan\,(f^3(x))\,\frac{d}{dx}\left(f^3(x)\right)$$

$$= 2\,sec^2\,(f^3(x))\,tan\,(f^3(x))\;\;3f^2(x)\,\frac{d}{dx}f(x).$$

$$= 6\,sec^2\,(f^3(x))\,tan\,(f^3(x))\,f^2(x) \cdot f'(x)$$

(v) $y = \sqrt{f(x)}$

$$\therefore\ \frac{dy}{dx} = \frac{1}{2}\left(f(x)\right)^{\frac{-1}{2}}\,\frac{d}{dx}f(x) = \frac{f'(x)}{2\sqrt{f(x)}}.$$

(vi) $y = f\left(\frac{1}{x}\right)$

$$\therefore\ \frac{dy}{dx} = f'\left(\frac{1}{x}\right)\frac{d}{dx}\left(\frac{1}{x}\right) = f'\left(\frac{1}{x}\right)\left(\frac{-1}{x^2}\right) = \frac{-f'\left(\frac{1}{x}\right)}{x^2}.$$

Illustration :

Find derivative of following functions ?

(i) $y = ln^3 tan^2(x^4)$ (ii) $y = cos\left(\frac{ax}{b}\right)$ (iii) $y = e^{\sqrt{sin(ln(x^2+7)^5)}}$

(iv) $y = sec\,x\sqrt{tan\,x}$ (v) $exp\,(cos^3\,(tan^{-1}x^3)^2\,)$

Sol.

(i) $y = ln^3\,(tan^2(x^4))$

$$\frac{dy}{dx} = 3\,ln^2\,(tan^2(x^4))\,\frac{d}{dx}\,ln\,(tan^2(x^4))$$

$$= 3\,ln^2\,(tan^2(x^4))\,\frac{1}{tan^2(x^4)}\,\frac{d}{dx}tan^2(x^4)$$

$$= \frac{3\,ln^2\left(tan^2(x^4)\right)}{tan^2(x^4)} \cdot 2\,tan\,(x^4) \cdot \frac{d}{dx}tan\left(x^4\right)$$

$$= \frac{6\,ln^2\left(tan^2(x^4)\right)}{tan(x^4)}\,sec^2(x^4)\,\frac{d}{dx}\left(x^4\right)$$

$$= \frac{24x^3\,ln^2\left(tan^2(x^4)\right)\,sec^2(x^4)}{tan\,(x^4)}$$

(ii) $y = \cos\left(\frac{ax}{b}\right)$

$$\frac{dy}{dx} = -\sin\left(\frac{ax}{b}\right)\frac{d}{dx}\left(\frac{ax}{b}\right) = \frac{-a\sin\left(\frac{ax}{b}\right)}{b}$$

(iii) $y = e^{\sqrt{\sin\left(\ln(x^2+7)^5\right)}}$

$$\therefore\ y = e^{\sqrt{\sin\left(5\ln(x^2+7)\right)}}$$

$$\therefore\ \frac{dy}{dx} = e^{\sqrt{\sin\left(5\ln\left(x^2+7\right)\right)}} \cdot \frac{d}{dx}\sqrt{\sin\left(5\ln\left(x^2+7\right)\right)}$$

$$= e^{\sqrt{\sin\left(5\ln(x^2+7)\right)}} \cdot \frac{1}{2\sqrt{\sin\left(5\ln\left(x^2+7\right)\right)}} \cdot \frac{d}{dx}\sin\left(5\ln(x^2+7)\right)$$

$$= \frac{e^{\sqrt{\sin\left(5\ln(x^2+7)\right)}}}{2\sqrt{\sin\left(5\ln(x^2+7)\right)}}\cos\left(5\ln(x^2+7)\right) \cdot \frac{5}{(x^2+7)} \cdot \frac{d}{dx}(x^2+7)$$

$$= \frac{5x\cos\left(5\ln(x^2+7)\right)\ e^{\sqrt{\sin\left(5\ln(x^2+7)\right)}}}{(x^2+7)\sqrt{\sin\left(5\ln(x^2+7)\right)}}.$$

(iv) $y = \sec x\sqrt{\tan x}$

$$\frac{dy}{dx} = \sec x\frac{d}{dx}\sqrt{\tan x} + \sqrt{\tan x}\frac{d}{dx}(\sec x)$$

$$= \sec x\frac{1}{2\sqrt{\tan x}}\sec^2 x + \sqrt{\tan x} \cdot \sec x \cdot \tan x$$

$$= \left(\frac{\sec^3 x + 2\sec x\tan^2 x}{2\sqrt{\tan x}}\right)$$

(v) $y = \exp\left(\cos^3\left(\tan x^3\right)^2\right)$

$$\frac{dy}{dx} = \exp\left(\cos^3\left(\tan x^3\right)^2\right) \cdot \frac{d}{dx}\left(\cos^3\left(\tan x^3\right)^2\right)$$

$$= y \cdot 3\cos^2(\tan x^3)^2 \cdot \frac{d}{dx}\cos(\tan x^3)^2$$

$$= 3y\cos^2(\tan x^3)^2\,(-\sin(\tan x^3)^2) \cdot \frac{d}{dx}\left(\tan x^3\right)^2$$

$$= -3y\cos^2(\tan x^3)^2\sin(\tan x^3)^2\,(2\tan(x^3) \cdot \sec^2 x^3) \cdot 3x^2$$

$$= -18y\,x^2\cos^2(\tan x^3)^2 \cdot \sin(\tan x^3)^2\tan(x^3)\sec^2 x^3.$$

Illustration :

If $f(x) = (1 + x)(3 + x^2)^{1/2}(9 + x^3)^{1/3}$ *then* $f'(-1)$ *is equal to*

(A) 0 (B) $2\sqrt{2}$ (C) 4 (D) 6

Sol. $f(x) = (1 + x)(3 + x^2)^{1/2}(9 + x^3)^{1/3}$

$$f'(x) = (3 + x^2)^{1/2}(9 + x^3)^{1/3} + (1 + x)\frac{1}{2}(3 + x^2)^{-1/2} \cdot 2x\,(9 + x^3)^{1/3}$$

$$+ (1 + x)(3 + x^2)^{1/2}\left(\frac{1}{3}(9 + x^3)^{-2/3} \cdot 3x^2\right)$$

$f'(-1) = (3 + 1)^{1/2}(9 - 1)^{1/3} + 0 + 0$

$f'(-1) = 2 \cdot 2 = 4$

Practice Problem

Q.1 Differentiate the following functions with respect to x using first principle

(i) $\sqrt{\sin x}$ (ii) $\cos^3 x$ (iii) $\tan^{-1} x$ (iv) $\log_e x$

Q.2 Find the derivative of the following functions with respect to x

(i) $\sin(x^2)$ (ii) $\sin\left(\cos(x^2)\right)$ (iii) $\sin(x^2 + 5)$ (iv) $\cos(\sin x)$

(v) $\sin(ax + b)$ (vi) $\sec\left(\tan(\sqrt{x})\right)$ (vii) $\dfrac{\sin(ax + b)}{\cos(cx + d)}$ (viii) $\cos x^3 \cdot \sin^2(x^5)$

(ix) $2\sqrt{\cos(x^2)}$ (x) $\cos\left(\sqrt{x}\right)$

Q.3 Find the derivative of the following functions with respect to x

(i) e^{-x} (ii) $\sin(\log x), x > 0$ (iii) $e^{\cos x}$

(iv) $\dfrac{e^x}{\sin x}$ (v) $\sin(\tan^{-1} e^{-x})$ (vi) $\log(\cos e^x)$

(vii) $e^x + e^{x^2} + \ldots. + e^{x^5}$ (viii) $\sqrt{e^{\sqrt{x}}}, x > 0$ (ix) $\log(\log x), x > 1$

(x) $\dfrac{\cos x}{\log x}, x > 0$ (xi) $\cos(\log x + e^x), x > 0$

Q.4 Find the derivative of the following functions with respect to x

(i) $x^2 + 3x + 2$ (ii) x^{20} (iii) $x \cdot \cos x$ (iv) $\log x$

(v) $x^3 \log x$ (vi) $e^x \sin 5x$ (vii) $e^{6x} \cos 3x$ (viii) $\tan^{-1}x$

(ix) $\log(\log x)$ (x) $\sin(\log x)$ (xi) $\sqrt{3x + 2} + \dfrac{1}{\sqrt{2x^2 + 4}}$ (xii) $\log_7(\log x)$

Answer key

Q.1 (i) $\frac{\cos x}{2\sqrt{\sin x}}$ (ii) $-3\cos^2 x \sin x$ (iii) $\frac{1}{1+x^2}$ (iv) $\frac{1}{x}$

Q.2 (i) $2x\cos x^2$ (ii) $-\cos(\cos x^2)\sin(x^2)\cdot 2x$ (iii) $2x\cos(x^2+5)$

(iv) $-\sin(\sin x)\cos x$ (v) $a\cos(ax+b)$ (vi) $\frac{\sec(\tan\sqrt{x})\tan(\tan\sqrt{x})\sec^2\sqrt{x}}{2\sqrt{x}}$

(vii) $a\cos(ax+b)\sec(cx+d)+c\sin(ax+b)\tan(cx+d)\sec(cx+d)$

(viii) $-3x^2\sin x^3\sin^2 x^5$ (ix) $\frac{-2\sqrt{2}x}{\sin x^2\sqrt{\sin 2x^2}}$ (x) $\frac{-\sin\sqrt{x}}{2\sqrt{x}}$

Q.3 (i) $-e^{-x}$ (ii) $\frac{\cos(\log x)}{x}$ (iii) $-e^{\cos x}\sin x$ (iv) $\frac{e^x(\sin x-\cos x)}{\sin^2 x}$

(v) $\frac{-e^{-x}\cos(\tan^{-1}e^{-x})}{1+e^{-2x}}$ (vi) $-e^x\tan e^x$, $e^x \neq (2n+1)\frac{\pi}{2}$, $n \in N$

(vii) $e^x+2xe^{x^2}+\ldots\ldots+5x^4e^{x^5}$ (viii) $\frac{e^{\sqrt{x}}}{4\sqrt{xe^{\sqrt{x}}}}$, $x>0$ (ix) $\frac{1}{x\log x}$

(x) $\frac{(x\sin x\log x+\cot x)}{x(\log x)^2}$ (xi) $-\left(\frac{1}{x}+e^x\right)\sin(\log x+e^x)$, $x>0$

Q.4 (i) $2x+3$ (ii) $20x^{19}$ (iii) $\cos x - x\sin$ (iv) $\frac{1}{x}$ (v) $x^2+3x^2\log x$

(vi) $e^x(\sin 5x+5\cos 5x)$ (vii) $e^{6x}(6\cos 3x-3\sin 3x)$ (viii) $\frac{1}{x^2+1}$

(ix) $\frac{1}{x\log x}$ (x) $\frac{\cos(\log x)}{x}$ (ix) $\frac{3}{2\sqrt{(3x+2)}}-\frac{2x}{(2x^2+4)^{3/2}}$ (xii) $\frac{\log_7 e}{x\log x}$

T-6 LOGARITHMIC DIFFERENTIATION :

To find the derivative of :

(i) a function which is the product or quotient of a number of functions

$$y=f_1(x)\,f_2(x)\,f_3(x)\ldots\ldots\quad \text{or} \quad y=\frac{f_1(x)\,f_2(x)\,f_3(x)\ldots\ldots}{g_1(x)\,g_2(x)\,g_3(x)\ldots\ldots}.$$

(ii) a function of the form $[f(x)]^{g(x)}$ where f & g are both derivable, it will be found convenient to take the logarithm of the function first & then differentiate

OR

$y=(f(x))^{g(x)}=e^{g(x).\ln(f(x))}$ and then differentiate.

Illustration :

If $y = \sin x \cdot \sin 2x \cdot \sin 3x ....\sin nx$, find y'.

Sol. $y = \sin x \sin 2x \sin 3x ..... \sin n x$

$\ln y = \ln (\sin x) + \ln (\sin 2x) + \ln (\sin 3x) + ... + \ln (\sin x)$

$$\frac{1}{y}\frac{dy}{dx} = \frac{\cos x}{\sin x} + \frac{2\cos 2x}{\sin 2x} + \frac{3\cos 3x}{\sin 3x} + ... + \frac{n\cos n x}{\sin nx}$$

$y' = y (\cot x + 2 \cot 2x + 3 \cot 3x + ... + n \cot n x)$.

Illustration :

If $f(x) = (x + 1)(x + 2)(x + 3)........(x + n)$ then $f'(0)$ is

(A) $n!$ (B) $\frac{n(n+1)}{2}$ (C) $(n!)(\ln n!)$ (D) $n!\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+....+\frac{1}{n}\right)$

Sol. $f(x) = (x + 1) (x + 2) (x + 3) .... (x + n)$

$\ln f(x) = \ln (x + 1) + \ln (x + 2) + \ln (x + 3) + ... + \ln (x + n)$

$$\frac{f'(x)}{f(x)} = \frac{1}{(x+1)} + \frac{1}{(x+2)} + \frac{1}{(x+3)} + ... + \frac{1}{(x+n)}$$

$f(0) = n!$

$$f'(0) = n!\left(1+\frac{1}{2}+\frac{1}{3}+........+\frac{1}{n}\right).$$

Illustration :

If $f(x) = \prod_{n=1}^{100}(x-n)^{n(101-n)}$ then find $\frac{f(101)}{f'(101)}$.

Sol. $$f(x) = \prod_{n=1}^{100}(x-n)^{n(101-n)}$$

$f(x) = (x - 1)^{1\,(101-1)} (x - 2)^{2(101-2)} (x - 3)^{3(101-3)} ..... (x - 100)^{(100)\,(1)}$.

$\ln f(x) = 100 \ln (x - 1) + 2(101 - 2) \ln (x - 2) + 3(101 - 3) \ln (x - 3) + ... + 100 \ln (x - 100)$

$$\frac{f'(x)}{f(x)} = \frac{100}{(x-1)} + \frac{2(101-2)}{(x-2)} + \frac{3(101-3)}{(x-3)} + ... + \frac{100}{(x-100)}$$

$$\left.\frac{f'(x)}{f(x)}\right|_{x=101} = 1+2+3+...+100 = (100)\frac{(100+1)}{2} = (50)\,(101) = 5050$$

So $$\frac{f(101}{f'(101)} = \frac{1}{5050}.$$

Illustration :

Find the derivative of the following functions

(i) $y = (\sin x)\left(e^{\sqrt{\sin x}}\right)(\ln x)(x^x)$; (ii) $y = (x^{\ln x})(\sec x)^{3x}$; (iii) $y = \dfrac{(\ln x)^x}{2^{3x+1}}$

Sol.

(i) $y = (\sin x)\left(e^{\sqrt{\sin x}}\right)(\ln x)\left(x^x\right)$ *take logarithm on both sides*

$\ln y = \ln(\sin x) + \sqrt{\sin x} + \ln(\ln x) + x \ln x$

$$\frac{1}{y}\frac{dy}{dx} = \frac{\cos x}{\sin x} + \frac{(\cos x)}{2\sqrt{\sin x}} + \frac{1}{(\ln x)}\cdot\frac{1}{x} + \ln x + 1$$

$$\frac{dy}{dx} = y\left(\cot x + \frac{1}{2}\frac{\cos x}{\sqrt{\sin x}} + \frac{1}{x\ln x} + \ln x + 1\right)$$

(ii) $y = x^{\ln x}(\sec x)^{3x}$

$\ln y = (\ln x)^2 + 3x \ln(\sec x)$

$$\frac{1}{y}\frac{dy}{dx} = \frac{2\ln x}{x} + 3\ln(\sec x) + \frac{3x}{(\sec x)}(\sec x \tan x)$$

$$\frac{dy}{dx} = y\left(\frac{2\ln x}{x} + 3\ln(\sec x) + 3x\tan x\right).$$

(iii) $y = \dfrac{(\ln x)^x}{2^{3x+1}}$

$\ln y = x\ln(\ln x) - (3x+1)\ln 2$

$$\frac{1}{y}\frac{dy}{dx} = \ln(\ln x) + \frac{x}{(\ln x)}\cdot\frac{1}{x} - 3\ln 2$$

$$\frac{dy}{dx} = y\left(\ln(\ln x) + \frac{1}{(\ln x)} - 3\ln 2\right).$$

Illustration :

If $y = (\sin x)^{\ln x}\,\text{cosec}\,(e^x(a+bx))$ *and* $a + b = \dfrac{\pi}{2e}$ *then the value of* $\dfrac{dy}{dx}$ *at* $x = 1$ *is*

(A) $(\sin 1)\ln\sin(1)$ (B) 0 (C) $\ln\sin(1)$ (D) $1 + \ln(\sin 1)$

Sol. $y = (\sin x)^{\ln x}\,\text{cosec}\,(e^x(a+bx))$

$\ln y = (\ln x)\ln(\sin x) - \ln\left(\sin\left(e^x(a+bx)\right)\right)$

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{x}\ln(\sin x) + (\ln x)\cot x - \frac{\cos\left(e^x(a+bx)\right)\left(ae^x + bxe^x + be^x\right)}{\sin\left(e^x(a+bx)\right)}$$

$$\left.\frac{1}{y}\frac{dy}{dx}\right|_{at\ x=1} = ln\,(sin\,1) + 0\,\frac{cos\left(\frac{\pi}{2}\right)(a+2b)e}{sin\left(\frac{\pi}{2}\right)} = ln\,(sin\,1) + 0 - 0$$

$\because\ a + b = \dfrac{\pi}{2e} \qquad \therefore\ y = 1\ \ at\ x = 1$

so $\quad \left.\dfrac{dy}{dx}\right|_{at\ x=1} = ln\,(sin\,1)\,y\,\Big|_{at\ x=1} = ln\,(sin\,1)\cdot 1 = ln\,(sin\,1)$

Illustration :

If $y = 2^{\log_2 x^{2x}} + \left(tan\dfrac{\pi\,x}{4}\right)^{\frac{4}{\pi\,x}}$ then $\left.\dfrac{dy}{dx}\right]_{x=1}$ is :

(A) 4 (B) 5/2 (C) 3 (D) not defined

Sol. $y = 2^{\log_2 x^{2x}} + \left(tan\dfrac{\pi x}{4}\right)^{\frac{4}{\pi x}}$

$$y = x^{2x} + \left(tan\frac{\pi x}{4}\right)^{\frac{4}{\pi x}}$$

Let $\quad u = x^{2x}$ so $ln\,u = 2x\,ln\,x$

$$\frac{1}{u}\frac{du}{dx} = 2\,(ln\,x + 1) \Rightarrow \left.\frac{du}{dx}\right|_{x=1} = (1)(2) = 2$$

Let $\quad v = \left(tan\dfrac{\pi x}{4}\right)^{\frac{4}{\pi x}}$

$$ln\,v = \frac{4}{\pi x}\,ln\left(tan\left(\frac{\pi x}{4}\right)\right)$$

$$\frac{1}{v}\frac{dv}{dx} = -\frac{4}{\pi x^2}\,ln\left(tan\left(\frac{\pi x}{4}\right)\right) + \frac{1}{x}\,\frac{sec^2\left(\frac{\pi}{4}x\right)}{tan\left(\frac{\pi}{4}x\right)}$$

$$\left.\frac{dv}{dx}\right|_{at\ x=1} = 0 + 2$$

So $\quad \dfrac{dy}{dx} = \dfrac{du}{dx} + \dfrac{dv}{dx} = 2 + 2 = 4$.

Illustration :

Find $\frac{dy}{dx}$ for (i) $y = (\sin x)^{\log x}$ (ii) $y = x^{\tan x} + (\sin x)^{\cos x}$

Sol.

(i) Given $y = (\sin x)^{\log x}$.

Then, $y = e^{\log x \log \sin x}$

Diff. both sides w.r.t. x, we get

$$\frac{dy}{dx} = e^{\log x \log \sin x} \frac{d}{dx} \{\log x \log \sin x\}$$

$$\Rightarrow \frac{dy}{dx} = (\sin x)^{\log x} \left\{\log \sin x \frac{d}{dx}(\log x) + \log x \frac{d}{dx}(\log \sin x)\right\}$$

$$\Rightarrow \frac{dy}{dx} = (\sin x)^{\log x}\left\{\frac{\log \sin x}{x} + \log x \frac{1}{\sin x} \cos x\right\}$$

$$\Rightarrow \frac{dy}{dx} = (\sin x)^{\log x}\left\{\frac{\log \sin x}{x} + \cot x \log x\right\}.$$

(ii) $y = x^{\tan x} + (\sin x)^{\cos x}$

Let $u = x^{\tan x}$, $v = (\sin x)^{\cos x}$

$\therefore$ $\ln u = \tan x \ln x$

$$\therefore \quad \frac{d}{dx}(\ln u) = \frac{d}{dx}(\tan x \cdot \ln x)$$

$$\frac{1}{u} \cdot \frac{du}{dx} = \tan x\left(\frac{1}{x}\right) + \sec^2 x \cdot \ln x$$

$$\frac{du}{dx} = x^{\tan x}\left(\frac{\tan x}{x} + \sec^2 x \cdot \ln x\right)$$

$v = (\sin x)^{\cos x}$

$\ln v = \cos x \cdot \ln \sin x$

$$\Rightarrow \quad \frac{d}{dx} \ln v = \frac{d}{dx}(\cos x \ln \sin x)$$

$$\frac{1}{v}\frac{dv}{dx} = \left(\cos x \cdot \frac{\cos x}{\sin x}\right) + \ln \sin x \cdot (-\sin x)$$

$$\frac{dv}{dx} = (\sin x)^{\cos x}\left(\frac{\cos^2 x}{\sin x} - \sin x \ln(\sin x)\right) \qquad ....(ii)$$

$y = u + v$

$$\therefore \quad \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}.$$

From (i) and (ii)

$$\Rightarrow \quad \frac{dy}{dx} = x^{\tan x}\left(\frac{\tan x}{x} + \sec^2 x \cdot \ln x\right) + (\sin x)^{\cos x}\left(\frac{\cos^2 x}{\sin x} - \sin x \cdot \ln(\sin x)\right).$$

Illustration :

Find the derivative of $\dfrac{\sqrt{x}\,(x+4)^{3/2}}{(4x-3)^{4/3}}$ *w.r.t. x,*

Sol. *Let* $y = \dfrac{\sqrt{x}\,(x+4)^{3/2}}{(4x-3)^{4/3}}$

Taking log of both sides, we get

$$\log y = \frac{1}{2}\log x + \frac{3}{2}\log(x+4) - \frac{4}{3}\log(4x-3)$$

Diff. both sides w.r.t. x, we get

$$\frac{1}{y}\cdot\frac{dy}{dx} = \frac{1}{2x} + \frac{3}{2x+4}\times 1 - \frac{4}{3}\times\frac{1}{4x-3}\times 4$$

$$\Rightarrow \quad \frac{dy}{dx} = y\left\{\frac{1}{2x} + \frac{3}{2(x+4)} - \frac{16}{3(4x-3)}\right\}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{\sqrt{x}\,(x+4)^{3/2}}{(4x-3)^{4/3}}\left\{\frac{1}{2x} + \frac{3}{2(x+4)} - \frac{16}{3(4x-3)}\right\}$$

Illustration :

Find $\dfrac{dy}{dx}$ *for* $y = x^x$.

Sol. $y = x^x$

$$\ln y = x\ln x \qquad \Rightarrow \quad \frac{d}{dx}(\ln y) = \frac{d}{dx}(x\ln x)$$

$$\Rightarrow \quad \frac{1}{y}\frac{dy}{dx} = x\cdot\left(\frac{1}{x}\right) + \ln x = 1 + \ln x \qquad \Rightarrow \quad \frac{dy}{dx} = x^x(1+\ln x).$$

Illustration :

If $y^x = x^y$, *then find* $\dfrac{dy}{dx}$.

Sol. $y^x = x^y$

$x\ln y = y\ln x$

Differentiating w.r.t. x.

$$\frac{d}{dx}(x\ln y) = \frac{d}{dx}(y\ln x)$$

$$x\cdot\left(\frac{1}{y}\cdot\frac{dy}{dx}\right) + \ln y = y\cdot\frac{1}{x} + \ln x\cdot\frac{dy}{dx} \qquad \Rightarrow \quad \frac{dy}{dx}\left(\frac{x}{y} - \ln x\right) = \frac{y}{x} - \ln y$$

$$\frac{dy}{dx} = \frac{\dfrac{x}{y} - \ln y}{\dfrac{x}{y} - \ln x} = \frac{y(y - x\ln y)}{x(x - y\ln x)}$$

T-7 IMPLICIT FUNCTIONS :

If the variable x and y are connected by a relation of the form $f(x, y) = 0$ and it is not possible to express y as a function of x in the form $y = \phi(x)$, then such functions are said to be implicit functions.
For example,

(i) $x^2 + xy + y^3 = 1$ (ii) $x + y + \sin(xy) = 1$ (iii) $x^y + y^x = 1$

(iv) $16^{x^2+y} + 16^{x+y^2} = 1$

DERIVATIVE OF IMPLICIT FUNCTION :

(i) In order to find dy/dx, in the case of implicit functions, we differentiate each term w.r.t. x regarding y as a functions of x & then collect terms in dy/dx together on one side to finally find dy/dx.

(ii) Corresponding to every curve represented by an implicit equation, there exist one or more explicit functions representing that equation. It can be shown that dy/dx at any point on the curve remains the same whether the process of differentiation is done explictly or implicitly.

$$\frac{d}{dx}f(y) = \frac{d}{dy}f(y)\cdot\frac{dy}{dx} = f'(y)\cdot\frac{dy}{dx}$$

For example :

(i) $\frac{d}{dx}(\sin y) = \frac{d}{dy}(\sin y)\cdot\frac{dy}{dx} = \cos y\left(\frac{dy}{dx}\right)$ (ii) $\frac{d}{dx}(y^3) = \frac{d}{dy}(y^3)\cdot\frac{dy}{dx} = 3y^2\frac{dy}{dx}$

A DIRECT FORMULA FOR IMPLICIT FUNCTIONS :

Let $f(x, y) = 0$. Take all the terms of left side and put left side equal to $f(x, y)$.

Then $\frac{dy}{dx} = -\frac{\text{diff. of } f \text{ w.r.t. x keeping y as constant}}{\text{diff. of } f \text{ w.r.t. y keeping x as constant}}$

Illustration :

If $x^2 + 2xy + y^3 = 4$, find $\frac{dy}{dx}$.

Sol. *We have $x^2 + 2xy + y^3 = 4$*
Diff. both sides w.r.t. x,

$$\Rightarrow \frac{d}{dx}(x^2) + 2\frac{d}{dx}(xy) + \frac{d}{dx}(y^3) = \frac{d}{dx}(4)$$

$$\Rightarrow 2x + 2\left(x\frac{dy}{dx} + y.1\right) + 3y^2\frac{dy}{dx} = 0 \qquad \Rightarrow \frac{dy}{dx} = -\frac{2(x+y)}{(2x+3y^2)}$$

Alternative Method :

$$\frac{dy}{dx} = -\frac{\text{diff. of } f \text{ w.r.t. x keeping y as constant}}{\text{diff. of } f \text{ w.r.t. y keeping x as constant}} = -\frac{2x+2y}{2x+3y^2}$$

Illustration :

If $y = x + \cfrac{1}{x + \cfrac{1}{x + \cfrac{1}{x + \ldots}}}$, *prove that* $\dfrac{dy}{dx} = \dfrac{y}{2y - x}$.

Sol. *We have,* $y = x + \cfrac{1}{x + \cfrac{1}{x + \cfrac{1}{x + \ldots}}}$,

$$\Rightarrow \quad y = x + \frac{1}{y} \qquad \Rightarrow \quad y^2 = xy + 1$$

$$\Rightarrow \quad 2y\frac{dy}{dx} = y + x\frac{dy}{dx} + 0 \qquad \text{[Diff. both sides with respect to } x\text{]}$$

$$\Rightarrow \quad \frac{dy}{dx}(2y - x) = y \qquad \Rightarrow \quad \frac{dy}{dx} = \frac{y}{2y - x}.$$

Illustration :

If $\sqrt{x} + \sqrt{y} = 4$, *then find* $\dfrac{dx}{dy}$ *at* $y = 1$.

Sol. *Diff. both sides of the given equation w.r.t. y, we get*

$$\frac{1}{2\sqrt{x}}\frac{dx}{dy} + \frac{1}{2\sqrt{y}} = 0 \quad \Rightarrow \quad \frac{dx}{dy} = -\frac{\sqrt{x}}{\sqrt{y}} = \frac{\sqrt{y} - 4}{\sqrt{y}} \quad \Rightarrow \left[\frac{dx}{dy}\right]_{y=1} = \frac{1-4}{1} = -3.$$

Illustration :

If $x^y = e^{x-y}$ *then prove that* $\dfrac{dy}{dx} = \dfrac{\ln x}{(1 + \ln x)^2}$.

Sol. $x^y = e^{x-y}$

$y \ln x = (x - y)$ ....(i)

now differentiating with respect to x.

$$\frac{y}{x} + \ln x \cdot \frac{dy}{dx} = 1 - \frac{dy}{dx}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1 - \frac{y}{x}}{1 + \ln x} = \frac{x - y}{x(1 + \sin x)} = \frac{y \ln x}{x(1 + \ln x)} \qquad \left(\text{From (i)}, \ y = \frac{x}{1 + \ln x}\right)$$

$$\therefore \quad \frac{dy}{dx} = \frac{\ln x}{(1 + \ln x)^2}.$$

Illustration :

If $\sin y = x\sin(a+y)$ *then prove that* $\dfrac{dy}{dx} = \dfrac{\sin^2(a+y)}{\sin a}$.

Sol. $\sin y = x\sin(a+y)$ *differentiating both sides w.r.t.* x

$$\cos y\cdot\frac{dy}{dx} = \sin(a+y) + x\cos(a+y)\cdot\left(0+\frac{dy}{dx}\right).$$

$$\Rightarrow \quad \frac{dy}{dx}\left(\cos y - x\cos(a+y)\right) = \sin(a+y)$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{\sin(a+y)}{\cos y - x\cos(a+y)} = \frac{\sin(a+y)}{\cos y - \dfrac{\sin y}{\sin(a+y)}\cos(a+y)}$$

$$= \frac{\sin^2(a+y)}{\sin(a+y)\cos y - \sin y\cos(a+y)} = \frac{\sin^2(a+y)}{\sin a}$$

Illustration :

If $x\sqrt{1+y} + y\sqrt{1+x} = 0$, *then prove that* $\dfrac{dy}{dx} = -\dfrac{1}{(1+x)^2}$

Sol. $x\sqrt{1+y} + y\sqrt{1+x} = 0$

$\Rightarrow \quad x^2(1+y) = y^2(1+x) \quad \Rightarrow \quad (x^2-y^2) + x^2y - y^2x = 0$

$\Rightarrow \quad (x-y)(x+y+xy) = 0 \quad \Rightarrow \quad x - y = 0$ *or* $x + y + xy = 0$

$\therefore \quad y = x$ *or* $\dfrac{-x}{x+1}$. $\quad \Rightarrow \quad \dfrac{dy}{dx} = 1$ *or* $\dfrac{-1}{(x+1)^2}$

Illustration :

If $y = \sqrt{\sin x + \sqrt{\sin x + \sqrt{\sin x + \ldots\ldots\infty}}}$ *find* $\dfrac{dy}{dx}$ $(\sin x > 0)$.

Sol. $y = \sqrt{\sin x + \sqrt{\sin x + \sqrt{\sin x + \ldots\ldots\infty}}} \quad \Rightarrow \quad y = \sqrt{\sin x + y}$

$$\Rightarrow \quad y^2 - y = \sin x \Rightarrow 2y\frac{dy}{dx} - \frac{dy}{dx} = \cos x \Rightarrow \quad \frac{dy}{dx} = \left(\frac{\cos x}{2y-1}\right).$$

Illustration :

If $ax^2 + 2hxy + by^2 = 0$ *then prove that* $\dfrac{dy}{dx} = -\dfrac{ax+hy}{hx+by} = \dfrac{y}{x}$

Sol. $ax^2 + 2hxy + by^2 = 0 \quad \ldots\ldots\ldots(1)$

differentiating w.r.t. x.

$$2ax + 2h\left(x\frac{dy}{dx} + y\right) + 2by\frac{dy}{dx} = 0 \quad \Rightarrow \quad (hx+by)\frac{dy}{dx} = -(ax+hy)$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{(ax+hy)}{(hx+by)} \quad ....(2)$$

From (1) $ax^2 + 2hxy + by^2 = 0$

$$\Rightarrow \quad x(ax + hy) + y\,(hx + by) = 0$$

$$\Rightarrow \quad \frac{ax+hy}{hx+by} = \frac{-y}{x}$$

From (ii) $\dfrac{dy}{dx} = \dfrac{-(ax+hy)}{(hx+by)} = \dfrac{y}{x}$.

Illustration :

A curve is discribed by the relation $ln(x + y) = xe^y$. Find the tangent to the curve at (0, 1).

Sol. $ln\,(x + y) = xe^y$

differentiating w.r.t. to x

$$\frac{1}{(x+y)}\left(1+\frac{dy}{dx}\right) = xe^y\,\frac{dy}{dx} + e^y \text{ at point } (0, 1),$$

$$\frac{1}{(0+1)}\left(1+\frac{dy}{dx}\right) = 0 + e^1$$

$$\Rightarrow \quad \left.\frac{dy}{dx}\right|_{x=0,\,y=1} = e - 1$$

Equation of tangent $y - 1 = (e - 1)(x - 0) \Rightarrow (1 - e)\,x + y = 1$.

Illustration :

(i) $y = x^{x^{x^{\cdot^{\cdot^{\infty}}}}}$ (ii) $y = (ln\,x)^{(ln\,x)^{(ln\,x)^{\cdot^{\cdot^{\infty}}}}}$

Sol.

(i) $y = x^{x^{x\ldots\infty}}$

$$\Rightarrow \quad y = x^y \quad \Rightarrow \quad ln\,y = y\,ln\,x \quad \Rightarrow \quad \frac{1}{y}\cdot\frac{dy}{dx} = \frac{y}{x} + ln\,x\,\frac{dy}{dx}$$

$$\Rightarrow \quad \frac{dy}{dx}\left(\frac{1}{y} - ln\,x\right) = \frac{y}{x} \qquad \Rightarrow \quad \frac{dy}{dx} = \frac{\frac{y}{x}}{\frac{(1-y\,ln\,x)}{y}} = \frac{y^2}{x(1-y\,ln\,x)}$$

(ii) $y = (ln\,x)^y$

$$\Rightarrow \quad ln\,y = y\,ln(ln\,x) \qquad \Rightarrow \quad \frac{1}{y}\cdot\frac{dy}{dx} = \frac{y}{x\,ln\,x} + ln\,ln\,x\,\frac{dy}{dx}$$

$$\Rightarrow \quad \frac{dy}{dx}\left(\frac{1}{y} - ln\,ln\,x\right) = \frac{y}{x\,ln\,x} \qquad \Rightarrow \quad \frac{dy}{dx} = \frac{\frac{y}{x\,ln\,x}}{\frac{(1-y\,ln\,ln\,x)}{y}} = \frac{y^2}{x\,ln\,x\,(1-y\,ln\,ln\,x)}.$$

Illustration :

If $y^5 + xy^2 + x^3 = 4x + 3$, *then find* $\frac{dy}{dx}$ *at (2, 1)*

Sol. $y^5 + xy^2 + x^3 = 4x + 3$

differentiating w.r.t. x

$$5y^4\frac{dy}{dx} + \left(x\,2y\frac{dy}{dx} + y^2\right) + 3x^2 = 4 \qquad \Rightarrow \qquad \frac{dy}{dx} = \frac{4-3x^2-y^2}{(5y^4+2xy)}$$

$$\left.\frac{dy}{dx}\right|_{x=2,\,y=1} = \frac{4-3(2)^2-(1)^2}{5(1)^4+2(2)(1)} = \frac{4-12-1}{5+4} = \frac{-9}{9} = -1.$$

Illustration :

If $y = \sqrt{x\log_e x}$, *then find* $\frac{dy}{dx}$ *at* $x = e$.

Sol. $$\frac{dy}{dx} = \frac{1}{2\sqrt{x\log_e x}}\,\frac{d}{dx}[x\log_e x] = \frac{1}{2\sqrt{x\log_e x}}\left[x\cdot\frac{1}{x} + 1\cdot\log_e x\right]$$

$$\Rightarrow \quad \left[\frac{dy}{dx}\right]_{x=e} = \frac{1}{2\sqrt{e\cdot 1}}(1+1) = \frac{1}{\sqrt{e}} \qquad (\because \log_e e = 1)$$

Practice Problem

Q.1 Find the derivative of the following functions with respect to x :

(i) $\sqrt{\frac{(x-3)(x^2+4)}{3x^2+4x+5}}$ (ii) $x^{\sin x}, x > 0$ (iii) $y^x + x^y + x^x = a^b$

(iv) $\cos x \cdot \cos 2x \cdot \cos 3x$ (v) $\sqrt{\frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}}$ (vi) $(\log x)^{\cos x}$

(vii) $x^x - 2^{\sin x}$ (viii) $(x+3)^2 \cdot (x+4)^3 \cdot (x+5)^4$ (ix) $\left(x+\frac{1}{x}\right)^x + x^{\left(1+\frac{1}{x}\right)}$

(x) $(\log x)^x + x^{\log x}$ (xi) $x^{x\cos x} + \frac{x^2+1}{x^2-1}$ (xii) $x^{\sin x} + (\sin x)^{\cos x}$

Q.2 Find $\frac{dy}{dx}$ of the following functions :

(i) $(x\cos x)^x + (x\sin x)^{\frac{1}{x}}$ (ii) $x^y + y^x = 1$

(iii) $y^x = x^y$ (iv) $(\cos x)^y = (\cos y)^x$

(v) $xy = e^{(x-y)}$ (vi) $y = (x^2 - 5x + 8)(x^3 + 7x + 9)$

(vii) $y = (1+x)(1+x^2)(1+x^4)(1+x^8)$

Q.3 Find $\frac{dy}{dx}$ of the following functions :

(i) $2x + 3y = \sin x$

(ii) $2x + 3y = \sin y$

(iii) $ax + by^2 = \cos y$

(iv) $xy + y^2 = \tan x + y$

(v) $x^2 + xy + y^2 = 100$

(vi) $x^3 + x^2y + xy^2 + y^3 = 81$

(vii) $\sin^2 y + \cos xy = k$

(viii) $x^2 + xy + \cos^2 y = 1$

(ix) $y + \sin y = \cos x$

Q.4 Find the derivative of the following functions with respect to x :

(i) $(\sin x)^{\sin x}, 0 < x < \pi.$

(ii) $(\sin x - \cos x)^{(\sin x - \cos x)}, \frac{\pi}{4} < x < \frac{3\pi}{4}$

(iii) $x^x + x^a + a^x + a^a$, for some fixed $a > 0$ and $x > 0$

(iv) $x^{x^2-3} + (x-3)^{x^2}$, for $x > 3$

(v) $\log\left\{e^x\left(\frac{x-2}{x+2}\right)^{\frac{3}{4}}\right\}$

Answer key

Q.1

(i) $\frac{1}{2}\sqrt{\frac{(x-3)(x^2+4)}{3x^2+4x+5}}\left[\frac{1}{x-3}+\frac{2x}{x^2+4}-\frac{6x+4}{3x^2+4x+5}\right]$

(ii) $x^{\sin x - 1}\sin x + x^{\sin x}\cdot\cos x\log x$

(iii) $\frac{-[y^x\log y + y\cdot x^{y-1} + x^x(1+\log x)]}{x\cdot y^{x-1} + x^y\log x}$

(iv) $-\cos x\,\cos 2x\,\cos 3x\,[\tan x + 2\tan 2x + 3\tan 3x]$

(v) $\frac{1}{2}\sqrt{\frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}}\left[\frac{1}{x-1}+\frac{1}{x-2}-\frac{1}{x-3}-\frac{1}{x-4}-\frac{1}{x-5}\right]$

(vi) $(\log x)^{\cos x}\left[\frac{\cos x}{x\log x} - \sin x\log\log x\right]$

(vii) $x^x(1+\log x) - 2^{\sin x}\cos x\log 2$

(viii) $(x+3)(x+4)^2(x+5)^3(9x^2+70x+133)$

(ix) $\left(x+\frac{1}{x}\right)^x\left[\frac{x^2-1}{x^2+1}+\log\left(x+\frac{1}{x}\right)\right] + x^{1+\frac{1}{x}}\left(\frac{x+1-\log x}{x^2}\right)$

(x) $(\log x)^{x-1}[1+\log x\cdot\log\log x] + 2x^{\log x - 1}\log x$

(xi) $x^{x\cos x}[\cos x(1+\log x) - x\sin x\log x] - \frac{4x}{(x^2-1)^2}$

(xii) $x^{\sin x}\left[\frac{\sin x}{x}+\cos x\log x\right] + (\sin x)^{\cos x}[\cos x\cot x - \sin x\log\sin x]$

Q.2 (i) $(x\cos x)^x[1 - x\tan x + \log(x\cos x)] + (x\sin x)^{1/x}\left[\frac{x\cot x + 1 - \log(x\sin x)}{x^2}\right]$

(ii) $-\left(\frac{yx^{y-1} + y^x\log y}{x^y\log x + xy^{x-1}}\right)$ (iii) $\frac{y}{x}\left(\frac{y - x\log y}{x - y\log x}\right)$ (iv) $\frac{y\tan x + \log\cos y}{x\tan y + \log\cos x}$

(v) $\frac{y(x-1)}{x(y+1)}$ (vi) $5x^4 - 20x^3 + 45x^2 - 52x + 11$

(vii) $(1+x)(1+x^2)(1+x^4)(1+x^8)\left[\frac{1}{1+x} + \frac{2x}{1+x^2} + \frac{4x^3}{1+x^4} + \frac{8x^7}{1+x^8}\right]$

Q.3 (i) $\frac{\cos x - 2}{3}$ (ii) $\frac{2}{\cos y - 3}$ (iii) $\frac{-9}{2by + \sin y}$ (iv) $\frac{\sec^2 x - y}{x + 2y - 1}$

(v) $-\frac{(2x+y)}{(x+2y)}$ (vi) $\frac{-(3x^2 + 2xy + y^2)}{(x^2 + 2xy + 3y^2)}$ (vii) $\frac{y\sin(xy)}{\sin 2y - x\sin xy}$ (viii) $\frac{2x+y}{\sin 2y + x}$

(ix) $\frac{-\sin x}{1 + \cos y}$

Q.4 (i) $(1 + \log\sin x)(\sin x)^{\sin x}\cos x$

(ii) $(\sin x - \cos x)^{\sin - \cos x}(\cos x + \sin x)[1 + \log(\sin x - \cos x)]$, $\sin x > \cos x$

(iii) $x^x(1 + \log x) + ax^{a-1} + a^x\log a$

(iv) $x^{x^2-3}\left[\frac{x^2-3}{x} + 2x\log x\right] + (x^2-3)^{x^2}\left[\frac{x^2}{x-3} + 2x\log(x-3)\right]$ (v) $\frac{x^2-1}{x^2-4}$

T-8 DERIVATIVE OF INVERSE FUNCTION :

Theorem :

Let the functions $f(x)$ and $g(x)$ be inverse of each other then

$$f(g(x)) = g(f(x)) = x$$

$$\therefore \quad f'(g(x))\,g'(x) = g'(f(x))\,f'(x) = 1$$

If $\frac{dy}{dx}$ exists and $\frac{dy}{dx} \neq 0$, then $\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}$ or $\frac{dy}{dx}.\frac{dx}{dy} = 1$ or $\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$ $\left[\frac{dx}{dy} \neq 0\right]$

Illustration :

(a) *If* $y = f(x) = x^3 + x^5$ *and g is the inverse of f find* $g'(2)$

(b) *Let* $f(x) = \exp(x^3 + x^2 + x)$ *for any real number x, and let g be the inverse function for f. The value of* $g'(e^3)$ *is*

(A) $\frac{1}{6e^3}$ (B) $\frac{1}{6}$ (C) $\frac{1}{34e^{39}}$ (D) 6

(c) *If g is the inverse of f and* $f'(x) = \frac{1}{1+x^n}$, *prove that* $g'(x) = 1 + (g(x))^n$

Sol.

(a) $y = f(x) = x^3 + 5$

Let $g(x)$ *be the inverse of* $f(x)$ *i.e.,* $g(x) = f^{-1}(x)$

$f(g(x)) = g(f(x)) = x$

differentiating w.r.t. x

$f'(g(x)) \cdot g'(x) = 1$

$$\Rightarrow \quad g'(x) = \frac{1}{f'(g(2))} \qquad \Rightarrow \quad g(2) = f^{-1}(2) = y \text{ (say)}$$

$$\Rightarrow \quad f(y) = 2 \quad \Rightarrow \quad y^3 + y^5 = 2 \Rightarrow y = 1$$

$$\Rightarrow \quad g'(2) = \frac{1}{f'(1)} = \frac{1}{(3x^2 + 5x^4)_{x=1}} = \frac{1}{8}.$$

(b) $f(g(x)) = x$

$$\Rightarrow \quad f'(g(x))g'(x) = 1 \quad \Rightarrow \quad g'(x) = \frac{1}{f'(g(x))} \quad \Rightarrow \quad g'(e^3) = \frac{1}{f'(g(e^3))}$$

$$\Rightarrow \quad e^{x^3+x^2+x} = e^3 \quad \Rightarrow \quad x = 1.$$

$$\Rightarrow \quad g'(e^3) = \frac{1}{f'(1)} = \frac{1}{\left[e^{x^3+x^2+x}(3x^2+2x+1)\right]_{x=1}} = \frac{1}{e^3(3+2+1)} = \frac{1}{6e^3}.$$

(c) $f(g(x)) = x$

$$\Rightarrow \quad f'(g(x))g'(x) = 1 \quad \Rightarrow \quad g'(x) = \frac{1}{f'(g(x))} = \frac{1}{\frac{1}{1+(g(x))^n}}$$

$$\Rightarrow \quad g'(x) = 1 + (g(x))^n.$$

DIFFERENTIATION OF INVERSE TRIGONOMETRIC FUNCTIONS :

(i) $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{(1-x^2)}}$ (ii) $\frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{(1-x^2)}}$

(iii) $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$ (iv) $\frac{d}{dx}(\cot^{-1} x) = \frac{-1}{1+x^2}$

(v) $\frac{d}{dx}(\sec^{-1} x) = \frac{1}{|x|\sqrt{x^2-1}}$ (vi) $\frac{d}{dx}(\text{cosec}^{-1} x) = \frac{-1}{|x|\sqrt{x^2-1}}$

Proof :

(i) Proof of derivative of $f(x) = \sin^{-1} x$:
Let $y = \sin^{-1} x$. Then, $x = \sin y$.
Differentiating both sides w.r.t. x, we get

$$1 = \cos y \frac{dy}{dx}$$

which implies that $\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\cos(\sin^{-1} x)}$

this is defined only for $\cos y \neq 0$, i.e., $\sin^{-1} x \neq -\frac{\pi}{2}, \frac{\pi}{2}$, i.e., $x \neq -1, 1$, i.e., $x \in (-1, 1)$

Recall that for $x \in (-1, 1)$, $\sin(\sin^{-1} x) = x$ and hence

$$\cos^2 y = 1 - (\sin y)^2 = 1 - (\sin(\sin^{-1} x))^2 = 1 - x^2$$

Also, since $y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, cos y is positive and hence $\cos y = \sqrt{1-x^2}$

Thus, for $x \in (-1, 1)$,

$$\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1-x^2}}$$

(ii) Proof of derivative of $f(x) = \cos^{-1} x$

$$\frac{d}{dx}(\cos^{-1} x) = \frac{d}{dx}\left(\frac{\pi}{2} - \sin^{-1} x\right) \qquad \left[\because \cos^{-1} x + \sin x = \frac{\pi}{2}\right]$$

$$= 0 - \frac{d}{dx}(\sin^{-1} x) \quad = -\frac{1}{\sqrt{(1-x^2)}}$$

(iii) Proof of derivative of $f(x) = \tan^{-1} x$;
Let $y = \tan^{-1} x$. Then, $x = \tan y$,
Differentiating both sides w.r.t. x, we get

$$1 - \sec^2 y \frac{dy}{dx}$$

which implies that

$$\frac{dy}{dx} = \frac{1}{\sec^2 y} = \frac{1}{1+\tan y} = \frac{1}{1+(\tan(\tan^{-1} x))^2} = \frac{1}{1+x^2}$$

(iv) Proof of derivative of $f(x) = \cot^{-1} x$.

$$\frac{d}{dx}(\cot^{-1} x) = \frac{d}{dx}\left(\frac{\pi}{2} - \tan^{-1} x\right) = 0 - \frac{d}{dx}(\tan^{-1} x)$$

$$= -\frac{1}{(1+x^2)}$$

Note :- $\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}$

(v) Proof of derivative of $f(x) = \sec^{-1} x$;

Let $y = \sec^{-1} x$; $\quad x = \sec y$

Differentiating both sides w.r.t. x, we get

$$1 = \sec y \tan y \frac{dy}{dx} \quad \Rightarrow \quad \frac{dy}{dx} = \frac{1}{\sec y \tan y} = \frac{1}{|x|\sqrt{(x^2-1)}}$$

(vi) Proof of derivative of $f(x) = \text{cosec}^{-1} x$.

Let $y = \text{cosec}^{-1} x$; $\quad x = \text{cosec}\, y$.

Differentiating both sides w.r.t. x, we get

$$1 = -\text{cosec}\, y \cot y \frac{dy}{dx} \quad \Rightarrow \quad \frac{dy}{dx} = -\frac{1}{\text{cosec}\, y \cot y} = -\frac{1}{x\sqrt{(x^2-1)}}$$

Finding of the derivativesz of other inverse trigonometric function is left exercise. The following table gives the derivatives of the remaining inverse trigonometric functions.

f(x)	$\sin^{-1} x$	$\cos^{-1} x$	$\tan^{-1} x$	$\cot^{-1} x$	$\sec^{-1} x$	$\text{cosec}^{-1} x$
f'(x)	$\frac{1}{\sqrt{(1-x^2)}}$	$\frac{-1}{\sqrt{1-x^2}}$	$\frac{1}{(1+x^2)}$	$\frac{1}{1+x^2}$	$\frac{1}{\lvert x\rvert\sqrt{x^2-1}}$	$\frac{-1}{\lvert x\rvert\sqrt{x^2-1}}$
Domain of f'	$(-1,1)$	$(-1,1)$	R	R	$(-\infty,-1)\cup(1,\infty)$	$(-\infty,-1)\cup(1,\infty)$

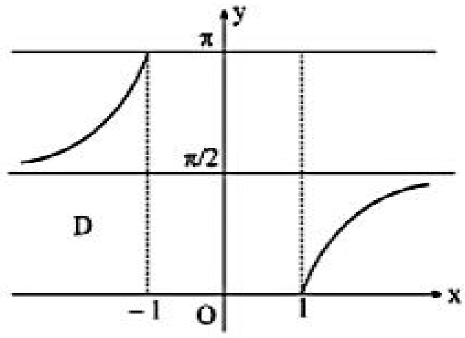
Note : Some Standard Substitutions

Expressions	**Substitution**
$\sqrt{a^2 - x^2}$	$x = a \sin\theta$ or $a\cos\theta$
$\sqrt{a^2 + x^2}$	$x = a\tan\theta$ or $a\cot\theta$
$\sqrt{x^2 - a^2}$	$x = a\sec\theta$ or $a\,\text{cosec}\,\theta$
$\sqrt{\frac{a+x}{a-x}}$ or $\sqrt{\frac{a-x}{a+x}}$	$x = a\cos\theta$ or $a\cos 2\theta$

Illustration :

Prove that derivative of $sec^{-1}x$ *is* $\dfrac{1}{|x|\sqrt{x^2-1}}$ *if* $|x| > 1$*, using first principle method.*

Sol. *Let,* $y = sec^{-1}x;\ \ |x| \ge 1\ \ ;\ y \in \left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$



$sec\ y = x\ \ ;\ \ sec\ (y + \Delta y) = x + \Delta x$

$\therefore \quad \Delta x = sec(y + \Delta y) - sec\ y$

$$\frac{\Delta x}{\Delta y} = \frac{sec(y+\Delta y) - sec\ y}{\Delta y}$$

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta y \to 0} \frac{1}{\dfrac{sec(y+\Delta y) - sec\ y}{\Delta y}} = \frac{1}{sec\ y\ tan\ y}$$

if $x > 1$ *then* $y \in \left(0, \frac{\pi}{2}\right)$ *(at* $x = 1$, $y = 0$ *and* $tan\ y = 0$ *therefore* $\frac{dy}{dx}$ *does not exist)*

$\Rightarrow$ *sign of tan y is + ve*

$$\therefore \quad \frac{dy}{dx} = \frac{1}{sec\ y\sqrt{sec^2 y - 1}} = \frac{1}{x\sqrt{x^2-1}},\ if\ x > 1 \qquad ....(1)$$

and if $x < -1$*, then* $y \in \left(\frac{\pi}{2}, \pi\right)$ *(at* $x = -1$, $y = \pi$ *and* $tan\ y = 0$ *and* $\frac{dy}{dx}$ *does not exist)*

hence sign of tan y is – ve

$$\therefore \quad \frac{dy}{dx} = -\frac{1}{sec\ y\sqrt{sec^2 y - 1}} = -\frac{1}{x\sqrt{x^2-1}},\ if\ x < -1 \qquad ....(2)$$

from (1) and (2)

$$D(sec^{-1}x) = \begin{cases} \dfrac{1}{|x|\sqrt{x^2-1}}, & if\ |x| > 1 \\ does\ not\ exist, & if\ |x| = 1 \end{cases}$$

Illustration :

Find $\frac{dy}{dx}$ *for* $y = tan^{-1}\left\{\frac{1 - cos\ x}{sin\ x}\right\}$*, where* $-\pi < x < \pi$.

Sol. $y = tan^{-1}\left\{\dfrac{1 - cos\ x}{sin\ x}\right\} = tan^{-1}\left\{\dfrac{2\sin\frac{x}{2}}{2\sin\frac{x}{2}\cos\frac{x}{2}}\right\}$

$$= tan^{-1}\left(tan\frac{x}{2}\right) = \frac{x}{2} \qquad \left(\because -\pi < x < \pi \Rightarrow -\frac{\pi}{2} < \frac{x}{2} < \frac{\pi}{2}\right)$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{2}.$$

Illustration :

$$D\left(\tan^{-1}x+\tan^{-1}\left(\frac{1}{x}\right)\right)=\_\_\_\_$$

Sol. $\tan^{-1}x + \tan^{-1}\left(\frac{1}{x}\right) = \begin{cases} \frac{\pi}{2}, & x>0 \\ \frac{-\pi}{2}, & x<0 \end{cases}$

$$D\left(\tan^{-1}+\tan^{-1}\left(\frac{1}{x}\right)\right)=0\ \forall x\in R-\{0\}$$

Illustration :

If $y=\dfrac{\tan^{-1}x-\cot^{-1}x}{\tan^{-1}x+\cot^{-1}x}$, find $\left.\dfrac{dy}{dx}\right]_{x=-1}$;

(A) 0 (B) 1 (C) $2/\pi$ (D) -1

Sol. $y=\dfrac{\tan^{-1}x-\cot^{-1}x}{\tan^{-1}x+\cot^{-1}x}=\dfrac{2}{\pi}\left(\tan^{-1}x-\cot^{-1}x\right)$

$$\therefore \quad \frac{dy}{dx}=\frac{2}{\pi}\left(\frac{1}{1+x}+\frac{1}{1+x^2}\right)=\frac{4}{\pi(1+x^2)}$$

$$\left.\frac{dy}{dx}\right|_{x=-1}\frac{4}{\pi(1+(-1)^2)}=\frac{2}{\pi}$$

Illustration :

If $y=\tan^{-1}\left(\dfrac{x}{1\cdot2+x^2}\right)+\tan^{-1}\left(\dfrac{x}{2\cdot3+x^2}\right)+\tan^{-1}\left(\dfrac{x}{3\cdot4+x^2}\right)+\ ....$ up to n times. Find $\dfrac{dy}{dx}$ expressing your answer in two terms and also find $\dfrac{dy}{dx}$ when $n\to\infty$.

Sol. $y=\tan^{-1}\left(\dfrac{x}{1\cdot2+x^2}\right)+\tan^{-1}\left(\dfrac{x}{2\cdot3+x^2}\right)+\tan^{-1}\left(\dfrac{x}{3\cdot4+x^2}\right)+......+\tan^{-1}\left(\dfrac{x}{n(n+1)+x^2}\right)$

$$T_r=\tan^{-1}\left(\frac{x}{r(r+1)+x^2}\right)\quad\Rightarrow\quad T_r=\tan^{-1}\left(\frac{\frac{x}{r(r+1)}}{1+\frac{x^2}{r(r+1)}}\right)$$

$$T_r=\tan^{-1}\left(\frac{\frac{x}{r}-\frac{x}{r+1}}{1+\frac{x}{r}\cdot\frac{x}{r+1}}\right)=\tan^{-1}\left(\frac{x}{r}\right)-\tan^{-1}\left(\frac{x}{r+1}\right).$$

$$y = S_n = \sum_{r=1}^{n} T_r = \sum_{r=1}^{n} tan^{-1}\left(\frac{x}{r}\right) - tan^{-1}\left(\frac{x}{r+1}\right)$$

$$= \left(tan^{-1}(x) - tan^{-1}\left(\frac{x}{2}\right) + \left(tan^{-1}\left(\frac{x}{2}\right) - tan^{-1}\left(\frac{x}{3}\right)\right)\right)$$

$$\left(tan^{-1}(x) - tan^{-1}\left(\frac{x}{2}\right) + \left(tan^{-1}\left(\frac{x}{2}\right) - tan^{-1}\left(\frac{x}{3}\right)\right)\right) + \left(tan^{-1}\left(\frac{x}{3}\right) - tan^{-1}\left(\frac{x}{4}\right)\right) +$$

$$\ldots\ldots + \left(tan^{-1}\left(\frac{x}{n}\right) - tan^{-1}\left(\frac{x}{n+1}\right)\right)$$

$$= tan^{-1}(x) - tan^{-1}\left(\frac{x}{n+1}\right).$$

$$\frac{dy}{dx} = \frac{1}{1+x^2} - \frac{1}{1+\left(\frac{x}{n+1}\right)^2} \cdot \left(\frac{1}{n+1}\right)$$

when $n \to \infty$

$$\frac{dy}{dx} = \frac{1}{1+x^2} - 0 = \frac{1}{1+x^2}.$$

Illustration :

(i) $y = tan^{-1}\left(\frac{\sqrt{1+x} - \sqrt{1-x}}{\sqrt{1+x} + \sqrt{1-x}}\right)$ (ii) $y = tan^{-1}\left(\frac{\sqrt{1+x^2} - 1}{x}\right)$

(iii) If $f(x) = sin^{-1}\left(\frac{2^{x+1}}{1+4^x}\right)$, find $f'(0)$.

Sol.

(i) $y = tan^{-1}\left(\frac{\sqrt{1+x} - \sqrt{1-x}}{\sqrt{1+x} + \sqrt{1-x}}\right)$

Let $x = \cos 2\theta \Rightarrow \theta = \frac{1}{2}\cos^{-1}(x)$

$$y = tan^{-1}\left(\frac{\sqrt{1+\cos 2\theta} - \sqrt{1-\cos 2\theta}}{\sqrt{1+\cos 2\theta} + \sqrt{1-\cos 2\theta}}\right) = tan^{-1}\left(\frac{\sqrt{2}\cos\theta - \sqrt{2}\sin\theta}{\sqrt{2}\cos\theta + \sqrt{2}\sin\theta}\right) = tan^{-1}\left(\frac{1-\tan\theta}{1+\tan\theta}\right)$$

$$y = tan^{-1}\left(tan\left(\frac{\pi}{4} - \theta\right)\right) = \frac{\pi}{4} - \theta$$

$$y = \frac{\pi}{4} - \frac{1}{2}\cos^{-1}x$$

$$\therefore \frac{dy}{dx} = \frac{1}{2\sqrt{1-x^2}}.$$

(ii) $y = \tan^{-1}\left(\dfrac{\sqrt{1+x^2}-1}{x}\right)$

Put $x = \tan\theta \therefore \theta = \tan^{-1}x$

$$y = \tan^{-1}\left(\frac{\sec\theta - 1}{\tan\theta}\right) = \tan^{-1}\left(\frac{1-\cos\theta}{\sin\theta}\right) = \tan^{-1}\left(\frac{2\sin^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}\right) = \tan^{-1}\left(\tan\frac{\theta}{2}\right)$$

$$\Rightarrow y = \frac{\theta}{2} = \frac{1}{2}\tan^{-1}x \qquad \Rightarrow \frac{dy}{dx} = \frac{1}{2(1+x^2)}.$$

(iii) $f(x) = \sin^{-1}\left(\dfrac{2^{x+1}}{1+2^{2x}}\right)$

Let $2^x = \tan\theta$

$$f(x) = \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right) = \sin^{-1}(\sin 2\theta)$$

$$f(x) = 2\theta = 2\tan^{-1}(2^x) \quad \Rightarrow \quad f'(x) = \frac{1}{1+2^{2x}}(2^x \ln 2)$$

$$f'(0) = \frac{2}{1+1}(2^0 \ln 2) = \ln 2.$$

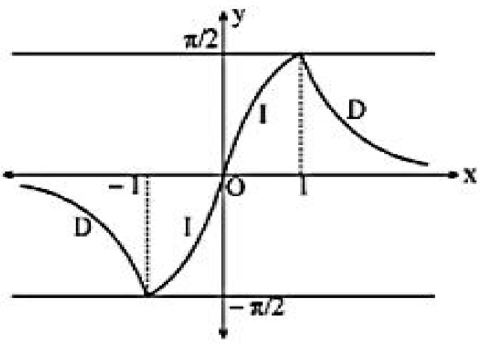
Derivative of $\sin^{-1}\left(\frac{2x}{1+x^2}\right)$; $\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$; $\tan^{-1}\left(\frac{2x}{1-x^2}\right)$; $\sin^{-1}(3x - 4x^3)$; $\cos^{-1}(4x^3 - 3x)$:

(i) $$y = f(x) = \sin^{-1}\left(\frac{2x}{1+x^2}\right) = \begin{bmatrix} 2\tan^{-1}x & |x| \le 1 \\ \pi - 2\tan^{-1}x & x > 1 \\ -(\pi + 2\tan^{-1}x) & x < -1 \end{bmatrix}$$

Highlights :

(a) Domain is $x \in R$ & range is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

Note: f is odd, aperiodic bound



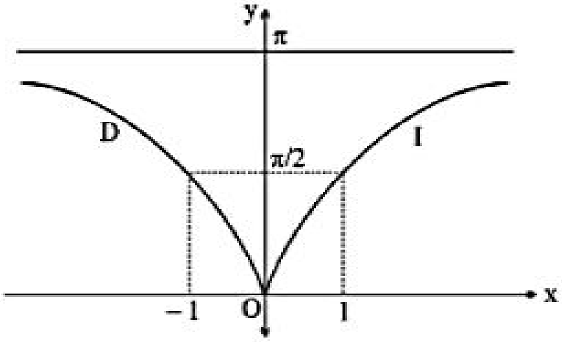
(b) f is continuous for all x but not diff. at x = 1 , - 1

(c) $$\frac{dy}{dx} = \begin{bmatrix} \frac{2}{1+x^2} & \text{for } |x| < 1 \\ \text{nonexistent} & \text{for } |x| = 1 \\ -\frac{2}{1+x^2} & \text{for } |x| > 1 \end{bmatrix}$$

(d) Increasing in (- 1 , 1) & decreasing in $(-\infty, -1) \cup (1, \infty)$

(ii) Consider $y = f(x) = \cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right) = \begin{cases} 2\tan^{-1}x & \text{if } x \geq 0 \\ -2\tan^{-1}x & \text{if } x < 0 \end{cases}$

Highlights :



(a) Domain is $x \in R$ & range is $(0, \pi)$

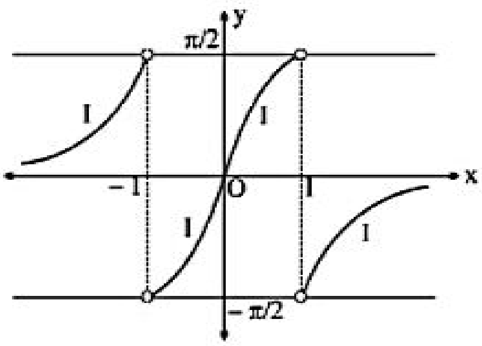
(b) Continuous for all x
but not differentiable at $x = 0$

(c) $\dfrac{dy}{dx} = \begin{cases} \dfrac{2}{1+x^2} & \text{for } x > 0 \\ \text{nonexistent} & \text{for } x = 0 \\ -\dfrac{2}{1+x^2} & \text{for } x < 0 \end{cases}$

(d) Increasing in $(0, \infty)$ & decreasing in $(-\infty, 0)$ Note: f is even, a periodic bound.

(iii) $y = f(x) = \tan^{-1}\dfrac{2x}{1-x^2} = \begin{cases} 2\tan^{-1}x & |x| < 1 \\ \pi + 2\tan^{-1}x & x < -1 \\ -(\pi - 2\tan^{-1}x) & x > 1 \end{cases}$

Highlights :



(a) Domain is $R - \{1, -1\}$ & range is $\left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$

(b) f is neither continuous
nor differentiable at $x = 1, -1$

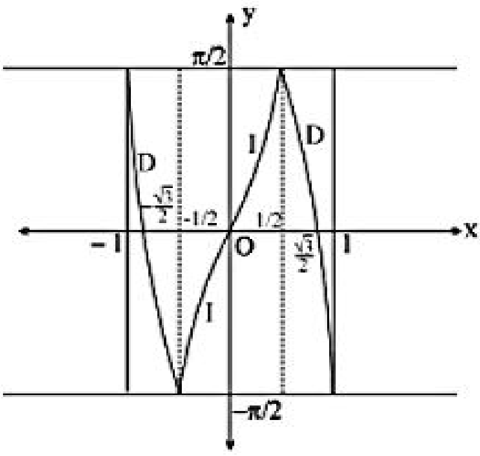
(c) $\dfrac{dy}{dx} = \begin{cases} \dfrac{2}{1+x^2} & |x| \neq 1 \\ \text{nonexistent} & |x| = 1 \end{cases}$

(d) Increasing $\forall$ x in its domain

(e) It is bound for all x

(iv) $y = f(x) = \sin^{-1}(3x - 4x^3) = \begin{cases} -(\pi + 3\sin^{-1}x) & \text{if } -1 \leq x \leq -\frac{1}{2} \\ 3\sin^{-1}x & \text{if } -\frac{1}{2} \leq x \leq \frac{1}{2} \\ \pi - 3\sin^{-1}x & \text{if } \frac{1}{2} \leq x \leq 1 \end{cases}$

Highlights :



(a) Domain is $x \in [-1, 1]$ & range is $\left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$

(b) Not derivable at $|x| = \dfrac{1}{2}$

(c) $\dfrac{dy}{dx} = \begin{cases} \dfrac{3}{\sqrt{1-x^2}} & \text{if } x \in \left(-\frac{1}{2}, \frac{1}{2}\right) \\ -\dfrac{3}{\sqrt{1-x^2}} & \text{if } x \in \left(-1, -\frac{1}{2}\right) \cup \left(\frac{1}{2}, 1\right) \end{cases}$

(d) Continuous everywhere in its domain

(v) $y = f(x) = \cos^{-1}(4x^3 - 3x) = \begin{bmatrix} 3\cos^{-1}x - 2\pi & \text{if} & -1 \le x \le -\frac{1}{2} \\ 2\pi - 3\cos^{-1}x & \text{if} & -\frac{1}{2} \le x \le \frac{1}{2} \\ 3\cos^{-1}x & \text{if} & \frac{1}{2} \le x \le 1 \end{bmatrix}$

Highlights :

(a) Domain is $x \in [-1, 1]$ & range is $[0, \pi]$

(b) Continuous everywhere in its domain

but not derivable at $x = \frac{1}{2}, -\frac{1}{2}$

(c) Increasing in $\left(-\frac{1}{2}, \frac{1}{2}\right)$ &

Decreasing in $\left(\frac{1}{2}, 1\right] \cup \left[-1, -\frac{1}{2}\right)$

(d) $\frac{dy}{dx} = \begin{bmatrix} \frac{3}{\sqrt{1-x^2}} & \text{if } x \in \left(-\frac{1}{2}, \frac{1}{2}\right) \\ -\frac{3}{\sqrt{1-x^2}} & \text{if } x \in \left(-1, -\frac{1}{2}\right) \cup \left(\frac{1}{2}, 1\right) \end{bmatrix}$

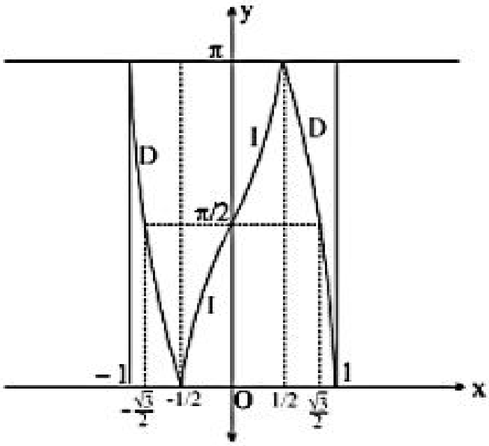


Illustration :

If $y = \sin^{-1}\left[x\sqrt{1-x} - \sqrt{x}\sqrt{1-x^2}\right]$, *and* $0 < x < 1$, *then find* $\frac{dy}{dx}$

Sol. $y = \sin^{-1}\left[x\sqrt{1-x} - \sqrt{x}\sqrt{1-x^2}\right]$, *where* $0 < x < 1$

$$= \sin^{-1}\left[x\sqrt{1-(\sqrt{x})^2} - \sqrt{x}\sqrt{1-x^2}\right] = \sin^{-1}x - \sin^{-1}\sqrt{x}$$

Differentiating w.r.t. x, we get

$$\frac{d}{dx} = \frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-(\sqrt{x})^2}}\frac{d}{dx}(\sqrt{x}) = \frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-x}}\cdot\frac{1}{2\sqrt{x}} = \frac{1}{\sqrt{1-x^2}} - \frac{1}{2\sqrt{x-x^2}}$$

Illustration :

Find $\frac{dy}{dx}$ *for* $y = \tan^{-1}\sqrt{\frac{a-x}{a+x}}$, *where* $-a < x < a$.

Sol. $y = \tan^{-1}\left\{\sqrt{\frac{a-x}{a+x}}\right\}$, *where* $-a < x < a$

Substituting $x = a\cos\theta$, *we have*

$$y = \tan^{-1}\left\{\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}\right\} = \tan^{-1}\left\{\sqrt{\tan^2\frac{\theta}{2}}\right\} = \tan^{-1}\left|\tan\frac{\theta}{2}\right|.$$

Also for $-a < x < a,\ -1 < \cos\theta < 1$

$$\Rightarrow \quad \theta \in (0, \pi) \Rightarrow \frac{\theta}{2} \in \left(0, \frac{\pi}{2}\right)$$

$$\therefore \quad y = \tan^{-1}\left|\tan\frac{\theta}{2}\right| = \tan^{-1}\left(\tan\frac{\theta}{2}\right) = \frac{\theta}{2} = \frac{1}{2}\cos^{-1}\left(\frac{x}{a}\right)$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{1}{2}\cdot\frac{1}{\sqrt{1-\frac{x^2}{a^2}}}\frac{d}{dx}\left(\frac{x}{a}\right) = -\frac{1}{2\sqrt{a^2-x^2}}$$

Practice Problem

Q.1 The function $f(x) = e^x + x$, being differentiable and one to one, has a differentiable inverse $f^{-1}(x)$. The value of $\frac{d}{dx}(f^{-1})$ at the point $f(\log 2)$ is

(A) $\frac{1}{\ln 2}$ (B) $\frac{1}{3}$ (C) $\frac{1}{4}$ (D) None of these

Q.2 Let g(x) be the inverse of an invertible function f(x) which is differentiable at $x = c$, then $g'(f(c))$ equals

(A) $f'(c)$ (B) $\frac{1}{f'(c)}$ (C) $f(C)$ (D) None of these

Q.3 If $f(x) = x + \tan x$ and f is inverse of g, then $g'(x)$ equals

(A) $\frac{1}{1+[g(x)-x]^2}$ (B) $\frac{1}{2-[g(x)-x]^2}$ (C) $\frac{1}{2+[g(x)-x]^2}$ (D) None of these

Q.4 Find $\frac{dy}{dx}$ of the following functions :

(i) $y = \sec^{-1}\left(\frac{\sqrt{x}+1}{\sqrt{x}-1}\right) + \sin^{-1}\left(\frac{\sqrt{x}-1}{\sqrt{x}+1}\right)$ (ii) $y = \tan^{-1}\frac{4x}{1+5x^2} + \tan^{-1}\frac{2+3x}{3-2x}$.

(iii) $y = \tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$, where $x \neq 0$ (iv) $y = \tan^{-1}\frac{3a^2x - x^3}{a(a^2-3x^2)}$.

(v) $y = \sin^{-1}\left(\frac{5x+12\sqrt{(1-x)^2}}{13}\right)$. (vi) $y = \tan^{-1}\left(\frac{x}{1+\sqrt{1-x^2}}\right)$.

(vii) $y = \tan^{-1}\left(\frac{a\cos x - b\sin x}{b\cos x + a\sin x}\right)$, where $-\frac{\pi}{2} < x < \frac{\pi}{2}$ and $\frac{a}{b}\tan x > -1$.

(viii) $y = \tan^{-1}\left(\dfrac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}\right)$, where $-1 < x < 1, x \neq 0$.

(ix) $y = \sin^{-1}\left(\dfrac{x}{\sqrt{1+x^2}}\right) + \cos^{-1}\left(\dfrac{1}{\sqrt{1+x^2}}\right), 0 < x < \infty$.

(x) $y = (\sin x)^x + \sin^{-1}\sqrt{x}$

Q.5 Find the derivative of the following functions with respect to x :

(i) $y = \sin^{-1}\left(\dfrac{2x}{1+x^2}\right)$ (ii) $y = \tan^{-1}\left(\dfrac{3x-x^3}{1-3x^2}\right), \dfrac{-1}{\sqrt{3}} < x < \dfrac{1}{\sqrt{3}}$

(iii) $y = \cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right), 0 < x < 1$ (iv) $y = \sin^{-1}\left(\dfrac{1-x^2}{1+x^2}\right), 0 < x < 1$

(v) $y = \cos^{-1}\left(\dfrac{2x}{1+x^2}\right), -1 < x < 1$ (vi) $y = \sin^{-1}\left(2x\sqrt{1-x^2}\right), -\dfrac{1}{\sqrt{2}} < x < \dfrac{1}{\sqrt{2}}$

(vii) $y = \sec^{-1}\left(\dfrac{1}{2x^2-1}\right), 0 < x < \dfrac{1}{\sqrt{2}}$

Q.6 Find the derivative of the following functions with respect to x :

(i) $\cos^{-1}(\sin x)$ (ii) $\tan^{-1}\left(\dfrac{\sin x}{1+\cos x}\right)$ (iii) $\sin^{-1}\left(\dfrac{2^{x+1}}{1+4^x}\right)$

Answer key

Q.1 B **Q.2** B **Q.3** C

Q.4 (i) 0 (ii) $\dfrac{1}{1+25x^2}$ (iii) $\dfrac{1}{2(1+x^2)}$ (iv) $\dfrac{3a}{(a^2+x^2)}$

(v) $\dfrac{1}{\sqrt{(1-x^2)}}$ (vi) $\dfrac{1}{2\sqrt{(1-x^2)}}$ (vii) -1 (viii) $-\dfrac{x}{\sqrt{(1-x^4)}}$

(ix) $\dfrac{2}{(1+x^2)}$ (x) $(\sin x)^x(x\cot x + \ln\sin x) + \dfrac{1}{2\sqrt{x(1-x)}}$

T-9 PARAMETRIC DIFFERENTIATION :

In some situation curves are represented by the equations e.g. $x = \sin t$ & $y = \cos t$
If $x = f(t)$ and $y = g(t)$ then

$$\frac{dy}{dx} = \frac{dy}{dt}\cdot\frac{dt}{dx} = \frac{g'(t)}{f'(t)}$$

Illustration :

Find $\frac{dy}{dx}$ *if* $x = a(\theta - \sin\theta)$ *and* $y = a(1 - \cos\theta)$.

Sol. *We have,*

$x = a(\theta - \sin\theta)$ and $y = a(1 - \cos\theta)$

$$\Rightarrow \quad \frac{dy}{dx} = a(1 - \cos\theta) \text{ and } \frac{dy}{d\theta} = a\sin\theta$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{a\sin\theta}{a(1-\cos\theta)} = \frac{2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)}{2\sin^2\left(\frac{\theta}{2}\right)} = \cot\frac{\theta}{2}.$$

Illustration :

If $x = a\sec^3\theta$ *and* $y = a\tan^3\theta$, *find* $\frac{dy}{dx}$ *at* $\theta = \frac{\pi}{3}$.

Sol. *We have* $x = a\sec^3\theta$ and $y = a\tan^3\theta$

$$\frac{dx}{d\theta} = 3a\sec^2\theta\frac{d}{d\theta}(\tan\theta) = 3a\tan^2\theta\sec^2\theta$$

$$\Rightarrow \quad \frac{dt}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{3q\tan^2\theta\sec^2\theta}{3a\sin^3\theta\tan\theta} = \frac{\tan\theta}{\sec\theta} = \sin\theta$$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)_{\theta=\frac{\pi}{3}} = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}.$$

Illustration :

Find $\frac{dy}{dx}$ *if*

(i) $x = a(\cos t + t\sin t)$ and $y = a(\sin t - t\cos t)$ (ii) $x = \frac{3at}{1+t^3}$; $y = \frac{3at^2}{1+t^3}$

(iii) $x = a\sec^2\theta$; $y = a\tan^3\theta$.

(iv) $x = a\sqrt{\cos 2t}\cos t$ and $y = a\sqrt{\cos 2t}\sin t$ then, find $\left.\frac{dy}{dx}\right|_{t=\pi/6}$.

(v) If $x = \sec\theta - \cos\theta$ & $y = \sec^n\theta - \cos^n\theta$, then show that $(x^2+4)\left(\frac{dy}{dx}\right)^2 = n^2(y^2+4)$.

Sol.

(i) $x = a(\cos t + t\sin t)$ and $y = a(\sin t - t\cos t)$

$$\frac{dx}{dt} = a(-\sin t + \sin t + t\cos t) = at\cos t$$

$$\frac{dy}{dt} = a(\cos t - \sin t + t\sin t) = at\sin t$$

$$\Rightarrow \frac{dy}{dx} = \tan t$$

(ii) $x = \dfrac{3at}{1+t^3}$; $y = \dfrac{3at^2}{1+t^3}$

$$\frac{dx}{dt} = \frac{3a\left((1+t^3)\cdot 1 - t(3t^2)\right)}{\left(1+t^3\right)^2} = \frac{3a(1-2t^3)}{\left(1+t^3\right)^2}$$

$$\frac{dy}{dt} = \frac{3a\left((1+t^3)2t - t^2(3t^2)\right)}{\left(1+t^3\right)^2} = \frac{3a(2t-t^4)}{\left(1+t^3\right)^2}$$

$$\frac{dy}{dx} = \frac{(2-t^3)t}{(1-2t^3)}$$

(iii) $x = a\sec^2\theta$; $y = a\tan^3\theta$

$$\frac{dx}{d\theta} = 2a\sec\theta(\sec\theta\tan\theta)$$

$$\frac{dy}{d\theta} = 3a\tan^2\theta\sec^2\theta$$

$$\frac{dy}{dx} = \frac{3\tan\theta}{2}.$$

(iv) $x = a\sqrt{\cos 2t}\,\cos t$ and $y = a\sqrt{\cos 2t}\,\sin t$

$$\frac{dx}{dt} = a\left(\sqrt{\cos 2t}(-\sin t) + \frac{\cos t(-2\sin 2t)}{2\sqrt{\cos 2t}}\right)$$

$$\frac{dx}{dt} = a\left(\frac{\sin t\cos 2t + \cos t\sin 2t}{\sqrt{\cos 2t}}\right)$$

$$\frac{dx}{dt} = -a\frac{\sin 3t}{\sqrt{\cos 2t}} \Rightarrow \left.\frac{dx}{dt}\right|_{\pi/6} = -\sqrt{2}\,a$$

$$\frac{dy}{dt} = a\left(\sqrt{\cos 2t}\,\cos t + \frac{\sin t}{2\sqrt{\cos 2t}}(-2\sin 2t)\right) = a\left(\frac{\cos 3t}{\sqrt{\cos 2t}}\right) \Rightarrow \left.\frac{dy}{dt}\right|_{\pi/6} = 0$$

$$\left.\frac{dy}{dx}\right|_{\pi/6} = 0.$$

(v) $x = \sec\theta - \cos\theta\,;\ y = \sec^n\theta - \cos^n\theta$

$$(x^2+4)\left(\frac{dy}{dx}\right)^2 = n^2\,(y^2+4)$$

$$\frac{dx}{d\theta} = \sec\theta\tan\theta + \sin\theta = \sin\theta\,(\sec^2\theta + 1)$$

$$\frac{dy}{d\theta} = n\sec^{n-1}\theta\sec\theta\tan\theta + n\cos^{n-1}\theta\sin\theta\ = n\sec^n\theta\tan\theta + n\cos^n\theta\tan\theta$$

$$\frac{dy}{d\theta} = n\tan\theta\,(\sec^n\theta + \cos^n\theta)$$

$$\frac{dy}{dx} = \frac{n\tan\theta\,(\sec^n\theta+\cos^n\theta)}{\sin\theta\,(\sec^2\theta-1)} = \frac{n\,(\sec^n\theta+\cos^n\theta)}{\cos\theta\,(\sec^2\theta+1)} = \frac{n\,(\sec^n\theta+\cos^n\theta)}{(\sec\theta+\cos\theta)}$$

$$\left(\frac{dy}{dx}\right)^2 = \frac{n^2\,(\sec^{2n}\theta+\cos^{2n}\theta+2)}{(\sec^2\theta+\cos^2\theta+2)} = \frac{n^2\,(y^2+4)}{(x^2+4)}.$$

Illustration :

For the curve represented parametrically indicate the relation between the parameter t and the angle α between the tangent to the given curve and the x-axis.

(i) $\begin{cases} x = \cos t + t\sin t - \dfrac{t^2}{2}\cos t \\ y = \sin t - t\cos t - \dfrac{t^2}{2}\sin t \end{cases}$; (ii) $x = a\cos^3 t,\ y = a\sin^3 t$

Sol.

(i) $x = \cos t + t\sin t - \dfrac{t^2}{2}\cos t$

$$\frac{dx}{dt} = -\sin t + t\cos t + \sin t - \left(\frac{t^2}{2}\sin t + t\cos t\right) = \frac{t^2\sin t}{2}$$

$$y = \sin t - t\cos t - \frac{t^2}{2}\sin t$$

$$\frac{dy}{dt} = \cos t - (\cos t - t\sin t) - \frac{1}{2}\,(2t\sin t + t^2\cos t) = \frac{-t^2\cos t}{2}$$

$$\therefore\ \frac{dy}{dx} = \frac{(dy/dt)}{(dx/dt)} = -\cot t = \tan\alpha \Rightarrow \tan\left(\frac{\pi}{2}-t\right) = \tan(-\alpha)$$

$$\Rightarrow \frac{\pi}{2} - t = -\alpha \Rightarrow t = \frac{\pi}{2} + \alpha.$$

(ii) $x = a\cos^3 t \Rightarrow \dfrac{dx}{dt} = -3a\cos^2 t\sin t$

$y = a\sin^3 t \Rightarrow \dfrac{dy}{dt} = 3a\sin^2 t\cos t$

$$\therefore \frac{dy}{dx} = \frac{(dy/dt)}{(dx/dt)} = \frac{3a\sin^2 t\cos t}{-3a\cos^2 t\sin t} = -\tan t = \tan\alpha$$

$\Rightarrow \tan(\pi - t) = \tan\alpha \Rightarrow \pi - t = \alpha \Rightarrow t = \pi - \alpha.$

T-10 DIFFERENTIATION OF ONE FUNCTION W.R.T. OTHER FUNCTION :

If $y = f(x)$ and $z = g(x)$ then derivative of $f(x)$ w.r.t. $g(x)$ is given by

$$\frac{dy}{dz} = \frac{dy}{dx}\cdot\frac{dx}{dz} = \frac{f'(x)}{g'(x)}$$

$$\therefore \quad \text{Differential coefficient of } f(x) \text{ w.r.t. } g(x) = \frac{\text{derivative of } f(x) \text{ w.r.t. } x}{\text{derivative of } g(x) \text{ w.r.t. } x} = \frac{f'(x)}{g'(x)}$$

Illustration :

Differentiate $\log \sin x$ *w.r.t.* $\sqrt{\cos x}$.

Sol. Let $u = \log\sin x$ and $v = \sqrt{\cos x}$

Then, $\dfrac{du}{dx} = \cot x$ and $\dfrac{dv}{dx} = -\dfrac{\sin x}{2\sqrt{\cos x}}$

$$\Rightarrow \quad \frac{du}{dv} = \frac{du/dx}{dv/dx} = \frac{\cot x}{-\dfrac{\sin x}{2\sqrt{\cos x}}} = -2\sqrt{\cos x}\,\cot x\,\operatorname{cosec} x.$$

Illustration :

Differentiate $\tan^{-1}\left(\dfrac{\sqrt{1+x^2}-1}{x}\right)$ *w.r.t.* $\tan^{-1}x$,

Sol. Let $u = \tan^{-1}\left(\dfrac{\sqrt{1+x^2}-1}{x}\right)$ and $v = \tan^{-1}x$.

Putting $x = \tan\theta$,

we get $u = \tan^{-1}\left(\dfrac{\sqrt{1+x^2}-1}{x}\right)$

$$= \tan^{-1}\left(\frac{\sec\theta - 1}{\tan\theta}\right) = \tan^{-1}\left(\frac{1-\cos\theta}{\sin\theta}\right) = \tan^{-1}\left(\tan\frac{\theta}{2}\right) = \frac{1}{2}\theta = \frac{1}{2}\tan^{-1}x.$$

Thus, we have $u = \frac{1}{2}\tan^{-1} x$ and $v = \tan^{-1} x$.

$$\Rightarrow \quad \frac{du}{dx} = \frac{1}{2} \times \frac{1}{1+x^2} \text{ and } \frac{dv}{dx} = \frac{1}{1+x^2}$$

$$\Rightarrow \quad \frac{du}{dv} = \frac{du/dx}{dv/dx} = \frac{1}{2(1+x^2)}\,(1 + x^2) = \frac{1}{2}.$$

Illustration :

Find derivative of $(\ln x)^{\tan x}$ w.r.t. x^x.

Sol. $u = (\ln x)^{\tan x}$; $v = x^x$

$\ln(u) = \tan x \ln(\ln x)$

$$\Rightarrow \frac{1}{u}\left(\frac{du}{dx}\right) = (\sec^2 x)\ln(\ln x) + \tan x\left(\frac{1}{\ln x}\cdot\frac{1}{x}\right)$$

$$\frac{du}{dx} = \frac{u\left((x\ln x)\ln(\ln x)\sec^2 x + \tan x\right)}{(x\ln x)}$$

$\ln v = x \ln x$

$$\Rightarrow \frac{1}{v}\left(\frac{dv}{dx}\right) = (\ln x + 1)$$

$$\frac{du}{dx} = \frac{(\ln x)^{\tan x}}{x^x}\left(\frac{x\ln x\ln(\ln x)\sec^2 x + \tan x}{(x\ln x)(\ln x + 1)}\right).$$

Illustration :

Derivative of $\cos^{-1}(2x^2 - 1)$ w.r.t. $\sqrt{1-x^2}$ when $x = \frac{1}{2}$.

Sol. $u = \cos^{-1}(2x^2 - 1)$; $v = \sqrt{(1-x^2)}$

put $x = \cos\theta$ as $x = \frac{1}{2}$ so $\theta = \frac{\pi}{3}$

$u = \cos^{-1}(\cos 2\theta) = 2\theta$ put $v = \sin\theta$

$$\frac{du}{dv} = \frac{du/d\theta}{dv/d\theta} = \frac{2}{\cos\theta} = 4.$$

Illustration :

Define derivative of $\sin^{-1}\left(\frac{2x}{1+x^2}\right)$ w.r.t. $\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$ $\forall x \in R$.

Sol. Let $u = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$ and $v = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

Let $x = \tan\theta$

$u = \sin^{-1}(\sin 2\theta)$ *and* $v = \cos^{-1}(\cos 2\theta)$

$$u = \begin{cases} 2\theta, \dfrac{-\pi}{2} \le 2\theta \le \dfrac{\pi}{2} \Rightarrow \dfrac{-\pi}{4} \le \theta \le \dfrac{\pi}{4} \Rightarrow -1 \le x \le 1 \\ (-\pi - 2\theta), \dfrac{\pi}{2} \le 2\theta \le \dfrac{3\pi}{2} \Rightarrow \dfrac{\pi}{4} \le \theta \le \dfrac{3\pi}{4} \Rightarrow 1 \le x \le \propto \\ (\pi - 2\theta), \dfrac{3\pi}{2} \le 2\theta \le \dfrac{-\pi}{2} \Rightarrow \dfrac{-3\pi}{4} \le \theta \le \dfrac{-\pi}{4} \Rightarrow -\infty \le x \le -1 \end{cases}$$

$$v = \cos^{-1}(\cos 2\theta)$$

$$v = \begin{cases} 2\theta, 0 < 2\theta \le \pi \Rightarrow 0 < \theta \le \dfrac{\pi}{2} \Rightarrow 0 < x < \propto \\ -2\theta, -\pi < 2\theta \le 0 \Rightarrow \dfrac{-\pi}{2} \le \theta \le 0 \Rightarrow -\propto < x < 0 \end{cases}$$

$$\frac{du}{d\theta} = \begin{cases} 2, & -1 \le x \le 1 \\ -2, & 1 \le x < \infty \\ -2, & -\infty < x \le -1 \end{cases}$$

$$\frac{dv}{d\theta} = \begin{cases} 2, & 0 < x < \infty \\ -2, & -\infty < x < 0 \end{cases}$$

$$\frac{du}{dv} = \begin{cases} \left(\dfrac{-2}{-2}\right) = 1, & -\infty < x < -1 \\ \left(\dfrac{2}{-2}\right) = -1, & -1 < x < 0 \\ \left(\dfrac{2}{2}\right) = 1, & 0 < x < 1 \\ \left(\dfrac{-2}{2}\right) = -1, & 1 < x < \infty \end{cases}$$

$$\frac{du}{dv} = \begin{cases} 1 \ \forall x \in (-\infty, -1) \cup (0, 1) \\ -1 \ \forall x \in (-1, 0) \cup (1, \infty) \\ \text{not exists at } x = -1, 0, 1 \end{cases}$$

Illustration :

(i) Differential coefficient of $e^{\sin^{-1}x}$ *w.r.t.* $e^{-\cos^{-1}x}$ *is independent of x. (T or F)*

(ii) Find the derivative of f(tan x) w.r.t. g (sec x) at $x = \frac{\pi}{4}$, *where* $f'(1) = 2$ *and* $g'(\sqrt{2}) = 4$.

Sol.

(i) Let $u = e^{\sin^{-1}x}$ and $v = e^{-\cos^{-1}x}$

$$u = e^{\frac{\pi}{2} - \cos^{-1}x} \qquad [\because \sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}]$$

$$u = e^{\frac{\pi}{2}} e^{-\cos^{-1}x} = e^{\frac{\pi}{2}} v$$

So, $\frac{du}{dv} = e^{\frac{\pi}{2}}$. *(True)*

(ii) Let $u = f(\tan x)$ and $v = g(\sec x)$

$$\Rightarrow \quad \frac{du}{dx} = f'(\tan x)\sec^2 x \quad \text{and} \quad \frac{dv}{dx} = g'(\sec x)\sec x \tan x$$

$$\Rightarrow \quad \frac{du}{dv} = \frac{du/dx}{dv/dx} = \frac{f'(\tan x)\sec^2 x}{g'(\sec x)\sec x \tan x}$$

$$\Rightarrow \quad \left[\frac{du}{dv}\right]_{x=\frac{\pi}{4}} = \frac{f'\left(\tan\frac{\pi}{4}\right)}{g'\left(\sec\frac{\pi}{4}\right)\sin\frac{\pi}{4}} = \frac{f'(1)\sqrt{2}}{g'(\sqrt{2})} = \frac{2\sqrt{2}}{4} = \frac{1}{\sqrt{2}}.$$

General Note :

Concavity in each case is decided by the sign of 2nd derivative as :

$$\frac{d^2y}{dx^2} > 0 \Rightarrow \text{Concave upwards} \quad ; \quad \frac{d^2y}{dx^2} < 0 \Rightarrow \text{Concave downwards}$$

D = Decreasing; I = Increasing

T-11 SUCCESSIVE DIFFERENTIATION :

$y = f(x)$; the popular symbols used to denote the derivatives are $\frac{dy}{dx} = Dy = f'(x) = y_1 = y'$. Higher order derivatives are denoted as $\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2} = D^2y = f''(x) = y_2$ or y'' etc.

Note : A homogeneous equation of degree n represents '*n*' straight lines passing through the origin, hence

$$\frac{dy}{dx} = \frac{y}{x} \qquad \text{and} \qquad \frac{d^2y}{dx^2} = 0$$

e.g. If $x^3 + 3x^2y - 6xy^2 + 2y^3 = 0$, then $\left.\frac{d^2y}{dx^2}\right]_{(1,1)} = 0$

Illustration :

If $y = \sin(\sin x)$ *then prove that* $y_2 + (\tan x)y_1 + y\cos^2 x = 0.$

Sol. $y = \sin(\sin x)$

$y_1 = \cos(\sin x) \cdot \cos x$

$y_2 = -\cos(\sin x)\sin x - \sin(\sin x)\cos^2 x$

$\therefore\ y_2 + (\tan x)\,y_1 + y\cos^2 x$

$= -\cos(\sin x)\cdot\sin x - \sin(\sin x)\cos^2 x + \tan x\,(\cos(\sin x)\cdot\cos x) + \sin(\sin x)\cos^2 x$

$= 0.$

Illustration :

(i) *If* $y = a\cos(\ln x) + b\sin(\ln x)$ *then prove that* $x^2y_3 + 3xy_2 + 2y_1 = 0$

(ii) *If* $y = e^{a\cos^{-1}x}$, *then prove that* $(1 - x^2)y_2 - xy_1 - a^2y = 0.$

(iii) *If* $y^{\frac{1}{m}} + y^{-\frac{1}{m}} = 2x$ *then prove that* $(x^2 - 1)y_3 + 3xy_2 + (1 - m^2)y_1 = 0$

Sol. (i) $y = a\cos(\ln x) + b\sin(\ln x)$

$$y_1 = \frac{-a\sin(\ln x)}{x} + \frac{b\cos(\ln x)}{x}$$

$\Rightarrow\ xy_1 = -a\sin(\ln x) + b\cos(\ln x)\ \Rightarrow\ xy_2 + y_1 = \dfrac{-a\cos(\ln x)}{x} - \dfrac{b\sin(\ln x)}{x}$

$\Rightarrow\ x^2y_2 + xy_1 = -a\cos(\ln x) - b\sin(\ln x) = -y\ \Rightarrow\ x^2y_2 + xy_1 + y = 0$

$\Rightarrow\ (x^2y_3 + 2xy_2) + (xy_2 + y_1) + y_1 = 0$

$\Rightarrow\ x^2y_3 + 3xy_2 + 2y_1 = 0.$

(ii) $y = e^{a\cos^{-1}x}$

$$\frac{dy}{dx} = e^{a\cos^{-1}x}\left(\frac{-a}{\sqrt{1-x^2}}\right) = y_1$$

$\Rightarrow y_1 = \dfrac{-ay}{\sqrt{1-x^2}}\ \Rightarrow\ (1 - x^2)(y_1)^2 = a^2y^2$

now differentiating w.r.t. to x

$(1 - x^2)\cdot 2y_1y_2 - 2x(y_1)^2 = a^2(2yy_1)$

$\Rightarrow (1 - x^2)y_2 - xy_1 = a^2y.$

(iii) $y^{\frac{1}{m}} + y^{\frac{-1}{m}} = 2x$

$$\Rightarrow \quad y^{2/m} - 2x\, y^{1/m} + 1 = 0 \quad \Rightarrow \quad y^{1/m} = \frac{2x \pm \sqrt{4x^2 - 4}}{2} = \left(x \pm \sqrt{x^2 - 1} \right)$$

$$\Rightarrow y = \left(x \pm \sqrt{x^2 - 1} \right)^m \quad \Rightarrow y_1 = m\left(x \pm \sqrt{x^2 - 1} \right)^{m-1} \left(1 \pm \frac{x}{\sqrt{x^2 - 1}} \right)$$

$$\Rightarrow y_1 = \frac{m}{\sqrt{x^2 - 1}} \left(x \pm \sqrt{x^2 - 1} \right)^{m-1} \left(\sqrt{x^2 - 1} \pm x \right)$$

$$\Rightarrow y_1 = \pm \frac{my}{\sqrt{x^2 - 1}}$$

$\Rightarrow (x^2 - 1)\,(y_1)^2 = m^2 y^2 \;\Rightarrow\; (x^2 - 1)\, 2y_1 y_2 + (y_1)^2\, 2x = m^2\, yy_1$

$\Rightarrow (x^2 - 1)\, y_2 + x\, y_1 = m^2 y$

$\Rightarrow (x^2 - 1)\, y_3 + 2x\, y_2 + x\, y_2 + y_1 = m^2\, y_1 \;\Rightarrow\; (x^2 - 1)\, y_3 + 3xy_2 + (1 - m^2)\, y_1 = 0.$

Illustration :

If $(x - a)^2 + (y - b)^2 = c^2 \;\; (c > 0)$ then $\dfrac{\left[1 + \left(\dfrac{dy}{dx}\right)^2\right]^{3/2}}{\dfrac{d^2y}{dx^2}}$ equals

(A) c (B) c^2 (C) c^3 (D) c^4

Sol. $(x - a)^2 + (y - b)^2 = c^2$ $(c > 0)$ ....(1)

Now differentiating w.r.t. to x

$2\,(x - a) + 2\,(y - b)\, y_1 = 0$

$\Rightarrow (x - a) + (y - b)\, y_1 = 0$ ....(2)

$\Rightarrow 1 + (y - b)\, y_2 + (y_1)^2 = 0$ ....(3)

$1 + (y_1)^2 = 1 + \dfrac{(x-a)^2}{(y-b)^2}$ from (2)

$1 + (y_1)^2 = \dfrac{c^2}{(y-b)^2}$

$$\frac{\left(1 + (y_1)^2\right)^{3/2}}{y_2} = \frac{\left(\dfrac{c^2}{(y-b)^2}\right)^{3/2}}{\dfrac{-\left(1 + (y_1)^2\right)}{(y-b)}} \quad \text{from (3)}$$

$$= \frac{\left| c^3/(y-b)^3 \right|}{-c^2/(y-b)^3} = \pm\, c.$$

Illustration :

Use the substitution $x = \tan\theta$ to show that the equation,

$$\frac{d^2y}{dx^2}+\frac{2x}{1+x^2}\frac{dy}{dx}+\frac{y}{(1+x^2)^2} = 0 \text{ changes to } \frac{d^2y}{d\theta^2}+y=0.$$

Sol. $$\frac{d^2y}{dx^2}+\frac{2x}{1+x^2}\left(\frac{dy}{dx}\right)+\frac{y}{\left(1+x^2\right)^2} = 0 \qquad ....(1)$$

$$x = \tan\theta \Rightarrow dx = \sec^2\theta\, d\theta \Rightarrow \frac{d\theta}{dx} = \frac{1}{\sec^2\theta} = \cos^2\theta$$

$$\frac{dy}{dx} = \frac{dy}{d\theta}\left(\frac{d\theta}{dx}\right) = \frac{1}{\sec^2\theta}\left(\frac{dy}{d\theta}\right) = \cos^2\theta\left(\frac{dy}{d\theta}\right)$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{1}{\sec^2\theta}\cdot\frac{dy}{d\theta}\right) = \frac{d}{d\theta}\left(\cos^2\theta\frac{dy}{d\theta}\right)\cdot\frac{d\theta}{dx} = \left(\cos^2\theta\frac{d^2y}{d\theta^2}-2\cos\theta\sin\theta\frac{dy}{d\theta}\right)\cos^2\theta$$

From equation (1) putting $x = \tan\theta$

$$\cos^2\theta\left(\cos^2\theta\frac{d^2y}{d\theta^2}-2\cos\theta\sin\theta\frac{dy}{d\theta}\right)+\left(\frac{2\tan\theta}{1+\tan^2\theta}\right)\left(\cos^2\theta\frac{dy}{d\theta}\right)+\left(\frac{y}{\left(1+\tan^2\theta\right)^2}\right) = 0$$

$$\Rightarrow \left(\frac{d^2y}{d\theta^2}-2\tan\theta\frac{dy}{d\theta}\right)+\left(2\tan\theta\frac{dy}{d\theta}\right)+y=0 \Rightarrow \frac{d^2y}{d\theta^2}+y=0.$$

Illustration :

Starting with $\dfrac{dx}{dy} = \dfrac{1}{dy/dx}$. *Prove that* $\dfrac{d^2x}{dy^2} = -\dfrac{d^2y/dx^2}{(dy/dx)^3}$ *and deduce that for the parabola*

$y^2 = 4ax,\quad \dfrac{d^2y}{dx^2}\cdot\dfrac{d^2x}{dy^2} = -\dfrac{2a}{y^3}$

Sol. $$\frac{dx}{dy} = \frac{1}{\left(\frac{dy}{dx}\right)}$$

$$\frac{d^2x}{dy^2} = \frac{d}{dy}\left(\frac{1}{\left(\frac{dy}{dx}\right)}\right) = \frac{1}{dx}\left(\left(\frac{dy}{dx}\right)^{-1}\right)\cdot\frac{dx}{dy} = -1\left(\frac{dy}{dx}\right)^{-2}\left(\frac{d^2y}{dx^2}\right)\cdot\frac{1}{\left(\frac{dy}{dx}\right)} = \frac{-\left(\frac{d^2y}{dx^2}\right)}{\left(\frac{dy}{dx}\right)^3}$$

Given $y^2 = 4ax$

$$\Rightarrow 2y\,\frac{dy}{dx} = 4a \Rightarrow y\frac{dy}{dx} = 2a \Rightarrow \frac{dy}{dx} = \frac{2a}{y} \Rightarrow \frac{d^2y}{dx^2} = \frac{-2a}{y^2}\left(\frac{dy}{dx}\right) = \frac{-4a^2}{y^3}$$

$$\left(\frac{d^2y}{dx^2}\right)\left(\frac{d^2x}{dy^2}\right) = \left(\frac{d^2y}{dx^2}\right)\left(\frac{-\left(\frac{d^2y}{dx^2}\right)}{\left(\frac{dy}{dx}\right)^3}\right) = \frac{-\left(\frac{4a^2}{y^3}\right)^2}{\left(\frac{2a}{y}\right)^3} = \frac{-2a}{y^3}.$$

Illustration :

If $y = \left(\frac{1}{x}\right)^x$ then prove that $y_2(1) = 0$ i.e. $\frac{d^2y}{dx^2} = 0$

Sol. $y = x^{-x} \Rightarrow \ln y = -x \ln x$

$$\frac{1}{y}\left(\frac{dy}{dx}\right) = -(1 + \ln x) \Rightarrow \frac{dy}{dx} = -y(1 + \ln x) \qquad \text{so } \left.\frac{dy}{dx}\right|_{x=1} = -y$$

$$\frac{d^2y}{dx^2} = -\left(\frac{dy}{dx}\right)(1 + \ln x) - \frac{y}{x}$$

$$\left.\frac{d^2y}{dx^2}\right|_{at\ x=1} = -\left.\frac{dy}{dx}\right|_{x=1} - y = y - y = 0.$$

Illustration :

If $e^{x+y} = y^2$ then prove that $y'' = \frac{2y}{(2-y)^3}$.

Sol. $e^{x+4} = y^2 \Rightarrow x + y = 2 \ln y$

$$1 + \frac{dy}{dx} = \frac{2}{y}\left(\frac{dy}{dx}\right)$$

$$1 = \left(\frac{2-y}{y}\right)\frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{y}{2-y}$$

$$\frac{d^2y}{dx^2} = \frac{(2-y)\frac{dy}{dx} - y\left(\frac{-dy}{dx}\right)}{(2-y)^2} = \frac{2}{(2-y)^2}\left(\frac{dy}{dx}\right) = \frac{2y}{(2-y)^3}.$$

Illustration :

(i) Use induction to prove that $D(x^n) = n\,x^{n-1} \;\; \forall\; n \in N$

(ii) If $y = e^{\tan^{-1}x}$ then prove that $(1 + x^2)y_{n+1} + (2nx - 1)y_n + n(n - 1)y_{n-1} = 0$.

(iii) If $y = \sin(m \sin^{-1}x)$ then show that $(1 - x^2)y_{n+2} = (2n + 1)xy_{n+1} + (n^2 - m^2)y_n$.

Sol.

(i) $D(x^n) = n\,x^{n-1}$

$D(x^{n+1}) = D(x^n \cdot x) = D(x^n)\,x + x^n\,D(x)$

$= nx^{n-1}\,x + x^n = (n + 1)\,x^n$

(ii) $y = e^{\tan^{-1}x}$ then $(1 + x^2)\,y_{n+1} + (2nx - 1)\,y_n + n\,(n - 1)\,y_{n-1} = 0$

$$\frac{dy}{dx} = \frac{e^{\tan^{-1}x}}{(1+x^2)} \Rightarrow (1 + x^2)\,y_1 = e^{\tan^{-1}x} = y$$

Differentiating again

$(1 + x^2)\,y_2 + 2xy_1 = y_1$

so at $n = 1$; $(1 + x^2)\,y_2 + (2x - 1)\,y_1 + 0 = 0.$

Let this equation is true for n ; then

$(1 + x^2)\,y_{n+1} + (2nx - 1)\,y_n + n\,(n - 1)\,y_{n-1} = 0$

Now again differentiating w.r.t. to x.

$(1 + x^2)\,y_{n+2} + 2x\,y_{n+1} + (2n\,x - 1)\,y_{n+1} + 2n\,y_n + n\,(n - 1)\,y_n = 0$

$(1 + x^2)\,y_{n+2} + [2x\,(n + 1) - 1]\,y_{n+1} + n\,(n - 1)\,y_n = 0$

so the equation is valid for $(n + 1)$ also

So from M.I. $(1 + x^2)\,y_{n+1} + (2nx - 1)\,y_n + n\,(n + 1)\,y_{n-1} = 0$ is valid for all n.

(iii) $y = \sin(m \sin^{-1}x)$ then $(1 - x^2)\,y_{n+2} = (2n + 1)\,xy_{n+1} + (n^2 - m^2)\,y_n$

Now, differentiating w.r.t. to x

$$\frac{dy}{dx} = \cos(m \sin^{-1}x)\,\frac{m}{\sqrt{(1-x^2)}}$$

$$y_1 = \sqrt{(1-y^2)}\,\frac{m}{\sqrt{(1-x^2)}} \qquad \text{Now, squaring we get}$$

$$y_1^2 = m^2\,\frac{1-y^2}{1-x^2} \Rightarrow (1 - x^2)\,y_1^2 = m^2\,(1 - y^2)$$

$$(1 - x^2)\,2y\,y_2 - 2x\,y_1^2 = -m^2\,2yy_1 \Rightarrow (1 - x^2)\,y_2 - xy_1 + m^2y = 0$$

So for $n = 0$; $(1 - x^2)\,y_2 - x\,y_1 + m^2y = 0$

Let the equation $(1 - x^2)\,y_{n+2} = (2n + 1)\,xy_{n+1} + (n^2 - m^2)\,y_n$ is true for any n then again differentiating

$(1 - x^2)\,y_{n+3} - 2x\,y_{n+2} = (2n + 1)\,xy_{n+2} + (2n + 1)\,y_{n+1} + (n^2 - m^2)\,y_{n+1}$

$\Rightarrow (1 - x^2)\,y_{n+3} - 2x\,y_{n+2} - (2n + 1)\,xy_{n+2} = [(n + 1)^2 - m^2]\,y_{n+1}$

$\Rightarrow (1 - x^2)\,y_{n+3} = [2\,(n + 1) + 1]\,xy_{n+2} + [(n + 1)^2 - m^2]\,y_{n+1}$.

Practice Problem

Q.1 Find $\frac{dy}{dx}$, if

(i) $x = a\cos\theta, y = a\sin\theta$

(ii) $x = at^2, y = 2at$

(iii) $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$

(iv) $x = 2at^2, y = at^4$

(v) $x = \frac{\sin^3 t}{\sqrt{\cos 2t}}, y = \frac{\cos^3 t}{\sqrt{\cos 2t}}$

(vi) $x = a\cos\theta, y = b\cos\theta$

(vii) $x = \cos\theta - \cos 2\theta, y = \sin\theta - \sin 2\theta$

(viii) $x = a(\theta - \sin\theta), y = a(1 + \cos\theta)$

(ix) $x = a\left(\cos t + \log\tan\frac{t}{2}\right), \; y = a\sin t$

(x) $x = a\sec\theta, y = b\tan\theta$

(xi) $x = a(\cos\theta + \theta\sin\theta), y = a(\sin\theta - \theta\cos\theta)$

Q.2 If $x = \sqrt{a^{\sin^{-1}t}}, y = \sqrt{a^{\cos^{-1}t}}$, show that $\frac{dy}{dx} = -\frac{y}{x}$..

Q.3 Find $\frac{d^2y}{dx^2}$, if $y = x^3 + \tan x$.

Q.4 (i) If $y = 3e^{2x} + 2e^{3x}$, prove that $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0$.

(ii) If $y = \sin^{-1}x$, show that $(1 - x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} = 0$.

(iii) If $y = 5\cos x - 3\sin x$, prove that $\frac{d^2y}{dx^2} + y = 0$.

(iv) If $y = 3\cos(\log x) + 4\sin(\log x)$, show that $x^2y_2 + xy_1 + y = 0$

(v) If $y = Ae^{mx} + Be^{nx}$, show that $\frac{d^2y}{dx^2} - (m+n)\frac{dy}{dx} + mny = 0$

(vi) If $e^y(x + 1) = 1$. Show that $\frac{d^2y}{dx^2} = \left(\frac{dy}{dx}\right)^2$

(vii) If $y = (\tan^{-1}x)^2$, show that $(x^2 + 1)^2 y_2 + 2x(x^2 + 1)y_1 = 2$.

Q.6 For a positive constant a find $\frac{dy}{dx}$, where $y = a^{t+\frac{1}{t}}$ and $x = \left(t + \frac{1}{t}\right)^a$.

Q.7 Differentiate $\sin^2 x$ w.r.t. $e^{\cos x}$.

Q.8 Find $\frac{dy}{dx}$, if $y = 12(1 - \cos t), x = 10(t - \sin t), -\frac{\pi}{2} < t < \frac{\pi}{2}$.

Q.9 Find $\frac{dy}{dx}$, if $y = \sin^{-1}x + \sin^{-1}\sqrt{1-x^2}, -1 \le x \le 1$

Q10 If $x\sqrt{1+y} + y\sqrt{1+x} = 0$, for, $-1 < x < 1$, prove that $\frac{dy}{dx} = -\frac{1}{(1+x)^2}$.

Q.11 If $\cos y = x \cos(a+y)$, with $\cos a \ne \pm 1$, prove that $\frac{dy}{dx} = \frac{\cos^2(a+y)}{\sin a}$.

Q.12 If $x = a(\cos t + t \sin t)$ and $y = a(\sin t - t\cos t)$, find $\frac{d^2y}{dx^2}$.

Q.13 If $y = e^{a\cos^{-1}x}, -1 \le x \le 1$, show that $(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - a^2y = 0$.

Q.14 If $y = \begin{vmatrix} \sin x & \cos x & \sin x \\ \cos x & -\sin x & \cos x \\ x & 1 & 1 \end{vmatrix}$, find $\frac{dy}{dx}$.

Q.15 If $f(x) = \begin{vmatrix} x^n & n! & 2 \\ \cos x & \cos\frac{n\pi}{2} & 4 \\ \sin x & \sin\frac{n\pi}{2} & 8 \end{vmatrix}$, then find the value of $\frac{d^n}{dx^n}[f(x)]_{x=0}$

Answer key

Q.1 (i) $-\cot\theta$ (ii) $\frac{1}{t}$ (iii) $-\left(\frac{y}{x}\right)^{1/3}$ (iv) $t^2(v)$ $-\cot 3t$

(vi) $\frac{b}{a}$ (vii) $\frac{(\cos\theta - 2\cos 2\theta)}{(2\sin 2\theta - \sin\theta)}$ (viii) $-\cot\left(\frac{\theta}{2}\right)$ (ix) $\tan t$

(x) $\frac{b}{a}\ \mathrm{cosec}\,\theta$ (xi) $\tan\theta$

Q.3 $6x + 2\sec^2 x \tan x$ **Q.6** $\frac{a^{t+\frac{1}{t}}\log a}{a\left(t+\frac{1}{t}\right)^{a-1}}$ **Q.7** $-\frac{2\cos x}{e^{\cos x}}$ **Q.8** $\frac{6}{5}\cot\left(\frac{t}{2}\right)$

Q.9 0 **Q.12** $\frac{\sec^3 t}{at}, 0 < t < \frac{\pi}{2}$ **Q.14** 1 **Q.15** 0

DEDUCTION OF NEW IDENTITIES BY DIFFERENTIATING A GIVEN IDENTITY :

Illustration :

If $\cos\frac{x}{2}\cdot\cos\frac{x}{2^2}\cdot\cos\frac{x}{2^3}.......\cos\frac{x}{2^n}=\frac{\sin x}{2^n\sin\left(x/2^n\right)}$, then prove that

$$\sum_{r=1}^{n}\frac{1}{2^r}\tan\frac{x}{2^r}=\frac{1}{2^n}\cot\frac{x}{2^n}-\cot x$$

Sol. $\cos\frac{x}{2}\cdot\cos\frac{x}{2^2}\cdot\cos\frac{x}{2^3}.......\cos\frac{x}{2^n} = \frac{\sin x}{2^n\sin\left(x/2^n\right)}$ ......(1)

Taking logarithm of both sides

$$\log\left(\cos\frac{x}{2}\right)+\log\left(\cos\frac{x}{2^2}\right)+\log\left(\cos\frac{x}{2^3}\right)+.......+\log\left(\cos\frac{x}{2^n}\right)=\log(\sin x)-\log\left(2^n\sin\left(\frac{x}{2^n}\right)\right)$$

Differentiating w.r.t. to x

$$-\left(\frac{1}{2}\frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}+\frac{1}{2^2}\frac{\sin\left(\frac{x}{2^2}\right)}{\cos\left(\frac{x}{2^2}\right)}+.......+\frac{1}{2^n}\frac{\sin\left(\frac{x}{2^n}\right)}{\cos\left(\frac{x}{2^n}\right)}\right)=\cot x-\frac{1}{2^n\sin\left(\frac{x}{2^n}\right)}\cdot\cos\left(\frac{x}{2^n}\right)$$

so $$\sum_{r=1}^{n}\frac{1}{2^r}\tan\left(\frac{x}{2^r}\right) = \frac{1}{2^n}\cot\left(\frac{x}{2^r}\right)-\cot x.$$

Illustration :

If $(1+x)^n = C_0 + C_1x + C_2x^2 + ... + C_nx^n$, then prove that

$C_1 + 2C_2 + 3C_3 + ... + nC_n = n2^{n-1}$

and $C_0 + 2C_1 + 3C_2 + ... + (n+1)C_n = (n+2)2^{n-1}$

Sol. Given $(1+x)^n = C_0 + C_1x + C_2x^2 + ...+C_nx^n$ ....(1)

(i) Differentiating equation (1) w.r. to x

$n(1+x)^{n-1} = C_1 + 2C_2x + 3C_3x^2 + ... + nC_nx^{n-1}$.

put $x = 1$

$n2^{n-1} = C_1 + 2C_2 + 3C_3 + ... + nC_n$

(ii) Now multiply x on both sides of equation (1)

$x(1+x)^n = C_0x + C_1x^2 + C_2x^3 + ... + C_nx^{n+1}$

Differentiating equation (1) w.r.t to x,

$(1+x)^n + nx(1+x)^{n-1} = C_0 + 2C_1x + 3C_2x^2 + ... + (n+1)C_nx^n$

put $x = 1$

$2^n + n\,2^{n-1} = C_0 + 2C_1 + 3C_2 + ... + (n+1)C_n$

$2^{n-1}(n+2) = C_0 + 2C_1 + 3C_2 + ... + (n+1)\,C_n$

DERIVATIVE OF FUNCTIONS EXPRESSED IN THE DETERMINANT FORM :

Let $F(x) = \begin{vmatrix} f & g & h \\ u & v & w \\ l & m & n \end{vmatrix}$ where all functions are differentiable then

$$D'(x) = \begin{vmatrix} f' & g' & h' \\ u & v & w \\ l & m & n \end{vmatrix} + \begin{vmatrix} f & g & h \\ u' & v' & w' \\ l & m & n \end{vmatrix} + \begin{vmatrix} f & g & h \\ u & v & w \\ l' & m' & n' \end{vmatrix}$$

This result may be proved by first principle and the same operation can also be done column wise.

Illustration :

If f, g & h are differentiable functions of x & $D = \begin{vmatrix} f & g & h \\ (xf)' & (xg)' & (xh)' \\ (x^2f)'' & (x^2g)'' & (x^2h)'' \end{vmatrix}$

prove that $D' = \begin{vmatrix} f & g & h \\ f' & g' & h' \\ (x^3f'')' & (x^3g'')' & (x^3h'')' \end{vmatrix}$

Sol. $D = \begin{vmatrix} f & g & h \\ (xf)' & (xg)' & (xh)' \\ (x^2f)'' & (x^2g)'' & (x^2h)'' \end{vmatrix}$

$$D = \begin{vmatrix} f & g & h \\ f+xf' & g+xg' & h+xh' \\ 2f+4xf'+x^2f'' & 2g+4xg'+x^2g'' & 2h+4xh'+x^2h'' \end{vmatrix}$$

After dowing row and column operation. This determinant simplifies to

$$D = \begin{vmatrix} f & g & h \\ xf' & xg' & xh' \\ x^2f'' & x^2g'' & x^2h'' \end{vmatrix} = x^3 \begin{vmatrix} f & g & h \\ f' & g' & h' \\ f'' & g'' & h'' \end{vmatrix}$$

$$D = \begin{vmatrix} f & g & h \\ f' & g' & h' \\ x^3f'' & x^3g'' & x^3h'' \end{vmatrix}$$

$$D' = \begin{vmatrix} f & g & h \\ f' & g' & h' \\ (x^3f'')' & (x^3g'')' & (x^3h'')' \end{vmatrix}$$

Illustration :

$$\text{If } f(x) = \begin{vmatrix} \cos(x+x^2) & \sin(x+x^2) & -\cos(x+x^2) \\ \sin(x-x^2) & \cos(x-x^2) & \sin(x-x^2) \\ \sin 2x & 0 & \sin 2x^2 \end{vmatrix} \text{ then find } f'(x).$$

Sol. $$f(x) = \begin{vmatrix} \cos(x+x^2) & \sin(x+x^2) & -\cos(x+x^2) \\ \sin(x-x^2) & \cos(x-x^2) & \sin(x-x^2) \\ \sin 2x & 0 & \sin 2x^2 \end{vmatrix}$$

Expending the determinant

$$f(x) = \cos(x + x^2)\cos(x - x^2)\sin 2x^2 - \sin(x + x^2)$$

$$\left(\sin(x-x^2)\right)\left(\sin 2x^2 - \sin 2x\right) + \cos(x+x^2)\cos(x-x^2)\sin 2x$$

$$= \sin 2x^2 \cos 2x + \sin 2x \cos(2x^2) = \sin(2x^2 + 2x)$$

$$f'(x) = \cos(2x^2 + 2x)\, 2(2x + 1)$$

Illustration :

If α be a repeated root of a quadratic equation $f(x) = 0$ & $A(x)$, $B(x)$, $C(x)$ be the polynomials of degree 3, 4 & 5 respectively, then show that $\begin{vmatrix} A(x) & B(x) & C(x) \\ A(\alpha) & B(\alpha) & C(\alpha) \\ A'(\alpha) & B'(\alpha) & C'(\alpha) \end{vmatrix}$ *is divisible by $f(x)$, where dash denotes the derivative.*

Sol. $$g(x) = \begin{vmatrix} A(x) & B(x) & C(x) \\ A(\alpha) & B(\alpha) & C(\alpha) \\ A'(\alpha) & B'(\alpha) & C'(\alpha) \end{vmatrix} \quad \text{and} \quad g'(x) = \begin{vmatrix} A'(x) & B'(x) & C'(x) \\ A(\alpha) & B(\alpha) & C(\alpha) \\ A'(\alpha) & B'(\alpha) & C'(\alpha) \end{vmatrix}$$

so $g(\alpha) = 0$ *and* $g'(\alpha) = 0$

so α *is a repeated root of* $g(x)$.

so $g(x) = (x - \alpha)^2 h(x) = f(x)\, h(x)$

so $g(x)$ *is divisible by* $f(x)$.

Note : If $(x - r)$ is a factor of the polynomial $f(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_0$ repeated m times where $1 \le m \le n$ then r is a root of the equation $f'(x) = 0$ repeated $(m - 1)$ times.

General things to remember :

Illustration :

Prove that the derivative of an even differentiable function is an odd functions and the derivative of an odd differentiable function is an even function.

Sol. *Let $f(x)$ be even function*

$f(-x) = f(x)$

$f'(-x)\,(-1) = f'(x)$

$\Rightarrow$ $f'(-x) = -f'(x) \Rightarrow f'(x)$ *is an odd function.*

Let $f(x)$ be odd function

$f(-x) = -f(x)$

$f'(-x)(-1) = -f'(x)$

$\Rightarrow$ $f'(-x) = f'(x) \Rightarrow f'(x)$ *is an even function.*

3.0 L'HOSPITAL'S RULE (0^0 / ∞^0):

e.g. $f(x) = x^x$ or $\left(-\frac{1}{x}\right)^{\sin x}$

If $f(x)$ and $g(x)$ are two function such that

(i) $\lim\limits_{x\to a} f(x) = 0$ and $\lim\limits_{x\to a} g(x) = 0$

(ii) f and g are derivable / continuous at $x = a$

i.e. $\lim\limits_{x\to a} f(x) = f(a) = 0$; $\lim\limits_{x\to a} g(x) = g(a) = 0$

(iii) $f'(x)$ and $g'(x)$ are continuous at $x = a$, then

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)} = \lim_{x\to a}\frac{f''(x)}{g''(x)}$$

Illustration :

Evaluate $\lim\limits_{x\to 0}\dfrac{x\cos x - \ln(1+x)}{x^2}$.

Sol. $\lim\limits_{x\to 0}\dfrac{x\cos x - \ln(1+x)}{x^2}$ $\quad\left(\frac{0}{0}\right)$ *form*

$$= \lim_{x\to 0}\frac{(\cos x - x\sin x) - \frac{1}{1+x}}{2x} \quad \left(\frac{0}{0}\right) \text{ form}$$

$$= \lim_{x\to 0}\frac{-\sin x - (\sin x + x\cos x) + \frac{1}{(1+x)^2}}{2} = \frac{-0-(0+0)+1}{2} = \frac{1}{2}.$$

Illustration :

Evaluate find a and b if $\lim\limits_{x\to 0}\dfrac{x(1+a\cos x) - b\sin x}{x^3} = 1$

Sol. *Let* $L = \lim\limits_{x\to 0}\dfrac{x(1+a\cos x) - b\sin x}{x^3}$ $\quad\left(\frac{0}{0}\right)$ *form*

$$L = \lim_{x\to 0}\frac{(1+a\cos x) - ax\sin x - b\cos x}{3x^2} \quad \left(\frac{0}{0}\right) \text{ form}$$

$1 + a - 0 - b = 0 \Rightarrow b - a = 1$ ....(1)

$$L = \lim_{x\to 0}\frac{-a\sin x - a(x\cos x + \sin x) + b\sin x}{6x}$$

$$= \lim_{x\to 0}\frac{1}{6}\left[-a\left(\frac{\sin x}{x}\right) - a\cos x - \frac{a\sin x}{x} + \frac{b\sin x}{x}\right] = \frac{1}{6}(-a-a-a+b) = 1$$

$\Rightarrow$ $-3a + b = 6$ ....(2)

From equation (1) and (2)

$$a = \frac{-5}{2},\ b = \frac{-3}{2}.$$

Illustration :

$$\lim_{x\to 0}\frac{e^x\sin x - x - x^2}{x^2+x-\ln(1-x)}$$

Sol. $L = \lim\limits_{x\to 0}\dfrac{e^x\sin x - x - x^2}{x^2+x-\ln(1-x)}$ $\qquad \left(\dfrac{0}{0}\right)$ *form*

$$= \lim_{x\to 0}\frac{e^x(\cos x+\sin x)-1-2x}{2x+1+\dfrac{1}{(1-x)}} \qquad \left(\frac{0}{0}\right) \textit{form}$$

$$= \frac{1-1-0}{0+1+1} = 0.$$

Illustration :

Evaluate $\lim\limits_{x\to 0}\dfrac{1+\sin x-\cos x+\ln(1-x)}{x\tan^2 x}$

(A) $-\dfrac{1}{2}$ (B) $-\dfrac{1}{3}$ (C) $\dfrac{1}{6}$ (D) DNE

Sol. $\lim\limits_{x\to 0}\dfrac{1+\sin x-\cos x+\ln(1-x)}{x\tan^2 x}$

$$= \lim_{x\to 0}\frac{1+\sin x-\cos x+\ln(1-x)}{x^3\left(\dfrac{\tan^2 x}{x^2}\right)}$$

$$= \lim_{x\to 0}\frac{1+\sin x-\cos x+\ln(1-x)}{x^3} \qquad \left(\frac{0}{0}\right) \textit{form}$$

$$= \lim_{x\to 0}\frac{\cos x+\sin x-\dfrac{1}{1-x}}{3x^2} \qquad \left(\frac{0}{0}\right) \textit{form}$$

$$= \lim_{x\to 0}\frac{-\sin x+\cos x-\dfrac{1}{(1-x)^2}}{6x} \qquad \left(\frac{0}{0}\right) \textit{form}$$

$$= \lim_{x\to 0}\frac{-\cos x+\sin x-\dfrac{2}{(1-x)^3}}{6} = \frac{-1+0-2}{6} = \frac{-1}{2}.$$

Illustration :

Evaluate $\lim_{x\to 0} \frac{\log_{\sec\frac{x}{2}}(\cos x)}{\log_{\sec x}(\cos(x/2))}$

(A) 1 (B) 16 (C) 4 (D) 2

Sol. $\lim_{x\to 0} \frac{\log_{\sec\left(\frac{x}{2}\right)} \cos x}{\log_{\sec x} \cos\left(\frac{x}{2}\right)} = \lim_{x\to 0} \frac{\left(\frac{\ln \cos x}{\ln \sec \frac{x}{2}}\right)}{\left(\frac{\ln \cos \frac{x}{2}}{\ln \sec x}\right)} = \lim_{x\to 0} \left(\frac{\ln \cos x}{\ln \cos \frac{x}{2}}\right)^2$

$= \left(\lim_{x\to 0} \frac{\ln \cos x}{\ln \cos\left(\frac{x}{2}\right)}\right)^2 \left(\frac{0}{0}\right) \text{form} = \left(\lim_{x\to 0} \frac{-\tan x}{\frac{-1}{2}\tan\frac{x}{2}}\right)^2 = \left(\lim_{x\to 0} 4\frac{\frac{\tan x}{x}}{\frac{\tan\frac{x}{2}}{\left(\frac{x}{2}\right)}}\right)^2 = 4^2 = 16.$

Illustration :

Evaluate $\lim_{x\to 1} \frac{x^x - x}{x - 1 - \ln x}$

Sol. $\lim_{x\to 1} \frac{x^x - 1}{x - 1 - \ln x} = \lim_{x\to 1} \frac{e^{x\ln x} - x}{x - 1 - \ln x} \left(\frac{0}{0}\right) \text{form}$

$\lim_{x\to 1} \frac{e^{x\ln x}(1+\ln x) - 1}{\left(1 - \frac{1}{x}\right)} \left(\frac{0}{0}\right) \text{form}$

$= \lim_{x\to 1} \frac{e^{x\ln x}(1+\ln x)^2 + e^{x\ln x}\cdot\frac{1}{x} - 0}{\left(\frac{1}{x^2}\right)} \left(\frac{0}{0}\right) \text{form}$

$= 1 + 1 - 0 = 2.$

Illustration :

Evaluate $\lim_{x\to 0} \frac{1 - \cos x\cdot\cos 2x\cdot\cos 3x}{x^2}$

Sol. $\lim_{x\to 0} \frac{1 - \cos x \cos 2x \cos 3x}{x^2} = \lim_{x\to 0} \frac{1 - e^{(\ln\cos x + \ln\cos 2x + \ln\cos 3x)}}{x^2}$

$= \lim_{x\to 0} \frac{-e^{\ln(\cos x) + \ln\cos 2x + \ln\cos 3x}\cdot(-\tan x - 2\tan 2x - 3\tan 3x)}{2x} = \frac{e^0}{2}(1 + 4 + 9) = 7.$ ***Ans.***

Illustration :

Evaluate $\lim\limits_{x\to 0}\dfrac{(\cos ax)^{1/m}-(\cos bx)^{1/n}}{x^2}$

Sol. $\lim\limits_{x\to 0}\dfrac{(\cos ax)^{1/m}-(\cos bx)^{1/n}}{x^2}$ $\left(\dfrac{0}{0}\right)$ *form*

$$= \lim_{x\to 0}\frac{\frac{1}{m}(\cos ax)^{\frac{1}{m}-1}\,(-a\sin ax)-\frac{1}{n}(\cos bx)^{\frac{1}{n}-1}(-b\sin bx)}{2x}$$

$$= \frac{1}{2}\left[\frac{-a}{m}\lim_{x\to 0}(\cos ax)^{\frac{1}{m}-1}\left(\frac{\sin ax}{x}\right)+\frac{b}{n}\lim_{x\to 0}(\cos bx)^{\frac{1}{n}-1}\frac{\sin bx}{x}\right]$$

$$= \frac{1}{2}\left[\frac{-a}{m}\cdot a+\frac{b}{n}\cdot b\right]=\frac{1}{2}\left(\frac{b^2}{n}-\frac{a^2}{m}\right)=\left(\frac{mb^2-na^2}{2mn}\right).$$

Note : If $\lim\limits_{x\to a} f(x)\to\infty$ and $\lim\limits_{x\to a} g(x)\to\infty$ then also $\lim\limits_{x\to a}\dfrac{f(x)}{g(x)}=\lim\limits_{x\to a}\dfrac{f'(x)}{g'(x)}$

Illustration :

Evaluate the following limits

(i) $\lim\limits_{x\to 0^+}(\operatorname{cosec} x)^{\frac{1}{\ln x}}$ (ii) $\lim\limits_{x\to \pi/2}(\sec x)^{\cot x}$ (iii) $\lim\limits_{x\to 0^+}(\cot x)^{\frac{1}{\ln x}}$

(iv) $\lim\limits_{x\to 0}\left(\dfrac{1}{x}\right)^{\tan x}$ (v) $\lim\limits_{x\to 0} x^x$ (vi) $\lim\limits_{x\to 1}(1-x^2)^{\frac{1}{\ln(1-x)}}$

(vii) $\lim\limits_{x\to 4}\dfrac{\sqrt{2x+1}+\sqrt{x-3}-4}{\sqrt{(3x+4)}+\sqrt{5x+5}-9}$

Sol.

(i) $L=\lim\limits_{x\to 0^+}(\operatorname{cosec} x)^{\frac{1}{\ln x}}$

$$\ln L=\lim_{x\to 0^+}\frac{\ln\operatorname{cosec} x}{\ln x}\ \left(\frac{\infty}{\infty}\right)\text{ form}$$

$$=\lim_{x\to 0^+}\frac{\frac{1}{\operatorname{cosec} x}(-\operatorname{cosec} x\cdot\cot x)}{\left(\frac{1}{x}\right)}=\lim_{x\to 0^+}-\left(\frac{x}{\tan x}\right)=-1.$$

$= L = e^{-1}$

(ii) $L = \lim_{x \to \frac{\pi}{2}} (\sec x)^{\cot x}$

$$\ln L = \lim_{x \to \frac{\pi}{2}} \frac{\ln \sec x}{\tan x} = \lim_{x \to \frac{\pi}{2}} \frac{\frac{1}{\sec x} \cdot (\sec x \cdot \tan x)}{\sec^2 x} = \lim_{x \to \frac{\pi}{2}} \sin x \cdot \cos^2 x = 0$$

$L = e^0 = 1.$

(iii) $L = \lim_{x \to 0^+} (\cot x)^{\frac{1}{\ln x}} \quad \Rightarrow \quad \ln L = \lim_{x \to 0^+} \frac{\ln(\cot x)}{\ln x}$

$$\ln L = \lim_{x \to 0^+} \frac{\frac{1}{\cot x}(-\operatorname{cosec}^2 x)}{\left(\frac{1}{x}\right)} = \lim_{x \to 0^+} \frac{-x}{\left(\frac{\cos x}{\sin x} \cdot \sin^2 x\right)} = \lim_{x \to 0^+} \frac{-2x}{\sin 2x} = -1$$

$\Rightarrow \quad L = \frac{1}{e}.$

(iv) $L = \lim_{x \to 0} \left(\frac{1}{x}\right)^{\tan x}$

$$\ln L = \lim_{x \to 0} \tan x(-\ln x) = \lim_{x \to 0} \frac{-\ln x}{\cot x} \left(\frac{\infty}{\infty}\right) \text{ form}$$

$$= \lim_{x \to 0} \frac{-\frac{1}{x}}{-\operatorname{cosec}^2 x} = \lim_{x \to 0} \left(\frac{\sin x}{x}\right) \cdot \sin x = 0$$

$L = 1.$

(v) $L = \lim_{x \to 0} x^x$

$$\ln L = \lim_{x \to 0} x \ln x = \lim_{x \to 0} \frac{\ln x}{(1/x)} \left(\frac{\infty}{\infty}\right) \text{ form}$$

$$= \lim_{x \to 0} \frac{\frac{1}{x}}{\left(\frac{-1}{x^2}\right)} = \lim_{x \to 0} (-x) = 0$$

$L = 1.$

(vi) $L = \lim_{x \to 1} (1 - x^2)^{\frac{1}{\ln(1-x)}}$

$$\ln L = \lim_{x \to 1} \frac{\ln(1 - x^2)}{\ln(1 - x)} \qquad \left(\frac{\infty}{\infty}\right) \text{ form}$$

$$= \lim_{x \to 1} \frac{\left(\frac{-2x}{1 - x^2}\right)}{\left(\frac{-1}{1 - x}\right)} = \lim_{x \to 1} \frac{2x}{1 + x} = 1.$$

$L = e$

(vii) $\lim_{x \to 4} \dfrac{\sqrt{2x+1}+\sqrt{x-3}-4}{\sqrt{3x+4}+\sqrt{5x+5}-9}$ $\left(\dfrac{0}{0}\right)$ form

$$= \lim_{x \to 4} \frac{\dfrac{2}{2\sqrt{2x+1}}+\dfrac{1}{2\sqrt{x-3}}-0}{\dfrac{3}{2\sqrt{3x+4}}+\dfrac{5}{2\sqrt{5x+5}}-0}\left(\frac{0}{0}\right) form$$

$$= \frac{\dfrac{2}{3}+\dfrac{1}{1}}{\dfrac{3}{4}+1} = \frac{\dfrac{5}{3}}{\dfrac{7}{4}} = \left(\frac{20}{21}\right).$$

Practice Problem

Q.1 Evaluate the following limits using L'hospital's Rule

(i) $\lim_{x \to \frac{\pi}{2}} \tan x \log \sin x$

(ii) $\lim_{x \to 0} \dfrac{\log \cos x}{x}$

(iii) $\lim_{x \to 0} \dfrac{2^x - 1}{(1+x)^{1/2} - 1}$

(iv) $\lim_{x \to \frac{\pi}{4}} (2 - \tan x)^{\frac{1}{ln(\tan x)}}$

Q.2 If $\lim_{x \to a} \dfrac{a^x - x^a}{x^x - a^a} = -1$ and $a > 0$, then find the value of a.

Q.3 If $f(x) = \dfrac{ln(x^2+e^x)}{ln(x^4+e^{2x})}$. Then $\lim_{n \to \infty} f(x)$ is equal to

(A) 1 (B) 1/2 (C) 2 (D) None of these

Q.4 Let $f(x+y) = f(x) \cdot f(y)$ for all x and y. Suppose $f(5) = 2$ and $f'(0) = 3$, find $f'(5)$.

Q.5 Let $f(xy) = f(x)\, f(y)\ \forall\, x, y \in R$ and f is differentiable at $x = 1$ such that $f'(1) = 1$ also $f(1) \neq 0$, $f(2) = 3$, then find $f'(2)$.

Q.6 If $f\left(\dfrac{x+2y}{3}\right) = \dfrac{f(x)+2f(y)}{3}\ \forall\, x, y \in R$ and $f'(0) = 1$, $f(0) = 2$, then find $f(x)$.

Answer key

Q.1	(i)	0	(ii)	0	(iii)	$ln\, 4$	(iv)	e^{-1}	
Q.2	$a = 1$				**Q.3**	B		**Q.4**	6
Q.5	6				**Q.6**	$x + 2$			

INDEFINITE INTEGRATION

DIFFERENTIALS:

Up to this point in our work, for $y = f(x)$ we have regarded dy/dx as a composite symbol for the derivative $f'(x)$, whose component parts, dy and dx, had no meaning by themselves. It is now convenient to modify this point of view and attach meaning to dy and dx, so that thereafter we can treat dy/dx as though it were a fraction in fact as well as in appearance. We shall not however enter into any discussions on it. We shall only state that,

for a function of a single variable $y = f(x)$, the diffrential of y denoted by dy is the product of the derivative of y (with respect to x) and the diffrential of x denoted by dx. Thus,

Diffrential of $y = f(x)$ is $dy = f'(x)dx$.
For $y = x^4$, $dy = 4x^3dx$, or simply $d(x^4) = 4x^3\,dx$. Thus
$d(\sin x) = \cos x\,dx$, $d(y^2) = 2y\,dy$, $d(\tan u) = \sec^2 u\,du$.

INTEGRATION AS ANTI-DERIVATIVE :

Simplest way to define integration is as an antiderivative the inverse of a derivative. Derivative of sin x is cos x then we may say that integral of cos x is sin x.

In general, if we consider

$$\frac{d}{dx} f(x) = \phi(x)$$

or, using differentials $d\,f(x) = \phi(x)\,dx$;
then an integral of $\phi(x)$ with respect to x or an integral of $\phi(x)\,dx$ is f (x) and symbolically, we write,

$$\int \phi(x)\,dx = f(x)$$

where the symbol $\int$ which is an elongated S (the first letter of the word sum, or, of the Latin word Summa) is known as the sign of integration. Now we come to some formal definitions:

The actual process of finding the function, when its derivative or its differential is known, is called Integration as anti-derivative; the function to which the integration is applied is called Integrand and the function obtained as a result of integration is said to be Integral. In the above case, $\phi(x)$ is the integrand and f (x) is the integral.

The process of integrating many ordinary functions is simple, but in general, integration is more involved than differentiation, as will be evident from future discussions.

Summary:

If $\frac{d}{dx}[F(x)+C] = f(x)$ then $F(x) + C$ is called an antiderivative of f (x) on [a, b] and is written as $\int f(x)\,dx = F(x) + C$.

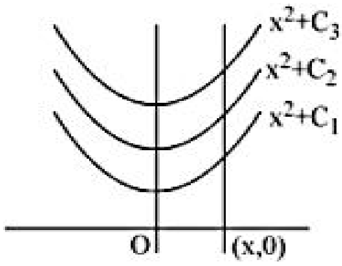
In this case we say that the function f(x) is integrable on [a, b]. Note that every function is not integrable.

e.g. $f(x)=\begin{cases} 0 & \text{if } x\in Q \\ 1 & \text{if } x\notin Q \end{cases}$ is not integrable in [0, 1]. Every function which is continuous on a closed and bounded interval is integrable.

However for integrability function f(x) may only be piece wise continuous in (a, b)

Notes on indefinite integration :

(1) **Geometrical interpretation :**



$$y=\int 2x\,dx=\frac{x^2}{2}+C$$

$$y=\int f(x)dx = F(x)+C$$

$\Rightarrow$ $F'(x)=f(x)$; $F'(x_1)=f(x_1)$

Hence $y=\int f(x)dx$ denotes a family of curves such that the slope of the tangent at $x=x_1$ on every member is same. i.e. $F'(x_1)=f(x)$ (when x_1 lies in the domain of $f(x)$) hence antiderivative of a function is not unique. If $g_1(x)$ and $g_2(x)$ are two antiderivatives of a function f(x) on [a, b] then they differ only by a constant i.e. $g_1(x)-g_2(x)=C$

(2) Antiderivative of a continuous function is differentiable

i.e. If f(x) is continuous then $\int f(x)dx=F(x)+C \Rightarrow F'(x)=f(x) \Rightarrow F'(x)$ is always exists

$\Rightarrow$ F (x) is differentiable

(3) If integrand is discontinuous at $x=x_1$ then its antiderivative at $x=x_1$ need not be discontinuous.

i.e. e.g. $\int x^{-1/3}dx$. Here $x^{-1/3}$ is discontinuous at $x=0$.

but $\int x^{-1/3}dx=\frac{3}{2}x^{2/3}+C$ is continuous at $x=0$

(4) If $\frac{d}{dx}(F(x)+C)=f(x) \Rightarrow \int f(x)dx=F(x)+C$ then only we say that f(x) is integrable.

(5) Antiderivative of a periodic function need not be a periodic function

e.g. $f(x)=\cos x+1$ is periodic but $\int(\cos x+1)dx = \sin x+x+C$ is aperiodic.

Problems based on Indefinite integral as antiderivative :

Some times it is possible to convert given integral as a loving integral (Standrad integral) after simple manipulation

Evaluate the following integrals :

Illustration :

$$\int 2^x . e^x \, dx$$

Sol. $\int 2^x e^x dx = \int (2e)^x dx = \frac{(2e)^x}{\ln(2e)} + c$

Illustration :

$$\int \frac{1-\tan^2 x}{1+\tan^2 x} \, dx$$

Sol. $\int \frac{1-\tan^2 x}{1+\tan^2 x} \, dx = \int \frac{1-\tan^2 x}{\sec^2 x} dx = \int (\cos^2 x - \sin^2 x) dx = \int \cos 2x \, dx = \frac{1}{2} \sin 2x + c$

Illustration :

$$\int \frac{(\sqrt{x}+1)(x^2-\sqrt{x})}{x\sqrt{x}+x+\sqrt{x}} \, dx$$

Sol. $\int \frac{(\sqrt{x}+1)(x^2-\sqrt{x})}{x\sqrt{x}+x+\sqrt{x}} = \int \frac{(\sqrt{x}+1)\sqrt{x}\left((\sqrt{x})^3-1\right)}{\sqrt{x}\ (x+\sqrt{x}+1)} dx$

$$= \int \frac{(\sqrt{x}+1)(\sqrt{x}-1)(x+1+\sqrt{x})}{(x+\sqrt{x}+1)} dx = \int (x-1) dx = \frac{x^2}{2} - x + c$$

Illustration :

$$\int \frac{(x^2+\sin^2 x)\sec^2 x}{1+x^2} \, dx$$

Sol. $\int \frac{(x^2+\sin^2 x)\sec^2 x}{1+x^2} dx = \int \left(\sec^2 x - \frac{1}{1+x^2}\right) dx = \tan x - \tan^{-1} x + c$

Illustration :

$$\int \frac{\sin 2x - \sin 2k}{\sin x - \sin k + \cos x - \cos k} dx$$

Sol. $\int \frac{\sin 2x - \sin 2k}{\sin x - \sin k + \cos x - \cos k} dx$

$$= \int \frac{(\sin x + \cos x + \sin k + \cos k)(\sin x + \cos x - \sin k - \cos k)}{\sin x - \sin k + \cos x - \cos k} dx$$

$$= \int (\sin x + \cos x + \sin k + \cos k) dx \quad = (\sin x - \cos x) + (\sin k + \cos k)x + C$$

Practice Problem

Q.1 $\int \frac{1+\cos^2 x}{1+\cos 2x}\,dx$ Q.2 $\int \frac{1+\tan^2 x}{1+\cot^2 x}\,dx$ Q.3 $\int \frac{x^4+x^2+1}{2\left(1+x^2\right)}\,dx$

Q.4 $\int \frac{\sin 2x+\sin 5x-\sin 3x}{\cos x+1-2\sin^2 2x}\,dx$ Q.5 $\int \frac{2+3x^2}{x^2\left(1+x^2\right)}\,dx$

Answer key

Q.1 $\frac{1}{2}(\tan x+x)+C$ Q.2 $\tan x - x + C$ Q.3 $\frac{1}{2}\left[\frac{x^3}{3}+\tan^{-1}x\right]+C$

Q.4 $-2\cos x + C$ Q.5 $-\frac{2}{x}+\tan^{-1}x+C$

TECHNIQUES OF INTERGRATION :

Often it is not possible to convert an integral into loving integral just by simple manipulation. Then required some techniques to convert an integral into loving integral. This techniques are following.

- Substitution
- By part (product rule)
- Partial (fraction)
- Kuturputur & Misc.

SUBSTITUTION :

Theory : $I = \int f(x)\,dx$ and let $x=\phi(z)$

$$\frac{dI}{dx}=f(x) \;;\quad \frac{dx}{dz}=\phi'(z)$$

$$\Rightarrow \quad \frac{dI}{dz}=\frac{dI}{dx}\cdot\frac{dx}{dz}=f(x)\,.\,\phi'(z) \text{ or } \frac{dI}{dz}=f(\phi(z))\;\phi'(z)$$

$$\boxed{\text{Hence } I=\int f(\phi(z))\phi'(z)\,dz} \qquad ....(1)$$

Substitution is said to be appropriate if the integrand in (1) is a loving one . (standard integral)

If $\int [f(x)]^n f'(x)\, dx$ or $\int \frac{f'(x)}{[f(x)]^n}\, dx$

Start with $f(x)=t$

$$\boxed{\begin{array}{l}\int(\tan x)dx = ln\sec x + C = -ln(\cos x)+C;\\ \int(\cot x)dx = ln(\sin x)\ \text{(loving integrals)}\end{array}}$$

Proof : $\int \tan x\, dx = \int \frac{\sin x}{\cos x} dx$

Put $\cos x = t$ to get $\int \frac{-dt}{t} = -ln\, t + c = -ln(\cos x)+c = ln(\sec x)+c$

Illustration :

$$\int \frac{\cos(ln\, x)}{x} dx$$

Sol. *Put* $ln\, x = t \Rightarrow \frac{1}{x}dx = dt$

Integral becomes $\int \cos dt = \sin t + c = \sin(ln\, x) + c$

Illustration :

$$\int \frac{x^3 dx}{1+x^8}$$

Sol. $x^4 = t \Rightarrow 4x^3\, dx = dt$

Integral becomes $\int \frac{\frac{1}{4}dt}{1+t^2} = \frac{1}{4}\tan^{-1} t + c = \frac{1}{4}\tan^{-1}(x^4) + c$

Illustration :

$$\int \frac{ln\left(x+\sqrt{1+x^2}\right)}{\sqrt{1+x^2}} dx$$

Sol. *Put* $ln\left(x+\sqrt{1+x^2}\right) = t \Rightarrow \frac{1+\frac{2x}{2\sqrt{1+x^2}}}{x+\sqrt{1+x^2}} dx = dt$ *or* $\frac{1}{\sqrt{1+x^2}} dx = dt$

Integral become $\int t\, dt = \frac{t^2}{2} + c = \frac{1}{2}\left\{ln\left(x+\sqrt{1+x^2}\right)\right\}^2 + c$

Illustration :

$$\int \frac{x^2 tan^{-1} x^3}{1+x^6} dx$$

Sol. *Put* $tan^{-1} x^3 = t \Rightarrow \frac{3x^2}{1+x^6} dx = dt$

Integral becomes $\int \frac{1}{3} t\, dt = \frac{1}{6} t^2 + c = \frac{1}{6} (tan^{-1} x^3)^2 + c$

Illustration :

$$\int \frac{tan\sqrt{x}\, sec^2 \sqrt{x}}{\sqrt{x}} dx$$

Sol. *Put* $tan\sqrt{x} = t \Rightarrow \frac{sec^2 \sqrt{x}}{2\sqrt{x}} dx = dt$

Integral becomes $\int 2t\, dt = t^2 + c = \left(tan\sqrt{x}\right)^2 + c$

$$\left.\begin{array}{l} \int sec\, x\, dx = ln(sec\, x + tan\, x) + C \text{ or } ln\, tan\left(\frac{\pi}{4} + \frac{x}{2}\right) + C; \\ \int cosec\, x\, dx = ln(cosec\, x - cot\, x) \text{ or } ln\, tan\frac{x}{2} + C \end{array}\right\} \text{(loving integrands)}$$

Proof : $\int sec\, x\, dx = \int \frac{sec\, x(sec\, x + tan\, x)}{sec\, x + tan\, x} dx$

Put $sec\, x + tan\, x = t \Rightarrow (sec\, x\, tan\, x + sec^2 x)dx = dt$

Integral becomes $\int \frac{dt}{t} = ln\, t$

Illustration :

$$\int \frac{cosec(tan^{-1} x)}{1+x^2} dx$$

Sol. *Put* $tan^{-1} x = t \Rightarrow \frac{1}{1+x^2} dx = dt$

integral becomes $\int \text{cosec t dt} = ln\ (cosec\, t - cot\, c) + c$

$= ln\, [cosec\, (tan^{-1} x) - cot\, (tan^{-1} x)] + c = ln\left[\frac{1+x^2}{x} - \frac{1}{x}\right] + c$

Illustration :

$$\int \frac{\cos 2x}{\sin x} dx$$

Sol. $\int \frac{1-2\sin^2 x}{\sin x} dx = \int (\text{cosec } x - 2\sin x) dx = \ln(\text{cosec } x - \cot x) + 2\cos x + c$

Illustration :

$$\int \frac{e^x(1+x)}{\cos(xe^x)} dx$$

Sol. *Put* $xe^x = t \Rightarrow e^x(1+x)dx = dt$

Integral becomes $\int \frac{dt}{\cos t} = \int \sec t \, dt = \ln(\sec t + \tan t) + c = \ln(\sec(xe^x) + \tan(xe^x)) + c$

General Substitution :

$\sqrt{a^2-x^2}$; $x = a\sin\theta$

$\sqrt{a^2+x^2}$; $x = a\tan\theta$

$\sqrt{x^2-a^2}$; $x = a\sec\theta$

$\sqrt{\frac{a^2-x^2}{a^2+x^2}}$; $x^2 = a^2\cos 2\theta$

Examples :

$\sqrt{\frac{x}{a-x}}$; $\cos\left(2\cot^{-1}\sqrt{\frac{1-x}{1+x}}\right)$ [Ans. $\frac{x^2}{2}$] ; $\frac{1}{\sqrt{-2x^2+3x+5}}$;

$\frac{dx}{(x^2+4)\sqrt{4x^2+1}}$; $\int \frac{\sqrt{(9-x^2)^3}}{x^6} dx$; $\int \frac{\sqrt{x}\, dx}{\sqrt{a^3-x^3}}$

$$\int \frac{x^2}{\sqrt{a^6-x^6}} dx$$

note that $\int \sqrt{a^2+x^2}\, dx$ & $\int \sqrt{x^2-a^2}\, dx$

to be executed by parts.

$$\boxed{\int \frac{dx}{\sqrt{x^2+a^2}} = \ln\left(x+\sqrt{x^2+a^2}\right) \ \& \int \frac{dx}{\sqrt{x^2-a^2}} = \ln\left(x+\sqrt{x^2-a^2}\right) \text{ (loving integrals)}}$$

Loving Integrals:-

$\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + c$; $\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c$;

$\int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a}\sec^{-1}\left(\frac{x}{a}\right) + c$; $\int \frac{dx}{\sqrt{x^2-a^2}} = \ln\left(x+\sqrt{x^2-a}\right) + c$;

$\int \frac{dx}{\sqrt{x^2+a^2}} = \ln\left(x+\sqrt{x^2-a^2}\right) + c$

Illustration :

$$\int \frac{\sin 2x}{\sqrt{9-\sin^4 x}}\,dx$$

Sol. *Put* $\sin^2 x = t \Rightarrow \sin 2x\,dx = dt$

Integral becomes $\int \frac{dt}{\sqrt{9-t^2}} = \sin^{-1}\left(\frac{t}{3}\right)+c = \sin^{-1}\left(\frac{\sin^2 x}{3}\right)+c$

Illustration :

$$\int \frac{e^x dx}{\sqrt{e^{2x}-1}}$$

Sol. *Put* $e^x = t \Rightarrow e^x\,dx = dt$

Integral becomes $\int \frac{dt}{\sqrt{t^2-1}} = \ln\left(t+\sqrt{t^2-1}\right)+c = \ln\left(e^x+\sqrt{e^{2x}-1}\right)+c$

Illustration :

$$\int \frac{e^x}{4+e^{2x}}\,dx$$

Sol. *Put* $e^x = t$ *to get* $\int \frac{dt}{4+t^2} = \frac{1}{2}\tan^{-1}\left(\frac{t}{2}\right)+c \qquad = \frac{1}{2}\tan^{-1}\left(\frac{e^x}{2}\right)+c$

NOTE :

For Integration of type $\int \frac{dx}{ax^2+bx+c}$ and $\int \frac{dx}{\sqrt{ax^2+bx+c}}$ make ax^2+bx+c as perfect square.

for Integration of type $\int \frac{px+q}{ax^2+bx+c}dx$ and $\int \frac{px+q}{\sqrt{ax^2+bx+c}}dx$ write $px+q=\lambda(2ax+b)+\mu$

Illustration :

$$\int \frac{e^x\,dx}{\sqrt{5-4e^x+e^{2x}}}$$

Sol. *Put* $e^x = t$ *to get* $\int \frac{dt}{\sqrt{5-4t+t^2}} = \int \frac{dt}{\sqrt{(t-2)^2+1}} = \ln\left(t-2+\sqrt{t^2-4+5}\right)+c$

Illustration :

$$\int \frac{4x+3}{3x^2+3x+1}dx$$

Sol. $4x + 3 = A\,(6x + 3) + B$ *by equating cofficients*

$$A = \frac{2}{3}\ \&\ B = 1 \quad \Rightarrow \quad \int \frac{4x+3}{3x^2+3x+1}dx = \frac{2}{3}\int \frac{6x+3}{3x^2+3x+1} + \int \frac{dx}{3x^2+3x+1}$$

$$= \frac{2}{3} ln\,(3x^2 + 3x + 1) + \frac{1}{3}\int \frac{dx}{\left(x+\frac{1}{2}\right)^2 + \frac{1}{12}} = \frac{2}{3} ln\,(3x^2 + 3x + 1) + \frac{1}{3}\frac{1}{\sqrt{\frac{1}{12}}} tan^{-1}\left(\frac{x+\frac{1}{2}}{\frac{1}{\sqrt{12}}}\right)$$

Illustration :

$$\int \frac{x\ dx}{x^4+x^2+1}$$

Sol. *Put* $x^2 = t$ *to get* $\displaystyle\int \frac{\frac{1}{2}dt}{t^2+t+1} = \frac{1}{2}\int \frac{dt}{\left(t+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$

$$= \frac{1}{2}\frac{1}{\sqrt{3/2}}\cdot tan^{-1}\left(t+\frac{1}{2}\right) + c = \frac{1}{\sqrt{3}} tan^{-1}\left(x^2+\frac{1}{2}\right) + c$$

Illustration :

$$\int \frac{5x+4}{\sqrt{x^2+2x+5}}dx.$$

Sol. *Let* $I = \displaystyle\int \frac{5x+4}{\sqrt{x^2+2x+5}}dx$

Let $5x + 4 = \lambda\,(2x + 2) + \mu$. *Comparing the coefficient's, we have*

$2\lambda = 5$ *and* $2\lambda + \mu = 4$ *gives* $\lambda = \frac{5}{2}$ *and* $\mu = -1$.

Hence, we have

$$I = \frac{5}{2}\int \frac{2x+2}{\sqrt{x^2+2x+5}}dx - \int \frac{dx}{\sqrt{x^2+2x+5}} = 5\sqrt{x^2+2x+5} - \int \frac{dx}{\sqrt{(x-1)^2+2^2}}$$

$$= 5\sqrt{x^2+2x+5} - ln\left| x+1+\sqrt{x^2+2x+5}\right| + C.$$

Practice Problem

Q.1 $\int \frac{x^2}{\sqrt{1-4x^6}}dx = \frac{1}{6}\sin^{-1}\{f(x)\}+c$. Find f'(2).

Q.2 $\int \frac{dx}{x^2(1+x^5)^{4/5}} = -\left(1+\frac{1}{x^m}\right)^n + c$. Find mn?

Q.3 $\int \frac{x+3}{\sqrt{4x^2+4x-3}}dx = \frac{1}{4}\sqrt{f(x)}+\frac{5}{2}\int\frac{dx}{\sqrt{4x^2+4x-3}}$ then find minimum value of f(x)?

Evaluate following integrals :

Q.4 $\int \frac{1}{x(\ln x)^3}dx$ Q.5 $\int \frac{x+2}{\sqrt{x-3}}dx$ Q.6 $\int \frac{\sqrt{x}}{a\sqrt{x}+b}dx$

Answer key

Q.1 24 Q.2 1 Q.3 –4 Q.4 $\frac{-1}{2\ln^2 x}+C$

Q.5 $\frac{2}{3}(x-3)^{3/2}+10\sqrt{x-3}+C$ Q.6 $\frac{1}{a^3}\left(a\sqrt{x}+b\right)^2-\frac{4b}{a^3}\left(a\sqrt{x}+b\right)+\frac{2b^2}{a^3}\ln\left|a\sqrt{x}+b\right|+C$

INTEGRATION BY PARTS :

Theory: If f (x) and g (x) are derivable functions then

$$\frac{d}{dx}[f(x).g(x)] = f(x).g'(x)+g(x).f'(x)$$

$$\therefore \quad \int \underset{\text{I}}{f(x)}.\underset{\text{II}}{g'(x)}\,dx = f(x).g(x) - \int g(x).f'(x)\,dx$$

$$I = \int \underbrace{f(x)}_{\text{I}}\cdot\underbrace{g(x)}_{\text{II}}dx$$

$$= 1^{st} \text{ function} \times \text{integral of } 2^{nd} - \int (\text{diff. co-eff. of } 1^{st}) \times (\text{integral of } 2^{nd})\,dx$$

Remember **ILATE** for deciding the choice of the first and second function which is not arbitrary.

Here I for inverse trigonometric function
L for Logarithmic function
A for Algebraic function
T for Trigonometric function
E for Exponential Function

Illustration :

$$\int x\cos x\,dx$$

Sol. *Take x as first and cos x as 2^{nd} function.*

$$\int x\cos x\,dx = x\sin x - \int 1+\sin x\,dx = x\sin x + \cos x + c.$$

Illustration :

$$\int x\tan^{-1} x\,dx$$

Sol. $$\int \underbrace{x}_{II}\ \underbrace{\tan^{-1} x}_{I}\,dx = (\tan^{-1} x)\ \frac{x^2}{2} - \int \frac{x^2}{2(1+x^2)}dx$$

$$= \frac{1}{2}x^2 (\tan^{-1} x) - \frac{1}{2}\int \frac{1+x^2-1}{1+x^2}dx = \frac{1}{2}x^2 (\tan^{-1} x) - \frac{1}{2}x + \frac{1}{2}\tan^{-1} x + c$$

Illustration :

$$\int \frac{\sin^{-1} x}{(1-x^2)^{3/2}}dx$$

Sol. *Put* $\sin^{-1} x = \theta$ *or* $x = \sin\theta \Rightarrow dx = \cos\theta\,d\theta$

Integral becomes $\int \frac{\theta.\cos\theta\,d\theta}{\cos^3\theta} = \int \underbrace{\theta}_{I}.\underbrace{\sec^2\theta}_{II}\,d\theta$

$$= \theta.\tan\theta - \int 1.\tan\theta\,d\theta = \theta\tan\theta - \ln(\sec\theta) + c. = (\sin^{-1} x).\frac{x}{\sqrt{1-x^2}} + \ln(\sqrt{1-x^2}) + c$$

Illustration :

$$\int \sin(\ln x)dx$$

Sol. *Put* $\ln x = t$ *to get* $I = \int \underbrace{e^t}_{I}\ \underbrace{\sin t}_{II}\,dt = e^t(-\cos t) - \int e^t(-\cos t)dt$

$$= -\cos t\cdot e^t + \int e^t\cos t\,dt = -\cos t\cdot e^t + \left[e^t\sin t - \int e^t\sin t\,dt\right]$$

$$\Rightarrow \quad I = -\cos t\cdot e^t + e^t\sin t - I \quad \text{or} \quad I = \frac{1}{2}e^t(\sin t - \cos t) + c$$

$$= \frac{1}{2}x(\sin(\ln x) - \cos(\ln x)) + c$$

Illustration :

$$\int x^2 e^{3x} dx$$

Sol. $I = \int \underset{I}{x^2} \underset{II}{e^{3x}} dx = \frac{x^2 e^{3x}}{3} - \int 2x \cdot \frac{e^{3x}}{3} dx$

$$= \frac{1}{3} x^2 e^{3x} - \frac{2}{3} \int \underset{I}{x} \underset{II}{e^{3x}} dx = \frac{1}{3} x^2 e^{3x} - \frac{2}{3}\left[x \cdot \frac{e^{3x}}{3} - \int \frac{e^{3x}}{3} dx\right]$$

$$= \frac{1}{3} x^2 e^{3x} - \frac{2}{9} x e^{3x} + \frac{2}{27} e^{3x} + c$$

Example : $I = \underset{I}{e^{ax}} \underset{II}{\cos(bx+c)} = e^{ax} \frac{\sin(bx+c)}{b} - \int a e^{ax} \frac{\sin(bx+c)}{b} dx$

$$= e^{ax} \frac{\sin(bx+c)}{b} - \frac{a}{b} \int \underset{I}{e^{ax}} \underset{II}{\sin(bx+c)} dx$$

$$\Rightarrow I = \frac{1}{b} e^{ax} \sin(b+c) + \frac{a}{b^2} e^{ax} \cos(bx+c) - \frac{a^2}{b^2} \int e^{ax} \cos(bx+c) dx$$

$$\text{or } I = \frac{1}{b} e^{ax} \sin(bx+c) + \frac{a}{b^2} e^{ax} \cos(bx+c) - \frac{a^2}{b^2} I$$

$$I = \frac{1}{a^2+b^2} e^{ax} [a \cos(bx+c) + b \sin(bx+c)]$$

Two Classic Integrands :

(a) $\int e^x (f(x) + f'(x)) dx = e^x f(x) + C$ **&** **(b)** $\int (f(x) + x f'(x)) dx = x\ f(x) + C$

Proof :

(a) $\int e^x (f(x) + f'(x)) dx = \int e^x f(x) dx + \int \underset{I}{e^x} \underset{II}{f'(x)} dx$

$$= \int e^x f(x) dx + e^x f(x) - \int e^x f(x) dx + c = e^x f(x) + c$$

(b) $\int f(x) dx + x f'(x) dx = \int f(x) dx + \int \underset{I}{x} \underset{II}{f'(x)} dx$

$$= \int f(x) dx + x f(x) - \int f(x) dx = x f(x) + c$$

Illustration :

$$\int \frac{xe^x}{(1+x)^2} dx$$

Sol. $\int \frac{xe^x}{(1+x)^2} dx = \int e^x \frac{1+x-1}{(1+x)^2} dx = \int e^x . \left\{\frac{1}{1+x} + \frac{-1}{(1+x)^2}\right\} = e^x \frac{1}{1+x} + c$

Illustration :

$$\int [\sin(\ln x) + \cos(\ln x)dx$$

Sol. Put $\ln x = t$ to get $\int e^t(\sin t + \cos t)dt = e^t .\sin t + c = x\sin(\ln x) + c$

Illustration :

$$\int \frac{e^x}{x}(1 + x \cdot \ln x)dx$$

Sol. $\int \frac{e^x}{x}(1 + x \cdot \ln x)dx = \int e^x\left(\frac{1}{x} + \ell n x\right)dx = e^x \ln x + c.$

Illustration :

$$\int \frac{x^2 e^x}{(x+2)^2}dx$$

Sol. $\int e^x \frac{x^2}{(x+2)^2}dx = \int e^x \frac{x^2 - 4 + 4}{(x+2)^2}dx = \int e^x\left\{\frac{x^2}{(x+2)^2} + \frac{4}{(x+2)^2}\right\}dx$

$$= \int e^x\left\{\frac{x-2}{x+2} + \frac{4}{(x+2)^2}\right\}dx = \int e^x\left(\frac{x-2}{x+2}\right) + c$$

Illustration :

$$\int(\sin x + x\cos x)dx$$

Sol. $\int(\sin x + x\cos x)dx = x\sin x + c$

Illustration :

$$\int(2\ln x + (\ln x)^2)dx$$

Sol. $\int(2\ln x + (\ln x)^2)dx = \int\left(x \cdot \frac{2\ln x}{x} + (\ln x)^2\right)dx = x \cdot (\ln n)^2 + c$

Illustration :

$$\int\left(\ell n(\ell n x) + \frac{1}{\ell n^2 x}\right)dx$$

Sol. Put $\ln x = t$ to get $\int e^t\left(\ln t + \frac{1}{t^2}\right)dt = \int e^t\left(\ln t - \frac{1}{t} + \frac{1}{t} + \frac{1}{t^2}\right)$

$$= e^t\left(\ln t - \frac{1}{t}\right) + c = e^{\ln x}\left(\ln(\ln x) - \frac{1}{\ln x}\right) + c = x\left[\ln(\ln x) - \frac{1}{\ln x}\right] + c$$

Practice Problem

Q.1 $\int \tan^{-1} x\, dx = x\, g(x) - \frac{1}{2} \ln(1+x^2) + c$. Find number of point of discontinuties of $g(x)$?

Q.2 $\int (\ln x)^2 dx = Ax(\ln x)^2 - Bx \ln x + cx + D$. Find value of $A+B+C$?

Evaluate the following integrals :

Q.3 $\int \frac{\ln(x^2+a^2)}{x^2} dx$ Q.4 $\int \frac{x}{1+\sin x} dx$ Q.5 $\int \frac{e^{\tan^{-1}x}(1+x+x^2)}{1+x^2} dx$

Q.6 $\int \frac{e^x(1+x+x^3)}{(1+x^2)^{3/2}} dx$

Answer key

Q.1 0 Q.2 5 Q.3 $\frac{-\ln(x^2+a^2)}{x} + \frac{2}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$

Q.4 $-x\tan\left(\frac{\pi}{4}-\frac{x}{2}\right) + 2\ln\ \ln\left|\cos\left(\frac{\pi}{4}-\frac{x}{2}\right)\right| + C$ Q.5 $x\, e^{\tan^{-1}x}$ Q.6 $\frac{e^x x}{\sqrt{x^2+1}}$

PARTIAL FRACTION :

This technique is used if a rational function is being integrated whose denomenator can be is factorised. If degree of numerator is greater then degree of denomenator then first devide numerator by denomenator

Loving Integrands $\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\ln\left(\frac{a+x}{a-x}\right)$ & $\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\ln\left(\frac{x-a}{x+a}\right)$

Illustration :

$$\int \frac{x^2+2}{(x+1)(x^2-1)} dx$$

Sol. *We have* $\frac{x^2+2}{(x+1)(x^2-1)} = \frac{x^2+2}{(x+1)^2(x-1)} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{x-1}$

i.e. $x^2 + 2 = A(x^2-1) + B(x-1) + C(x+1)^2$

Comparing the coeff. s, we have

$A + C = 1,\ B + 2C = 0$ *and* $-A - B + C = 2$ *gives* $A = \frac{1}{4}, B = \frac{-3}{2}$ *and* $C = \frac{3}{4}$

Hence, we have

$$I = \int \frac{x^2+2}{(x+1)(x^2-1)} dx = A\int\frac{dx}{x+1} + B\int\frac{dx}{(x+1)^2} + C\int\frac{dx}{x-1}$$

$$= \frac{1}{4}\ln|x+1| + \frac{3}{2(x+1)} + \frac{3}{4}\ln|x-1| + C$$

Illustration :

$$\int \frac{(1+x)^3}{(1-x)^3}\,dx$$

Sol. *We have* $\frac{(1+x)^3}{(1-x)^3} = \frac{x^3+3x^2+3x+1}{-x^3+3x^2-3x+1} + 1 - 1 = \frac{6x^2+2}{-x^3+3x^2-3x+1} - 1 = -1 - \frac{6x^2+2}{(x-1)^3}$

Note : Before decomposing into partial fractions, we must ensure that the degree of the numerator is less than the degree of the denominator. Take special note of the method of performing division. Adding 1 to the given fraction cancels out the x^3 them in the numerator, thereby reducing the degree of the numerator.

Now we have $\frac{6x^2+2}{(x-1)^3} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{(x-1)^3}$

i.e. $6x^2 + 2 = A(x-1)^2 + B(x-1) + C$

Comparing the coeff.s, we have

$A = 6, -2A + B = 0$ and $A - B + C = 2$ gives $A = 6, B = 12$ and $C = 8$

Hence, we have

$$I = \int \frac{(1+x)^3}{(1-x)^3}\,dx = -1\int dx - A\int \frac{dx}{x-1} - B\int \frac{dx}{(x-1)^2} - C\int \frac{dx}{(x-1)^3}$$

$$= -x - 6\,ln\,|x-1| + \frac{12}{x-1} + \frac{4}{(x-1)^2} + C.$$

Illustration :

$$\int \frac{1}{(x+1)(x^2+1)^2}\,dx$$

Sol. *We have* $\frac{1}{(x+1)(x^2+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2}$

i.e. $1 = A(x^2+1)^2 + (Bx+C)(x^3+x^2+x+1) + (Dx+E)(x+1)$

Comparing the coeff.s, we have

$A + B = 0,\ B + C = 0,\ 2A + B + C + D = 0$

$B + C + D + E = 0$ *and* $A + C + E = 1$

gives $A = \frac{1}{4}\ B = \frac{-1}{4}, C = \frac{1}{4}, D = \frac{-1}{2}$ *and* $E = \frac{1}{2}$

Hence, we have

$$I = \int \frac{1}{(x+1)(x^2+1)^2}\,dx = \frac{1}{4}\int \frac{dx}{x+1} - \frac{1}{4}\int \frac{x-1}{x^2+1}\,dx - \frac{1}{2}\int \frac{x-1}{(x^2+1)^2}\,dx$$

$$= \frac{1}{4}\,ln\,|x+1| - \frac{1}{8}\int \frac{2x}{x^2+1}\,dx + \frac{1}{4}\int \frac{dx}{x^2+1} - \frac{1}{4}\int \frac{2x}{(x^2+1)^2}\,dx + \frac{1}{2}\int \frac{dx}{(x^2+1)^2}$$

$$= \frac{1}{4}\,ln\,|x+1| - \frac{1}{8}\,ln\,(x^2+1) + \frac{1}{4}\tan^{-1}x + \frac{1}{4(x^2+1)} + \frac{1}{2}I_1$$

To evaluate I_1, put $x = \tan\theta$ and $dx = \sec^2\theta\, d\theta$. Thus, we have

$$I_1 = \int \frac{\sec^2\theta\, d\theta}{\sec^4\theta} = \int \frac{1+\cos 2\theta}{2} d\theta = \frac{1}{2}\left[\theta + \frac{\sin 2\theta}{2}\right] = \frac{1}{2}\left[\tan^{-1}x + \frac{x}{x^2+1}\right]$$

Hence, we have

$$I = \frac{1}{4}\ln|x+1| - \frac{1}{8}\ln(x^2+1) + \frac{1}{2}\tan^{-1}x + \frac{x+1}{4(x^2+1)} + C$$

Decomposition of Fractions Involving Even Powers of x only :

Illustration :

$$\int \frac{x^2+1}{(x^2+2)(x^2+3)}dx$$

Sol. *Before decomposing such fractions into partial fractions, it is more convenient to write them as*

$$\frac{y+1}{(y+2)(y+3)} = \frac{A}{y+2} + \frac{B}{y+3} \qquad [\text{writing } x^2 = y]$$

i.e. $\quad y + 1 = A(y+3) + B(y+2)$

Comparing the coeff.s, we have

$A + B = 1 \quad$ *and* $\quad 3A + 2B = 1 \qquad$ *gives* $\quad A = -1$ *and* $B = 2$

Thus, we have

$$f(x) = \frac{x^2+1}{(x^2+2)(x^2+3)} \quad \Rightarrow \quad \int \frac{x^2+1}{(x^2+2)(x^2+3)}dx$$

$$= \int \frac{-1}{x^2+2}dx + \int \frac{2}{x^2+3}dx = \frac{-1}{\sqrt{2}}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right) + \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right) + C$$

Substitution after manipulation (Kuturputur) :

Illustration :

$$\int \frac{dx}{x(x^n+1)}$$

Sol. $$\int \frac{dx}{x(x^n+1)} = \int \frac{dx}{x^{n+1}\left(1+\frac{1}{x^n}\right)}$$

Put $\quad 1 + \frac{1}{x^n} = t \quad$ *to get* $\quad \int \frac{-\frac{1}{n}dt}{t} = -\frac{1}{n}\ln t + c = -\frac{1}{n}\ln(1+x^{-n}) + c$

Illustration :

$$\int \frac{x^7}{(1-x^2)^5}\,dx$$

Sol. $$\int \frac{x^7 dx}{x^{10}\left(\frac{1}{x^2}-1\right)^5} = \int \frac{dx}{x^3\left(\frac{1}{x^2}-1\right)^5}$$

Put $\frac{1}{x^2}-1 = t$ *to get* $\int \frac{-\frac{1}{2}dt}{t^5}dt = \frac{-1}{2}\cdot\frac{-1}{4}t^4 + c = \frac{1}{8}\frac{1}{(x^{-2}-1)^4}+c$

Illustration :

$$\int \frac{xdx}{(1-x^4)^{3/2}}$$

Sol. $I = \int \frac{x\,dx}{x^6\left(\frac{1}{x^4}-1\right)^{3/2}} = \int \frac{dx}{x^5\left(\frac{1}{x^4}-1\right)^{3/2}}$ *Put* $\frac{1}{x^4}-1 = t \Rightarrow -4\frac{1}{x^5}dx = dt$

$$\Rightarrow I = \int \frac{-\frac{1}{4}dt}{t^{3/2}} = \frac{1}{2}\frac{1}{\sqrt{t}}+c = \frac{1}{2}\frac{1}{\sqrt{\frac{1}{x^4}-1}}$$

Practice Problem

Q.1 $\int \frac{x+1}{x(1+xe^x)}dx = ln|f(x)|+c$. Find value of $f(ln\,2)$?

Q.2 $\int \frac{x}{(x-1)(x^2+4)}dx = \frac{1}{5}\log(x-1)+\frac{1}{5}\int\frac{\lambda x+\mu}{x^2+4}dx$ Find value of $\lambda+\mu$.

Evaluate the following indefinite integrals :

Q.3 $\int \frac{dx}{x^4(x^3+1)^2}$

Q.4 $\int \frac{2x^2-3x-3}{(x-1)(x^2-2x+5)}dx$

Answer key

Q.1 $\log_{(4e)}4$

Q.2 3

Q.3 $-\frac{1}{3}\left(t-\frac{1}{t}-2\log t\right)+c$ where $t = \frac{1}{x^3}+1$

Q.4 $\frac{3}{2}ln\left(x^2-2x+5\right)+\frac{1}{2}\tan^{-1}\left(\frac{x-1}{2}\right)-ln|x-1|+C$

INTEGRALS OF TRIGONOMETRIC FUNCTIONS :

Type – 1 : $\int \frac{dx}{a+b\sin^2 x} \Big/ \int \frac{dx}{a+b\cos^2 x} \Big/ \int \frac{dx}{a\sin^2 x+b\cos^2 x+c\sin x\cos x} \Big/ \frac{dx}{(a\cos x+b\sin x)^2}$

Multiply N^r and D^r by $\sec^2 x$ or $\operatorname{cosec}^2 x$ and proceed

Type – 2 : $\int \frac{dx}{a+b\sin x} \Big/ \int \frac{dx}{a+b\cos x} \Big/ \int \frac{dx}{a+b\sin x+c\cos x}$

Convert sin x and cos x into their corresponding tangent to half the angles and

put $\tan\frac{x}{2}=t$

Type–3 : $\int \frac{a\sin x+b\cos x+c}{\ell\sin x+m\cos x+n}dx;\qquad N^r=A(D^r)+B\left(\frac{d}{dx}D^r\right)+C$

Type–4 : $\int \frac{x^2+1}{x^4+kx^2+1}dx$ **or** $\int \frac{x^2-1}{x^4+kx^2+1}dx$

Divide N^r and D^r by x^2 and take suitable substituion

Illustration :

$$\int \frac{dx}{4-5\sin^2 x}$$

Sol. $\int \frac{dx}{4-5\sin^2 x}=\int \frac{\operatorname{cosec}^2 x}{4\operatorname{cosec}^2 x-5}dx=\int \frac{\operatorname{cosec}^2 x}{4\cot^2 x-1}dx$

Put $\cot x=t$ *to get* $\int \frac{-dt}{4t^2-1}=\frac{-1}{4}\int \frac{dt}{t^2-\left(\frac{1}{2}\right)^2}=\frac{-1}{4}\ln\left(\frac{t-\frac{1}{2}}{t+\frac{1}{2}}\right)+c=-\frac{1}{4}\ln\left(\frac{2\cot x-1}{2\cot x+1}\right)+c$

Illustration :

$$\int \frac{dx}{5+4\cos x}$$

Sol. $\int \frac{dx}{5+4\cos x}=\int \frac{dx}{5+4\frac{\left(1-\tan^2\frac{x}{2}\right)}{\left(1+\tan^2\frac{x}{2}\right)}}=\int \frac{\sec^2\frac{x}{2}dx}{9+\tan^2\frac{x}{2}}$

Put $\tan\frac{x}{2}=t$ *to get* $\int \frac{2dt}{9+t^2}=\frac{2}{3}\tan^{-1}\left(\frac{t}{3}\right)+c$

Illustration :

$$\int \frac{1}{\sin x - 3\cos x - 1}dx$$

Sol. *Let* $I = \int \frac{1}{\sin x - 3\cos x - 1}dx$

Putting $\tan\frac{x}{2} = t$, *we have* $I = \int \frac{1}{\frac{2t}{1+t^2} - \frac{3(1-t^2)}{1+t^2}} \cdot \frac{2dt}{1+t^2} = \int \frac{2dt}{2t - 3(1-t^2) - (1+t^2)}$

$$= \int \frac{2dt}{2t^2 + 2t - 4} = \int \frac{dt}{t^2 + t - 2} = \int \frac{dt}{(t-1)(t+2)} = \frac{1}{3}\int \left[\frac{1}{t-1} - \frac{1}{t+2}\right]dt = \frac{1}{3}\ln\left|\frac{t-1}{t+2}\right| + C$$

$$= \frac{1}{3}\ln\left|\frac{\tan\frac{x}{2} - 1}{\tan\frac{x}{2} + 2}\right| + C$$

Illustration :

Evaluate the indefinite integral $\int \frac{\sin x + 3\cos x + 1}{\sin x - 3\cos x - 1}dx.$

Sol. *Let* $\sin x + 3\cos x + 1 = \lambda(\sin x - 3\cos x - 1) + \mu(\cos x + 3\sin x) + \nu$

Comparing the coefficient of $\sin x, \cos x$ *and constant term we have*

$$\lambda + 3\mu = 1,\ -3\lambda + \mu = 3 \text{ and } -\lambda + \nu = 1 \quad \text{kgives } \lambda = -\frac{4}{5},\ \mu = \frac{3}{5} \text{ and } \nu = \frac{1}{5}$$

Thus, we have

$$I = \frac{-4}{5}\int 1\,dx + \frac{3}{5}\int \frac{\cos x + 3\sin x}{\sin x - 3\cos x - 1}dx + \frac{1}{5}\int \frac{dx}{\sin x - 3\cos x - 1}$$

$$= \frac{-4}{5}x + \frac{3}{5}\ln|\sin x - 3\cos x - 1| + \frac{1}{5}I_1$$

Now, we have

$$I_1 = \int \frac{1}{\frac{2t}{1+t^2} - \frac{3(1-t^2)}{1+t^2} - 1} \cdot \frac{2dt}{1+t^2} \qquad \left[\text{Putting } \tan\frac{x}{2} = t\right]$$

$$= \int \frac{2dt}{2t - 3(1-t^2) - (1+t^2)} = \int \frac{2dt}{2t^2 + 2t - 4} = \int \frac{dt}{(t-1)(t+2)} = \frac{1}{3}\int\left(\frac{1}{t-1} - \frac{1}{t+2}\right)dt$$

$$= \frac{1}{3}\ln\left|\frac{t-1}{t+2}\right| = \frac{1}{3}\ln\left|\frac{\tan(x/2) - 1}{\tan(x/2) + 2}\right|$$

Hence, we have

$$I = \frac{-4}{5}x + \frac{3}{5}\ln|\sin x - 3\cos x - 1| + \frac{1}{15}\ln\left|\frac{\tan(x/2) - 1}{\tan(x/2) + 2}\right| + C$$

Illustration :

$$\int \frac{dx}{(3\sin x - 4\cos x)^2}$$

Sol. $I = \int \frac{dx}{(3\sin x - 4\cos x)^2} = \int \frac{\sec^2 x\, dx}{(3\tan x - 4)^2}$ Put $3\tan x - 4 = t \Rightarrow 3\sec^2 x\, dx = dt$

$$\Rightarrow I = \int \frac{\frac{1}{3}dt}{t^2} = -\frac{1}{3}\frac{1}{(3\tan x - 4)} + c$$

Illustration :

$$\int \frac{x^2+1}{x^4+7x^2+1}dx$$

Sol. $\int \frac{x^2+1}{x^4+7x^2+1}dx = \int \frac{\left(1+\frac{1}{x^2}\right)dx}{x^2+7+\frac{1}{x^2}}$ Put $x - \frac{1}{x} = t \Rightarrow \left(1+\frac{1}{x^2}\right)dx = dt$ & $x^2 + \frac{1}{x^2} = t^2 + 2$

To get $\int \frac{dt}{t^2+9} = \frac{1}{3}\tan^{-1}\left(\frac{t}{3}\right) + c = \frac{1}{3}\tan^{-1}\left(\frac{x^2-1}{3x}\right) + c$

Illustration :

$$\int \frac{x^2+1}{x^4+x^2+1}dx$$

Sol. $I = \int \frac{x^2+1}{x^4+x^2+1}dx = \int \frac{1+\frac{1}{x^2}}{x^2+\frac{1}{x^2}+1}dx = \int \frac{1+\frac{1}{x^2}}{\left(x-\frac{1}{x}\right)^2+3}dx$

$= \int \frac{dt}{t^2+3}$ $\left[\text{Putting } x - \frac{1}{x} = t \text{ and } \left(1+\frac{1}{x^2}\right)dx = dt\right]$

$= \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{t}{\sqrt{3}}\right) + C = \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{x^2-1}{\sqrt{3}x}\right) + C$

Illustration :

$$\int \frac{x^2+2}{x^4-5x^2+4}dx$$

Sol. $I = \int \frac{x^2+2}{x^4-5x^2+4}dx = \int \frac{1+\frac{2}{x^2}}{x^2+\frac{4}{x^2}-5}dx = \int \frac{1+\frac{2}{x^2}}{\left(x-\frac{2}{x}\right)-1}dx$

$= \int \frac{dt}{t^2-1}$ $\left[\text{Putting } x - \frac{2}{x} = t \text{ and } \left(1+\frac{2}{x^2}\right)dx = dt\right]$

$= \frac{1}{2}\ln\left|\frac{t-1}{t+1}\right| + C = \frac{1}{2}\ln\left|\frac{x^2-x-2}{x^2+x-2}\right| + C$

INTEGRATION OF IRRATIONAL ALGEBRAIC FUNCTION :

Type–1 : $\int \frac{dx}{(x-\alpha)\sqrt{(x-\alpha)(\beta-x)}}$ $(\beta > \alpha)$ (Start: $x = \alpha\cos^2\theta + \beta\sin^2\theta$)

Type – 2 : $\int \frac{dx}{(ax+b)\sqrt{px+q}}$; e.g. $\int \frac{dx}{(2x+1)\cdot\sqrt{4x+3}}$

Put $px + q = t^2$

Type – 3 : $\int \frac{dx}{(ax+b)\sqrt{px^2+qx+r}}$; e.g. $\int \frac{dx}{(x+1)\cdot\sqrt{1+x-x^2}}$

Put $ax + b = \frac{1}{t}$

Type – 4 : $\int \frac{dx}{(ax^2+bx+c)\sqrt{px+q}}$; Put $px + q = t^2$

e.g. $\int \frac{dx}{\left(x^2+5x+2\right)\sqrt{x-2}}$ this reduces to $2\int \frac{dt}{t^4+9t^2+16}$

Type – 5 : $\int \frac{dx}{(ax^2+bx+c)\sqrt{px^2+qx+r}}$

Case-I: When $(ax^2 + bx + c)$ breaks up into two linear factors, e.g.

$$I = \int \frac{dx}{(x^2-x-2)\sqrt{x^2+x+1}} \text{ then}$$

$$= \int\left(\frac{A}{x-2}+\frac{B}{x+1}\right)\frac{1}{\sqrt{x^2+x+1}}\,dx = A\underbrace{\int \frac{dx}{(x-2)\sqrt{x^2+x+1}}}_{\text{put } x-2=1/t} + B\underbrace{\int \frac{dx}{(x+1)\sqrt{x^2+x+1}}}_{\text{put } x+1=1/t}$$

Case-II: If $ax^2 + bx + c$ is a perfect square say $(lx + m)^2$ then put $lx + m = 1/t$

Case-III: If $b = 0$; $q = 0$ e.g. $\int \frac{dx}{(ax^2+b)\sqrt{px^2+r}}$ then put $x = \frac{1}{t}$ or the trigonometric substitution are also helpful.

e.g. $\int \frac{dx}{(x^2+4)\sqrt{4x^2+1}}$.

Illustration :

$$\int \frac{dx}{(x+2)\sqrt{x+1}}$$

Sol. $I = \int \frac{dx}{(x+2)\sqrt{x+1}}$ Put $x + 1 = t^2 \Rightarrow I = \int \frac{2t\,dt}{(t^2+1)t} = 2\tan^{-1} + c = 2\tan^{-1}\left(\sqrt{x+1}\right) + c$

Illustration :

$$\int \frac{dx}{(x^2+4)\sqrt{4x^2+1}}$$

Sol. $I = \int \frac{dx}{(x^2+4)\sqrt{4x^2+1}}$ Put $x = \frac{1}{t}$ to get $I = \int \frac{-\frac{1}{t^2}dt}{\left(\frac{1}{t^2}+4\right)\sqrt{\frac{4}{t^2}+1}} = \int \frac{-t\,dt}{(1+4t^2)\sqrt{4+t^2}}$

Again put $4 + t^2 = z^2 \Rightarrow tdt = zdz \Rightarrow I = \int \frac{-zdz}{\{1+4(z^2-4)\}z}$

$$= \int \frac{-dz}{4z^2-15} = \frac{1}{4}\int \frac{dz}{z^2-15/4} = -\frac{1}{4}\cdot\frac{1}{\sqrt{15}}\ln\left|\frac{2z-\sqrt{15}}{2z+\sqrt{15}}\right| + c$$

Illustration :

$$\int \frac{dx}{(x+1)\sqrt{1+x-x^2}}$$

Sol. $I = \int \frac{dx}{(x+1)\sqrt{1+x-x^2}}$ Put $x + 1 = \frac{1}{t}$ $\Rightarrow I = \int \frac{-\frac{1}{t^2}dt}{\frac{1}{t}\sqrt{\frac{1}{t}-\left(\frac{1}{t}-1\right)^2}} = \int \frac{-dt}{\sqrt{3t-1-t^2}}$

$$= \int \frac{-dt}{\sqrt{\frac{5}{4}-\left(t-\frac{3}{2}\right)^2}} = -\sin^{-1}\left(\frac{2t-3}{\sqrt{5}}\right) + c = -\sin^{-1}\left(\frac{\frac{2}{x+1}-3}{\sqrt{5}}\right) + c$$

$$= \sin^{-1}\left(\frac{3x+1}{\sqrt{5}(x+1)}\right) + c$$

Practice Problem

Q.1 $\int \frac{dx}{5\sin^2 x + 4} = \frac{1}{6}\tan^{-1} f(x) + c$. Find value of $f'(x)$ at $x = \frac{\pi}{4}$.

Q.2 $\int \frac{x^2 - 3}{x^4 + 2x^2 + 9}\, dx = \frac{1}{4}\ln|f(x)| + c$. Find $f(0)$.

Evaluate the following indefinite integrals :

Q.3 $\int \frac{1}{(\cos x + 2\sin x)^2}\, dx$

Q.4 $\int \frac{x^x(x^{2x}+1)(\ln x + 1)}{x^{4x}+1}\, dx$

Q.5 $\int \frac{\sin 2x}{\sin^4 x + \cos^4 x}\, dx$

Q.6 $\int \sqrt{\frac{5-x}{x-2}}\, dx$

Q.7 $\int \sqrt{2x^2 - x + 1}\, dx$

Answer key

Q.1 3

Q.2 1

Q.3 $\frac{-1}{2(1+2\tan x)} + C$

Q.4 $\frac{1}{\sqrt{2}}\left[\tan^{-1}\left(\frac{x^x - \frac{1}{x^x}}{\sqrt{2}}\right)\right]$

Q.5 $c - \tan^{-1}(\cot^2 x)$

Q.6 $\sqrt{(x-2)(5-x)} + 3\sin^{-1}\sqrt{\frac{x-2}{3}}$

Q.7 $\frac{1}{2}\left(x - \frac{1}{4}\right)\sqrt{2x^2 - x + 1} + \frac{7}{16\sqrt{2}}\ln\left|x - \frac{1}{4} + \frac{\sqrt{2x^2 - x + 1}}{\sqrt{2}}\right| + C$

STANDARD RESULTS (Must be memorised):

(i) $\int (ax+b)^n\, dx = \frac{(ax+b)^{n+1}}{a(n+1)} + c \quad n \neq -1$

(ii) $\int \frac{dx}{ax+b} = \frac{1}{a}\ln(ax+b) + c$

(iii) $\int e^{ax+b}\, dx = \frac{1}{a} e^{ax+b} + c$

(iv) $\int a^{px+q}\, dx = \frac{1}{p}\frac{a^{px+q}}{\ell n a}\ (a > 0) + c$

(v) $\int \sin(ax+b)\, dx = -\frac{1}{a}\cos(ax+b) + c$

(vi) $\int \cos(ax+b)\, dx = \frac{1}{a}\sin(ax+b) + c$

(vii) $\int \tan(ax+b)\, dx = \frac{1}{a}\ln\sec(ax+b) + c$

(viii) $\int \cot(ax+b)\, dx = \frac{1}{a}\ln\sin(ax+b) + c$

(ix) $\int \sec^2(ax+b)\, dx = \frac{1}{a}\tan(ax+b) + c$

(x) $\int \operatorname{cosec}^2(ax+b)\, dx = -\frac{1}{a}\cot(ax+b) + c$

(xi) $\int \sec(ax+b) \,.\, \tan(ax+b)\,dx = \frac{1}{a}\sec(ax+b)+c$

(xii) $\int \operatorname{cosec}(ax+b) \,.\, \cot(ax+b)\,dx = -\frac{1}{a}\operatorname{cosec}(ax+b)+c$

(xiii) $\int \sec x\,dx = \ln(\sec x + \tan x) + c$ **OR** $\ln\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)+c$

(xiv) $\int \operatorname{cosec} x\,dx = \ln(\operatorname{cosec} x - \cot x)+c$ **OR** $\ln\tan\frac{x}{2}+c$ **OR** $-\ln(\operatorname{cosec} x + \cot x)$

(xv) $\int \sinh x\,dx = \cosh x + c$

(xvi) $\int \cosh x\,dx = \sinh x + c$

(xvii) $\int \operatorname{sech}^2 x\,dx = \tanh x + c$

(xviii) $\int \operatorname{cosech}^2 x\,dx = -\coth x + c$

(xix) $\int \operatorname{sech} x \,.\, \tanh x\,dx = -\operatorname{sech} x + c$

(xx) $\int \operatorname{cosech} x \,.\, \coth x\,dx = -\operatorname{cosech} x + c$

(xxi) $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a}+c$

(xxii) $\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}+c$

(xxiii) $\int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a}\sec^{-1}\frac{x}{a}+c$

(xxiv) $\int \frac{dx}{\sqrt{x^2+a^2}} = \ln\left[x+\sqrt{x^2+a^2}\right]$ **OR** $\sinh^{-1}\frac{x}{a}+c$

(xxv) $\int \frac{dx}{\sqrt{x^2-a^2}} = \ln\left[x+\sqrt{x^2-a^2}\right]$ **OR** $\cosh^{-1}\frac{x}{a}+c$

(xxvi) $\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\ln\frac{a+x}{a-x}+c$

(xxvii) $\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\ln\frac{x-a}{x+a}+c$

(xxviii) $\int \sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}+c$

(xxix) $\int \sqrt{x^2+a^2}\,dx = \frac{x}{2}\sqrt{x^2+a^2}+\frac{a^2}{2}\sinh^{-1}\frac{x}{a}+c$

(xxx) $\int \sqrt{x^2-a^2}\,dx = \frac{x}{2}\sqrt{x^2-a^2}-\frac{a^2}{2}\cosh^{-1}\frac{x}{a}+c$

(xxxi) $\int e^{ax}.\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)+c$

(xxxii) $\int e^{ax}.\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)+c$

Solved Examples

Q.1 Evaluate the following indefinte integrals :

(i) $\int\left(\sqrt{x}+\frac{2}{x}-\sin x\right)dx$ (ii) $\int\left(\frac{1}{2x+3}+\sin(2x+3)\right)dx$

Sol.(i) $I=\int\left(\sqrt{x}+\frac{2}{x}-\sin x\right)dx=\int\sqrt{x}\,dx+2\int\frac{1}{x}dx-\int\sin x\,dx$

$=\frac{x^{1/2+1}}{1/2+1}+2\ln|x|-(-\cos x)+C=\frac{2}{3}x^{3/2}+2\ln|x|+\cos x+C.$

(ii) $I=\int\left(\frac{1}{2x+3}+\sin(2x+3)\right)dx=\int\frac{dx}{2x+3}+\int\sin(2x+3)dx$

$=\frac{\ln|2x+3|}{2}-\frac{\cos(2x+3)}{2}+C$

Q.2 Evaluate the following indefinite integrals :

(i) $\int\frac{\sqrt{x}}{x+1}dx$ (ii) $\int\frac{dx}{e^x+e^{-x}}$

Sol.(i) $\int\frac{\sqrt{x}}{x+1}dx=\int\frac{t}{t^2+1}\cdot 2t\,dt$ $\left[\text{Putting } \sqrt{x}=t \text{ and } dx=2t\,dt\right]$

$=\int\frac{2t^2}{t^2+1}dt=2\int\frac{t^2+1-1}{t^2+1}dt=2\int dt-2\int\frac{dt}{t^2+1}=2t-2\tan^{-1}t+C=2\sqrt{x}-2\tan^{-1}\sqrt{x}+C.$

(ii) $\int\frac{dx}{e^x+e^{-x}}=\int\frac{1}{t+1/t}\cdot\frac{dt}{t}$ $\left[\text{Putting } e^x=t \text{ and } dx=\frac{dt}{t}\right]$

$=\int\frac{dt}{t^2+1}=\tan^{-1}t+C=\tan^{-1}(e^x)+C.$

Q.3 Evaluate the following integrals :

(i) $\int x^2\sin x\,dx$ (ii) $\int(x^2+5x)e^{2x}dx$

Sol.(i) We have

$I=\int x^2\sin x\,dx=x^2(-\cos x)-\int(-\cos x)2x\,dx$ [integrating by parts]

$=-x^2\cos x+2\int x\cos x\,dx$

Using parts again, we have $\int x\cos x\,dx=x\sin x-\int\sin x\,dx=x\sin x+\cos x$

Hence, we have $I=-x^2\cos x+2x\sin x+2\cos x+C.$

(ii) $I = \int (x^2+5x)e^{2x}\,dx = (x^2+5x)\frac{e^{2x}}{2} - \int \frac{e^{2x}}{2}(2x+5)\,dx$

$$= \frac{(x^2+5x)e^{2x}}{2} - (2x+5)\frac{e^{2x}}{4} + \int \frac{e^{2x}}{4} 2\,dx$$

$$= \frac{(x^2+5x)e^{2x}}{2} - \frac{(2x+5)e^{2x}}{4} + \frac{e^{2x}}{4} + C$$

Q.4 Evaluate the following indefinte integrals :

(i) $\int \frac{(x-1)^3}{\sqrt{x}}dx$ (ii) $\int \frac{\sin^3 x + \cos^3 x}{\sin^2 x \cos^2 x}dx$

Sol.(i) $\int \frac{(x-1)^3}{\sqrt{x}}dx = \int \frac{x^3-3x^2+3x-1}{\sqrt{x}}dx = \int (x^{5/2} - 3x^{3/2} + 3x^{1/2} - x^{-1/2})dx$

$$= \frac{x^{7/2}}{7/2} - \frac{3x^{5/2}}{5/2} + \frac{3x^{3/2}}{3/2} - \frac{x^{1/2}}{1/2} + C = \frac{2}{7}x^{7/2} - \frac{6}{5}x^{5/2} + 2x^{3/2} - 2x^{1/2} + c$$

(ii) $\int \frac{\sin^3 x + \cos^3 x}{\sin^2 x \cos^2 x}dx = \int \tan x \sec x\,dx + \int \cot x \csc x\,dx = \sec x - \csc x + C.$

Q.5 Evaluate the following indefinite integrals :

(i) $\int \sin x \sin 2x \sin 3x\,dx$ (ii) $\int \sin^3 x \cos 3x\,dx$

Sol.(i) $I = \int \sin x \sin 2x \sin 3x\,dx = \int \left(\frac{\cos x - \cos 3x}{2}\right)\sin 3x\,dx$

$$= \frac{1}{2}\int (\cos x \sin 3x - \sin 3x \cos 3x)\,dx = \frac{1}{4}\int (\sin 4x + \sin 2x - \sin 6x)\,dx$$

$$= \frac{1}{4}\left(\frac{-\cos 4x}{4} - \frac{\cos 2x}{2} + \frac{\cos 6x}{6}\right) + C$$

(ii) $I = \int \sin^3 x \cos 3x\,dx = \int \left(\frac{3\sin x - \sin 3x}{4}\right)\cos 3x\,dx$ $[\sin 3x = 3\sin x - 4\sin^3 x]$

$$= \frac{3}{4}\int \sin x \cos 3x\,dx - \frac{1}{4}\int \sin 3x \cos 3x\,dx$$

$$= \frac{3}{8}\int (\sin 4x - \sin 2x)\,dx - \frac{1}{8}\int \sin 6x\,dx$$

$$= \frac{3}{8}\left(\frac{-\cos 4x}{4}\right) + \frac{3}{8}\left(\frac{\cos 2x}{2}\right) + \frac{1}{8}\left(\frac{\cos 6x}{6}\right) + C$$

Q.6 Evaluate the following indefinite integral $\int \frac{\sin 2x}{a^2 + b^2 \sin^2 x} dx$.

Sol. $I = \int \frac{\sin 2x}{a^2 + b^2 \sin^2 x} dx$ Put $\sin^2 x = t \Rightarrow \sin 2x\, dx = dt$

$$\Rightarrow I = \int \frac{dt}{a^2 + b^2 t^2} = \frac{1}{b^2} \int \frac{dt}{t^2 + (a/b)^2} = \frac{1}{b^2}\left(\frac{1}{a/b}\right)\tan^{-1}\left(\frac{t}{a/b}\right) + c$$

$$= \frac{1}{ab}\tan^{-1}\left(\frac{b \sin^2 x}{a}\right) + c$$

Q.7 Evaluate the indefinite integral $\int x^2 \cos x\, dx$.

Sol. $I = \int x^2 \cos x\, dx = x^2 \sin x - \int \sin x \cdot 2x\, dx = x^2 \sin x - 2I_1$

where $I_1 = \int x \sin dx = x(-\cos x) - \int (-\cos x) \cdot 1\, dx = -x \cos x + \sin x$

Hence, we have $I = x^2 \sin x + 2x \cos x - 2 \sin x + C$

Q.8 Evaluate the following indefinite integrals:

(i) $\int e^x \left(\frac{1-x}{1+x^2}\right)^2 dx$ (ii) $\int x(1+x^2) e^{x^2} dx$

Sol.(i) $I = \int e^x \left(\frac{1-x}{1+x^2}\right)^2 dx = \int e^x \frac{1+x^2-2x}{(1+x^2)} dx$

$$= \int e^x \left[\frac{1}{1+x^2} + \frac{-2x}{(1+x^2)^2}\right] dx = \frac{e^x}{1+x^2} + C$$

(ii) $I = \int x(1+x^2) e^{x^2} dx = \frac{1}{2}\int (t+1) e^t\, dt$ [Putting $x^2 = t$ and $2x\, dx = dt$]

$$= \frac{1}{2} t e^t + C = \frac{1}{2} x^2 e^{x^2} + C$$

Q.9 Evaluate the indefinite integral $\int \frac{1}{2 + \sin 2x + \cos 2x}$.

Sol. Let $I = \int \frac{1}{2 + \sin 2x + \cos 2x} dx$

Putting $\tan x = t$, we have

$$I = \int \frac{1}{2 + \frac{2t}{1+t^2} + \frac{1-t^2}{1+t^2}} \cdot \frac{dt}{1+t^2} = \int \frac{1}{t^2 + 2t + 3} dt$$

$$= \int \frac{1}{(t+1)^2 + 2} dt = \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{t+1}{\sqrt{2}}\right) + C = \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{1+\tan x}{\sqrt{2}}\right) + C$$

Q.10 Evaluate the indefinite integral $\int \frac{dx}{(x+1)\sqrt{2x-3}}$

Sol. Let $I = \frac{dx}{(x+1)\sqrt{2x-3}}$

Putting $2x-3=t^2$, i.e. $x = \frac{t^2+3}{2}$ and $dx = t\,dt$, we have

$$I = \int \frac{t dt}{\left(\frac{t^2+3}{2}+1\right)t} = \int \frac{2dt}{t^2+5} = \frac{2}{\sqrt{5}}\tan^{-1}\left(\frac{t}{\sqrt{5}}\right)+C = \frac{2}{\sqrt{5}}\tan^{-1}\sqrt{\frac{2x-3}{5}}+C$$

Q.11 Evaluate the indefinite integral $\int \frac{\cos^4 x\, dx}{\sin^3 x\left(\sin^5 x+\cos^5 x\right)^{3/5}}$

Sol. Let $I = \int \frac{\cos^4 x\, dx}{\sin^3 x\left(\sin^5 x+\cos^5 x\right)^{3/5}} = \int \frac{\cot^4 x\, \mathrm{cosec}^2 x\, dx}{\left(1+\cot^5 x\right)^{3/5}}$

Putting $1+\cot^5 x = t$ and $-5\cot^4 x\,\mathrm{cosec}^2 x\, dx = dt$, we have

$$I = \int \frac{-dt}{5t^{3/5}} = \frac{-1}{2}t^{2/5}+C = \frac{-1}{2}\left(1+\cot^5 x\right)^{2/5}+C.$$

Q.12 Evaluate the indefinite integral $\int \cos 2x\, ln\left(\frac{\cos x+\sin x}{\cos x-\sin x}\right)dx$.

Sol. Let $I = \int \cos 2x\, ln\left(\frac{\cos x+\sin x}{\cos x-\sin x}\right)dx$

Integrating by parts, we have

$$I = \frac{\sin 2x}{2} ln\left(\frac{\cos x+\sin x}{\cos x-\sin x}\right) - \int \frac{\sin 2x}{2}\cdot\frac{\cos x-\sin x}{\cos x+\sin x}\cdot\frac{(\cos x-\sin x)^2+(\cos x+\sin x)^2}{(\cos x-\sin x)^2}dx$$

$$= \frac{\sin 2x}{2} ln\left(\frac{\cos x+\sin x}{\cos x-\sin x}\right) - \int \frac{\sin 2x}{\cos 2x}dx = \frac{\sin 2x}{2} ln\left(\frac{\cos x+\sin x}{\cos x-\sin x}\right) + \frac{1}{2} ln|\cos 2x|+C.$$

Q.13 Evaluate the indefinite integral $\int \frac{\sec x\, dx}{\sqrt{\sin(2x+a)+\sin a}}$.

Sol. Let $I = \int \frac{\sec x\, dx}{\sqrt{\sin(2x+a)+\sin a}} = \int \frac{\sec x\, dx}{\sqrt{\sin 2x \cos a+(1+\cos 2x)\sin a}}$

$$= \int \frac{\sec x\, dx}{\sqrt{2\cos x(\sin x\cos a+\cos x\sin a)}} = \int \frac{\sec x\, dx}{\sqrt{2\cos^2 x\cos a\,(\tan x+\tan a)}}$$

$$= \frac{1}{\sqrt{2\cos a}}\int \frac{\sec^2 x\, dx}{\sqrt{\tan x+\tan a}}$$

Putting $\tan x + \tan a = t$ and $\sec^2 x\, dx = dt$, we have

$$I = \frac{1}{\sqrt{2\cos a}}\int \frac{dt}{\sqrt{t}} = \frac{1}{\sqrt{2\cos a}}\cdot 2\sqrt{t} + C = \sqrt{\frac{2}{\cos a}}\,\sqrt{\tan x + \tan a} + C.$$

Q.14 Evaluate the integral $\int \frac{1}{x}\sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}}\,dx$

Sol. Let $I = \int\left(\frac{1-\sqrt{x}}{1+\sqrt{x}}\right)^{1/2}\frac{dx}{x} = \int\left(\frac{1-\cos 2\theta}{1+\cos 2\theta}\right)^{1/2}\frac{d\left(\cos^2 2\theta\right)}{\cos^2 2\theta}$

Putting $x = \cos^2 2\theta$ and $dx = 2\cos 2\theta\,(-2\sin 2\theta)\,d\theta$, we have

$$I = \int \frac{\sin\theta}{\cos\theta}\cdot\frac{-8\sin\theta\cos\theta\,d\theta}{\cos 2\theta} = -8\int\frac{\sin^2\theta}{\cos 2\theta}\,d\theta$$

$$= 4\int\left(\frac{\cos 2\theta - 1}{\cos 2\theta}\right)d\theta = 4\int\left(1-\sec 2\theta\right)d\theta = 4\theta - 2\ln\left|\sec 2\theta + \tan 2\theta\right| + C$$

$$= 2\cos^{-1}\sqrt{x} - 2\,ln\left|\frac{1}{\sqrt{x}} + \sqrt{\frac{1}{x}-1}\right| + C = 2\cos^{-1}\sqrt{x} - 2\,ln\left|\frac{1+\sqrt{1-x}}{\sqrt{x}}\right| + C.$$

Q.15 Evaluate the integral $\int \frac{dx}{x^{11}\sqrt{1+x^4}}$.

Sol. Let $I = \int \frac{dx}{x^{11}\sqrt{1+x^4}} = \int \frac{x\,dx}{x^{12}\sqrt{1+x^4}}$

Putting $x^2 = \tan\theta$ and $2x\,dx = \sec^2\theta\,d\theta$, we have

$$I = \frac{1}{2}\int\frac{\sec^2\theta\,d\theta}{\tan^6\theta\sqrt{1+\tan^2\theta}} = \frac{1}{2}\int\frac{\sec^2\theta\,d\theta}{\tan^6\theta\sec\theta} = \frac{1}{2}\int\frac{\cos^5\theta}{\sin^6\theta}d\theta = \frac{1}{2}\int\frac{\cos^4\theta}{\sin^6\theta}.\cos\theta\,d\theta$$

Put $\sin\theta = t$

$$= \frac{1}{2}\int\frac{(1-t^2)^2}{t^6}dt = \frac{1}{2}\int\frac{t^2-2t^2+1}{t^6}dt = \frac{1}{2}\int(t^{-2}-2t^{-4}+t^{-6})dt$$

$$= \frac{-1}{2t} + \frac{1}{3t^3} - \frac{1}{10t^5} + C = \frac{-1}{2\sin\theta} + \frac{1}{3\sin^3\theta} - \frac{1}{10\sin^5\theta} + C$$

Here $\sin\theta = \frac{x^2}{\sqrt{1+x^4}}$.

Q.16 Evaluate the integral $\int \cos^{-1}\left(x+\sqrt{x^2+2}\right)dx$.

Sol. Let $I = \int \cos^{-1}\left(x+\sqrt{x^2+2}\right)dx$

Put $\quad x+\sqrt{x^2+2} = \cos t \quad$ and $\quad \left(1+\frac{x}{\sqrt{x^2+2}}\right)dx = \sin t\, dt$

i.e. $\quad dx = -\frac{\sqrt{x^2+2}}{x+\sqrt{x^2+2}} \cdot \sin t\, dt = \frac{\sqrt{x^2+2}}{\cos t} \cdot \sin t\, dt$

Now, rationalising LHS of equation (1), we have $\quad \sqrt{x^2+2} - = \frac{2}{\cos t}$

Adding equations (1) and (2), we have $\quad \sqrt{x^2+2} = \frac{1}{2}\left(\cos t + \frac{2}{\cos t}\right)$

Thus, we have $\quad dx = \frac{-1}{2}\left(\cos t + \frac{2}{\cos t}\right)\frac{\sin t}{\cos t}dt$

Hence, we have

$$I = \frac{-1}{2}\int t\left(\cos t + \frac{2}{\cos t}\right)\frac{\sin t}{\cos t}dt = \frac{-1}{2}\int t \sin t\, dt + \int t\left(\frac{-\sin t}{\cos^2 t}\right)dt$$

$$= \frac{-1}{2}\left[t(-\cos t) + \int \cos t\, dt\right] + \left[t\left(\frac{-1}{\cos t}\right) - \int \frac{-1}{\cos t}dt\right]$$

$$= \frac{1}{2}t\cos t - \frac{1}{2}\sin t - \frac{t}{\cos t} + ln\,|\sec t + \tan t| + C \quad \text{where} \quad \cos t = x+\sqrt{x^2+2}.$$

Q.17 Evaluate the integral $\int\left[\frac{1}{\sqrt[3]{x}+\sqrt[4]{x}} + \frac{ln\left(1+\sqrt[6]{x}\right)}{\sqrt[3]{x}+\sqrt{x}}\right]dx$

Sol. $I = \int\left[\frac{1}{\sqrt[3]{x}+\sqrt[4]{x}} + \frac{ln\left(1+\sqrt[6]{x}\right)}{\sqrt[3]{x}+\sqrt{x}}\right]dx$

The common denominator of $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}$ and $\frac{1}{6}$ is 12.

Putting $x = t^{12}$ and $dx = 12t^{11}\,dt$, we have

$$I = \int\left[\frac{1}{t^4+t^3} + \frac{ln\,(1+t^2)}{t^4+t^6}\right]12t^{11}dt \; = 12\left[\int\frac{t^8}{t+1}dt + \int\frac{t^7 ln(1+t^2)}{1+t^2}\right] = 12\,(I_1 + I_2)$$

Now, we have

$$I_1 = \int\frac{t^8-1+1}{t+1}dt = \int(t-1)\left(t^2+1\right)\left(t^4+1\right)dt + \int\frac{dt}{t+1}$$

$$= \int(t-1)\left(t^6+t^4+t^2+1\right)dt + ln\,|t+1| \; = \int\left(t^7-t^6+t^5-t^4+t^3+t-1\right)dt + ln\,|t+1|$$

$$= \frac{t^8}{8} - \frac{t^7}{7} + \frac{t^6}{6} + \frac{t^5}{5} + \frac{t^4}{4} - \frac{t^3}{3} + \frac{t^2}{2} - t + ln\,|t+1| \quad \text{and } I_2 = \int (t^2)^3 \ln\left(1+t^2\right)\frac{t\,dt}{1+t^2}$$

Putting $\ln(1+t^2) = u$ and $\frac{2t\,dt}{1+t^2} = du$, we have

$$I_2 = \frac{1}{2}\int \left(e^u - 1\right)^3 u\,du = \frac{1}{2}\int \left(e^{3u} - 3e^{2u} + 3e^u - 1\right) u\,du$$

$$= \frac{1}{2}\left(\frac{e^{3u}}{3} - \frac{3e^{2u}}{2} + 3e^u - u\right)u - \frac{1}{2}\int\left(\frac{e^{3u}}{3} - \frac{3e^{2u}}{2} + 3e^u - u\right)du \qquad \text{[integrating by parts]}$$

$$= \frac{u}{2}\left(\frac{e^{3u}}{3} - \frac{3e^{2u}}{2} + 3e^u - u\right) - \frac{1}{2}\left(\frac{e^{3u}}{9} - \frac{3e^{2u}}{4} + 3e^u - \frac{u^2}{2}\right)$$

$$= \frac{u}{12}\left(2e^{3u} - 9e^{2u} + 18e^u + 18e^u - 6u\right) - \frac{1}{36}\left(4e^{3u} - 27e^{2u} + 108e^u - 18u^2\right)$$

where $e^u = t^2 - 1 = x^{1/6} - 1$.

Q.18 Evaluate the following indefinite integrals $\int \sqrt{\frac{\sin(x-a)}{\sin(x+a)}}\,dx$

Sol. Let $I = \int \sqrt{\frac{\sin(x-a)}{\sin(x+a)}}\,dx = \int \frac{\sin(x-a)}{\sqrt{\sin(x+a)\sin(x-a)}}\,dx$

$$= \int \frac{\sin x \cos a - \cos x \sin a}{\sqrt{\sin^2 x - \sin^2 a}}\,dx = I_1 - I_2$$

Now, we have

$$I_1 = \int \frac{\cos a \sin x\,dx}{\sqrt{\sin^2 x - \sin^2 a}} = \cos a \int \frac{\sin x\,dx}{\sqrt{\cos^2 a - \cos^2 x}}$$

$$= \cos a \int \frac{-dt}{\sqrt{\cos^2 a - t^2}} \qquad \text{[putting } \cos x = t\text{]}$$

$$= \cos a \cos^{-1}\left(\frac{t}{\cos a}\right) = \cos a \cos^{-1}\left(\frac{\cos x}{\cos a}\right)$$

and $$I_2 = \int \frac{\sin a \cos x\,dx}{\sqrt{\sin^2 x - \sin^2 a}} = \sin a \int \frac{dt}{\sqrt{t^2 - \sin^2 a}} \qquad \text{[putting } \sin x = t\text{]}$$

$$= \sin a \ln\left|t + \sqrt{t^2 - \sin^2 a}\right| = \sin a \ln\left|\sin x + \sqrt{\sin^2 x - \sin^2 a}\right|$$

Hence, we have

$$I = \cos a \cos^{-1}\left(\frac{\cos x}{\cos a}\right) + \sin a\, ln\left|\sin x + \sqrt{\sin^2 x - \sin^2 a}\right| + C.$$

Q.19 Evaluate the following indefinite integral $\int e^x\left(\frac{1+\sin x}{1+\cos x}\right)dx$

Sol. $\int e^x\left(\frac{1+\sin x}{1+\cos x}\right)dx = \int e^x\left[\frac{1}{1+\cos x}+\frac{\sin x}{1+\cos x}\right]dx$

$$= \int e^x\left[\frac{1}{2}\sec^2\left(\frac{x}{2}\right)+\tan\left(\frac{x}{2}\right)\right]dx = e^x\tan\left(\frac{x}{2}\right)+C$$

Q.20 Evaluate the following indefinite integral $\int \frac{(x^2-1)\,dx}{x\sqrt{x^4+3x^2+1}}$

Sol. Let $I = \int \frac{(x^2-1)dx}{x\sqrt{x^4+3x^2+1}} = \int \frac{x\,dx}{\sqrt{x^4+3x^2+1}} - \int \frac{dx}{x\sqrt{x^4+3x^2+1}} = I_1 - I_2$

Putting $x^2 = t$ and $2x\,dx = dt$, we have

$$I_1 = \frac{1}{2}\int \frac{dt}{\sqrt{t^2+3t+1}} = \frac{1}{2}\int \frac{dt}{\sqrt{\left(t+\frac{3}{2}\right)^2-\left(\frac{\sqrt{5}}{2}\right)^2}}$$

$$= \frac{1}{2}\cosh^{-1}\left(\frac{2t+3}{\sqrt{5}}\right) = \frac{1}{2}\cosh^{-1}\left(\frac{2x^2+3}{\sqrt{5}}\right) \quad \text{and} \quad I_2 = \int \frac{dx}{x^3\sqrt{1+\frac{3}{x^2}+\frac{1}{x^4}}}$$

$$= \frac{-1}{2}\int \frac{dt}{\sqrt{t^2+3t+1}} \qquad \left[\text{putting } \frac{1}{x^2} = t \text{ and } \frac{-2dx}{x^3} = dt\right]$$

$$= \frac{-1}{2}\cosh^{-1}\left(\frac{2t+3}{\sqrt{5}}\right) = \frac{-1}{2}\cosh^{-1}\left(\frac{3x^2+2}{\sqrt{5}x^2}\right)$$

Hence, we have

$$I = \frac{-1}{2}\cosh^{-1}\left(\frac{x^3+3}{\sqrt{5}}\right) + \frac{1}{2}\cosh^{-1}\left(\frac{3x^2+2}{\sqrt{5}x^2}\right) + C.$$

DEFINITE INTEGRATION

Definition:

$\int_a^b f(x)\,dx = [F(x)]_a^b = F(b) - F(a)$ is called the definite integral of f(x) between the limits a and b.

where $\frac{d}{dx}(F(x)) = f(x)$

Note : The word limit here is quite different as used in differential calculus.

Important Points:

(I) If $\int_a^b f(x)\,dx = 0$, then the equation $f(x) = 0$ has atleast one root in (a, b) provided f is continuous in (a, b).

Note that the converse is not true.

Illustration :

$\int_0^1 e^x(ax^2+bx+c)\,dx = 0 \Rightarrow e^x(ax^2+bx+c) = 0$ *has at least one root in (0, 1)*

$\Rightarrow ax^2+bx+c = 0$ *has at least one root in (0, 1) [e^x is always positive]*

(II) $\lim\limits_{n\to\infty}\left(\int_a^b f_n(x)\,dx\right) = \int_a^b\left(\lim\limits_{n\to\infty} f_n(x)\right)dx$

Illustration :

If $\lim\limits_{n\to\infty}\int_{-\sqrt[3]{a}}^{\sqrt[3]{a}}\left(1-\frac{t^3}{n}\right)^n t^2\,dt = \frac{2\sqrt{2}}{3}$ *$(n \in N)$, then the value of find 'a'.*

Sol. *L.H.S.* $= \int_{-a^{1/3}}^{a^{1/3}} \lim\limits_{n\to\infty}\left(1-\frac{t^3}{n}\right)^n t^2\,dt = \int_{-a^{1/3}}^{a^{1/3}} e^{-t^3}\,t^2\,dt$

$$= \left[-\frac{1}{3}e^{-t^3}\right]_{-a^{1/3}}^{a^{1/3}} = \frac{1}{3}\left[e^a - e^{-a}\right] \Rightarrow e^a - e^{-a} = 2\sqrt{2} \;\; or \;\; a = ln\left(\sqrt{2}+\sqrt{3}\right)$$

(III) $\int_a^b f(x)\cdot d(g(x)) = \int_{g^{-1}(a)}^{g^{-1}(b)} f(x)\cdot g'(x)\,dx$.

(IV) If f(x) is continuous in (a, b), Then $\int_a^b \frac{d}{dx}(f(x)) = [f(x)]_a^b$ and if f(x) is discontinuous in (a, b) at $x = c \in (a, b)$, then $\int_a^b \frac{d}{dx}(f(x)) = [f(x)]_a^{c^-} + [f(x)]_{c^+}^b$

Illustration :

$$\int_{-1}^{1}\left(\frac{d}{dx}\left(\cot^{-1}\frac{1}{x}\right)\right)=\cot^{-1}\frac{1}{x}\Bigg]_{-1}^{0^-}+\cot^{-1}\frac{1}{x}\Bigg]_{0^+}^{1}=\pi-\left(\frac{3\pi}{4}\right)+\frac{\pi}{4}=\frac{\pi}{2}$$

(V) If g (x) is the inverse of f (x) and f (x) has domain $x \in [a, b]$ where f (a) = c and f (b) = d then the value of $\int_a^b f(x)dx + \int_c^d g(y)dy = (bd - ac)$

Illustration :

Evaluate : $\int_0^1 e^{\sqrt{e^x}}\,dx + 2\int_e^{e^{\sqrt{e}}} \ln(\ln x)dx$

Sol. Consider $f : [0, 1] \to [e,\ e^{\sqrt{e}}]$, $f(x) = e^{\sqrt{e^x}}$ then $f^{-1}(x) = 2\ln(\ln x)$

$I = \int_0^1 e^{\sqrt{e^x}}\,dx + 2\int_e^{e^{\sqrt{e}}} \ln(\ln x)dx$ hence $I = 1\cdot e^{\sqrt{e}} - 0\cdot e = e^{\sqrt{e}}$

Evaluating definite integrals by finding antiderivatives :

Illustration :

Evaluate : $\int_3^8 \frac{\sin\sqrt{x+1}}{\sqrt{x+1}}\,dx$

Sol. $\int \frac{\sin\sqrt{x+1}}{\sqrt{x+1}}dx = -2\cos\sqrt{x+1}$

$\Rightarrow n\int_3^8 \frac{\sin\sqrt{x+1}}{\sqrt{x+1}}dx = \left[-2\cos\sqrt{x+1}\right]_3^8 = 2(\cos 2 - \cos 3)$

Illustration :

Evaluate : $\int_0^{\pi/4} \cos 2x\sqrt{4-\sin 2x}\,dx$

Sol. $\int \cos 2x\sqrt{4-\sin 2x}\,dx$, Put $4 - \sin 2x = t \Rightarrow\ = -2\cos 2x\,dx = dt$

Integral becomes $\int -\frac{1}{2}\sqrt{t}\,dt = -\frac{1}{3}(t)^{3/2} = -\frac{1}{3}(4-\sin 2x)^{3/2}$

$\Rightarrow \int_0^{\pi/4} \cos 2x\sqrt{4-\sin 2x}\,dx = \left[-\frac{1}{3}(4-\sin 2x)^{3/2}\right]_0^{\pi/4} = \frac{8-3\sqrt{3}}{3}$

Illustration :

Evaluate : $\int_0^1 x\ell n(1+2x)\ dx = \frac{3\ln 3}{8}$

Sol. $\int \underset{II}{x} \ln(\underset{I}{1+2x})dx = \ln(1+2x)\cdot\frac{x^2}{2} - \int \frac{2}{1+x}\cdot\frac{x^2}{2}dx = \frac{x^2}{2}\ln(1+2x) - \int\left(\frac{x}{2}-\frac{1}{4}\right)+\frac{1}{4}\left(\frac{1}{1+2x}\right)dx$

$= \frac{x^2}{2}\ln(1+2x) - \frac{x^2}{4}+\frac{1}{4}x-\frac{1}{8}\ln(1+2x) \Rightarrow \int_0^1 x\ln(1+2x)dx = \frac{3}{8}\ln 3$

Illustration :

The value of the integral $\int_0^{2008}\left(3x^2-8028x+(2007)^2+\frac{1}{2008}\right)dx$ *equals*

(A) $(2008)^2$ (B) $(2009)^2$ (C) 2009 (D) 1

Sol. $\int\left(3x^2-8028x+(2007)^2+\frac{1}{2008}\right)dx = x^3 - 4014x^2 + (2007)^2x + \frac{x}{2008}$

$\Rightarrow$ value of integral $= \left[x^3-4014x^2+(2007)^2x+\frac{x}{2008}\right]_0^{2008} = 2009$

Illustration :

Evaluate : $\int_2^4 \frac{\sqrt{x^2-4}}{x^4}\ dx$

Sol. $\int\frac{\sqrt{x^2-4}}{x^2}dx = \int\frac{\sqrt{1-4/x^2}}{x^3}dx$ Put $1-\frac{4}{x^2}=t \Rightarrow 8x^{-3}dx = dt$ integral becomes

$\int\frac{1}{8}t^{1/2}dt = \frac{1}{12}t^{3/2} = \frac{1}{12}\left(1-\frac{4}{x^2}\right)^{3/2} \Rightarrow \int_2^4\frac{\sqrt{x^2-4}}{x^4}dx = \left[\frac{1}{12}\left(1-\frac{4}{x^2}\right)^{3/2}\right]_2^4 = \frac{\sqrt{3}}{32}$

Illustration :

Evaluate : $\int_0^{\pi/4} x\sin^2 x\,dx$

Sol. Let $I = \int_0^{\pi/4} x\sin^2 x\,dx = \int_0^{\pi/4}\frac{x}{2}(1-\cos 2x)dx$

Integrating by parts, we have

$$I = \left[\frac{x}{2}\left(x-\frac{\sin 2x}{2}\right)\right]_0^{\pi/4} - \int_0^{\pi/4}\frac{1}{2}\left(x-\frac{\sin 2x}{2}\right)dx = \frac{\pi}{8}\left(\frac{\pi}{4}-\frac{1}{2}\right)-\left[\frac{x^2}{4}+\frac{\cos 2x}{8}\right]_0^{\pi/4}$$

$$= \left(\frac{\pi^2}{32}-\frac{\pi}{16}\right)-\left(\frac{\pi^2}{64}-\frac{1}{8}\right) = \frac{\pi^2+8-4\pi}{64}.$$

Illustration :

Find the value of $\int_{0}^{1/2} \frac{x\cos^{-1}x}{\sqrt{1-x^2}}dx$.

Sol. *Let* $I = \int_{0}^{1/2} \frac{x\cos^{-1}x}{\sqrt{1-x^2}}dx$

Let us put x = cos t, dx = – sin t dt. Also, when x = 0, then $t = \frac{\pi}{2}$ *and when* $x = \frac{1}{2}$, *then* $t = \frac{\pi}{3}$.

Thus, we have

$$I = \int_{\pi/2}^{\pi/3} \frac{t\cos t}{\sin t}(-\sin t\,dt) - \int_{\pi/2}^{\pi/3} t\cos t\,dt$$

$$= \left[-t\sin t\right]_{\pi/2}^{\pi/3} + \int_{\pi/2}^{\pi/3} \sin t\,dt = \frac{\pi}{2} - \frac{\pi}{3}\cdot\frac{\sqrt{3}}{2} - \left[\cos t\right]_{\pi/2}^{\pi/3} = \frac{\pi(\sqrt{3}-1)}{2\sqrt{3}} - \frac{1}{2}.$$

Practice Problem

Q.1 Let $\int_0^1 \frac{dx}{\sqrt{16+9x^2}} + \int_0^2 \frac{dx}{\sqrt{9+4x^2}} = \ln a$. Find a.

Q.2 Evaluate: $\int_0^{\ln 2} x e^{-x} dx$

Q.3 Evaluate: $\int_0^{\pi/4} \frac{\sin^2 x \cdot \cos^2 x}{(\sin^3 x + \cos^3 x)^2} dx$

Q.4 Evaluate: $\int_0^{\pi/2} \frac{dx}{1+\cos\theta\cdot\cos x}$ $\theta \in (0, \pi)$

Q.5 Suppose that the function f, g, f ' and g ' are continuous over [0, 1], $g(x) \neq 0$ for $x \in [0, 1]$, $f(0) = 0$, $g(0) = \pi$, $f(1) = \frac{2009}{2}$ and $g(1) = 1$. Find the value of the definite integral,

$$\int_0^1 \frac{f(x)\cdot g'(x)\,\{g^2(x)-1\} + f'(x)\cdot g(x)\,\{g^2(x)+1\}}{g^2(x)} dx.$$

Answer key

Q.1 $(2^{1/3}\cdot 3^{1/2})$ Q.2 $1 - \frac{2}{e}$ Q.3 $\frac{1}{6}$ Q.4 $\left(\frac{\theta}{\sin\theta}\right)$

Q.5 2009

PROPERTIES OF DEFINITE INTEGRAL :

(A) PROPERTIES :

P – 1 : $\int_a^b f(x)\,dx = \int_a^b f(t)dt$; **P – 2 :** $\int_a^b f(x)\,dx = -\int_b^a f(x)dx$

P – 3 : $\int_a^b f(x) = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$ provided f has a piece wise continuity

or when f is not uniformly defined in (a, b)

Integral is broken at points of discontinuity or at the points where definition of 'f' changes.

Illustration :

Evaluate : $\int_0^{\pi}\sqrt{\dfrac{1+\cos 2x}{2}}\,dx$.

Sol. $\int_0^{\pi}|\cos x|dx = \int_0^{\pi/2}\cos x\,dx + \int_{\pi/2}^{\pi}-\cos x\,dx = [\sin x]_0^{\pi/2} + [-\sin x]_{\pi/2}^{\pi} = 2$

Illustration :

Evaluate : $\int_0^3 |5x-9|dx$.

Sol. $\int_0^3 |5x-9|dx = \int_0^{9/5}(9-5x)dx + \int_{9/5}^3(5x-9)dx = \left[9x-\frac{5}{2}x^2\right]_0^{9/5} + \left[\frac{5x^2}{2}-9x\right]_{9/5}^3 = \frac{15}{2}$

Illustration :

Evaluate : $\int_{-1}^3 \left[x+\frac{1}{2}\right]dx$

Sol. $I = \int_{-1}^{-1/2}\left[x+\frac{1}{2}\right]dx + \int_{-1/2}^{1/2}\left[x+\frac{1}{2}\right]dx + \int_{1/2}^{3/2}\left[x+\frac{1}{2}\right]dx + \int_{3/2}^{5/2}\left[x+\frac{1}{2}\right]dx + \int_{5/2}^{3}\left[x+\frac{1}{2}\right]dx$

$= \int_{-1}^{-1/2}-1\,dx\ \ dx + \int_{-1/2}^{1/2}0\,dx + \int_{1/2}^{3/2}1\,dx + \int_{3/2}^{5/2}2\,dx + \int_{5/2}^{3}3\,dx = 4$

Illustration :

Evaluate : $\int_1^4 \ln[x]\,dx$, *[·] is the greatest integer function.*

Sol. *Let* $I = \int_1^4 \ln[x]dx = \int_1^2 \ln[x]dx + \int_2^3 \ln[x]dx + \int_3^4 \ln[x]dx$

$= 0 + \ln 2\int_2^3 dx + \ln 3\int_2^3 dx = \ln 2 + \ln 3 = \ln 6$

Illustration :

Find the value of following integrals

(a) $\int_{-3}^{1} |x+1|\,dx$ (b) $\int_{0}^{\pi} |\cos x - \sin x|\,dx$.

Sol.

(a) We have $|x+1| = -(x+1)$ for $x \le -1$; $= x+1$ for $x > -1$

Hence, we have

$$\int_{-3}^{1} |x+1|\,dx = \int_{-3}^{-1} -(x+1)\,dx + \int_{-1}^{1} (x+1)\,dx = \left[\frac{(x+1)^2}{2}\right]_{-3}^{-1} + \left[\frac{(x+1)^2}{2}\right]_{-1}^{1}$$

$$= -(0-2) + (2-0) = 4.$$

(b) We have

$$|\cos x - \sin x| = \sqrt{2}\left|\sin x\left(x - \frac{\pi}{4}\right)\right| = -\sqrt{2}\,\sin\left(x - \frac{\pi}{4}\right) \qquad 0 \le x \le \frac{\pi}{4}$$

$$= \sqrt{2}\,\sin\left(x - \frac{\pi}{4}\right) \qquad \frac{\pi}{4} < x \le \pi$$

Hence, we have

$$\int_{0}^{\pi} |\cos x - \sin x|\,dx = -\sqrt{2}\int_{0}^{\pi/4} \sin\left(x - \frac{\pi}{4}\right)dx + \sqrt{2}\int_{\pi/4}^{\pi} \sin\left(x - \frac{\pi}{4}\right)dx$$

$$= \sqrt{2}\left[\cos\left(x - \frac{\pi}{4}\right)\right]_{0}^{\pi/4} - \sqrt{2}\left[\cos\left(x - \frac{\pi}{4}\right)\right]_{\pi/4}^{\pi}$$

$$= \sqrt{2}\left[\cos 0 - \cos\frac{\pi}{4}\right] - \sqrt{2}\left[\cos\left(x - \frac{\pi}{4}\right) - \cos 0\right]$$

$$= \sqrt{2}\left(1 - \frac{1}{\sqrt{2}}\right) - \sqrt{2}\left(\frac{1}{\sqrt{2}} - 1\right) = \sqrt{2} - 1 + 1 + \sqrt{2} = 2\sqrt{2}.$$

P – 4 : $\int_{-a}^{a} f(x)\,dx = \int_{0}^{a} (f(x) + f(-x))\,dx = \begin{cases} 0 & \text{if } f(x) \text{ is odd} \\ 2\int_{0}^{a} f(x)\,dx & \text{if } f(x) \text{ is even} \end{cases}$

Proof: $I = \int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx$ Put $x = -t$ in first integral.

$$= \int_{a}^{0} f(-t)(-dt) + \int_{0}^{a} f(x)\,dx = \int_{0}^{a} f(-t)\,dt + \int_{0}^{a} f(x)\,dx = \int_{0}^{a} f(-x)\,dx + \int_{0}^{a} f(x)\,dx$$

$$= \int_{0}^{a} \{f(x) + f(-x)\}\,dx$$

Illustration :

Show that $\int_{-1/2}^{1/2} \sec x \ln \frac{1-x}{1+x}\, dx = 0$.

Sol. Let $f(x) = \sec x \ln\left(\frac{1-x}{1+x}\right)$ then $f(-x) = \sec(-x) \ln\left(\frac{1+x}{1-x}\right) = -f(x)$

Illustration :

Show that $\int_{-1/2}^{1/2}\left([x] + \ln\left(\frac{1+x}{1-x}\right)\right)dx = -\frac{1}{2}$.

Sol. $I = \int_{-1/2}^{1/2}\left([x] + \ln\left(\frac{1+x}{1-x}\right)\right)dx = \int_{-1/2}^{1/2} [x]\, dx + \int_{-1/2}^{1/2} \ln\left(\frac{1+x}{1-x}\right)dx$

$= \int_{-1/2}^{1/2} [x]\, dx + 0 \qquad \left(\ln\left(\frac{1+x}{1-x}\right) \text{ is an odd function}\right)$

$= \int_{-1/2}^{0} -1\, dx + \int_{0}^{1/2} 0\, dx = -\frac{1}{2}$

Illustration :

The value of $\int_{-1}^{3}\left(\tan^{-1}\frac{x}{x^2+1} + \tan^{-1}\frac{x^2+1}{x}\right)dx$ is

(A) π (B) 2π (C) 3π (D) $5\pi/2$

Sol. $I = \int_{-1}^{3}\left(\tan^{-1}\frac{x}{x^2+1} + \tan^{-1}\frac{x^2+1}{x}\right)dx$

$= \int_{-1}^{1}\left(\tan^{-1}\frac{x}{x^2+1} + \tan^{-1}\frac{x^2+1}{x}\right)dx + \int_{1}^{3}\left(\tan^{-1}\frac{x}{x^2+1} + \tan^{-1}\frac{x^2+1}{x}\right)dx$

$= 0 + \int_{1}^{3} \tan^{-1}\left(\frac{x}{x^2+1}\right) + \cot^{-1}\left(\frac{x}{x^2+1}\right)dx = 0 + \int_{1}^{3}\frac{\pi}{2}dx = 2 \times \frac{\pi}{2} = \pi$ ***Ans.***

Illustration :

Evaluate : $\int_{-1}^{1} x^3 \tan(x^2)\, dx$

Sol. Since $x^3 \tan(x^2)$, $x \in [-1, 1]$ is an odd function, hence by property (4), we have

$$\int_{-1}^{1} x^3 \tan(x^2)\, dx = 0.$$

Practice Problem

Evaluate the following definite integral

Q.1 $\int_{-\pi/2}^{0} |\sin x + \cos x|\, dx$

Q.2 $\int_{-\sqrt{2}}^{\sqrt{2}} \frac{2x^7+3x^6-10x^5-7x^3-12x^2+x+1}{x^2+2}\, dx$

Q.3 $\int_{-2}^{2} \frac{x^2-x}{\sqrt{x^2+4}}\, dx$

Q.4 $\int_{0}^{\sqrt{3}} \sin^{-1}\frac{2x}{1+x^2}\, dx$

Q.5 $\int_{-1}^{1} \frac{(2x^{332}+x^{998}+4x^{1668}\cdot\sin x^{691})}{1+x^{666}}\, dx$

Answer key

Q.1 $2\left(\sqrt{2}-1\right).$

Q.2 $\frac{\pi}{2\sqrt{2}} - \frac{16\sqrt{2}}{5}$

Q.3 $4\sqrt{2}-4\,ln(\sqrt{2}+1)$

Q.4 $\frac{\pi\sqrt{3}}{3}$

Q.5 $\frac{\pi+4}{666}$

P – 5 : $\int_{a}^{b} f(x)\,dx = \int_{a}^{b} f(a+b-x)\,dx$ or $\int_{0}^{a} f(x)\,dx = \int_{0}^{a} f(a-x)\,dx$

Proof : $I = \int_a^b (a+b-x)dx$ Put $a+b-x=t \Rightarrow -dx = dt$ & $I = \int_b^a f(t)(-dt)$

$= \int_a^b f(t)\,dt = \int_a^b f(x)dx$

P – 6 : $\int_{0}^{2a} f(x)\,dx = \int_{0}^{a} f(x)\,dx + \int_{0}^{a} f(2a-x)\,dx \Rightarrow \left[\begin{array}{ll} 0 & \text{if } f(2a-x) = -f(x) \\ 2\int_0^a f(x)\,dx & \text{if } f(2a-x) = f(x) \end{array}\right.$

Proof : $I = \int_0^{2a} f(x)dx = \int_0^a f(x)dx + \int_a^{2a} f(x)dx$ Put $x = 2a-t$ in 2nd integral

$\Rightarrow I = \int_0^a f(x)dx + \int_a^0 f(2a-t)(-dt) = \int_0^a f(x)dx + \int_0^a f(2a-x)dx$

$= \int_0^a \{f(x)+f(2a-x)\}\,dx$

Illustration :

Evaluate : $\int_{50}^{100} \frac{\ell nx}{\ell nx+\ell n(150-x)} dx$

Sol. $I = \int_{50}^{100} \frac{\ell nx}{\ell nx+\ell n(150-x)} dx$ using P-5 $[x \to 100 + 50 - x]$

$I = \int_{50}^{100} \frac{ln(150-x)}{ln(150-x)+ln\, x} dx$

$I + I = \int_{50}^{100} 1\, dx = 50 \Rightarrow I = 25$

Illustration :

Evaluate : $\int_{0}^{\pi} \frac{dx}{1+2^{tanx}}$

Sol. $I = \int_{0}^{\pi} \frac{dx}{1+2^{tanx}}$ using P-5 $[x \to \pi - x]$

$I = \int_{0}^{\pi} \frac{dx}{1+2^{-tan\, x}} = \int_{0}^{\pi} \frac{2^{\tan x}}{1+2^{\tan x}} dx$

$\Rightarrow I + I = \int_{0}^{\pi} 1\, dx = \pi$ or $I = \frac{\pi}{2}$

Illustration :

Evaluate : $\int_{-\pi/4}^{\pi/4} \frac{tan^2 x}{1+e^x} dx$

Sol. $I = \int_{-\pi/4}^{\pi/4} \frac{tan^2 x}{1+e^x} dx$ using P-5 $[x \to 0 - x]$

$\Rightarrow I = \int_{-\pi/4}^{\pi/4} \frac{tan^2 x}{1+e^{-x}} dx$ or $I + I = \int_{-\pi/4}^{\pi/4} tan^2 x\, dx = 2\int_{0}^{\pi/4} (sec^2 x - 1) dx$

$\Rightarrow I = [tan\, x - x]_0^{\pi/4} = 1 - \frac{\pi}{4}$

Illustration :

Find the value of following integral

(a) $\displaystyle\int_0^{\pi/2} \frac{1}{1+\sqrt{\tan x}}\,dx$ (b) $\displaystyle\int_2^{3} \frac{\sqrt{5-x}}{\sqrt{x}+\sqrt{5-x}}\,dx$

Sol.

(a) $$\int_0^{\pi/2} \frac{1}{1+\sqrt{\tan x}}\,dx = \int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}}\,dx$$

Also, we have by property P-5

$$I = \int_0^{\pi/2} \frac{\sqrt{\cos(\pi/2-x)}}{\sqrt{\cos(\pi/2-x)}+\sqrt{\sin(\pi/2-x)}}\,dx = \int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\,dx$$

Adding the above integrals, we have

$$2I = \int_0^{\pi/2} \frac{\sqrt{\sin x}+\sqrt{\cos x}}{\sqrt{\sin x}+\sqrt{\cos x}}\,dx = \int_0^{\pi/2} 1\,dx = [x]_0^{\pi/2} = \frac{\pi}{2}$$

i.e. $I = \dfrac{\pi}{4}$.

(b) *Let* $I = \displaystyle\int_2^{3} \frac{\sqrt{5-x}}{\sqrt{x}+\sqrt{5-x}}\,dx$

Also, we have by property P-5

$$I = \int_2^{3} \frac{\sqrt{5-(5-x)}}{\sqrt{5-x}+\sqrt{5-(5-x)}}\,dx = \int_2^{3} \frac{\sqrt{x}}{\sqrt{5-x}+\sqrt{x}}\,dx$$

Adding the above integrals, we have

$$2I = \int_2^{3} \frac{\sqrt{5-x}\ \sqrt{x}}{\sqrt{5-x}+\sqrt{x}}\,dx = \int_2^{3} 1\,dx = [x]_2^3 = 1$$

i.e. $I = \dfrac{1}{2}$.

Illustration :

Evaluate : $\displaystyle\int_0^{\pi/2} \frac{dx}{1+\sin x}$

Sol. $$\int_0^{\pi/2} \frac{dx}{1+\sin x} = \int_0^{\pi/2} \frac{dx}{1+\sin(\pi/2-x)} = \int_0^{\pi/2} \frac{dx}{1+\cos x}$$

$$= \frac{1}{2}\int_0^{\pi/2} \sec^2\frac{x}{2}\,dx = \frac{1}{2}\left[\frac{\tan x/2}{1/2}\right]_0^{\pi/2} = 1.$$

Illustration :

$$I=\int_0^{2\pi} \sin^4 x\,dx = k\int_0^{\pi/2}\cos^4 x\,dx \quad \text{find value of } k?$$

Sol. Let $f(x)=\sin^4 x$ then $f(2\pi-x)=\sin^4(2\pi-x)=f(x)$

using P-6 $\quad I=2\int_0^{\pi}\sin^4 x\,dx$, again using P-6 $\quad I=4\int_0^{\pi/2}\sin^4 x\,dx$

using P-5 $\quad I=4\int_0^{\pi/2}\cos^4 x\,dx$ **Ans. 4**

Illustration :

Evaluate : $\quad I=\int_0^{2\pi} x\cdot\cos^5 x\,dx$

Sol. Using P-5 $\quad I=\int_0^{2\pi}(2\pi-x)\cos^5 x\,dx$

$$\Rightarrow \quad I+I=\int_0^{2\pi}(x+2\pi-x)\cos^5 x\,dx \quad \text{or } I=\int_0^{2\pi}\pi\cos^5 x\,dx = 2\pi\int_0^{\pi}\cos^5 x\,dx \quad \text{[using P-5]}$$

$= 2\pi\times 0$ $\quad$ [using P-6 as $\cos^5(\pi-x)=-\cos^5 x$]

Illustration :

Evaluate : $\displaystyle\int_0^{\pi}\frac{\pi-x}{a^2\cos^2 x+b^2\sin^2 x}\,dx$

Sol. Let $\displaystyle I=\int_0^{\pi}\frac{\pi-x}{a^2\cos^2 x+b^2\sin^2 x}\,dx$ $\quad$ using P-5

we have

$$I=\int_0^{\pi}\frac{\pi-(\pi-x)}{a^2\cos^2(\pi-x)+b^2\sin^2(\pi-x)}\,dx = \int_0^{\pi}\frac{x}{a^2\cos^2 x+b^2\sin^2 x}\,dx$$

Adding the above integrals, we have

$$2I=\int_0^{\pi}\frac{\pi}{a^2\cos^2 x+b^2\sin^2 x}\,dx \quad \text{i.e.} \quad I=\frac{\pi}{2}\int_0^{\pi}\frac{\sec^2 x}{a^2+b^2\tan^2 x}\,dx$$

Hence, we have

$$I = \pi \int_0^{\pi/2} \frac{\sec^2 x}{a^2 + b^2 \tan^2 x} dx \qquad \text{[using P-6]}$$

Putting $\tan x = t$ and $\sec^2 x\, dx = dt$, we have

$$I = \pi \int_0^{\infty} \frac{dt}{a^2 + b^2 t^2} = \frac{\pi}{ab} \lim_{t \to \infty} \left[\tan^{-1}\left(\frac{bt}{a}\right) \right]_0^t = \frac{\pi}{ab} \cdot \frac{\pi}{2} = \frac{\pi^2}{ab}.$$

Practice Problem

Q.1 Prove that $\int_0^{\pi/4} \ln(1 + \tan x)\, dx = \frac{\pi}{8} \ln 2$

Q.2 $\int_{\pi/4}^{3\pi/4} \frac{x \sin x}{1 + \sin x} dx$

Q.3 $\int_0^{\pi} \frac{(ax+b)\sec x \tan x}{4 + \tan^2 x} dx$ (a,b>0)

Q.4 $\int_0^{\pi} \frac{(2x+3)\sin x}{(1+\cos^2 x)} dx$

Q.5 $\pi \int_0^{\pi} \frac{x^2 \sin 2x \cdot \sin\left(\frac{\pi}{2}\cos x\right)}{2x - \pi} dx$

Answer key

Q.2 $\pi[\frac{\pi}{4} - (\sqrt{2} - 1)]$ Q.3 $\frac{(a\pi + 2b)\pi}{3\sqrt{3}}$ Q.4 $\frac{\pi(\pi+3)}{2}$ Q.5 8

P–7 : $\int_0^{nT} f(x)\, dx = n \int_0^{T} f(x)\, dx$ where $f(T + x) = f(x)\ n \in I$

Illustration :

Show that $\int_0^{1000} e^{x-[x]} dx = 1000(e-1)$

Sol. $I = \int_0^{1000} e^{x-[x]} dx = 1000 \int_0^{1} e^{x-[x]} dx$ $\left(e^{x-[x]} \text{ has period } 1\right)$

$$= 1000 \int_0^1 e^x dx = 1000\left[e^x\right]_0^1 = 1000\,(e-1)$$

Illustration :

Prove that : $\displaystyle\int_0^{n\pi+v} |\cos x|\,dx = (2n+2-\sin v)$; *where* $\frac{\pi}{2} < v < \pi$ & $n \in N$.

Sol. $$I = \int_0^{n\pi+v} |\cos x|\,dx = \int_0^{n\pi} |\cos x|\,dx + \int_{n\pi}^{n\pi+v} |\cos x|\,dx \quad (Put\ x - n\pi = t \ \ in\ \ 2^{nd}\ integral)$$

$$= n\int_0^{\pi} |\cos x|\,dx + \int_0^{v} |\cos(n\pi+t)|\,dt = 2n\int_0^{\pi/2} |\cos x|\,dx + \int_0^{v} |\cos t|\,dt$$

$$= 2n + \int_0^{\pi/2} \cos t\,dt + \int_{\pi/2}^{v} -\cos t\,dt = 2n + 1 + 1 - \sin v = 2n + 2 - \sin v$$

Illustration :

Evaluate : $\displaystyle\int_0^{10} \sqrt{1-\cos \pi x}\,dx$

Sol. *We have* $\sqrt{1-\cos \pi x} = \sqrt{2}\left|\sin\frac{\pi x}{2}\right|$

which is a periodic function, having period $T = \frac{1}{2} \times \frac{2\pi}{\pi/2} = 2$.

Hence, we have

$$\int_0^{10} \sqrt{1-\cos \pi x}\,dx = \int_0^{10} \sqrt{2}\left|\sin\frac{\pi x}{2}\right| dx = 5\sqrt{2}\int_0^{2} \left|\sin\frac{\pi x}{2}\right| dx$$

$$= 5\sqrt{2}\int_0^{2} \sin\left(\frac{\pi x}{2}\right) dx = 5\sqrt{2}\left[\frac{-\cos(\pi x/2)}{\pi/2}\right]_0^2 = \frac{20\sqrt{2}}{\pi}.$$

(B) DERIVATIVES OF ANTIDERIVATIVES (LEIBNITZ RULE) :

If f is continuous then

$$\frac{d}{dx}\int_{g(x)}^{h(x)} f(t)\,dt = f(h(x))\cdot h'(x) - f(g(x)).\,g'(x)$$ (integral of a continuous function is always differentiable)

Proof : Let $\int f(t)dt = F(t) + c$ then $\int_{g(x)}^{h(x)} f(t)dt = F(h(x)) - F(g(x))$

$$\Rightarrow \frac{d}{dx}\int_{g(x)}^{h(x)} f(t)dt = F'(h(x))\,h'(x) - F'(g(x))\,g'(x) = f(h(x))\,h'(x) - f(g(x))\,g'(x)$$

Illustration :

Let $G(x) = \int_2^{x^2} \frac{dt}{1+\sqrt{t}}$ $(x > 0)$. Find $G'(9)$.

Sol. $G'(x) = \frac{1}{1+\sqrt{x^2}} . 2x - 0 = \frac{2x}{1+x}$ $\Rightarrow$ $G'(9) = \frac{2\times 9}{1+9} = \frac{9}{5}$

Illustration :

If $f(x) = \int_{e^{2x}}^{e^{3x}} \frac{t}{\ell nt} dt$ $x > 0$. Find derivative of $f(x)$ w.r.t. $\ln x$ when $x = \ln 2$.

Sol. $f'(x) = \frac{e^{3x}}{\ln(e^{3x})} \cdot 3e^{3x} - \frac{e^{2x}}{\ln(e^{2x})} \cdot 2e^{2x} = \frac{e^{6x}}{x} - \frac{e^{4x}}{x}$

$$f'(\ln 2) = \frac{e^{6\ln 2} - e^{4\ln 2}}{\ln 2} = \frac{2^6 - 2^4}{\ln 2} = \frac{48}{\ln 2}$$

Illustration :

Evaluate : $\lim_{x\to 0} \frac{\int_0^x (1-\cos 2x)\,dx}{x\int_0^x \tan x\,dx}$.

Sol. $\lim_{x\to 0} \frac{\int_0^x (1-\cos 2x)\,dx}{x\int_0^x \tan x\,dx} = \lim_{x\to 0} \frac{\int_0^x (1-\cos 2x)\,dx}{x^3} \cdot \frac{x^2}{\int_0^x \tan x\,dx}$

Now, we have

$$\lim_{x\to 0} \frac{\int_0^x (1-\cos 2x)\,dx}{x^3} \left(\frac{0}{0}\right) = \lim_{x\to 0} \frac{1-\cos 2x}{3x^2} = \frac{2}{3}$$

and $\lim_{x\to 0} \frac{x^2}{\int_0^x \tan x\,dx} \left(\frac{0}{0}\right) = \lim_{x\to 0} \frac{2x}{\tan x} = 2.$

Hence, we have $L = \frac{4}{3}$.

Practice Problem

Q.1 Evaluate $\lim\limits_{x\to 0}\dfrac{\int_0^{x^2}\cos t^2\,dt}{x\sin x}$

Q.2 Evaluate $\int_0^{200\pi}\sqrt{1+\cos x}\,dx$

Q.3 Value of $\lim\limits_{x\to 0}\dfrac{\int_0^{x} x e^{t^2}\,dt}{1-e^{x^2}}$ is

(A) -1 (B) $-\frac{1}{2}$ (C) 0 (D) -2

Q.4 If $\int_0^{n\pi}\dfrac{x|\sin x|}{1+|\cos x|}dx$ $(n\in N)$ is equal to $100\pi\ln 2$, then find the value of n.

Q.5 If $y = x^{\int_1^x \ln t\,dt}$, find $\dfrac{dy}{dx}$ at $x=e$.

Q.6 Let $g(x)=x^c\,.\,e^{2x}$ & let $f(x)=\int_0^x e^{2t}\,.\,(3t^2+1)^{1/2}\,dt$. For a certain value of 'c', the limit of $\dfrac{f'(x)}{g'(x)}$ as $x\to\infty$ is finite and non zero. Determine the value of 'c' and the limit.

Answer key

Q.1 1 Q.2 $400\sqrt{2}$ Q.3 A Q.4 10

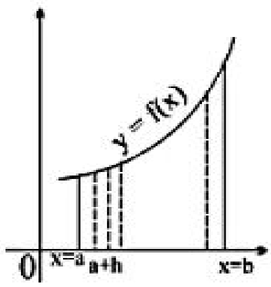
Q.5 $1+e$ Q.6 $c=1$ and $\lim\limits_{x\to\infty}$ will be $\dfrac{\sqrt{3}}{2}$

(C) DEFINITE INTEGRAL AS A LIMIT OF SUM :

Fundamental theorem of integral calculus :

$$\int_a^b f(x)\,dx = \lim_{\substack{h\to 0\\ n\to\infty}} h\,[f(a)+f(a+h)+f(a+2h)+......+f(a+\overline{n-1}\,h)]$$

or $$\int_a^b f(x)\,dx = \lim_{\substack{h\to 0\\ n\to\infty}} h\sum_{r=0}^{n-1} f(a+rh) \quad \text{where} \quad b-a=nh$$



Note: Evaluating a definite integral by evaluating the limit of a sum is called evaluating definite integral by first principle or by a b initio method.

Put $a=0$ & $b=1$ $\Rightarrow$ $nh=1$, we have

$$\int_0^1 f(x)\,dx = \frac{1}{n}\sum_{r=0}^{n-1} f\left(\frac{r}{n}\right);\quad \text{replace } \frac{1}{n}\to dx;\ \Sigma\to\int\ ;\ \frac{r}{n}\to x$$

Illustration :

Evaluate $\int_a^b \cos x\, dx$ *as the limit of a sum.*

Sol. *We have* $\int_a^b \cos x\, dx = \lim_{h \to 0} \sum_{r=1}^{n} h\, f(a+rh) = \lim_{h \to 0} \sum_{r=1}^{n} h \cos(a+rh)$

$$= \lim_{h \to 0} h\left[\cos(a+h)+\cos(a+2h)+....+\cos(a+nh)\right]$$

Now, let $S = \cos(a+h) + \cos(a+2h) + ... + \cos(a+nh)$. *Multiplying both sides by* $2\sin\frac{h}{2}$,

we have

$$\left(2\sin\frac{h}{2}\right)S = 2\sin\frac{h}{2}\cos(a+h) + 2\sin\frac{h}{2}\cos(a+2h) + ... + 2\sin\frac{h}{2}\cos(a+nh)$$

$$= \sin\left(a+\frac{3}{2}h\right) - \sin\left(a+\frac{1}{2}h\right) + \sin\left(a+\frac{5}{2}h\right) - \sin\left(a+\frac{3}{2}h\right)$$

$$+ ......+ \sin\left(a+\frac{2n+1}{2}h\right) - \sin\left(a+\frac{2n-1}{2}h\right)$$

$$= \sin\left(a+\frac{2n+1}{2}h\right) - \sin\left(a+\frac{1}{2}h\right)$$

$$= \sin\left(a+nh+\frac{h}{2}\right) - \sin\left(a+\frac{h}{2}\right) = \sin\left(b+\frac{h}{2}\right) - \sin\left(a+\frac{h}{2}\right)$$

Hence, we have

$$\int_a^b \cos x\, dx = \lim_{h \to 0} \frac{h\left[\sin\left(b+\frac{h}{2}\right)\right] - \sin\left(a+\frac{h}{2}\right)}{2\sin\left(\frac{h}{2}\right)} = \sin b - \sin a.$$

Illustration :

Find the value of $\lim_{n \to \infty} \frac{1}{n} + \frac{1}{n+1} + \frac{1}{n+2} + ... + \frac{1}{4n}$

Sol. $S = \lim_{n \to \infty} \sum_{r=0}^{3n} \frac{1}{n+r} = \lim_{n \to \infty} \frac{1}{n} \sum_{r=0}^{3n} \frac{1}{1+\left(\frac{r}{n}\right)} = \int_0^3 \frac{1}{1+x} d = \left[\ln(1+x)\right]_0^3 = \ln 4$

Illustration :

Evaluate $\lim\limits_{n \to \infty} \prod\limits_{r=1}^{n} \left(\frac{n+r}{n}\right)^{1/n}$.

Sol. We have

$$S = \lim_{n \to \infty} \prod_{r=1}^{n} \left(\frac{n+r}{n}\right)^{1/n} = \lim\left[\frac{n+1}{n} \cdot \frac{n+2}{n} \ldots\ldots \frac{n+n}{n}\right]^{1/n}$$

Taking in both sides, we have

$$\ln S = \lim_{n \to \infty} \frac{1}{n}\left[\ln\left(\frac{n+1}{n}\right) + \ln\left(\frac{n+2}{n}\right) + \ldots\ldots + \ln\left(\frac{n+n}{n}\right)\right]$$

$$= \lim_{n \to \infty} \frac{1}{n} \sum_{n=1}^{n} \ln\left(1 + \frac{r}{n}\right) = \int_0^1 \ln(1+x)\,dx$$

$$= \ln 2 - (1 - \ln 2) = \ln 4 - 1 = \ln\left(\frac{4}{e}\right)$$

$\therefore \quad S = \frac{4}{e}$.

Illustration :

If $n \to \infty$, then find the limit of $\frac{1}{n}\sum\limits_{r=1}^{n} \sin^{2k}\left(\frac{r\pi}{2n}\right)$.

Sol. Let $\quad P = \lim\limits_{n\to\infty} \frac{1}{n} \sum\limits_{r=1}^{n} \sin^{2k}\left(\frac{r\pi}{2n}\right) = \int_0^1 \sin^{2k}\left(\frac{\pi}{2}x\right)dx$

Put $\quad \frac{\pi}{2}x = t \qquad \therefore \qquad dx = \frac{2}{\pi}dt$

$$= \frac{2}{\pi}\int_0^{\pi/2} \sin^{2k} t\,dt \quad = \frac{2}{\pi} \frac{(2k-1)(2k-3)(2k-5)\ldots\ldots 3.1}{2k(2k-2)(2k-4)\ldots\ldots 4.2} \cdot \frac{\pi}{2}$$

$$= \frac{2k(2k-1)(2k-2)(2k-3)\ldots\ldots 3.2.1}{[2k(2k-2)(2k-4)\ldots\ldots 4.2]^2}$$

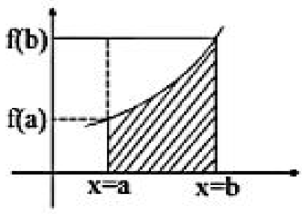
Hence $P = \frac{2k!}{(2^k k!)^2} = \frac{2k!}{2^{2k}(k!)^2}$

(D) ESTIMATION OF DEFINITE INTEGRAL AND GENERAL INEQUALITIES IN INTEGRATION:

Not all integrals can be evaluated using the technique discussed so far. In this situation we try to obtain the interval in which value of integral may lie by using following method.

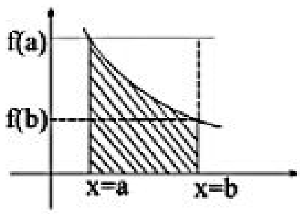
(a) For a monotonic increasing function in (a, b)

$$(b-a)\,f(a) < \int_a^b f(x)\,dx < (b-a)\,f(b)$$



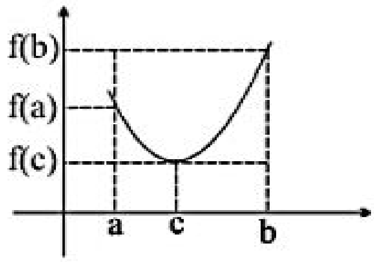
(b) For a monotonic decreasing function in (a, b)

$$f(b).\,(b-a) < \int_a^b f(x)\,dx < (b-a)\,f(a)$$



(c) For a non monotonic function in (a, b)

$$f(c)\,(b-a) < \int_a^b f(x)\,dx < (b-a)\,f(b)$$



(d) In addition to this note that

$\left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx$ inequality holds when f(x) lies completely above the x-axis

(e) If $h(x) \le f(x) \le g(x)\ \forall\ x \in [a, b]$ then $\int h(x)dx < \int f(x)dx < \int g(x)dx$

Illustration :

Show that $1 < \int_0^{\pi/2} \frac{\sin x}{x}dx < \frac{\pi}{2}$.

Sol. $f(x) = \frac{\sin x}{x}$ or $f'(x) = \frac{x\cos x - \sin x}{x^2} = \frac{\cos x}{x^2}[x - \tan x]$

$\Rightarrow f'(x) < 0$ hence $f(x)_{min} = \frac{2}{\pi}$ $f(x)_{max} = 1$.

$\Rightarrow \frac{2}{\pi}\left(\frac{\pi}{2} - 0\right) < I < 1\left(\frac{\pi}{2} - 0\right)$ or $1 < I < \frac{\pi}{2}$.

Illustration :

Show that $\frac{1}{4} \le \int_0^1 \frac{dx}{1+x^2+2x^5} \le 1$.

Sol. *Consider the following function* $f(x) = \frac{1}{1+x^2+2x^5}$, $x \in [0, 1]$

In the interval [0, 1], f(x) is strictly decreasing, therefore we have

$$f(1) \le f(x) \le f(0) \quad \text{i.e.} \quad \frac{1}{4} \le f(x) \le 1$$

Hence, we have

$$(1-0)\,\frac{1}{4} \le \int_0^1 f(x)\,dx \le (1-0)\,1 \qquad \text{[by property (7)]}$$

i.e. $\frac{1}{4} \le \int_0^1 f(x)\,dx \le 1$ *which is the desired result.*

Illustration :

Prove that $\frac{\pi}{6} < \int_0^1 \frac{dx}{\sqrt{4-x^2-x^3}} < \frac{\pi\sqrt{2}}{8}$

Sol. $4-2x^2 \le 4-x^2-x^3 \le 4-x^2$

$$\Rightarrow \frac{1}{\sqrt{4-2x^2}} \ge \frac{1}{\sqrt{4-x^2-x^3}} \ge \frac{1}{\sqrt{4-x^2}} \quad \Rightarrow \int_0^1 \frac{dx}{\sqrt{4-x^2}} < \int_0^1 \frac{dx}{\sqrt{4-x^2-x^3}} < \int_0^1 \frac{dx}{\sqrt{4-2x^2}}$$

$$\Rightarrow \left[\sin^{-1}\left(\frac{x}{2}\right)\right]_0^1 < I < \frac{1}{\sqrt{2}}\left[\sin^{-1}\frac{x}{\sqrt{2}}\right]_0^1 \quad \text{or} \quad \frac{\pi}{6} < I < \frac{\sqrt{2}}{8}\pi$$

Practice Problem

Q.1 $\lim\limits_{n\to\infty} \frac{\sqrt{1}+\sqrt{2}+\sqrt{3}+\ldots\ldots\ldots+2\sqrt{n}}{n\sqrt{n}}$.

Q.2 $\lim\limits_{n\to\infty} \frac{[(n+1)(n+2)\ldots\ldots(n+n)]^{1/n}}{n}$

Q.3 Prove that $\frac{e-1}{3} < \int_1^e \frac{dx}{2+\ln x} < \frac{e-1}{2}$

Q.4 Prove the inequalities: $2\,e^{-1/4} < \int_0^2 e^{x^2-x}\,dx < 2e^2$.

Answer key

Q.1 $\frac{16}{3}$

Q.2 $\frac{4}{e}$

(E) WALLI'S THEORM :

$$\int_0^{\pi/2} \sin^n x \cos^m x \, dx = \frac{[(n-1)(n-3)....1 \text{ or } 2][(m-1)(m-3)...1 \text{ or } 2]}{(m+n)(m+n-2)...1 \text{ or } 2} K$$

(m, n are non-negative integer)

$$\text{where } K = \begin{cases} \frac{\pi}{2} & \text{if } m, n \text{ both are even} \\ 1 & \text{otherwise} \end{cases}$$

Illustration :

Evaluate : $\int_0^{2\pi} x \sin^6 x \cos^4 x dx$

Sol. $I = \int_0^{2\pi} x \sin^6 x \cos^4 x dx$ *using P5*

$$I = \int_0^{2\pi} (2\pi - x) \sin^6 x \cos^4 x \, dx \Rightarrow I + I = \int_0^{2\pi} 2\pi \sin^6 x \cos^4 x \, dx$$

$$\text{or } I = \pi \int_0^{2\pi} \sin^6 x \cos^4 x \, dx \qquad \textit{Using P6 twice}$$

$$I = 4\pi \int_0^{\pi/2} \sin^6 x \cos^4 x \, dx = 4\pi \frac{(5\cdot 3\cdot 1)(3\cdot 1)}{10\cdot 8\cdot 6\cdot 4\cdot 2} = \frac{3\pi^2}{128} \qquad \textbf{Ans.}$$

Illustration :

Evaluate : $\int_0^{3\pi/2} \cos^4 3x \cdot \sin^2 6x dx$

Sol. $I = \int_0^{3\pi/2} \cos^4 3x \cdot \sin^2 6x dx = \int_0^{3\pi/2} 4 \sin^2(3x) \cos^6(3x) \, dx$

Put $3x = t$ *to get*

$$I = \int_0^{9\pi/2} 4 \cdot \sin^2 t \cdot \cos^6 t \, \frac{1}{3} dt = \frac{4}{3}\left[\int_0^{4\pi} \sin^2 t \cos^6 t \, dt + \int_{4\pi}^{9\pi/2} \sin^2 t \cos^6 t \, dt\right]$$

$$= \frac{4}{3}\left[4\int_0^{\pi} \sin^2 t \cos^6 t \, dt + \int_{4\pi}^{9\pi/2} \sin^2 t \cos^6 t \, dt\right] \qquad (\sin^2 t \cos^6 t \text{ has period } \pi)$$

$$= \frac{4}{3}\left[8\int_0^{\pi/2} \sin^2 t \cos^6 t \, dt + \int_0^{\pi/2} \sin^2 t \cos^6 t \, dt\right] \qquad (\text{using P-6 in } 1^{st} \text{ integral and } t - 4\pi = z \text{ in } 2^{nd})$$

Illustration :

Evaluate : $\int_0^{\pi/2} \cos^7 x\, dx$

Sol. $I = \int_0^{\pi/2} \cos^7 x\, dx = \frac{6 \cdot 4 \cdot 2}{7 \cdot 5 \cdot 3 \cdot 1}$.

REDUCTION METHOD :

For integration of type $\int_a^b (f(x))^n dx$

where 'n' is big natural number it is possible to reduce 'n' by some methods specially by parts.

Illustration :

Let $I_n = \int_0^1 (1-x^a)^n dx$. *Find the ratio* I_n/I_{n+1}.

Sol. *We have* $I_{n+1} = \int_0^1 (1-x^a)^n dx$

$= \left[x(1-x^a)^{n+1}\right]_0^1 + (n+1)a \int_0^1 x^a(1-x^a)^n dx$

[taking 1 as one function and integrating by parts]

$= (n+1)a \int_0^1 (x^a - 1 + 1)(1-x^a)^n dx = (n+1)a \int_0^1 (1-x^a)^n dx - (n+1)a \int_0^1 (1-x^a)^{n+1} dx$

$= (n+1)a\, I_n - (n+1)\, a\, I_{n+1}$

Simplifying, we have $\frac{I_n}{I_{n+1}} = 1 + \frac{1}{(n+1)a}$.

Illustration :

Given $I_n = \int_0^{\pi/4} (\tan x)^n dx \quad (n \in N)$

Prove that $I_n + I_{n-1} = \frac{1}{n-1}$ $(n \geq 3)$. *Hence find value of* I_6.

Sol. $I_n = \int_0^{\pi/4} (\tan x)^n dx$, $\quad I_{n-2} = \int_0^{\pi/4} (\tan x)^{x-2} dx$

$I_n + I_{n-1} = \int_0^{\pi/4} (\tan x)^n + (\tan x)^{n-2} dx$

Put $\tan x = t$ *to get* $I_n + I_{n-2} = \int_0^1 t^{n-2}\,dt = \left.\frac{t^{n-1}}{n-1}\right|_0^1 = \frac{1}{n-1}$

Also $I_2 = \int_0^{\pi/4} \tan^2 x\,dx = \int_0^{\pi/4} (\sec^2 x - 1)dx = [\tan x - x]_0^{\pi/4} = 1 - \frac{\pi}{4}$

Using $I_n + I_{n-2} = \frac{1}{n-1}$, $I_4 + I_2 = \frac{1}{3}$ *and* $I_6 + I_4 = \frac{1}{5}$

$\Rightarrow I_6 = I_2 - \frac{2}{\sqrt{5}}$ *or* $I_6 = 1 - \frac{\pi}{4} - \frac{2}{15} = \frac{13}{15} - \frac{\pi}{4}$

(F) SOME INTEGRALS WHICH CANNOT BE FOUND IN TERMS OF KNOWN ELEMENTRY FUNCTIONS :

(1) $\int \frac{\sin x}{x}\,dx$ (2) $\int \frac{\cos x}{x}\,dx$ (3) $\int \sqrt{\sin x}\,dx$ (4) $\int \sin x^2\,dx$

(5) $\int \cos x^2\,dx$

(6) $\int x \tan x\,dx$ (7) $\int e^{-x^2}\,dx$ (8) $\int e^{x^2}\,dx$ (9) $\int \frac{x^3}{1+x^5}\,dx$

(10) $\int (1+x^2)^{1/3}\,dx$ (11) $\int \frac{dx}{\ln x}$ (12) $\int \sqrt{1+k^2\sin^2 x}\,dx \quad k \in R$

Practice Problem

Q1 Let $U_{10} = \int_0^{\frac{\pi}{2}} x \sin^{10} x\,dx$, then find the value of $\left(\frac{100U_{10} - 1}{U_8}\right)$.

Q.2 If $U_n = \int_0^{\pi/2} \frac{\sin^2 nx}{\sin^2 x}\,dx$, then show that $U_1, U_2, U_3, \ldots, U_n$ constitute an AP. Hence or otherwise find the value of U_n.

Q.3 Evaluate : $\int_0^{4\pi} \sin^4 x \cos^2 x\,dx$

Answer key

Q1 90 Q.2 $U_n = \frac{n\pi}{2}$ Q.3 $\frac{\pi}{4}$

Solved Examples

Q.1 Prove that $\int_0^{\infty} \frac{dx}{\left[x+\sqrt{(1+x^2)}\right]^2} = \frac{n}{n^2-1} (n > 1)$

Sol. L.H.S. $= \int_0^{\infty} \frac{dx}{\left(x+\sqrt{1+x^2}\right)^n}$ Put $x+\sqrt{(1+x^2)} = t$ ......(i)

$\therefore \quad \left(1+\frac{x}{\sqrt{1+x^2}}\right)dx = dt$ when $x \to 0$ then $t \to 1$ $\quad x \to \infty$ then $t \to \infty$

$\Rightarrow \quad \frac{t\,dx}{\sqrt{1+x^2}} = dt \quad \Rightarrow \quad dx = \frac{\sqrt{(1+x^2)}}{t}dt \quad \Rightarrow \quad \sqrt{1+x^2} - x = \frac{1}{t}$ ......(ii)

Adding (i) & (ii) we get $2\sqrt{1+x^2} = t + \frac{1}{t}$

$\Rightarrow \quad \sqrt{(1+x^2)} = \frac{t^2+1}{2t} \quad \therefore \quad dx = \frac{(t^2+1)}{2t^2}dt$

Hence L.H.S. $= \int_1^{\infty} \frac{(t^2+1)dt}{2t^2 t^n} = \frac{1}{2}\int_1^{\infty}\left(\frac{1}{t^n}+\frac{1}{t^{n+2}}\right)dt = \frac{1}{2}\left[-\frac{1}{(n-1)t^{n-1}} - \frac{1}{(n+1)t^{n+1}}\right]_1^{\infty}$

$= \frac{1}{2}\left[(-0-0)-\left(-\frac{1}{n-1}-\frac{1}{n+1}\right)\right] = \frac{1}{2}\left[\frac{1}{n-1}+\frac{1}{n+1}\right] \qquad (\because\ n > 1)$

$= \frac{n}{n^2-1} = \text{R.H.S.}$

Q.2 If $f(x) = \int_0^x \frac{e^t}{t}dt, x > 0$. Prove that $\int_1^x \frac{e^t dt}{(t+a)} = e^{-a}[f(x+a) - f(1+a)]$.

Sol. R.H.S. $= e^{-a}[f(x+a) - f(1+a)] = e^{-a}\left[\int_0^{x+a}\frac{e^t}{t}dt - \int_0^{1+a}\frac{e^t}{t}dt\right] = e^{-a}\left[\int_0^{x+a}\frac{e^t}{t}dt + \int_{1+a}^{0}\frac{e^t}{t}dt\right]$

$= e^{-a}\int_{1+a}^{a+x}\frac{e^t}{t}dt \qquad$ Put $\quad t = a+y \quad \therefore \quad dt = dy$

$= e^{-a}\int_1^x \frac{e^{a+y}}{(a+y)}dy = e^{-a}\cdot e^{a}\int_1^x \frac{e^y}{(a+y)}dy = \int_1^x \frac{e^y}{a+y}dy = \int_1^x \frac{e^y}{(a+t)}dt$ (by Prop.)

$=$ L.H.S.

Q.3 Evaluate $\int_{-\pi}^{\pi}\left|x\sin[x^2-\pi]\right|dx$, $[\cdot]$ is the greatest integer function.

Sol. Let $I=\int_{-\pi}^{\pi}\left|x\sin[x^2-\pi]\right|dx \;=2\int_{0}^{\pi}\left|x\sin[x^2-\pi]\right|dx$ [it is even function]

$$I=2\left[\int_0^{\sqrt{\pi-3}}\left|x\sin[x^2-\pi]\right|dx+\int_{\sqrt{\pi-3}}^{\sqrt{\pi-2}}\left|x\sin[x^2-\pi]\right|dx+\int_{\sqrt{\pi-2}}^{\sqrt{\pi-1}}\left|x\sin[x^2-\pi]\right|\right.$$

$$\left.+\int_{\sqrt{\pi-1}}^{\sqrt{\pi}}\left|x\sin[x^2-\pi]\right|dx+\ldots\ldots+\int_{\sqrt{\pi+6}}^{\pi}\left|x\sin[x^2-\pi]\right|dx\right]$$

$$=2\left[\int_0^{\sqrt{\pi-3}}x\sin 4\,dx+\int_{\sqrt{\pi-3}}^{\sqrt{\pi-2}}x\sin 3\,dx+\int_{\sqrt{\pi-2}}^{\sqrt{\pi-1}}x\sin 2\,dx+\int_{\sqrt{\pi-1}}^{\sqrt{\pi}}x\sin 1\,dx+0\right.$$

$$\left.+\int_{\sqrt{\pi-1}}^{\sqrt{\pi+2}}x\sin 1\,dx+0+\int_{\sqrt{\pi-1}}^{\sqrt{\pi+2}}x\sin 1\,dx+\ldots\ldots+\int_{\sqrt{\pi-6}}^{\pi}x\sin 6\,dx\right]$$

$$=\{\sin 4(\pi-3)+\sin 3(1)+\sin 2(1)+\sin 1(1)+\ldots\ldots+\sin 1(1)+\ldots\ldots$$
$$+\sin 1(1)+\ldots\ldots+\sin 6(\pi^2-\pi-6)\}$$
$$=2\sin 1+2\sin 2+3\sin 3+(\pi-2)\sin 4+\sin 5+(\pi^2-\pi-6)\sin 6$$

Q.4 Show that $\frac{\pi}{3\sqrt{3}}\le\int_0^1\frac{dx}{1+x^2+2x^5}\le\frac{\pi}{4}$.

Sol. We have $1+x^2+2x^5\ge 1+x^2$

and $1+x^2+2x^5\le 1+x^2+2x^2=1+3x^2$ $[x^5<x^2 \text{ on } [0,1]]$

Hence, we have $\frac{1}{1+3x^2}\le\frac{1}{1+x^2+2x^5}\le\frac{1}{1+x^2}$

i.e. $\int_0^1\frac{dx}{1+3x^2}\le\int_0^1\frac{dx}{1+x^2+2x^5}\le\int_0^1\frac{dx}{1+x^2}$ [by property (8)]

i.e. $\left[\frac{\tan^{-1}\sqrt{3}\,x}{\sqrt{3}}\right]_0^1\le\int_0^1\frac{dx}{1+x^2+2x^5}\le\left[\tan^{-1}x\right]_0^1$ i.e. $\frac{\pi}{3\sqrt{3}}\le\int_0^1\frac{dx}{1+x^2+2x^5}\le\frac{\pi}{4}$

which is the desired result.

Q.5 Evaluate $\int_{-1}^{1}\frac{|\sin x|}{\sin x}dx$

Sol. We have $\frac{|\sin x|}{\sin x}=-1,\; -1\le x\le 0 \;=1,\; 0\le x\le 1\Rightarrow\frac{|\sin x|}{\sin x}$, $x\in[-1,1]$ is an odd function

Hence, by property (4), we have $\int_{-1}^{1}\frac{|\sin x|}{\sin x}=0$.

Q.6 Evaluate : $\int_0^2 \left[x^2-1\right]dx$ where [x] represents integral part of x.

Sol. We have $[x^2-1] = -1,\ 0 \le x < 1 \quad = 0,\ 1 \le x < \sqrt{2} \quad = 1,\ \sqrt{2} \le x < \sqrt{3} \quad = 2,\ \sqrt{3} \le x < 2.$

Hence, we have $\int_0^2 \left[x^2-1\right]dx = \int_0^1 -1\,dx + \int_0^{\sqrt{2}} 0\,dx + \int_{\sqrt{2}}^{\sqrt{3}} 1\,dx + \int_{\sqrt{3}}^{2} 1\,dx$

$= [-x]_0^1 + 0 + [x]_{\sqrt{2}}^{\sqrt{3}} + [x]_{\sqrt{3}}^{2} = -1 + \sqrt{3} - \sqrt{2} + 2 - \sqrt{3} = 1 - \sqrt{2}.$

Q.7 Evaluate : $\int_1^3 \frac{dx}{\sqrt{x+1}-\sqrt{x-1}}$

Sol. We have $\int_1^3 \frac{dx}{\sqrt{x+1}-\sqrt{x-1}} = \int_1^3 \frac{\sqrt{x+1}-\sqrt{x-1}}{2}dx = \frac{1}{2}\left[\frac{(x+1)^{3/2}}{3/2} + \frac{(x-1)^{3/2}}{3/2}\right]_1^3$

$= \frac{1}{3}\left[(x+1)^{3/2} + (x-1)^{3/2}\right]_1^3 = \frac{1}{3}\left[4^{3/2} + 2^{3/2} - 2^{3/2}\right] = \frac{8}{3}.$

Q.8 Evaluate the definite integral $\int_0^{\pi/2} \ln\left(\tan x + \cot x\right)dx$

Sol. Let $I = \int_0^{\pi/2} \ln\left(\tan x + \cot x\right)dx = \int_0^{\pi/2} \ln\left(\frac{\sin x}{\cos x} + \frac{\cos x}{\sin x}\right)dx = \int_0^{\pi/2} \ln\left(\frac{1}{\sin x \cos x}\right)dx$

$= \int_0^{\pi/2} \ln\left(\frac{2}{\sin 2x}\right)dx = \int_0^{\pi/2} \ln 2\,dx - \int_0^{\pi/2} \ln\left(\sin 2x\right)dx = \frac{\pi}{2}\ln 2 - \int_0^{\pi/2}\ln\left(\sin 2x\right)dx.$

Let us put $2x = y$ and $2\,dx = dy$ in the second integral on the RHS. Also, when $x = 0$, then $y = 0$ and when $x = \frac{\pi}{2}$, then $y = \pi$. Hence, we have $\int_0^{\pi/2} \ln\left(\sin 2x\right)dx = \frac{1}{2}\int_0^{\pi/2} \ln\left(\sin y\right)dy$

$= \frac{1}{2}\cdot 2\int_0^{\pi/2} \ln\left(\sin y\right)dy \qquad [\ln(\sin y) = \ln \sin \pi - y]$

$= -\frac{\pi}{2}\ln 2$

Hence, we have $I = \frac{\pi}{2}\ln 2 + \frac{\pi}{2}\ln 2 = \pi \ln 2.$

Q.9 Find f(x) if it satisfies the relation $f(x) = e^x + \int_0^1 \left(x + ye^x\right) f(y)\, dy$.

Sol. We have $f(x) = e^x + x\int_0^1 f(y)\, dy + e^x \int_0^1 y\, f(y)\, dy$

$$= e^x\left(1 + \int_0^1 y\, f(y)\, dy\right) + x\int_0^1 f(y)\, dy = ae^x + bx \text{ (say)}$$

where a, b are constants, given by $a = 1 + \int_0^1 y\, f(y)\, dy = 1 + \int_0^1 y\left(ae^y + by\right) dy$

$$= 1 + \left[(y-1)\, e^y\right]_0^1 + \left[\frac{by^3}{3}\right]_0^1 = 1 + a + \frac{b}{3}$$

and $b = \int_0^1 f(y)\, dy = \int_0^1 \left(ae^y + by\right) dy = \left[ae^y + \frac{by^2}{3}\right]_0^1 = a(e-1) + \frac{b}{2}$

Solving, we have $b = -2$ and $a = \dfrac{-3}{2(e-1)}$

Hence, we have $f(x) = \dfrac{-3e^x}{2(e-1)} - 3x$.

Q.10 If $b = \int_0^1 \frac{e^t}{t+1}\, dt$, then show that $\int_{a-1}^{a} \frac{e^{-t}}{t-a-1}\, dt = be^{-a}$.

Sol. We have $I = \int_{a-1}^{a} \frac{e^{-t}}{t-a-1}\, dt = \int_{a-1}^{-a} \frac{e^y}{-y-a-1}(-dy)$ [putting $t = -y$]

$$= e^{-a}\int_{a-1}^{-a} \frac{e^{y+a}}{y+a+1}\, dy = e^{-a}\int_{a-1}^{-a} \frac{e^u}{u+1}\, du \quad \text{[putting } y+a=u]$$

$$= -e^{-a}\int_0^1 \frac{e^u}{u+1}\, dx = -be^{-a} \qquad \left[\int_0^1 \frac{e^t}{t+1}\, dt = b \text{ given}\right]$$

Q.11 Evaluate the definite integral $\int_0^1 \frac{1-x^2}{1+x^2} \cdot \frac{dx}{\sqrt{1+x^4}}$

Sol. We have $I = \int_0^1 \frac{1-x^2}{1+x^2} \cdot \frac{dx}{\sqrt{1+x^4}} = \int_0^1 \frac{\frac{1}{x^2} - 1}{x + \frac{1}{x}} \cdot \frac{dx}{\sqrt{x^2 + \frac{1}{x^2}}} = \int_0^1 \frac{-d\left(x + \frac{1}{x}\right)}{\left(x + \frac{1}{x}\right)\sqrt{\left(x + \frac{1}{x}\right)^2 - 2}}$

$$= \int_{\infty}^{2} \frac{-dt}{t\sqrt{t^2-2}} \quad [\text{Puting } x + \frac{1}{x} = t] \qquad = \int_{2}^{\infty} \frac{t\,dt}{t^2\sqrt{t^2-2}} \quad [\text{Putting } t^2 - 2 = u^2]$$

$$= \int_{\sqrt{2}}^{\infty} \frac{u\,du}{u(u^2+2)} = \frac{1}{\sqrt{2}}\left[\tan^{-1}\frac{u}{\sqrt{2}}\right]_{\sqrt{2}}^{\infty} = \frac{1}{\sqrt{2}}\left(\frac{\pi}{2}-\frac{\pi}{4}\right) = \frac{\pi}{4\sqrt{2}}$$

Q.12 If $I_1 = \int_0^{n\pi} f(\sin^4 x)dx$ and $I_2 = \int_0^{\pi} f(\sin^4 x)dx$. Find the value of $\frac{I_1}{I_2}$.

Sol. We have

$$I = \frac{\int_0^{n\pi} f(\sin^4 x)\,dx}{\int_0^{\pi} f(\sin^4 x)\,dx} = \frac{\int_0^{2n\left(\frac{\pi}{2}\right)} f(\sin^4 x)\,dx}{\int_0^{2\left(\frac{\pi}{2}\right)} f(\sin^4 x)\,dx} = \frac{2n\int_0^{\pi/2} f(\sin^4 x)\,dx}{2\int_0^{\pi/2} f(\sin^4 x)\,dx} \; [\text{period of } \sin^4 x \text{ is } \frac{\pi}{2}] \; = n.$$

Q.13 Prove that $\int_{1/2}^{2} (\ln x)^2\,dx < \int_{1/2}^{2} |\ln x|\,dx$

Sol. In the interval $\left[\frac{1}{4}, \frac{1}{2}\right]$, $|\ln x|$ is a fraction. hence, we have $(\ln x)^2 < |\ln x|$

i.e. $\int_{1/2}^{2} (\ln x)^2\,dx < \int_{1/2}^{2} |\ln x|\,dx$.

Q.14 Show that $\int_0^{k\pi} \sin\left[\frac{2x}{\pi}\right]dx = \frac{\pi}{2}\cdot\frac{\sin k \sin(k+1/2)}{\sin(1/2)}$.

Sol. We have $I = \int_0^{k\pi} \sin\left[\frac{2x}{\pi}\right]dx = \int_0^{\pi/2} \sin 0\,dx + \int_{\pi/2}^{2\pi/2} \sin 1\,dx + \int_{2\pi/2}^{3\pi/2} \sin 2\,dx + \ldots + \int_{2\pi/2}^{2k\pi/2} \sin(2k-1)\,dx$

$$= \frac{\pi}{2}[\sin 1 + \sin 2 + \sin 3 + \ldots + \sin(2k-1)]$$

$$= \frac{\pi}{2}\,\frac{\left[\sin\frac{1}{2}\sin 1 + \sin\frac{1}{2}\sin 2 + \sin\frac{1}{2}\sin 3 + \ldots + \sin\frac{1}{2}\sin(2k-1)\right]}{\sin\frac{1}{2}}$$

$$= \frac{\left[\cos\frac{1}{2} - \cos\frac{3}{2} + \cos\frac{3}{2} - \cos\frac{5}{2} + ... + \cos\left(2k - \frac{3}{2}\right) - \cos\left(2k + \frac{1}{2}\right)\right]}{2\sin\frac{1}{2}}$$

$$= \frac{\pi}{2} \cdot \frac{\cos\frac{1}{2} - \cos\left(2k + \frac{1}{2}\right)}{2\sin\frac{1}{2}} = \frac{\pi}{2} \cdot \frac{\sin k \sin(k + 1/2)}{\sin(1/2)}$$

Q.15 Let $I_n = \int_0^1 x^n \tan^{-1} x\,dx$. Show that $(n+1)I_{n-1} + (n-1)I_{n-2} = \frac{\pi}{2} - \frac{1}{n}$.

Sol. We have $I_n = \int_0^1 x^n \tan^{-1} x\,dx = \int_0^{\pi/4} \theta(\tan\theta)^n \sec^2\theta\,d\theta$ [Putting $x = \tan\theta$]

$$= \left[\frac{\theta(\tan\theta)^{n+1}}{n+1}\right]_0^{\pi/4} - \int_0^{\pi/4} \frac{(\tan\theta)^{n+1}}{n+1} d\theta = \frac{\pi/4}{n+1} - \int_0^{\pi/4} \frac{(\tan\theta)^{n-1}(\sec^2\theta - 1)}{n+1} d\theta$$

$$= \frac{\pi/4}{n+1} - \frac{1}{n+1}\int_0^{\pi/4} (\tan\theta)^{n-1}\sec^2\theta\,d\theta + \frac{1}{n+1}\int_0^{\pi/4} (\tan\theta)^{n-1}\,d\theta$$

$$= \frac{\pi/4}{n+1} - \frac{1}{n+1}\left[\frac{(\tan\theta)^n}{n}\right]_0^{\pi/4} + \left(\frac{1}{n+1}\right) I$$

$$= \frac{\pi/4}{n+1} - \frac{1}{n(n+1)} + \left(\frac{1}{n+1}\right) I \qquad \text{and} \qquad I_{n-2} = \int_0^{\pi/4} \theta(\tan\theta)^{n-2}\sec^2\theta\,d\theta$$

$$= \left[\frac{\theta(\tan\theta)^{n-1}}{n-1}\right]_0^{\pi/4} - \int_0^{\pi/4} \frac{(\tan\theta)^{n-1}}{n-1} d\theta = \frac{\pi/4}{n-1}\left(\frac{1}{n-1}\right) I$$

Eliminating I from equations (1) and (2), we have

$(n+1)I_{n-1} + (n-1)\,I_{n-2} = \frac{\pi}{2} - \frac{1}{n}$ which is the desired result.

Q.16 Evaluate : $\lim_{n\to\infty} \prod_{r=1}^{n} \frac{(n^2 + r^2)^{1/n}}{n^2}$

Sol. We have $S = \lim_{n\to\infty} \prod_{r=1}^{n} \frac{(n^2 + r^2)^{1/n}}{n^2}$

$$= \lim_{n\to\infty} \left[\left(\frac{n^2 + 1^2}{n^2}\right)\left(\frac{n^2 + 2^2}{n^2}\right)......\left(\frac{n^2 + n^2}{n^2}\right)\right]^{1/n}$$ Taking in both sides, we have

$$\ln S = \lim_{n\to\infty} \frac{1}{n}\left[\ln\left(1+\frac{1^2}{n^2}\right) + \ln\left(1+\frac{2^2}{n^2}\right) + ... + \ln\left(1+\frac{n^2}{n^2}\right)\right]$$

$$= \lim_{n\to\infty} \frac{1}{n} \ln\left(1+\frac{r^2}{n^2}\right) = \int_0^1 \ln\left(1+x^2\right)dx = \left[x \ln\left(1+x^2\right)\right]_0^1 - \int_0^1 x\left(\frac{2x}{1+x^2}\right)dx$$

$$= \ln 2 - - \int_0^1 2\left(1-\frac{1}{1+x^2}\right)dx = \ln 2 - 2\left[x - \tan^{1} x\right]_0^1$$

$$= \ln 2 - 2\left(1-\frac{\pi}{4}\right) = \ln 2 + \frac{\pi-4}{2} \quad \text{gives} \quad S = 2e^{\left(\frac{\pi-4}{2}\right)}.$$

Q.17 Evaluate the definite integrals $\int_0^{\pi/2} \sin x \ln(\cos x)\, dx$

Sol. We have

$$I = \int_0^{\pi/2} \sin x \ln(\cos x)\, dx = \left[-\cos x \ln(\cos x)\right]_0^{\pi/2} + \int_0^{\pi/2} \cos x . \frac{-\sin x}{\cos x} dx$$

$$= \lim_{x\to\pi/2} \frac{-\ln(\cos x)}{\sec x} + \left[\cos x\right]_0^{\pi/2} = \lim_{x\to\pi/2} \frac{\tan x}{\sec x \tan x} - 1$$

$$= \lim_{x\to\pi/2} \cos x - 1 = -1.$$

Q.18 Evaluate the following limits, using definite integral :

$$\lim_{n\to\infty}\left[\frac{1}{\sqrt{n^2}} + \frac{1}{\sqrt{n^2-1^2}} + \frac{1}{\sqrt{n^2-2^2}} + ... + \frac{1}{\sqrt{n^2(n-1)^2}}\right]$$

Sol. We have

$$\lim_{n\to\infty}\left[\frac{1}{\sqrt{n^2}} + \frac{1}{\sqrt{n^2-1^2}} + \frac{1}{\sqrt{n^2-2^2}} + ... + \frac{1}{\sqrt{n^2(n-1)^2}}\right]$$

$$= \lim_{n\to\infty} \sum_{r=0}^{n} \frac{1}{\sqrt{n^2-r^2}} = \lim_{n\to\infty} \frac{1}{n} \sum_{r=1}^{n} \frac{1}{\sqrt{1-(r/n)^2}} \qquad \text{[Omitting one term will not affect the limit]}$$

$$= \int_0^1 \frac{dx}{\sqrt{1-x^2}} = \left[\sin^{-1} x\right]_0^1 = \frac{\pi}{2}.$$

Q.19 Prove the following results, using definite integral :

$$\lim_{n\to\infty}\sum_{r=1}^{n}\frac{\sqrt{n}}{\sqrt{r}\,(a\sqrt{n}-b\sqrt{r})^2}=\frac{2}{a(a-b)}$$

Sol. We have

$$\lim_{n\to\infty}\sum_{r=1}^{n}\frac{\sqrt{n}}{\sqrt{r}\,(a\sqrt{n}-b\sqrt{r})^2}=\frac{2}{a(a-b)}$$

$$=\int_0^1\frac{dx}{\sqrt{x}\,(a-b\sqrt{x})^2}=\frac{-2}{b}\int_a^{a-b}\frac{dt}{t^2}\qquad [\text{Putting } a-b\sqrt{x}=t]$$

$$=\frac{2}{b}\left[\frac{1}{t}\right]_a^{a-b}=\frac{2}{b}\left(\frac{1}{a-b}-\frac{1}{a}\right)=\frac{2}{a(a-b)}$$

Q.20 Evaluate : $\lim_{n\to\infty}\left[\frac{\sqrt{n}}{\sqrt{n^3}}+\frac{\sqrt{n}}{\sqrt{(n+2)^3}}+\frac{\sqrt{n}}{\sqrt{(n+4)^3}}+\frac{\sqrt{n}}{\sqrt{(n+8)^3}}+.....\right]$

Sol. We have

$$\lim_{n\to\infty}\left[\frac{\sqrt{n}}{\sqrt{n^3}}+\frac{\sqrt{n}}{\sqrt{(n+2)^3}}+\frac{\sqrt{n}}{\sqrt{(n+4)^3}}+\frac{\sqrt{n}}{\sqrt{(n+8)^3}}+.....\right]$$

$$=\lim_{n\to\infty}\sum_{r=0}^{n}\frac{\sqrt{n}}{\sqrt{(n+2r)^3}}=\lim_{n\to\infty}\frac{\sqrt{n}}{\sqrt{(n+2r)^3}}\qquad [\text{omitting one term will not affect the limit}]$$

$$=\lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\frac{n^{3/2}}{(n+2r)^{3/2}}=\lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{\left\{1+2\left(\frac{r}{n}\right)\right\}^{3/2}}$$

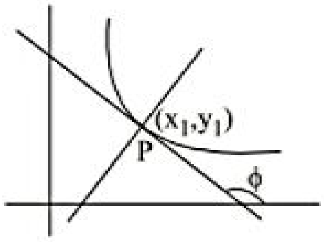
$$=\int_0^1\frac{1}{(1+2x)^{3/2}}dx=\left[\frac{(1+2x)^{-1/2}}{-1/2}\cdot\frac{1}{2}\right]_0^1=\left[\frac{1}{\sqrt{1+2x}}\right]_1^0=1-\frac{1}{\sqrt{3}}.$$

APPLICATION OF DERIVATIVE

TANGENT AND NORMAL

Define : $\tan\phi = \left.\frac{dy}{dx}\right|_P$

(1) Equation of a tangent at $P(x_1, y_1)$



$$y - y_1 = \left.\frac{dy}{dx}\right|_{x_1,y_1}(x - x_1)$$

(2) Equation of normal at (x_1, y_1)

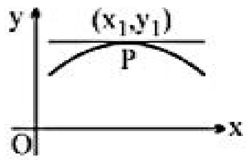
$$y - y_1 = -\frac{1}{\left(\frac{dy}{dx}\right)_{x_1,y_1}}(x - x_1)\ ,\ \text{if } \left.\frac{dy}{dx}\right]_{x_1,y_1} \text{ exists.}$$

However in some cases $\frac{dy}{dx}$ fails to exist but still a tangent can be drawn e.g. case of vertical tangent.

Note that the point (x_1, y_1) must lie on the curve for the equation of tangent and normal.

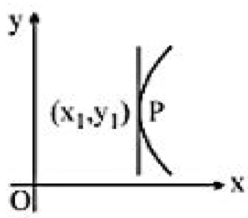
Important notes to remember:

(a) If $\left.\frac{dy}{dx}\right|_{x_1,y_1} = 0 \Rightarrow$ tangent is parallel to x-axis and converse.



If tangent is parallel to $ax + by + c = 0 \Rightarrow \frac{dy}{dx} = -\frac{a}{b}$

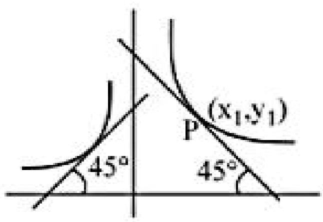
(b) If $\left.\frac{dy}{dx}\right|_{x_1,y_1} \to \infty$ or $\left.\frac{dx}{dy}\right|_{x_1,y_1} = 0 \Rightarrow$ tangent is perpendicular to x-axis.



If tangent with a finite slope is perpendicular to $ax + by + c = 0$

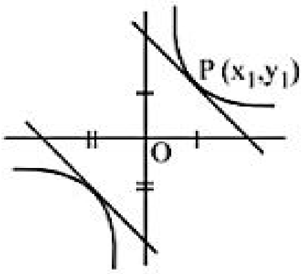
$$\Rightarrow \left.\frac{dy}{dx}\right|_{x_1,y_1} \cdot \left(-\frac{a}{b}\right) = -1.$$

(3) If the tangent at $P(x_1, y_1)$ on the curve is equally inclined to the coordinate axes



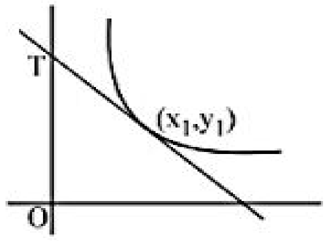
$$\Rightarrow \left.\frac{dy}{dx}\right|_{x_1,y_1} = \pm 1.$$

(4) If the tangent makes equal non zero intercept on the coordinate axes then $\left.\frac{dy}{dx}\right|_{x_1,y_1} = -1$



(5) If tangent cuts off from the coordinate axes equal distance from the origin $\Rightarrow \frac{dy}{dx} = \pm 1$.

(6) OT is called the initial ordinate of the tangent



$$Y - y = \frac{dy}{dx}(X - x)$$

put $X = 0$ to get

$\therefore \quad Y = OT = y - x\frac{dy}{dx}$ (It is the y intercept of a tangent at P)

(7) **Concept:** $F(x)=f(x)\cdot g(x)$ are such that $f(x)$ is continuous at $x = a$ and $g(x)$ is differentiable at $x = a$ with $g(a)=0$ then the product function $f(x)\cdot g(x)$ is diffenentiable at $x = a$.

Illustration :

Find the tangent and normal for $x^{2/3} + y^{2/3} = 2$ *at (1, 1).*

Sol. $x^{\frac{2}{3}} + y^{\frac{2}{3}} = 2 \Rightarrow \frac{2}{3}\left(x^{\frac{-1}{3}} + y^{\frac{-1}{3}}y'\right) = 0$ or $y' = -\left(\frac{x}{y}\right)^{\frac{-1}{3}}$

At (1, 1) $\quad y' = -1$

Equation of tangent $\quad y - 1 = -1(x - 1) \quad \Rightarrow \quad x + y = 2$

Equation of normal $\quad y - 1 = 1(x - 1) \quad \Rightarrow \quad x - y = 0$

Illustration :

Find tangent to $x = a\sin^3 t$ *and* $y = a\cos^3 t$ *at* $t = \pi/2$.

Sol. $x = a\sin^3 t; \quad y = a\cos^3 t$

$$\frac{dy}{dx} = \frac{-3a\cos^2 t\sin t}{3a\sin^2 t\cos t} = -\cot t$$

At $\quad t = \frac{\pi}{2}, \frac{dy}{dx} = 0 \quad$ *point is (a, 0)*

$\therefore \quad$ *Equation of tangent* $\Rightarrow y = 0$

Illustration :

A curve in the plane is defined by the parametric eqautions $x = e^{2t} + 2e^{-t}$ *and* $y = e^{2t} + e^{t}$. *An equation for the line tangent to the curve at the point* $t = \ln 2$ *is*

(A) $5x - 6y = 7$ (B) $5x - 3y = 7$ (C) $10x - 7y = 8$ (D) $3x - 2y = 3$

Sol. $x = e^{2t} + 2e^{-t}, y = e^{2t} + e^{t}$

At $t = \ln 2$ $x = 4 + 1 = 5,\ y = 4 + 2 = 6$

$$\frac{dy}{dx} = \frac{2e^{2t} + e^{t}}{2e^{2t} - 2e^{-t}} = \frac{8+2}{8-1} = \frac{10}{7} \Rightarrow \text{equation of tangent is } y - 6 = \frac{10}{7}(x - 5)$$

$7y - 42 = 10x - 50$ or $10x - 7y = 8$

Illustration :

Equation of the normal to the curve $x^2 = 4y$ *which passes through (1, 2).*

Sol. $x^2 = 4y$ $2x = 4y'$

$$y' = \frac{x_1}{2} \quad \& \quad y_1 = \frac{x_1^2}{4}$$

Normal : $y - y_1 = \dfrac{-2}{x_1}(x - x_1)$ or $y - \dfrac{x_1^2}{4} = \dfrac{-2}{x_1}(x - x_1)$

It passes through (1, 2)

$$2 - \frac{x_1^2}{4} = \frac{-2}{x_1}(1 - x_1) = -\frac{2}{x_1} + 2$$

$$x_1^3 = 8 \Rightarrow x_1 = 2 \quad \& \quad y_1 = \frac{x_1^2}{4} = 1$$

$$\therefore \text{ normal is } y - 1 = \frac{-2}{2}(x - 2) = 2 - x$$

$$x + y = 3$$

Illustration :

Curve $C_1 : y = ex \ln x$ *and* $C_2 : y = \dfrac{\ln x}{ex}$ *intersect at point 'P' whose abscissa is less than 1. Find equation of normal to curve* C_1 *at point P.*

Sol. For point of intesection $ex \ln x = \dfrac{\ln x}{ex} \Rightarrow \ln x = 0$ or $e^2x^2 = 1$

$$\Rightarrow x = 1 \text{ or } x = \pm\frac{1}{e} \quad \text{but} \quad 0 < x < 1$$

$$\Rightarrow \text{Point } P \text{ is } \left(\frac{1}{e}, -1\right)$$

For curve C_1, $\frac{dy}{dx} = e(1 + \ln x)$ $\Rightarrow$ *Slope of tangent at point P is equal to* $e\left(1+\ln\frac{1}{e}\right)=0$

$\Rightarrow$ *Equation of normal is* $x = \frac{1}{e}$

Illustration :

A line is drawn touching the curve $y - \frac{2}{3-x} = 0$*. Find the line if its slope/gradient is 2.*

Sol. $y = \frac{2}{3-x}$ $\Rightarrow$ $y' = \frac{2}{(3-x)^2}$ or $\frac{2}{(3-x_1)^2} = 2$

$\Rightarrow$ $(3 - x_1)^2 = 1$

or $3 - x_1 = 1, -1$

or $x_1 = 2, 4$ $\therefore$ *equation of line can be*

$\Rightarrow$ $y_1 = 2, -2$ $y - 2 = 2(x - 2)$

or $y + 2 = 2(x - 4)$

SOME COMMON PARAMETRIC COORDINATES ON A CURVE:

(a) for $x^{2/3} + y^{2/3} = a^{2/3}$ take parametric coordinate $x = a\cos^3\theta$ & $y = a\sin^3\theta$.

(b) for $\sqrt{x} + \sqrt{y} = \sqrt{a}$ take $x = a\cos^4\theta$ & $y = a\sin^4\theta$.

(c) $\frac{x^n}{a^n} + \frac{y^n}{b^n} = 1$ taken $x = a(\sin\theta)^{2/n}$ & $y = b(\sin\theta)^{2/n}$.

(d) for $c^2(x^2 + y^2) = x^2y^2$ take $x = c\sec\theta$ and $y = c\,\mathrm{cosec}\theta$.

(e) for $y^2 = x^3$, take $x = t^2$ and $y = t^3$.

Note: The tangent at P meeting the curve again at Q.

$\Rightarrow$ $\left.\frac{dy}{dx}\right|_P = \frac{y_2 - y_1}{x_2 - x_1}$

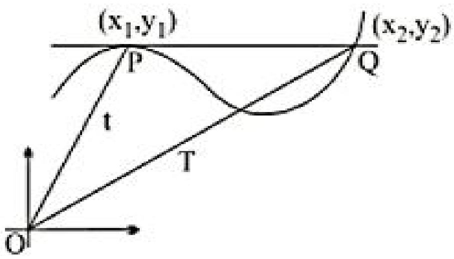


Illustration :

Tangent at point P on the curve $y^2 = x^3$ *meets the curve again at point Q. Find* $\frac{m_{OP}}{m_{OQ}}$*, where O is origin.*

Sol. *Take* $P(t^2, t^3)$ *and* $Q(T^2, T^3)$

$\frac{dy}{dx} = \frac{3x^2}{2y}$ or $\left(\frac{dy}{dx}\right) = \frac{3}{2}t$.

Slope line joining P and Q is = $\dfrac{T^3-t^3}{T^2-t^2}=\dfrac{T^2+t^2+Tt}{T+t}$

$\Rightarrow \quad \dfrac{3}{2}t=\dfrac{T^2+t^2+Tt}{T+t}$ or $3tT+3t^2=2T^2+2t^2+2Tt \Rightarrow T=\dfrac{-t}{2} \Rightarrow \dfrac{m_{OP}}{m_{OQ}}=-2.$

Illustration :

The equation(s) of the straight lines which is (are) tangents at one point and normal at another point of the curve $x = 3t^2$, $y = 2t^3$ *is(are)*

(A) $\sqrt{3}x+y=2\sqrt{2}$ (B) $\sqrt{3}x-y=2\sqrt{2}$ (C) $\sqrt{2}x-y=2\sqrt{2}$ (D) $\sqrt{2}x+y=2\sqrt{2}$

Sol. *Parametric equation of the given curve is* $x = 3t^2$, $y = 2t^3$ ....(1)

$\therefore \quad \dfrac{dx}{dt}=6t$ ....(2)

and $\quad \dfrac{dy}{dt}=6t^2$ ....(3)

$$\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=t.$$

Let $A \equiv (3t_1^2, 2t_1^3)$, $B \equiv (3t_2^2, 2t_2^3)$

Let line AB be a tangent to curve (1) at A and normal to the curve at B, then

$t_1=\dfrac{-1}{t_2}$ or $t_1t_2=-1$ ....(4)

Also slope of AB = $\dfrac{2(t_1^3-t_2^3)}{3(t_1^2-t_2^2)}=\dfrac{2}{3}\dfrac{t_1^2+t_2^2+t_1t_2}{t_1+t_2}$

Since AB is tangent to the curve at A,

$\therefore \quad \dfrac{2}{3}\left(\dfrac{t_1^2+t_2^2-1}{t_1+t_2}\right)=t_1 \Rightarrow 2t_1^2+2t_2^2-2=3t_1^2-3$

$\Rightarrow \quad t_1^2-2t_2^2-1=0 \Rightarrow t_1^2-\dfrac{2}{t_1^2}-1=0$ $[\because t_1t_2=-1]$

$\Rightarrow \quad k^2-k-2=0$

$\therefore \quad k=2,-1,$ where $k=t_1^2$

$\Rightarrow \quad t_1^2=2 \Rightarrow t_1=\pm\sqrt{2}$

$\therefore \quad t_2=\mp\dfrac{1}{\sqrt{2}}$

Hence, $t_1=\sqrt{2}$, $t_2=-\dfrac{1}{\sqrt{2}}$; $t_1=-\sqrt{2}$, $t_2=\dfrac{1}{\sqrt{2}}$

$\therefore \quad A\equiv\left(6,4\sqrt{2}\right)$ or $A\equiv\left(6,-4\sqrt{2}\right)$.

$\therefore$ Equation of AB i.e., equation of tangent at A is

$y-4\sqrt{2}=\sqrt{2}(x-6)$ i.e. $\sqrt{2}x-y-2\sqrt{2}=0$

or $\quad y+4\sqrt{2}=-\sqrt{2}(x-6)$ i.e. $\sqrt{2}x+y-2\sqrt{2}=0$.

Practice Problem

Q.1 Find the equation of tangent and normal to the curve $f(x)=\begin{cases} x-2 & \text{if } x<1 \\ x^2-x-1 & \text{if } x\geq 1\end{cases}$ at $x=1$ if it exists.

Q.2 Find the points on the curve $y=x^2-x^2-x+3$, where the tangent is parallel to x-axis.

Q.3 Show that the line $\frac{x}{a}+\frac{y}{b}=1$ is tangent to the curve, $y=be^{-x/a}$ at the point where the curve crosses the axis of y.

Q.4 For the curve $y=4x^3-2x^5$, find all points at which the tangent passes through the origin.

Q.5 The tangent at any point on the curve $x=a\cos^3\theta$, $y=a\sin^3\theta$ meets the axes in P and Q. Prove that the locus of the mid point of PQ is a circle.

Answer key

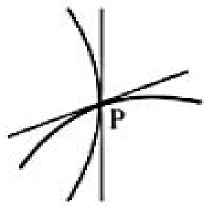
Q.1 T: $x-y-2=0$; $x+y=0$ Q.2 $(1, 2)$ and $\left(-\frac{1}{3}, 3\frac{5}{27}\right)$

Q.4 $(0, 0)$ and $(-1, -2)$

ANGLE OF INTERSECTION OF TWO CURVES :

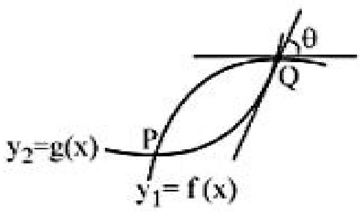
Definition :

The angle of intersection of two curves at a point P is defined as the angle between the two tangents to the curve at their point of intersection.



If the curves are orthogonal then

$\left(\frac{dy_1}{dx}\right)\left(\frac{dy_2}{dx}\right)=-1$ everywhere whereever they intersect.



If $\left(\frac{dy_1}{dx}\right)_P\left(\frac{dy_2}{dx}\right)_P=-1$ but $\left(\frac{dy_1}{dx}\right)_Q\left(\frac{dy_2}{dx}\right)_Q\neq-1$

then the two curves are orthogonal at P but not at Q hence they are not orthogonal.

e.g. $y^2=4ax$ & $y=e^{-x/2a}$; $xy=a^2$ & $x^2-y^2=b^2$ and $y=ax$ & $x^2+y^2=c^2$ are orthogonal but $y^2=4ax$ and $x^2=4by$ are not orthogonal.

Note : If the curves touch at $P(x_1, y_1)$ then $\theta=0$ hence $f'(x_1)=g'(x_1)$

Illustration :

Find the acute angle between the curve $y = \sin x$ & $y = \cos x$.

Sol. $y = \sin x$ & $y = \cos x$

Intersection point is $x = \frac{\pi}{4}$; $y = \frac{1}{\sqrt{2}}$

$y'_1 = \cos x = \frac{1}{\sqrt{2}}$ $\quad y'_2 = -\sin x = -\frac{1}{\sqrt{2}}$

$$\tan\theta = \left|\frac{\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}}{1 - \frac{1}{2}}\right| = 2\sqrt{2}$$

$$\theta = \tan^{-1}\left(2\sqrt{2}\right)$$

Illustration :

If θ *is the angle between* $y = x^2$ *and* $6y = 7 - x^3$ *at* (a, a). *Find* θ.

Sol. $y = x^2$ and $6y = 7 - x^3$ Point (a, a) is $(1, 1)$

$$y'_1 = 2x \quad \text{and} \quad y'_2 = \frac{-x^2}{2}$$

$$y'_1 \times y'_2 = -1$$

$$\therefore \quad \theta = \frac{\pi}{2}$$

Illustration :

Find the condition for the two concentric ellipses $a_1x^2 + b_1y^2 = 1$ *and* $a_2x^2 + b_2y^2 = 1$ *to intersect orthogonally.*

Sol. $a_1x^2 + b_1y^2 = 1$ $\quad$ *Let curves intersect at* (x_0, y_0) *then*

$a_2x^2 + b_2y^2 = 1$ $\quad$ $(a_1 - a_2)x_0^2 = (b_2 - b_1)y_0^2$ ...(1)

For 1^{st} *curve* $2a_1x + 2b_1yy' = 0$, $\quad y' = \frac{-a_1x_0}{b_1y_0}$

Similarly for 2^{nd} *curve* $y' = -\frac{a_2}{b_2}\frac{x_0}{y_0}$

For orthogonal intersection $\quad -\left(\frac{a_1}{b_1}\frac{a_2}{b_2}\frac{x_0^2}{y_0^2}\right) = -1$

$$\frac{a_1a_2}{b_1b_2}\left(\frac{b_2-b_1}{a_1-a_2}\right)=-1 \qquad \text{[From relation (1)]}$$

$$\frac{b_2-b_1}{b_1b_2}=\frac{a_2-a_1}{a_1b_2}$$

$$\therefore \qquad \frac{1}{b_1}-\frac{1}{b_2}=\frac{1}{a_1}-\frac{1}{a_2}$$

Illustration :

Match the following :

	Column-I		Column-II
(A)	If the parabola $y^2 = 4ax$, $a > 0$ cuts the hyperbola $xy = \sqrt{2}$ at right angles, then a =	(P)	$4\sqrt{2}$
(B)	If the curves $ay + x^2 = 7$, $a > 0$ and $x^3 = y$ cut orthogonally at (1, 1), then a =	(Q)	$2\sqrt{2}$
(C)	If the curves $y^2 = 4x$ and $xy = a$, $a > 0$ cut orthogonally, then a =	(R)	$\frac{1}{2}$
(D)	Curves $2x = y^2$ and $2xy = a$, $a > 0$ cut each other at right angles, then a =	(S)	6

[Ans. (A) R, (B) S, (C) P, (D) Q]

Sol.

(A) Given curves are, $y^2 = 4ax$ .....(1)

and $xy = \sqrt{2}$ .....(2)

From (1), $2y\frac{dy}{dx}=4a \quad \therefore \quad \frac{dy}{dx}=\frac{2a}{y}$ ......(3)

From (2), $y + x\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{y}{x}$ ....(4)

Putting the value of y from (2) in (1), we get $\frac{2}{x^2} = 4ax \Rightarrow x^3 = \frac{1}{2a}$ .....(5)

For curves (1) and (2) to cut at right angles, $\left(\frac{2a}{y}\right)\left(\frac{y}{x}\right) = -1$

$\Rightarrow \quad 2a = x \qquad 8a^3 = x^3 = \frac{1}{2a}$ [From (5)]

$\Rightarrow \quad 16a^4 = 1 \Rightarrow a = \frac{1}{2}$ [$\because a > 0$]

(B) Given curves are $ay + x^2 = 7$ ....(1)

and $y = x^3$ ....(2)

From (1), $\frac{dy}{dx} = \frac{-2x}{a}$ ....(3)

From (2), $\frac{dy}{dx} = 3x^2$ ....(4)

For curves (1) and (2) to cut each other orthogonally at (1, 1),

$$\left(-\frac{2}{a}\right)\cdot 3 = -1 \Rightarrow a = 6.$$

(C) Given curves are, $y^2 = 4x$ ....(1)

and $xy = a$ ....(2)

From (1), $2y\frac{dy}{dx} = 4$

$\therefore \quad \frac{dy}{dx} = \frac{2}{y}$ ......(3)

From (2), $1\cdot y + x\frac{dy}{dx} = 0 \quad \therefore \quad \frac{dy}{dx} = -\frac{y}{x}$ ....(4)

Putting the value of y from (2) in (1), we get

$$\frac{a^2}{x^2} = 4x \Rightarrow a^2 = 4x^3 \qquad \text{.......(5)}$$

From (2), $y = \frac{a}{x} \Rightarrow \frac{dy}{dx} = \frac{-a}{x^2}$ $\quad [\because \text{from (2)}, y = \frac{a}{x}]$

$$\left(\frac{dy}{dx}\right)_{any\ curve\ (1)}\cdot\left(\frac{dy}{dx}\right)_{for\ curve\ (2)} = \frac{2}{y}\cdot\frac{-a}{x^2} = \frac{-2}{x} \qquad \text{......(6)}$$

For curves (1) and (2) to cut each other orthogonally,

$$\frac{-2}{x} = -1 \Rightarrow x = 2. \qquad \text{[From (6)]}$$

$\therefore$ From (5), $a = 4\sqrt{2}$ $\quad [\because a > 1]$

(D) Given curves are, $y^2 = 2x$ ....(1)

and $xy = \frac{a}{2}$ ....(2)

From (1), $\frac{dy}{dx} = \frac{1}{y}$ ....(3)

From (2), $y + x\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{y}{x}$ ....(4)

$$\left(\frac{dy}{dx}\right)_{for\ curve\ (1)}\cdot\left(\frac{dy}{dx}\right)_{for\ curve\ (2)} = \frac{-1}{x} \qquad \text{.........(5)}$$

Putting the value of y from (2) in (1), we get

$$\frac{a^2}{4x^2} = 2x \Rightarrow 8x^3 = a^2 \qquad \text{.........(6)}$$

For the two curves to cut each other at ight angles,

$$-\frac{1}{x} = -1 \quad \Rightarrow x = 1$$

$\therefore$ *From (6),* $a^2 = 8 \;\Rightarrow\; a = 2\sqrt{2}$.

Illustration :

Find the angle of intersection of curves $y = [\,|\sin x| + |\cos x|\,]$ *and* $x^2 + y^2 = 5$, *where [ ] denotes the greatest integer function.*

Sol. *Given curves are* $y = [\,|\sin x| + |\cos x|\,]$ ......(1)
and $x^2 + y^2 = 5$ ......(2)

Let $|\sin x| = t$, *then* $|\cos x| = \sqrt{1-t^2}$ *and* $0 \le t \le 1$.

Now $|\sin x| + |\cos x| = t + \sqrt{1-t^2}$

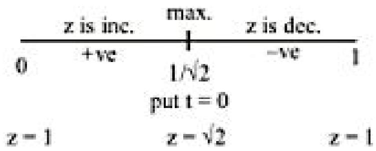
Let $z = t + \sqrt{1-t^2}$, $0 \le t \le 1$

Then $\dfrac{dz}{dt} = 1 - \dfrac{t}{\sqrt{1-t^2}} = \dfrac{\sqrt{1-t^2} - t}{\sqrt{1-t^2}}$

Since $\sqrt{1-t^2} > 0$, *therefore sign scheme for* $\dfrac{dz}{dt}$ *will be same as that of* $\sqrt{1-t^2} - t$.

Now $\sqrt{1-t^2} - t = 0 \Rightarrow 1 - t^2 = t^2 \Rightarrow t = \dfrac{1}{\sqrt{2}}$ $\qquad [\because t > 0]$

Sign scheme for $\left(\sqrt{1-t^2} - t\right)$ *is*



$\therefore$ *Greatest value of* $z = \sqrt{2}$
and least value of $z = 1$

$\therefore$ $1 \le z \le \sqrt{2}$ $\quad\therefore\quad [z] = 1$

$\therefore$ *curve (1) becomes* $y = 1$ ....(3)

Putting $y = 1$ *in (2), we get* $x^2 = 4$

$\therefore$ $x = \pm 2$

Hence points of intersection of curves (1) and (2) are P(– 2, 1) and Q(2, 1)

From (2), $\dfrac{dy}{dx} = -\dfrac{x}{y} = \begin{cases} 2, \text{ at } P(-2,1) \\ -2, \text{ at } Q(2,1) \end{cases}$

Clearly line (3) is parallel to x-axis. Its slope $m_1 = 0$
At P(– 2, 1), $m_1 = 0$ *and* $m_2 = 2$.

$\therefore$ *Acute angle between the two curves at P* $= \tan^{-1}\left|\dfrac{m_1 - m_2}{1 + m_1 m_2}\right| = \tan^{-1}2$.

Similarly acute angle between the two curves At Q(– 2, 0) is $\tan^{-1} 2$.

Practice Problem

Q.1 Find the angle between the curve $2y^2 = x^3$ and $y^2 = 32x$

Q.2 Show that the curves $x^3 - 3xy^2 = a$ and $3x^2y - y^3 = b$ cut each other orthogonally, where a and b are constants.

Q.3 Find the acute angles between the curves $y = |x^2 - 1|$ and $y = |x^2 - 3|$ at their points of intersection.

Q.4 Curves $\frac{x^2}{a^2} + \frac{y^2}{4} = 1$ and $y^3 = 16x$ cut each other at right angle then find the value of a^2.

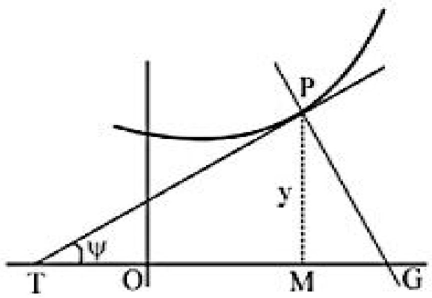
Answer key

Q.1 $\theta = \tan^{-1}\left(\frac{1}{2}\right)$ Q.3 $\theta = \tan^{-1}\left(\frac{4\sqrt{2}}{7}\right)$ Q.4 $a^2 = \frac{4}{3}$

LENGTH OF TANGENT, NORMAL, SUBTANGENT AND SUBNORMAL:

(i) **Tangent :**

$$PT = MP \operatorname{cosec} \psi = y\sqrt{1+\cot^2\psi} = \left| \frac{y\sqrt{1+\left(\frac{dy}{dx}\right)^2}}{\frac{dy}{dx}} \right|$$



(ii) **Subtangent :** $TM = MP \cot\psi = \left| \frac{y}{(dy/dx)} \right|$

(iii) **Normal :** $GP = MP \sec\psi = y\sqrt{1+\tan^2\psi} = \left| y\sqrt{1+\left(\frac{dy}{dx}\right)^2} \right|$

(iv) **Subnormal :** $MG = MP \tan y = \left| y\left(\frac{dy}{dx}\right) \right|$

Illustration :

Show that for the curve $by^2 = (x + a)^3$ the square of the subtangent varies as the subnormal.

Sol. $by^2 = (x + a)^3$ or $2byy' = 3(x + a)^2$

$$S.T. = \frac{y}{y'} = \frac{y}{3(x+a)^2}\,2by = \frac{2by^2}{3(x+a)^2} = \frac{2(x+a)}{3}$$

$$S.N. = yy' = y\frac{3(x+a)^2}{2by} = \frac{3(x+a)^2}{2b} \Rightarrow ST^2 \propto SN$$

Illustration :

Show that at any point on the hyperbola $xy = c^2$, the subtangent varies as the abscissa and the subnormal varies as the cube of the ordinate of the point of contact.

Sol. $xy = c^2 \Rightarrow xy' + y = 0$ or $y' = \frac{-y}{x}$

$$ST = \frac{y}{y'} = -x\,,\quad SN = yy' = \frac{-y^2}{x} = \frac{-y^2}{c^2}y = -\frac{-y^3}{c}$$

APPROXIMATION AND DIFFERENTIALS :

For the figure it is clear that if Δx and Δy are sufficiently small quantities then

$$\frac{\Delta y}{\Delta x} = \tan\psi \cong \frac{dy}{dx} = f\,'(x)$$

Hence approximate change in the value of y, called its differential is given by

$$\Delta y = f\,'(x)\cdot\Delta x \qquad ....(1)$$

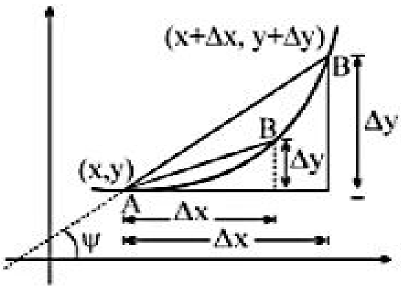


Illustration :

Use differential to approximate $\sqrt{101}$.

Sol. Let $f(x) = \sqrt{x}$ $\qquad f(100) = 10$ & $\qquad f\,'(x) = \frac{1}{2\sqrt{x}}$

$$\Delta y = \frac{1}{2\sqrt{100}} = \frac{1}{20} = 0.05 \Rightarrow f(101) = 10 + 0.05 = 10.05$$

INTERPRETATION OF $\frac{dy}{dt}$ AS A RATE MEASURE :

Recall that by the derivative $\frac{ds}{dt}$, we mean the rate of change of distance s with respect to the time t. In a similar fashion, whenever one quantity y varies with another quantity x, satisfying some rule $y = f(x)$, then $\frac{dy}{dx}$ (or $f'(x)$) represents the rate of change of y with respect to x and $\left(\frac{dy}{dx}\right)_{x=x_0}$ (or $f'(x_0)$) represents the rate of change of y with respect to x at $x = x_0$.

Illustration :

Displacement s' of a particle at time t is expressed as $s = \frac{1}{2}t^3 - 6t$, find the acceleration at the time when the velocity vanishes (i.e., velocity tends to zero).

Sol. $s = \frac{1}{2}t^3 - 6t$

Thus velocity, $v = \frac{ds}{dt} = \left(\frac{3t^2}{2} - 6\right)$

and acceleration, $a = \frac{dv}{dt} = \frac{d^2s}{dt^2} = 3t$

velocity vanishes when $\frac{3t^2}{2} - 6 = 0$

$\Rightarrow \quad t^2 = 4 \quad \Rightarrow \quad t = 2$

Thus, the acceleration when the velocity vanishes, is $a = 3t = 6$ *units.*

Illustration :

On the curve $x^3 = 12y$, find the interval of values of x for which the abscissa changes at a faster rate than the ordinate?

Sol. *Given* $x^3 = 12y$, *differentiating w.r.t. y*

$$3x^2 \frac{dx}{dy} = 12$$

$$\therefore \quad \frac{dy}{dx} = \frac{x^2}{4}$$

Now abscissa changes at a faster rate than the ordinate, then we must have $\left|\frac{dy}{dx}\right| < 1$

$\Rightarrow \quad |x^2| < 4, \; x \neq 0$

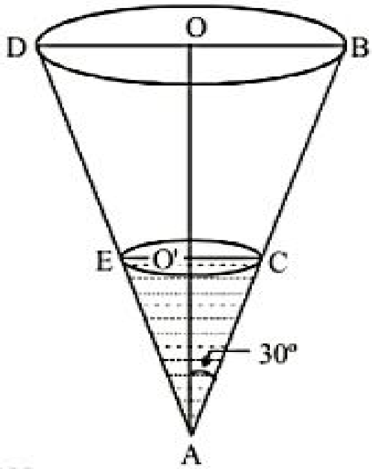
$\Rightarrow \quad -2 < x < 2, \; x \neq 0$

$\Rightarrow \quad x \in (-2, 2) - \{0\}$

Illustration :

If water is poured into an inverted hollow cone whose semi-vertical angle is 30°, such that its depth (measured along axis) increases at the rate of 1 cm per sec, find the rate at which the volume of water increases when the depth is 24 cm.

Sol. *Let A be the vertex and AO the axis of the cone.*
Let O'A = h be the depth of water in the cone.



In $\Delta AO'C$, $\tan 30^o = \dfrac{O'C}{h}$

$\Rightarrow \quad O'C = \dfrac{h}{\sqrt{3}}$

V = volumeof water in the cone $= \dfrac{1}{3}\pi (O'C)^2 \times AO'$

$$= \frac{1}{3}\pi\left(\frac{h^2}{3}\right)\times h$$

$\Rightarrow \quad V = \dfrac{\pi}{9}h^3 \quad \Rightarrow \quad \dfrac{dV}{dt} = \dfrac{\pi}{3}h^2\dfrac{dh}{dt} \quad$ ...(i)

But given that depth of water increases at the rate of 1 cm/sec

$\Rightarrow \quad \dfrac{dh}{dt} = 1$ cm/s $\quad$ ...(ii)

From (i) and (ii), $\quad \dfrac{dV}{dt} = \dfrac{\pi h^2}{3}$

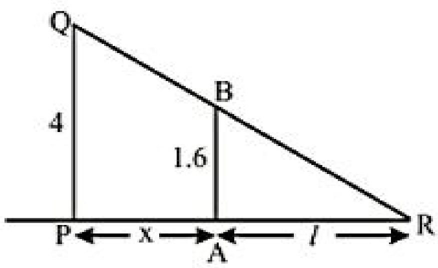
When $h = 24$cm, the rate of increase of volume $\dfrac{dV}{dt} = \dfrac{\pi(24)^2}{3} = 192$ cm^3/s

Illustration :

A man 1.6 m high walks at the rate of 30 metre per minute away from a lamp which is 4m above ground. How fast does the man's shadow lengthen?

Sol. *Let PQ = 4m be the height of pole and AB = 1.6 m be the height of the man.*
Let the end of a shadow is R and it is at a distance of l from A when the man is at a distance x from PQ at some instant.

Since, ΔPQR and ΔABR are similar, $\dfrac{PQ}{AB} = \dfrac{PR}{AR}$



$\Rightarrow \quad \dfrac{4}{1.6} = \dfrac{x+l}{l}$

$\Rightarrow \quad 2x = 3l$

$\Rightarrow \quad 2\dfrac{dx}{dt} = 3\dfrac{dl}{dt} \qquad \left[\text{given } \dfrac{dx}{dt} = 30\ m/min\right]$

$\Rightarrow \quad \dfrac{dl}{dt} = \dfrac{2}{3}.30$ m/min $= 20$ m/min.

Illustration :

A horse runs along a circle with a speed of 20 km/hr. A lantern is at the centre of the circle. A fence is along the tangent to the circle at the point at which the horse starts. The speed with which the shadow of the horse moves along the fence at the moment when it covers 1/8 of the circle in km/hr is

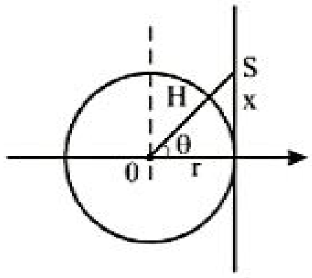
(A) 20 *(B) 40* *(C) 30* *(D) 60*

Sol. $\tan\theta = \dfrac{x}{r} \quad \Rightarrow \quad x = r\tan\theta$

H is position of horse and S is its shadow on fence

$$\Rightarrow \quad \frac{dx}{dt} = r\sec^2\theta\left(\frac{d\theta}{dt}\right) = r\omega\sec^2\theta = v\sec^2\theta$$

where $\theta = \dfrac{2\pi}{8}, \quad \dfrac{dx}{dt} = v\sec^2\left(\dfrac{\pi}{4}\right) = 2v = 40$ km/hr,



Practice Problem

Q.1 A particle moves along the curve $6y = x^3 + 2$. Find the points on the curve at which the y coordinate is changing 8 times as fast as the x coordinate.

Q.2 An open Can of oil is accidently dropped into a lake; assume the oil spreads over the surface as a circular disc of uniform thickness whose radius increases steadily at the rate of 10 cm/sec. At the moment when the radius is 1 meter, the thickness of the oil slick is decreasing at the rate of 4 mm/sec, how fast is it decreasing when the radius is 2 meters.

Q.3 A circular ink blot grows at the rate of 2 cm^2 per second. Find the rate at which the radius is increasing after $2\dfrac{6}{11}$ seconds. Use $\pi = \dfrac{22}{7}$.

Q.4 If in a triangle ABC, the side 'c' and the angle 'C' remain constant, while the remaining elements are changed slightly, show that $\dfrac{da}{\cos A} + \dfrac{db}{\cos B} = 0$.

Q.5 Using differentials, find the approximate value of $\sqrt{0.037}$, correct upto thre decimal places.

Answer key

Q.1 $(4, 11)$ & $(-4, -31/3)$

Q.2 0.05 cm/sec

Q.3 $\dfrac{1}{4}$ cm/sec.

Q.5 0.1924

Solved Examples

Q.1 Using differentials, find the approximate value of $(82)^{1/4}$ upto 3 places of decimal.

Sol. Let $f(x) = x^{1/4}$

$\therefore \quad f'(x) = \frac{1}{4}(x)^{-\frac{3}{4}} = \frac{1}{4x^{3/4}}$

Also, $f(x + \delta x) = (x + \delta x)^{1/4}$

Now, $f(x + \delta x) = f(x) + \delta x\, f'(x)$ (approximately)

$\Rightarrow \quad (x + \delta x)^{1/4} = x^{1/4} + \delta x \cdot \frac{1}{4x^{3/4}}$

We have to find $(82)^{1/4}$ and we know the value of $(81)^{1/4}$ which is equal to 3

$\therefore$ Putting $x = 81$, $x + \delta x = 82$ so that $dx = 1$ in (4), we get

$$(82)^{1/4} = (81)^{1/4} + 1 \cdot \frac{1}{4(81)^{3/4}} = 3 + \frac{1}{4 \times 3^3} = 3 + \frac{1}{108} = 3.009.$$

Q.2 The curve $y = ax^3 + bx^2 + cx + 5$ touches the x-axis at $P(-2, 0)$ and cuts the y axis at a point Q where its slope is 3. Find a, b and c.

Sol. Since the curve $y = ax^3 + bx^2 + cx + 5$ touches x-axis at $P(-2, 0)$ then x-axis is the tangent at $(-2, 0)$. The curve meets y-axis in (0, 5). We have

given where is $y = ax^3 + bx^2 + cx + 5$ ....(1)

$\therefore \quad \left(\frac{dy}{dx}\right)_{\text{at }(0,5)} = 3 \Rightarrow 0 + 0 + c = 3$

$\therefore \quad c = 3$ ....(2)

Since x-axis touches the curve at $P(-2, 0)$

$\therefore \quad \left(\frac{dy}{dx}\right)_{\text{at }(-2,0)} = 0$

$\Rightarrow \quad 12a - 4b + c = 0$

$\Rightarrow \quad 12a - 4b + 3 = 0$ [From (2)] ....(3)

Also $(-2, 0)$ lies on curve (1)

$\therefore \quad 0 = -8a + 4b - 2c + 5 \Rightarrow 0 = -8a + 4b - 1$ [$\because c = 3$]

$\Rightarrow \quad 8a - 4b + 1 = 0$ ....(4)

From (3) and (4) we get $a = -\frac{1}{2}$, $b = -\frac{3}{4}$.

Hence $a = -\frac{1}{2}$ $b = \frac{3}{4}$ anc $c = 3$.

Q.3 Show that angle between the tangents to the ellise $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, a > b and the circle $x^2+y^2=ab$ at a point of intersection is $\tan^{-1}\frac{a-b}{\sqrt{ab}}$.

Sol. Given curves are $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ ....(1)

and $x^2+y^2=ab$ ....(2)

Putting the value of y^2 from (2) in (1), we get

$$\frac{x^2}{a^2}+\frac{ab-x^2}{b^2}=1$$

or, $b^2x^2+a^3b-a^2x^2=a^2b^2$

or $(b^2-a^2)x^2=a^2b^2-a^3b=a^2b\,(b-a)$

or, $(b+a)x^2=a^2b$

$\therefore \quad x=\pm\sqrt{\frac{a^2b}{a+b}}=\pm a\sqrt{\frac{b}{a+b}}$

From (2), $y^2=ab-x^2=ab-\frac{a^2b}{a+b}=\frac{ab^2}{a+b}$

$\therefore \quad y=\pm b\sqrt{\frac{a}{a+b}}$

This points of intersection of curves (1) and (2) are

$P\left(a\sqrt{\frac{a}{a+b}}, b\sqrt{\frac{a}{a+b}}\right)$, $Q\left(-a\sqrt{\frac{a}{a+b}}, -b\sqrt{\frac{a}{a+b}}\right)$,

$R\left(-a\sqrt{\frac{b}{a+b}}, b\sqrt{\frac{a}{a+b}}\right)$ and $S\left(a\sqrt{\frac{b}{a+b}}, -b\sqrt{\frac{a}{a+b}}\right)$

Angle between the two curves at $P\left(a\sqrt{\frac{b}{a+b}}, b\sqrt{\frac{a}{a+b}}\right)$:

From (1), $\frac{dy}{dx}=-\frac{b^2}{a^2}\frac{x}{y}$; $\quad\therefore\quad \left(\frac{dy}{dx}\right)_{at\,P}=-\frac{b^2}{a^2}\cdot\frac{a}{b}\cdot\sqrt{\frac{b}{a}}=-\left(\frac{b}{a}\right)^{\frac{3}{2}}=m_1$ (say)

From (2), $\frac{dy}{dx}=-\frac{x}{y}$; $\quad\therefore\quad \left(\frac{dy}{dx}\right)_{at\,P}=-\frac{a}{b}\sqrt{\frac{b}{a}}=-\sqrt{\frac{a}{b}}=m_2$ (say)

If θ be the acute angle between curves (1) and (2) at P, then

$$\tan\theta=\left|\frac{m_1-m_2}{1+m_1m_2}\right|=\frac{-\left(\frac{b}{a}\right)^{\frac{3}{2}}+\sqrt{\frac{a}{b}}}{1+\frac{b}{a}}=\frac{a^2-b^2}{a^{\frac{3}{2}}\sqrt{b}}\,\frac{a}{a+b}=\frac{a-b}{\sqrt{ab}}$$

$\therefore \quad \theta = \tan^{-1}\dfrac{a-b}{\sqrt{ab}}$

Acute angle between the two curves at Q, R, S will be each $\tan^{-1}\dfrac{a-b}{\sqrt{ab}}$.

Q.4 Tangent at a point P_1 (other than (0, 0) on the curve $y = x^3$ meets the curve again at P_2. The tangent at P_2 meets the curve again at P_3 and so on. Show that the abscissae of $P_1, P_2, ..., P_n$ form a G.P. Also find the ratio $\dfrac{\text{area}(\Delta P_1P_2P_3)}{\text{area}(\Delta P_2P_3P_4)}$.

Sol. Given curve is $y = x^3$ ....(1)

$\therefore \quad \dfrac{dy}{dx} = 3x^2$ ....(2)

Let $P_1 (t_1, t_1^3)$ be a point on curve $y = x^3$

$\therefore \quad \left(\dfrac{dy}{dx}\right)_{\text{at}\left(t_1,\, t_1^3\right)} = 3t_1^2$

Equation of tangent at P_1 is $y - t_1^3 = 3t_1^2(x - t_1)$ .....(3)

Putting the value of y from (1) in (2), we get

$\Rightarrow \quad x^3 - t_1^3 = 3t_1^2(x - t_1)$

$\Rightarrow \quad (x - t_1)(x^2 + xt_1 + t_1^2) - 3t_1^2(x - t_1) = 0$

$\Rightarrow \quad (x - t_1)^2(x + 2t_1) = 0$

If $P_2 (t_2, t_2^3)$, then $(t_2 - t_1)^2(t_2 + 2t_1) = 0$

$\therefore \quad t_2 = -2t_1 (t_2 \neq t_1)$

Similarly, the tangent at P_2 will meet the curve at the point $P_3(t_3, t_3^3)$, where $t_3 = -2t_2 = 4t_1$ and so on.

Thus the abscissae of $P_1, P_2, ..., P_n$ are

$t_1, -2t_1, 4t_1, ..., (-2)^{n-1}t_1$, which are in G.P.

$\therefore \quad \dfrac{t_2}{t_1} = \dfrac{t_3}{t_2} = \dfrac{t_4}{t_3} = ... = -2(r \text{ say})$

$\therefore \quad t_2 = t_1 r,\ t_3 = t_2 r$ and $t_4 = t_3 r$

Now, area of $\Delta P_1P_2P_3 = \dfrac{1}{2}\begin{vmatrix} t_1 & t_1^3 & 1 \\ t_2 & t_2^3 & 1 \\ t_3 & t_3^3 & 1 \end{vmatrix}$ and $\Delta P_2P_3P_4 = \dfrac{1}{2}\begin{vmatrix} rt_1 & r^3 t_1^3 & 1 \\ rt_2 & r^3 t_2^3 & 1 \\ rt_3 & r^3 t_3^3 & 1 \end{vmatrix}$

$= r^4 (\text{Area of } (\Delta P_1P_2P_3))$

$\therefore \quad \dfrac{\text{Area of }(\Delta P_1P_2P_3)}{\text{Area of }(\Delta P_2P_3P_4)} = \dfrac{1}{r^4} = \dfrac{1}{(-2)^4} = \dfrac{1}{16}$.

Q.5 If the sum of the squares of the intercepts on the axes cut off by tangent to the curve $x^{1/3} + y^{1/3} = a^{1/3}$, $a > 0$ at $\left(\dfrac{a}{8}, \dfrac{a}{8}\right)$ is 2, then a =

(A) 1 (B) 2 (C) 4 (D) 8

Sol. Given, $x^{1/3} + y^{1/3} = a^{1/3}, a > 0$

$$\therefore \quad \frac{1}{3}x^{-2/3} + \frac{1}{3}y^{-2/3}\frac{dy}{dx} = 0$$

$$\therefore \quad \frac{dy}{dx} = -\left(\frac{y}{x}\right)^{2/3}$$

At $P\left(\frac{a}{8}, \frac{a}{8}\right), \frac{dy}{dx} = -1$

Equation of tangent at P is

$$y - \frac{a}{8} = -1\left(x - \frac{a}{8}\right) \quad \text{or,} \quad x + y = \frac{a}{4}$$

$\therefore$ It intercepts on the axes are $\frac{a}{4}, \frac{a}{4}$.

Given, $\frac{a^2}{16} + \frac{a^2}{16} = 2 \Rightarrow a^2 = 16 \Rightarrow a = 4.$ $(\because a > 0)$

Q.6 Tangents are drawn from the origin to the curve y = sin x, then their point of contact lie on the curve

(A) $\frac{1}{y^2} - \frac{1}{x^2} = 1$ (B) $\frac{1}{x^2} + \frac{1}{y^2} = 1$ (C) $x^2 - y^2 = 1$ (D) $x^2 + y^2 = 1$

Sol. Given curve is $y = \sin x$ ....(1)

Let the tangent to curve (1) at P (α, β) pass through (0, 0).

Equation of tangent at (α, β) is

$y - \beta = \cos\alpha(x - \alpha)$ ....(2)

Since (2) passes through (0, 0)

$\therefore$ $-\beta = -\alpha\cos\alpha$ or $\cos\alpha = \frac{\beta}{\alpha}$ ....(3)

Also, (α, β) lies on (1)

$\therefore$ $\sin\alpha = \beta$

From (3) and (4), $1 = \frac{\beta^2}{\alpha^2} + \beta^2$ ....(4)

or, $\alpha^2 - \beta^2 = \alpha^2\beta^2$ or, $\frac{\alpha^2 - \beta^2}{\alpha^2\beta^2} = 1$ or, $\frac{1}{\beta^2} - \frac{1}{\alpha^2} = 1$

$\therefore$ (α, β) lies on curve $\frac{1}{y^2} - \frac{1}{x^2} = 1$.

Q.7 The equation of the tangent(s) to the curve $y = \cos(x + y), -2\pi \le x \le 2\pi$ that is (are) parallel to the line $x + 2y = 0$ is (are)

(A) $2x + 4y + 3\pi = 0$ (B) $2x + 4y - \pi = 0$

(C) $x + 2y + \pi = 0$ (D) $2x + y + 3\pi = 0$

Sol. The given curve is $y = \cos(x + y)$

$\therefore$ $\frac{dy}{dx} = \sin(x + y)\left(1 + \frac{dy}{dx}\right)$ ....(1)

or, $\frac{dy}{dx} = -\frac{\sin(x+y)}{1+\sin(x+y)}$ ....(2)

The given line is $x + 2y = 0$ ....(3)

Its slope $= -\frac{1}{2}$

Since tangent is parallel to given line

$\therefore$ $\frac{dy}{dx}$ = slope of the tangent $= -\frac{1}{2}$.

From (2) and (3), $-\frac{\sin(x+y)}{1+\sin(x+y)} = -\frac{1}{2}$

or, $2\sin(x+y) = 1 + \sin(x+y)$

or, $\sin(x+y) = 1 \Rightarrow \cos(x+y) = 0$ ....(4)

From (1) and (4)

$\therefore$ $y = 0$

$\therefore$ From (1), $\cos(x+0) = 0$ or $\cos x = 0 = \cos\frac{\pi}{2}$

$\therefore$ $x = 2n\pi \pm \frac{\pi}{2}$, where $n = 0, \pm 1, \pm 2, ...$

$\therefore$ Values of x such that $-2\pi \le x \le 2\pi$ are

$\pm\frac{\pi}{2}, \pm\frac{3\pi}{2}$ of which only $\frac{\pi}{2}$ and $-\frac{3\pi}{2}$ satisfy equation (4)

Hence possible points are, $\left(\frac{\pi}{2}, 0\right)$ and $\left(-\frac{3\pi}{2}, 0\right)$

Equation of tangent at $\left(\frac{\pi}{2}, 0\right)$ is $y - 0 = -\frac{1}{2}\left(x - \frac{\pi}{2}\right)$ or $2x + 4y - \pi = 0$

Equation of tangent at $\left(-\frac{3\pi}{2}, 0\right)$ is $y - 0 = -\frac{1}{2}\left(x + \frac{3\pi}{2}\right)$ or $2x + 4y - 3\pi = 0$.

Q.8 The coordinates of the feet of the normals drawn from the point (14, 7) to the curve $y^2 - 16x - 8y = 0$ are

(A) (0, 0) (B) (3, 2) (C) (3, – 4) (D) (8, 16)

Sol. Given curve is $y^2 - 16x - 8y = 0$ ....(1)

Let $P \equiv (14, 7)$

Equation (1) can be written as $y^2 - 8y = 16x$

or, $y^2 - 8y + 16 = 16x + 16$

or, $(y-4)^2 = 16(x+1)$ ....(2)

This is of the form $(y-\beta)^2 = 4a(x-\alpha)$, where $a = 4, \alpha = -1, \beta = 4$

Let $(-1 + 4t^2, 4 + 8t)$ be any point on curve (2).

From (2), $2(y-4)\frac{dy}{dx} = 16$

$\therefore \quad \frac{dy}{dx} = \frac{8}{y-4}$

At $\quad (4t^2 - 1, 4 + 8t), \frac{dy}{dx} = \frac{8}{8t} = \frac{1}{t}$

$\therefore$ Equation of normal at $(-1 + 4t^2, 4 + 8t)$ is

$y - 4 - 8t = -t(x + 1 - 4t^2)$

or, $\quad tx + y - 4 - 8t + t - 4t^3 = 0 \qquad$ ....(3)

If line (3) passes through the point P(14, 7), then

$14t + 7 - 4t^3 - 7t - 4 = 0$

or, $\quad 4t^3 - 7t - 3 = 0$

or, $\quad (t + 1)(4t^2 - 4t - 3) = 0$

$\therefore \quad t = -1, \frac{4 \pm 8}{8} = 1, \frac{3}{2}, -\frac{1}{2}$

when $t = -1$, foot of the normal is $(3, -4)$

when $t = \frac{3}{2}$, foot of the normal is $(8, 16)$

when $t = -\frac{1}{2}$, foot of the normal is $(0, 0)$.

Q.9 Find the equations of tangents to the curve $x^2 + y^2 - 2x - 4y + 1 = 0$ which are parallel to the x-axis.

Sol. Equation of the curve is $x^2 + y^2 - 2x - 4y + 1 = 0$

$\therefore \quad 2x + 2y\frac{dy}{dx} - 2 - \frac{4dy}{dx} = 0 \qquad \Rightarrow \quad (2y - 4)\frac{dy}{dx} = 2 - 2x$

$\therefore \quad \frac{dy}{dx} = \frac{2-2x}{2y-4} = \frac{1-x}{y-2}$

Since tangents are parallel to the x-axis, slope of each of the tangents = 0.

$\therefore \quad \frac{1-x}{y-2} = 0 \quad \Rightarrow \quad 1 - x = 0 \quad \Rightarrow \quad x = 1$

At $x = 1$, $1^2 + y^2 - 2(1) - 4y + 1 = 0$

$\Rightarrow \quad y^2 - 4y = 0 \quad \Rightarrow y(y - 4) = 0 \quad \Rightarrow \quad y = 0 \quad$ or $\quad y = 4$

$\therefore$ the points are $(1, 0)$ and $(1, 4)$

$\therefore$ the equation of tangent through $(1, 0)$ and parallel to the x-axis is $y = 0$

and the equation of tangent through $(1, 4)$ and parallel to the x-axis is $y = 4$

$\therefore$ the equations of tangents are $y = 0$ and $y = 4$.

Q.10 Find the equation of tangent to the curve $\sqrt{x} + \sqrt{y} = \sqrt{a}$ at (x_0, y_0) and show that the sum of its intercepts on axes is constant.

Sol. The equation of the curve is $\sqrt{x} + \sqrt{y} = \sqrt{a}$

$\therefore \quad \frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}}\frac{dy}{dx} = 0 \quad \Rightarrow \quad \frac{dy}{dx} = -\frac{\sqrt{y}}{\sqrt{x}}$

At (x_0, y_0), $\dfrac{dy}{dx} = -\dfrac{\sqrt{y_0}}{\sqrt{x_0}}$

$\therefore$ slope of the tangent $= -\dfrac{\sqrt{y_0}}{\sqrt{x_0}}$

$\therefore$ equation of the tangent is $y - y_0 = m(x - x_0)$

$\Rightarrow$ $y - y_0 = -\dfrac{\sqrt{y_0}}{\sqrt{x_0}}(x - x_0)$ $\Rightarrow$ $y\sqrt{x_0} - y_0\sqrt{x_0} = --x\sqrt{y_0} + x_0\sqrt{y_0}$

$\Rightarrow$ $x\sqrt{y_0} + y\sqrt{x_0} = x_0\sqrt{y_0} + y_0\sqrt{x_0}$

$\Rightarrow$ $x\sqrt{y_0} + y\sqrt{x_0} = \sqrt{x_0 y_0}\left(\sqrt{x_0} + y_0\right)$ ......(i)

Since (x_0, y_0) is on the curve,

$$\sqrt{x_0} + \sqrt{y_0} = \sqrt{a} \qquad \text{......(ii)}$$

Putting this value in (i), we get the equation of tangent as

$$x\sqrt{y_0} + y\sqrt{x_0} = \sqrt{x_0 y_0}\sqrt{a} = \sqrt{a x_0 y_0}$$

$$y = 0 \Rightarrow x = \sqrt{a x_0} \Rightarrow \text{x-intercept} = \sqrt{a x_0}$$

$$x = 0 \Rightarrow y = \sqrt{a y_0} \Rightarrow \text{y-intercept} = \sqrt{a y_0}$$

$\therefore$ sum of intercepts $= \sqrt{a x_0} + \sqrt{a y_0} = \sqrt{a}\left(\sqrt{x_0} + \sqrt{y_0}\right)$

$= \sqrt{a} \cdot \sqrt{a}$ [from (ii)]

$= a =$ constant.

Q.11 Find the point on the parabola $y = (x - 3)^2$, where the tangent is parallel to the line joining (3, 0) and (4, 1).

Sol. Given curve is $y = (x - 3)^2$ ...(i)

Let $A \equiv (3, 0)$ and $B \equiv (4, 1)$

Slope of AB $= \dfrac{1 - 0}{4 - 3} = 1$ ...(ii)

From (i), $\dfrac{dy}{dx} = 2(x - 3)$ ...(iii)

Since tangent is parallel to line AB

$\therefore$ $\dfrac{dy}{dx} = 1$

From (ii) and (iii), we have $2(x - 3) = 1 \Rightarrow x = \dfrac{7}{2}$ ...(iv)

From (i), when $x = \dfrac{7}{2}, y = \left(\dfrac{7}{2} - 3\right)^2 = \dfrac{1}{4}$

Hence the required point is $\left(\dfrac{7}{2}, \dfrac{1}{4}\right)$

MONOTONICITY

GENERAL INTRODUCTION :

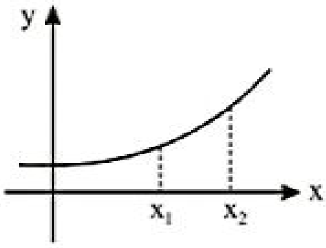
The most useful element taken into consideration amongst the total post mortuam activities of functions, is their monotonic behaviour.

(a) Monotonic function :

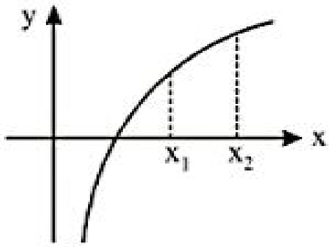
Functions are said to be monotonic if they are either increasing or decreasing in their entire domain *e.g.* $f(x) = e^x$; $f(x) = ln\,x$ & $f(x) = 2x + 3$ are some of the examples of functions which are increasing whereas $f(x) = -x^3$; $f(x) = e^{-x}$ and $f(x) = \cot^{-1}(x)$ are some of the examples of the functions which are decreasing.

Increasing function

$f(x) = e^x$



$f(x) = ln\,x$



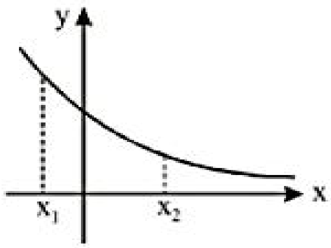
If $x_1 < x_2$ and $f(x_1) < f(x_2)$ then function is called increasing function or strictly increasing function.

Decreasing function

$f(x) = e^{-x}$

If $x_1 < x_2$

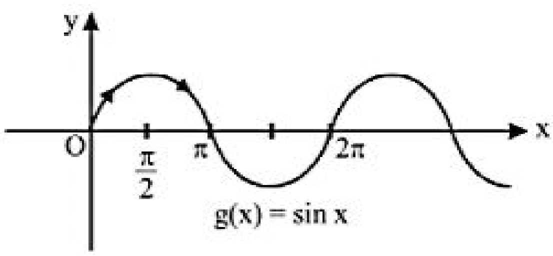
but $f(x_1) > f(x_2)$ in entire domain then fucntion is called decreasing function or strictly decreasing function.

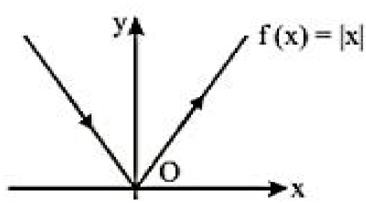


(b) Non Monotonic :

Functions which are increasing as well as decreasing in their domain are said to be non monotonic

e.g. $f(x) = \sin x$; $f(x) = ax^2 + bx + c$ and $f(x) = |x|$, however in the interval $\left[0, \frac{\pi}{2}\right]$, $f(x) = \sin x$ will be said to be increasing.

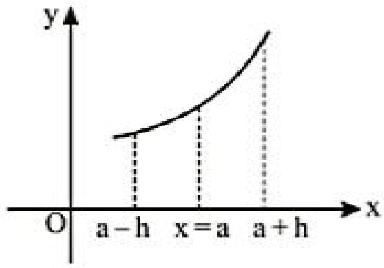




Monotonocity of a functon at a point :

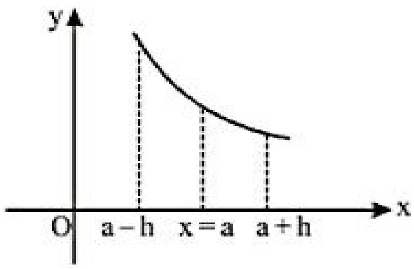
A function is said to be monotonically increasing $x = a$ if $f(x)$ satisfies

$$\left.\begin{array}{l} f(a+h) > f(a) \\ f(a-h) < f(a) \end{array}\right] \text{for a small Positive h.}$$

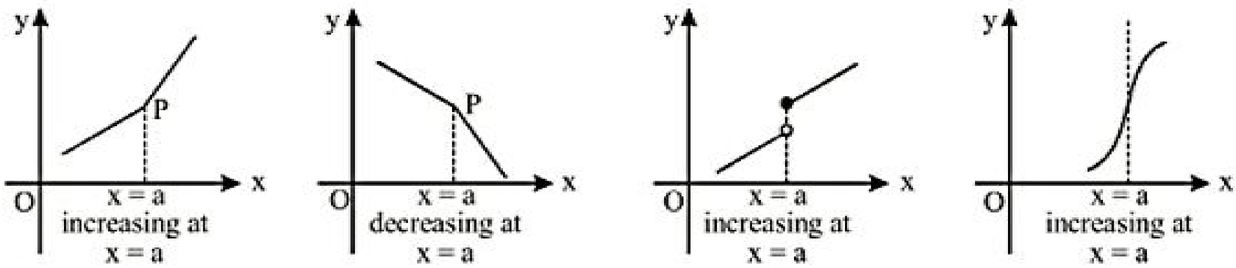


A function is said to be monotonically decreasing at $x = a$ if $f(x)$ satisfies

$$\left.\begin{array}{l} f(a+h) < f(a) \\ f(a-h) > f(a) \end{array}\right] \text{for a small Positive h.}$$



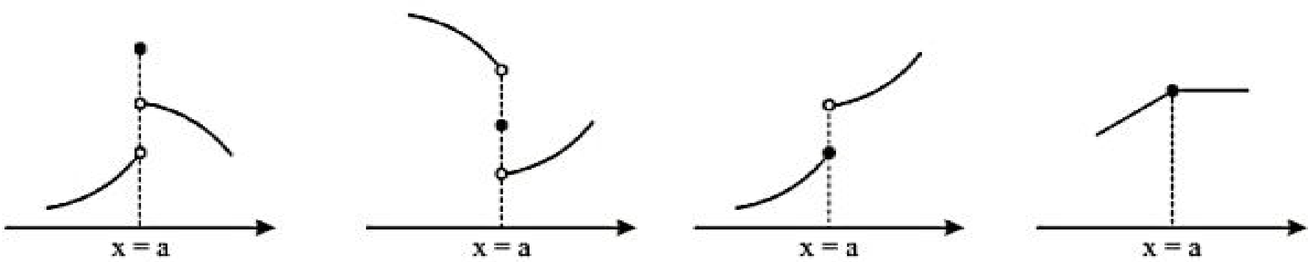
Note : It should be noted that we can talk of monotonocity of $f(x)$ at $x = a$ only if $x = a$ lies in the domain of $f(x)$, without any consideration of continuity or differentiability of $f(x)$ at $x = a$.



(a) (b) (c) (d)

Illustration :

For each of the following graph, comment whether f(x) is increasing or decreasing or neither incresing or decreasing at x = a.



(A) (B) (C) (D)

Sol.

(A) *Neither monotonically increasing nor decreasing as $f(a-h) < f(a)$ and $f(a+h) < f(a)$*

(B) *Monotonically decreasing as $f(a-h) < f(a) > f(a+h)$*

(C) *Monotonically increasing as $f(a-h) < f(a) < f(a+h)$*

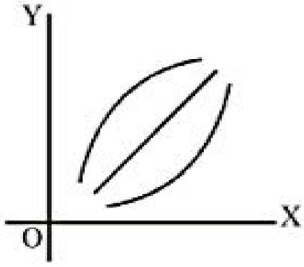
(D) *Neither monotonically increasing nor decreasing as $f(a-h) < f(a)$ but $f(a+h) = f(a)$*

MONOTONICITY IN AN INTERVAL :

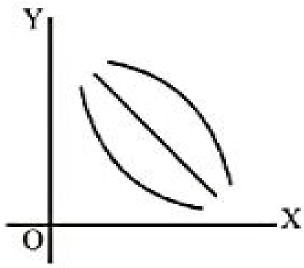
(a) For an increasing function in some interval,

if $\Delta x > 0 \Leftrightarrow \Delta y > 0$ or $\Delta x < 0 \Leftrightarrow \Delta y < 0$

then f is said to be monotonic (strictly) increasing in that interval. In other words if Δy and Δx have the same sign *i.e.* $\frac{dy}{dx} > 0$, for increasing function. Hence if $\frac{dy}{dx} > 0$ in some (interval) then y is said to be increasing function in that interval and conversely if $f(x)$ is increasing in some interval then $\frac{dy}{dx} > 0$ in that J.



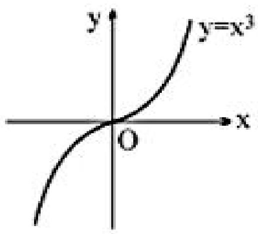
(b) Similarly if $\frac{dy}{dx} < 0$ in some interval then y is decreasing in that J and conversely.



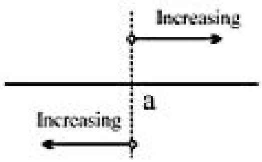
Note : Hence to find the intervals of monotonocity for a function $y = f(x)$ one has to find $\frac{dy}{dx}$ and solve the inequality, $\frac{dy}{dx} > 0$ or $\frac{dy}{dx} < 0$. The solution of this inequality gives the interval of monotonocity.

It should however be noted that

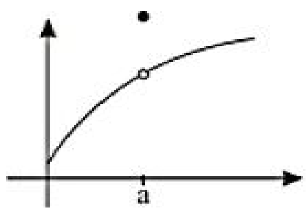
(a) $\frac{dy}{dx}$ at some point may be equal to zero but $f(x)$ may still be increasing at x = a. Consider $f(x) = x^3$ which is increasing at x = 0 although $f'(x)=0$. This is because $f(0+h) > f(0)$ and $f(0-h) < f(0)$. At all such points where $\frac{dy}{dx} = 0$ but y is still increasing or decreasing are known as **point of inflection**, which indicate the change of concavity of the curve.



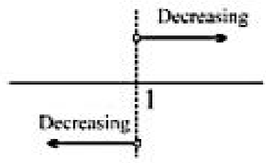
(b) If f is increasing for $x > a$ and f is also increasing for $x < a$ then f is also increasing as $x = a$ provided $f(x)$ is continuous at $x = a$.



(c) If f(x) is discontinuous at x = a then one can draw the graph as shown x = a is the point of maxima



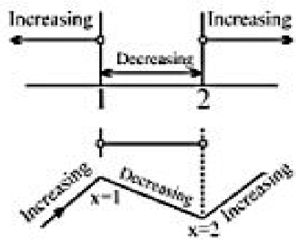
(d) Similarly if f is decreasing for $x > a$ and f is also decreasing for $x < a$ then f is also decreasing for $x = a$ provided $f(x)$ is continuous at $x = a$.



(e) However if $f(a)$ is not defined then monotonocity will not be indicated at $x = a$

e.g. $f(x) = \dfrac{1}{x-1}$ is decreasing for $x \in (-\infty, 1) \cup (1, \infty)$.

However if f is increasing and decreasing as shown then at $x = 1$ and $x = 2$, f will have extremum values, being maximum at $x = 1$ and minimum at $x = 2$.



Increasing and decreasing functions :

(i) A function f(x) is said to be monotonically increasing for all such interval (a, b) where $f'(x) \geq 0$ and equality may hold only for discrete values of x, i.e., $f'(x)$ does not identically become zero for $x \in (a, b)$ or any sub-interval.

(ii) f(x) is said to be monotonically decreasing for all such interval (a, b) where $f'(x) \leq 0$ and equility may hold only for discrete values of x.

Note : By discrete points, we mean those points where $f'(x) = 0$ does not form any interval.

Illustration :

Prove that $f(x) = x^3$ in an increasing function.

Sol. *Clearly $f'(x) = 3x^2 \geq 0$ in $(-\infty, \infty)$ and equility holds only at $x = 0$ and not in any interval, therefore $f(x) = x^3$ is an increasing fucnton in $(-\infty, \infty)$.*

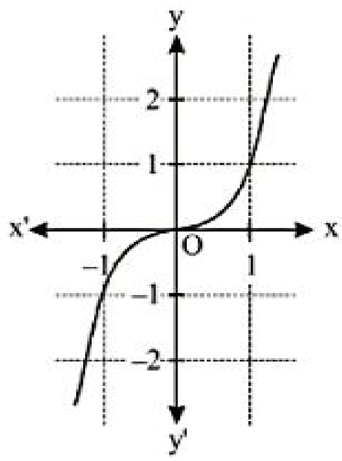


Illustration :

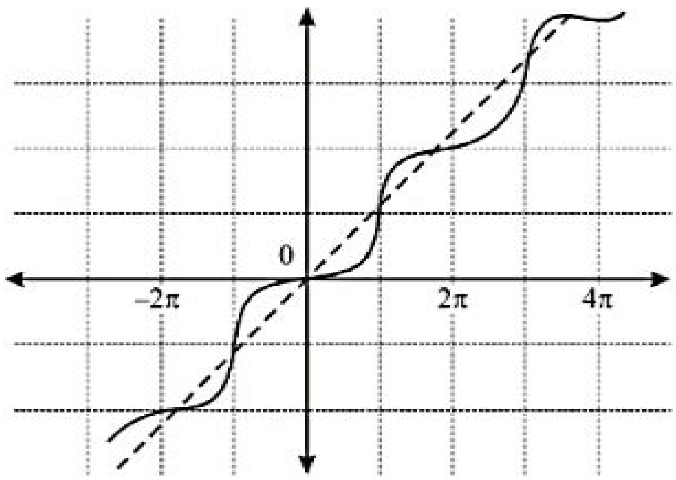
Prove that $f(x) = x - \sin x$ is an increasing function.

Sol. *$f(x) = x - \sin x$*

$\Rightarrow$ $f'(x) = 1 - \cos x$

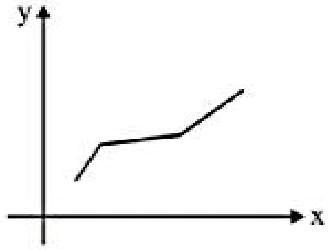
Now, $f'(x) > 0$ everywhere except at $x = 0, \pm 2\pi, \pm 4\pi$ etc. but all these points are discrete and do not form an interval hence we can conclude that f(x) is monotoically increasing for $x \in R$.

In fact, we can also see it graphically.



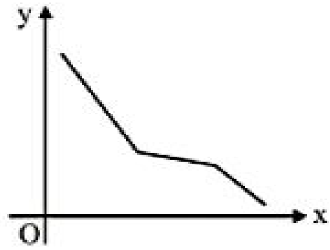
Non decreasing function :

f(x) is said to be non-decreasing in domain if for every $x_1, x_2 \in D_1$, $x_1 > x_2 \Rightarrow f(x_1) \geq f(x_2)$. It means that the value of f(x) would never decrease with an increase in the value of x (Figure).

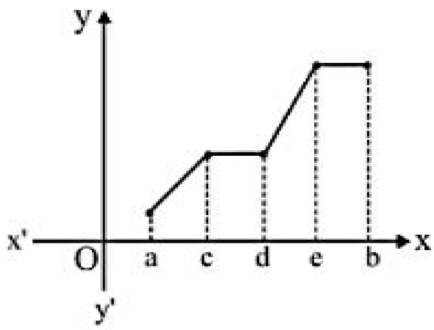


Non Increasing function :

f(x) is said to be non-increasing in domain if for every $x_1, x_2 \in D_1$, $x_1 > x_2$ $f(x_1) \leq f(x_2)$. It means that the value of f(x) would never increase with an increase in the value of x (Figure).



Let us consider another function whose graph is shown for $x \in (a, b)$.



Here also $f'(x) \geq 0$ for all $x \in (a, b)$ but note that in this case equality of $f'(x) = 0$ holds for all $x \in (c, d)$ and (e, b). Here f'(x) becomes identically zero and hence the given function cannot be assumed to be monotonically increasing for $x \in (a, b)$.

Note:

(i) If a function is monotonic at x = a it can not have extremum point at x = a and vice versa *i.e.* a point on the curve can not simultaneously be an extremum as well as monotonic point.

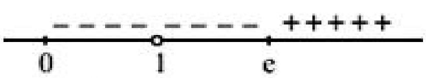
(ii) If f is increasing then nothing definite can be said about the function f'(x) w.r.t. its increasing or decreasing behaviour.

Illustration :

Find intervals of monotonicity of $f(x) = \dfrac{x}{\ln x}$

Sol. $f'(x) = \dfrac{\ln x - 1}{(\ln x)^2}$

sign of f'(x)



strictly increasing in (e, ∞) *and strictly decreasing in* $(0, 1) \cup (1, e)$

Illustration :

If the function $f(x) = (a + 2)x^3 - 3ax^2 + 9ax - 1$ is always decreasing $\forall\, x \in R$, find 'a'.

Sol. $f'(x) = 3(a + 2)\,x^2 - 6ax + 9a \le 0 \quad \forall\; x \in R$

$\Rightarrow\; 3(a + 2) < 0 \qquad \& \qquad 36a^2 - 4 \cdot 3\,(a + 2) \cdot 9a \le 0$

$\Rightarrow\; a < -2 \qquad \& \qquad a^2 - 3a(a + 2) \le 0$

$\Rightarrow\; a < -2 \;\;...(1) \qquad \& \qquad a^2 + 3a \ge 0 \quad \Rightarrow\; a \in (-\infty, -3] \cup [0, \infty) \;...\; (2)$

from (1) and (2)

$a \le -3$ *Ans.*

Illustration :

Find intervals of monotonicity of following functions :

(a) $f(x) = x^4 - 8x^3 + 22x^2 - 24x + 7$ (b) $f(x) = \dfrac{2x}{1+x^2}$

(c) $f(x) = \ln\,(x^2 - 2x)$ (d) $f(x) = \dfrac{|x-1|}{x^2}$

Sol.

(a) *We have*

$f(x) = x^4 - 8x^3 + 22x^2 - 24x + 7,\;\; x \in R$

and $f'(x) = 4x^3 - 24x^2 + 44x - 24 = 4(x - 1)\,(x - 2)\,(x - 3)$

From the sign scheme for $f'(x)$, we can see that $f(x)$

From the sign scheme for $f'(x)$, we can see that $f(x)$

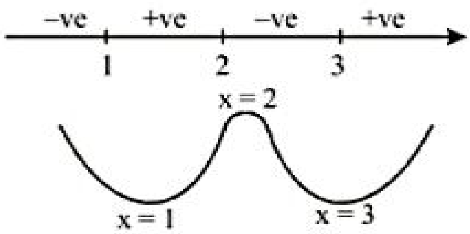
strictly decreases in $(-\infty, 1)$

strictly increases in $(1, 2)$

strictly decreases in $(2, 3)$

strictly increases in $(3, \infty)$.

The shape of the curve is drawn alongside.



(b) *We have* $f(x) = \dfrac{2x}{1+x^2},\; x \in R$

and $f'(x) = \dfrac{(1+x^2)2 - 2x(2x)}{(1+x^2)^2} = \dfrac{-2(x^2-1)}{(1+x^2)^2} = \dfrac{-2(x+1)(x-1)}{(1+x^2)^2}$

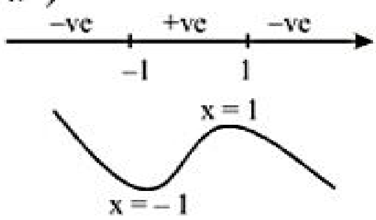
From the sign scheme for $f'(x)$, we can see that $f(x)$

strictly decreases in $(-\infty, 1)$

strictly increases in $(-1, 1)$

strictly decreases in $(1, \infty)$

The shape of the curve is shown alongside



(c) *We have* $f(x) = \ln\,(x^2 - 2x),\; x \in (-\infty, 0) \cup (2, \infty)$

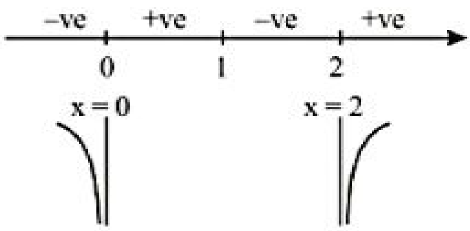
and $f'(x) = \dfrac{2x-2}{x^2-2x} = \dfrac{2(x-1)}{x(x-2)}$

From the sign scheme for $f'(x)$, we can see that $f(x)$

strictly decreases in $(-\infty, 0) \cup (1, 2)$

strictly increases in $(0, 1) \cup (2, \infty)$.

Also, we can see that $f(0^-) = -\infty$ and $f(2^+) = -\infty$.



(d) *We have* $f(x) = -\dfrac{(x-1)}{x^2}$, $x < 1$ *and* $f(x) = \dfrac{x-1}{x^2}$, $x \ge 1$

and $f'(x) = \dfrac{-2}{x^3} + \dfrac{1}{x^2} = \dfrac{x-2}{x^3}$, $x < 1$ *and* $f'(x) = \dfrac{2-x}{x^3}$, $x > 1$

Now, from the sign scheme for $f'(x)$, *we have*

$\Rightarrow$ *$f(x)$ strictly increases in $(-\infty, 0)$*

strictly decreases in $(0, 1)$

strictly increases in $(1, 2)$

strictly decreases in $(2, \infty)$.

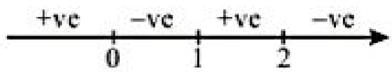


Illustration :

If $\phi(x) = f(x) + f(1-x)$ *and* $f''(x) < 0$ *in* $(-1, 1)$, *then show that* $\phi(x)$ *strictly increases in* $\left(0, \dfrac{1}{2}\right)$.

Sol. *We have* $\phi(x) = f(x) + f(1-x)$ *and* $\phi'(x) = f'(x) - f'(1-x)$

which vanishes at points given by $x = 1 - x$ *i.e.* $x = \dfrac{1}{2}$

$f''(x) < 0 \Rightarrow f'(x)$ *is decreasing for* $x \in \left(0, \dfrac{1}{2}\right)$ *i.e.* $1 - x > x \Rightarrow f'(1-x) < f'(x)$

hence $\phi'(x) > 0 \ \forall\ x \in \left(0, \dfrac{1}{2}\right)$

Hence, $\phi(x)$ *strictly increases in* $\left(0, \dfrac{1}{2}\right)$.

Practice Problem

Q.1 Compute the intervals of monotonicity for the following $f(x) = x^2 \cdot e^{-x}$

Q.2 Compute the intervals of monotonicity for the following $f(x) = x + ln(1 - 4x)$

Q.3 Find value of a so that $f(x) = ax - \sin x$ is monotonic.

Q.4 Prove that $f(x) = \dfrac{2}{3}x^9 - x^6 + 2x^3 - 3x^2 + 6x - 1$ is always increasing.

Answer key

Q.1 $\uparrow$ in $(0, 2)$ and $\downarrow$ in $(-\infty, 0) \cup (2, \infty)$

Q.2 $\uparrow$ in $(-\infty, -3/4)$ and $\downarrow$ in $(-3/4, 1/4)$

Q.3 $\uparrow$ for $a \ge 1$ & $\downarrow$ for $a \le -1$

GREATEST AND LEAST VALUE OF A FUNCTION :

Case-I :

If a continuous function $y = f(x)$ is strictly increasing in the closed interval [a, b] then $f(a)$ is the least value. (figure - 1) and f(b) is greatest value

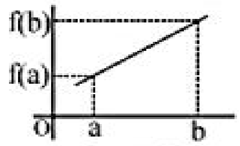


figure -1

Case-II :

If $f(x)$ is decreasing in [a, b] then $f(b)$ is the least and $f(a)$ is the greatest value of $f(x)$ in [a, b].(figure - 2)

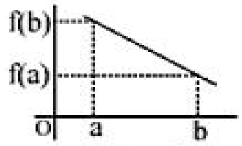


figure -2

Case-III :

However if $f(x)$ is non monotonic in [a, b] and is continuous then the greatest and least value of $f(x)$ in [a, b] are those where $f(x) = 0$ **or** $f'(x)$ does not exist or at the extreme values. (figure - 3)

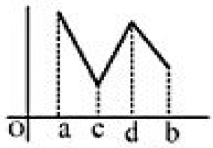


figure -3

Illustration :

Find least and greatest value of $f(x) = e^{x^2-4x+3}$ *in* $[-5, 5]$

Sol. $f(x) = e^{x^2-4x+3}$

For $f(x)$ *max* $\rightarrow x^2 - 4x + 3$ *be maximum in* $[-5, 5]$

$x^2 - 4x + 3$ *will be maximum at* $x = -5$ *in the given interval.*

i.e., $25 + 20 + 3 = 48$

$\therefore$ *Max* $f(x) = e^{48}$ *at* $x = -5$

$x^2 - 4x + 3$ *will be minimum at* $x = 2$ *i.e,* $4 - 8 + 3 = -1$

$\therefore$ *Min* $f(x) = e^{-1}$ *at* $x = 2$.

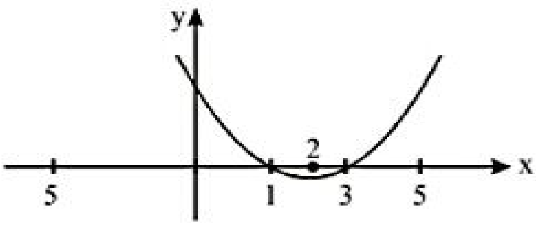


Illustration :

Find the image of interval $[-1, 3]$ *under the mapping specified by the function* $f(x) = 4x^3 - 12x$.

Sol. $f'(x) = 12x^2 - 12 = 12(x^2 - 1)$

$f'(x) = 0$ *at* $x = \pm 1$, $f(-1) = 8$, $f(1) = -8$

$f(3) = 72$ $\Rightarrow$ *greatest value is 72 and least value is* -8.

Illustration :

Find the range of the following functions $f(x) = \sqrt{x-3} + 2\sqrt{5-x}$.

Sol. *We have* $f(x) = \sqrt{x-3} + 2\sqrt{5-x}$

whose domain is $x \in [3, 5]$ *and its derivative is*

$$f'(x) = \frac{1}{2\sqrt{x-3}} - \frac{1}{\sqrt{5-x}} = \frac{\sqrt{5-x} - 2\sqrt{x-3}}{2\sqrt{x-3}\,\sqrt{5-x}}$$

Now, solving

$$\sqrt{5-x} > 2\sqrt{x-3} \qquad i.e. \qquad 5 - x > 4(x-3) \text{ given } x < \frac{17}{5}.$$

Hence, we have

$$f'(x) > 0 \ \forall x \in \left(3, \frac{17}{5}\right) \ \& \ f'(x) < 0 \ \forall x \in \left(\frac{17}{5}, 5\right)$$

$\Rightarrow$ $f(x)$ *strictly increases in* $\left(3, \frac{17}{5}\right)$ *and strictly decreases in* $\left(\frac{17}{5}, 5\right)$.

Now, we have

$$f(3) = 2\sqrt{2},\ f(5) = \sqrt{2} \qquad \text{and} \qquad f\left(\frac{17}{5}\right) = \sqrt{\frac{17}{5} - 3} + 2\sqrt{5 - \frac{17}{5}} = \sqrt{10}.$$

Hence, the range is $y \in \left[\sqrt{2}, \sqrt{10}\right]$

.

Illustration :

Find the range of the following functions $f(x) = \dfrac{x^4 - x^2 - 2x + 8}{x^4 - x^2 - 2x + 4}$

Sol. *We have*

$$f(x) = \frac{x^4 - x^2 - 2x + 8}{x^4 - x^2 - 2x + 4} = 1 + \frac{4}{x^4 - x^2 - 2x + 4} = 1 + \frac{4}{(x^2-1)^2 + (x-1)^2 + 2}.$$

Let $g(x) = (x^2 - 1)^2 + (x - 1)^2 + 2$, *whose least value* $= 2$

and *greatest value* $= \infty$

Thus, we have for $f(x)$ *greatest value* $= 1 + \frac{4}{2} = 3$ *and least value* $= 1 + \frac{4}{\infty} = 1$.

Also, $f(x)$ *is continuous and defined on R. Hence, the range of* $f(x)$, *is* $y \in (1, 3]$.

ESTABLISHING INEQUALITIES :

Notion of monotonocity helps is establishing variety of inequalities involving algebraic and transcidental function with much greater ease.

If $f(x) \geq g(x)$ or $f(x) \leq g(x)$ is to be shown in some interval we create a new function $h(x) = f(x) - g(x)$ and using monotonicity check which $h(x) \geq 0$ or $h(x) \leq 0$ in the given interval.

Illustration :

Prove that $2 \sin x + \tan x \geq 3x \quad \left(0 \leq x < \dfrac{\pi}{2}\right)$

Sol. $f(x) = 2 \sin x + \tan x - 3x$

$f'(x) = 2 \cos x + \sec^2 x - 3$

$f''(x) = -2\sin x + 2\sec^2 x \tan x \qquad = 2\sin x\,[\sec^3 x - 1] \geq 0 \ \forall x \in \left[0, \dfrac{\pi}{2}\right)$

$\Rightarrow f'(x) \uparrow \text{in} \left[0, \dfrac{\pi}{2}\right) \Rightarrow f'(x) \geq f'(0)$

or $f'(x) \geq 0 \Rightarrow f(x)\uparrow$ hence $f(x) \geq f(0)$

in $\left[0, \dfrac{\pi}{2}\right) \Rightarrow f(x) \geq 0 \Rightarrow 2\sin x + \tan x \geq 3x$

Illustration :

Find the set of values of x for which $\ln(1 + x) > \dfrac{x}{1+x}$ *[Ans.* $(-1, 0) \cup (0, \infty)$*]*

Sol. $f(x) = \ln(1 + x) - \dfrac{x}{1+x} = \ln(1 + x) + \dfrac{1}{1+x} - 1$

Domain : $x > -1$

$f'(x) = \dfrac{1}{1+x} - \dfrac{1}{(1+x)^2} = \dfrac{x}{(1+x)^2}$

$f'(x) \geq 0 \ \forall \ x \geq 0 \Rightarrow f(x)\uparrow$

$\&\ f'(x) \leq 0 \ \forall \ x \leq 0 \Rightarrow f(x)\downarrow$

$f'(0) = 0$

$\therefore f(x) > f(0) \ \forall \ x \in D_f - \{0\}$

$\therefore f(x) > 0 \ \forall \ x \in (-1, 0) \cup (0, \infty)$

Illustration :

Show that $\ln(1+x) > x - \dfrac{x^2}{2} \ \forall\ x \in (0, \infty)$

Sol. *Consider the function* $f(x) = \ln(1+x) - x + \dfrac{x^2}{2}, x \in (0, \infty)$

Then $f'(x) = \dfrac{1}{1+x} - 1 + x = \dfrac{x^2}{1+x} > 0 \ \forall x \in (0, \infty)$

$\Rightarrow$ $f(x)$ *strictly increases in* $(0, \infty)$

$\Rightarrow$ $f(x) > f(0^+) = 0$ *i.e.* $\ln(1+x) > x - \dfrac{x^2}{2}$ *which is the desired result.*

Illustration :

Show that the equation $x^5 - 3x - 1 = 0$ *has a unique root in* $[1, 2]$.

Sol. *Consider the function*

$f(x) = x^5 - 3x - 1,\ x \in [1, 2]$

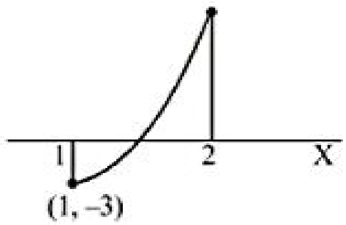
and $f'(x) = 5x^4 - 3 > 0\ \forall x \in [1, 2]$

$\Rightarrow$ $f(x)$ *stricIty increases in* $[1, 2]$

Also, we have

$f(1) = 1 - 3 - 1 = -3$

and $f(2) = 32 - 6 - 1 = 25$



From the shape of the curve shown alongside, we can see that the curve $y = f(x)$ *will cut the X-axis exactly once in* $[1, 2]$

i.e. $f(x)$ *will vanish exactly once in* $[1, 2]$

Illustration :

Prove that $\dfrac{x}{1+x} < \ln(1+x) < x\ \forall x > 0$

Sol. *Consider the function* $f(x) = \ln(1+x) - \dfrac{x}{1+x}, x > 0.$

Then $f'(x) = \dfrac{x}{1+x} - \dfrac{x}{(1+x)^2} = \dfrac{x}{(1+x)^2} > 0\ \forall x > 0$

$\Rightarrow$ $f(x)$ *strictly increases in* $(0, \infty)$

$\Rightarrow$ $f(x) > f(0^+) = 0$ *i.e.* $\ln(1+x) > \dfrac{x}{1+x}$ *which proves the LHI.*

Now, consider the function $g(x) = x - \ln(1+x), x > 0$

Then $g'(x) = 1 - \dfrac{1}{1+x} = \dfrac{x}{1+x} > 0\ \forall x > 0$

$\Rightarrow$ $g(x)$ *strictly increases in* $(0, \infty)$ $\Rightarrow g(x) > g(0^+) = 0$

i.e. $x > \ln(1+x)$ *which proves the RHI.*

Practice Problem

Q.1 $f(x) = \cos 3x - 15\cos x + 8$ in $\left[\frac{\pi}{3}, \frac{3\pi}{2}\right]$

Q.2 Use the function $f(x) = x^{\frac{1}{x}}$ $(x > 0)$ to ascertain whether π^e or e^π is greater.

Q.3 Prove that $e^x - e^{-x} - 2x > 0 \ \forall\ x > 0$. Hence, prove that $e^x + e^{-x} \ge x^2 + 2 \ \forall\ x \ge 0$.

Q.4 Prove that the function $f(x) = \frac{\ln x}{x}$, is strictly decreasing in (e, ∞). Hence, prove that $303^{202} < 202^{303}$.

Answer key

Q.1 Max. at $x = \pi = 22$ & Min. at $x = \frac{\pi}{3} = -\frac{1}{2}$ Q.2 $\pi^e < e^\pi$

ROLLE'S & MEAN VALUE THEOREM :

Rolle's Theorem :

Let f(x) be a function subject to the following conditions :

(i) f(x) is a continuous function in the closed interval of $a \le x \le b$.

(ii) $f'(x)$ exists for every point in the open interval $a < x < b$.

(iii) $f(a) = f(b)$.

Then there exists at least one point $x = c$ such that $a < c < b$ where $f'(c) = 0$.

Alternative statement:

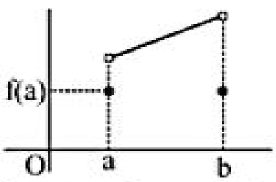
Rolles theorem states that between any two real zeroes of a differentiable real function f, lies at least one critical point of $f(x)$.

Remarks:

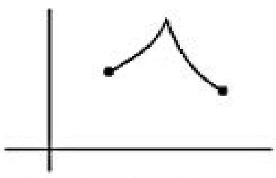
(i) Converse of Rolle's theorem is Not true i.e. $f'(x)$ may vanish at a point within (a, b) without satisfying all the three conditions of Rolle's Theorem.

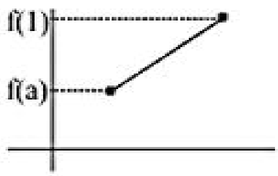
(ii) The three conditions are sufficient but not necessary for $f'(x) = 0$ for some x in (a, b)

(iii) If the function $y = f(x)$ defined over [a, b] does not satisfy even one of the 3 conditions then Rolle's Theorem fails i.e. there may or may not exist point in (a, b) where $f'(x) = 0$.



f is not continuous in [a, b] and Rolles theorem failing

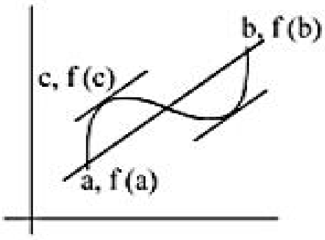

f is not derivable in open (a, b)



$f(a) \ne f(b)$

LMVT THEOREM (LAGRANGE'S MEAN VALUE THEOREM) :

Let f(x) be a function of x subject to the following conditions :



(i) f(x) is a continuous function of x in the closed interval of $a \le x \le b$.

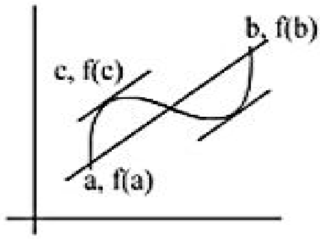
(ii) $f'(x)$ exists for every point in the open interval $a < x < b$.

(iii) $f(a) \ne f(b)$.

Then there exists at least one point $x = c$ such that $a < c < b$ where $f'(c) = \dfrac{f(b)-f(a)}{b-a}$

Geometrically, the slope of the secant line joining the curve at $x = a$ & $x = b$ is equal to the slope of the tangent line drawn to the curve at $x = c$. Note the following :

Note : Now $[f(b) - f(a)]$ is the change in the value of function f as x changes from a to b so that $[f(b) - f(a)] / (b - a)$ is the *average rate of change* of the function over the interval [a, b]. Also f'(c) is the instantaneous rate of change of the function at $x = c$. Thus, the theorem states that the average rate of change of a function over an interval is also the actual rate of change of the function at some point of the interval. In particular, for instance, the average velocity of a particle over an interval of time is equal to the velocity at some instant belonging to the interval.
This interpretation of the theorem justifies the name "Mean Value" for the theorem.



Rolles theorem is a special case of LMVT since

$$f(a) = f(b) \Rightarrow f'(c) = \frac{f(b)-f(a)}{b-a} = 0.$$

Alternative form of LMVT :

Another form of statement of Lagrange's Mean Value Theorem. If a function f is continuous in a closed interval [a, a + h] and derivable in the open interval] a, a + h [, then there exists at least one number 'θ' $\in (0, 1)$ such that

$$f(a+h) = f(a) + h f'(a + \theta h). \quad \theta \in (0, 1)$$

Proof:

We write $b - a = h$ so that h denotes the length of the interval [a, b] which may now be rewritten as [a, a + h]. The number, 'c' which lies between a and a + h, is greater than a and less than a + h so that we may write $c = a + \theta h$, where θ is some number between 0 and 1. Thus the equation (i) becomes

$$\frac{f(a+h) - f(a)}{h} = f'(a + \theta h) \qquad \Rightarrow \qquad f(a + h) = f(a) + h f'(a + \theta h)$$

Illustration :

Verify Rolle's Theorem for

(a) $f(x) = x(x+3)e^{-x/2}$ in $[-3, 0]$

(b) $f(x) = \dfrac{\sin x}{e^x}$ in $[0, \pi]$ $\qquad [c = \dfrac{\pi}{4}]$

(c) $f(x) = 1 - x^{2/3}$ in $[-1, 1]$ $\qquad$ ($f'(0)$ non existent)

Sol.

(a) $f(x) = x(x+3)e^{-x/2} \quad [-3, 0]$

$f(-3) = 0 \;\; f(0) = 0$

Clearly $f(x)$ is continuous and diffentiable function

there exists $c \in (-3, 0)$ s. t. $f'(c) = 0$

$$f'(c) = (2c+3)e^{-c/2} - \frac{1}{2}(c^2+3c)e^{-c/2} \quad \Rightarrow \quad e^{-c/2}\left[2c+3-\frac{(c^2+3c)}{2}\right] = 0$$

$4c + 6 - c^2 - 3c = 0$

$c^2 - c - 6 = 0$

$c = -2;\ c = 3 \notin (-3, 0)$

(b) $f(x) = \dfrac{\sin x}{e^x} \quad [0, \pi]$

$f(0) = f(\pi) = 0$

Continuity and differentiable function.

$f'(c) = e^{-c}\cos c - \sin c\, e^{-c} = 0$

$\tan c = 1 \Rightarrow c = \dfrac{\pi}{4} \in (0, \pi)$

(c) $f(x) = 1 - x^{2/3} \; [-1, 1]$

$f(-1) = f(1) = 0$

$f'(c) = \dfrac{-2}{3} c^{-1/3}$ $\qquad$ *but for* $c = 0$

$f'(c)$ is non-existent.

$\therefore$ *$f(x)$ is not differentiable in given interval.*

Hence Rolle's theorem not applicable.

Illustration :

Find c of LMVT $f(x) = \sqrt{x-1}$ *in [1, 3]*

Sol. $f(x) = \sqrt{x-1}$ $[1, 3]$

continuous and diff. in given interval $f'(c) = \dfrac{f(3)-f(1)}{3-1}$

$$\frac{1}{2\sqrt{c-1}} = \frac{\sqrt{2}}{2} \Rightarrow c = \frac{3}{2}$$

Illustration :

Using LMVT prove that $|\cos a - \cos b| \le |a-b|$

Sol. *Consider* $f(x) = \cos x$ *in* $[a, b]$

$$\left|\frac{\cos b - \cos a}{b-a}\right| = |-\sin c| \le 1 \quad \Rightarrow \quad |\cos b - \cos a| \le |b-a|$$

$|\cos a - \cos b| \le |a-b|$ *Hence proved.]*

Practice Problem

Q.1 Verify Rolles throrem for $f(x) = (x-a)^m (x-b)^n$ on $[a, b]$; m, n being positive integer.

Q.2 Let $f(x) = 4x^3 - 3x^2 - 2x + 1$, use Rolle's theorem to prove that there exist c, $0 < c < 1$ such that $f(c) = 0$.

Q.3 Let f be a twice differentiable function on $[0, 2]$ such that $f(0) = 0, f(1) = 2, f(2) = 4$, then prove that
(a) $f'(\alpha) = 2$ for some $\alpha \in (0, 1)$ (b) $f'(\beta) = 2$ for some $\beta \in (1, 2)$
(c) $f''(\gamma) = 0$ for some $\gamma \in (0, 2)$

Answer key

Q.1 $c = \dfrac{mb + na}{m + n}$ which lies between a & b

Solved Examples

Q.1 Find the values of a, if the equation $x - \sin x = a$ has a unique root in $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$.

Sol. Consider the function

$$f(x) = x - \sin x - a,\ x \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$$

Then $\quad f'(x) = 1 - \cos x = 2\sin^2\left(\frac{x}{2}\right) \ge 0\ \forall\ x \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$

$\Rightarrow$ f(x) increases in $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$

Also, we have $\quad f\left(\frac{-\pi}{2}\right) = \frac{-\pi}{2} + 1 - a \quad$ and $\quad f\left(\frac{\pi}{2}\right) = \frac{\pi}{2} - 1 - a$

The curve $y = f(x)$ will cut the X-axis exactly once, if $f\left(\frac{-\pi}{2}\right)$ is negative or zero and $f\left(\frac{\pi}{2}\right)$ is positive or zero.

i.e. $\quad \frac{-\pi}{2} + 1 - a \le 0 \quad$ and $\quad \frac{\pi}{2} - 1 - a \ge 0 \quad$ i.e. $\quad a \ge \frac{-\pi}{2} + 1 \quad$ and $\quad a \le \frac{\pi}{2} - 1$

Hence, we have $\quad a \in \left[1 - \frac{\pi}{2}, \frac{\pi}{2} - 1\right]$

Q.2 Find the intervals in which the given function increases or decreases $f(x) = \frac{x^2 + x + 1}{x^2 - x + 1}$.

Sol. We have $\quad f(x) = \frac{x^2 + x + 1}{x^2 - x + 1},\ x \in R$

and $\quad f'(x) = \frac{(x^2 - x + 1)(2x + 1) - (x^2 + x + 1)(2x - 1)}{(x^2 - x + 1)^2}$

$$= \frac{(2x^3 - x^2 + x + 1) - (2x^3 + x^2 + x - 1)}{(x^2 - x + 1)^2} = \frac{-2(x + 1)(x - 1)}{(x^2 - x + 1)^2}$$

Now, from the sign scheme for f '(x), we have

$$\underset{-\infty \quad\quad -1 \quad\quad +1 \quad\quad \infty}{\xrightarrow{\ -\text{ve}\ \quad +\text{ve}\ \quad -\text{ve}\ }}$$

$\Rightarrow$ f(x) strictly decreases in $(-\infty, -1)$
strictly increases in $(-1, 1)$
strictly decreases in $(1, \infty)$.

Q.3 Prove the following inequalities $1+\cot x \le \cot\frac{x}{2}\ \forall\, x \in (0,\pi)$

Sol. Consider the function $f(x)=\cot\left(\frac{x}{2}\right)-1-\cot x,\ x\in(0,\pi)$.

Then $f'(x)=\frac{-1}{2}\csc^2\left(\frac{x}{2}\right)+\csc^2 x=\frac{1}{\sin^2 x}-\frac{1}{2\sin^2(x/2)}$

$$=\frac{1}{2\sin^2(x/2)}\left[\frac{1}{2\cos^2(x/2)}-1\right]=\frac{-\cos x}{4\sin^2(x/2)\cos^2(x/2)}$$

$\Rightarrow$ $f'(x)=\frac{-\cos x}{\sin^2 x},\ f'(x)<0\ \forall\, x\in\left(0,\frac{\pi}{2}\right),\ f'(x)>0\ \forall\, x\in\left(\frac{\pi}{2},\pi\right)$

$\Rightarrow$ f(x) strictly decreases in $\left(0,\frac{\pi}{2}\right)$

strictly increases in $\left(\frac{\pi}{2},\pi\right)$

$\Rightarrow$ f(x) has least value at $x=\frac{\pi}{2}$

$\Rightarrow$ $f(x)\ge f\left(\frac{\pi}{2}\right)=0$ i.e. $\cot\left(\frac{x}{2}\right)\ge 1+\cot x$. which prove the desired result.

Q.4 Let $f(x)=x^3+ax^2+bx+5\sin^2 x$ be an increasing function on the set R. Then find the condition on a and b.

Sol. $f(x)=x^3+ax^2+bx+5\sin^2 x$ is increasing on R

$\Rightarrow$ $f'(x)>0$ for $x\in R$

$\Rightarrow$ $3x^2+2ax+b+5\sin 2x>0$ for all $x\in R$

$\Rightarrow$ $3x^2+2ax+(b-5)>0$ for all $x\in R$

$\Rightarrow$ $(2a)^2-4\times 3\times(b-5)<0$

$\Rightarrow$ $a^2-3b+15<0$

Q.5 Find the values of x where $f(x)=\sin(\ln x)-\cos(\ln x)$ is strictly increasing.

Sol. Since $f(x)=\sin(\ln x)-\cos(\ln x),\ x>0$

$$=\sqrt{2}\sin\left(\ln x-\frac{\pi}{4}\right)$$

$$\therefore\ f'(x)=\frac{\sqrt{2}}{x}\cos\left(\ln x-\frac{\pi}{4}\right)$$

$$=\frac{\sqrt{2}}{x}\sin\left(\frac{\pi}{2}+\ln x-\frac{\pi}{4}\right)=\frac{\sqrt{2}}{x}\sin\left(\frac{\pi}{4}+\ln x\right)>0\qquad(\because\ x>0)$$

or $\sin\left(\frac{\pi}{4}+\ln x\right)>0$ or $2n\pi<\frac{\pi}{4}+\ln x<(2n+1)\pi, n\in I$

or $2n\pi-\frac{\pi}{4}<\ln x<2n\pi+\frac{3\pi}{4}, n\in I$ $\Rightarrow$ $e^{2n\pi-\frac{\pi}{4}}<x<e^{2n\pi+\frac{3\pi}{4}}, \quad n\in I$

Therefore, f(x) is strictly increasing when $x\in\left(e^{2n\pi-\frac{\pi}{4}}, e^{2n\pi+\frac{3\pi}{4}}\right), \quad n\in I$

Q.6 Prove that $\sin^2\theta<\theta\sin(\sin\theta)$ for $0<\theta<\frac{\pi}{2}$.

Sol. In this problem, first we have to select an appropriate function.

Now by observation, given inequality can be set as $\frac{\sin(\sin\theta)}{\sin\theta}>\frac{\sin\theta}{\theta}$. This clearly gives indication that one has to study the function $f(x)=\frac{\sin x}{x}$

$\Rightarrow$ $f'(x)=\frac{(x\cos x-\sin x)}{x^2}=\frac{\cos x(x-\tan x)}{x^2}<0$ (as in first quadrant $x<\tan x$)

$\Rightarrow$ $f(x)$ is a decreasing function

Now, $\sin\theta<\theta$ for $0<\theta<\frac{\pi}{2}$

$\Rightarrow$ $f(\sin\theta)>f(\theta)\Rightarrow\frac{\sin(\sin\theta)}{\sin\theta}>\frac{\sin\theta}{\theta}$ [From (i)]

Hence, $\sin^2\theta<\theta\sin(\sin\theta)$ for $0<\theta<\frac{\pi}{2}$.

Q.7 Let $f:R\to R$ be a function such that $f(x)=ax+3\sin x+4\cos x$. Then $f(x)$ is invertible if
(A) $a\in(-5,5)$ (B) $a\in(-\infty,5)$ (C) $a\in(-5,+\infty)$ (D) None of these

Sol. $f'(x)=a+3\cos x-4\sin x=a+5\cos(x+\alpha)$, where $\cos\alpha=\frac{3}{5}$

For invertible, $f(x)$ must be monotonic

$\Rightarrow$ $f'(x)\ge 0\ \forall x$ or $f'(x)\le 0\ \forall x$

$\Rightarrow$ $a+5\cos(x+\alpha)\ge 0$ or $a+5\cos(x+\alpha)\le 0$

$\Rightarrow$ $a\ge -5\cos(x+\alpha)$ or $a\le -5\cos(x+\alpha)$

$\Rightarrow$ $a\ge 5$ or $a\le -5$

Q.8 $f(x)=(x-2)|x-3|$ is monotonically increasing in

(A) $\left(-\infty,\frac{5}{2}\right)\cup(3,\infty)$ (B) $\left(\frac{5}{2},\infty\right)$

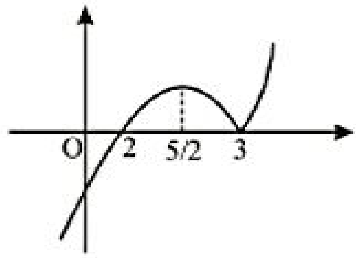
(C) $(2,\infty)$ (D) $(-\infty,3)$

Sol. $f(x)=(x-2)|x-3|$

For $x\ge 3$, $f(x)=(x-2)(x-3)=x^2-5x+6$

$$f'(x) = 2x - 5 = 0 \Rightarrow x = \frac{5}{2}$$

Now, the graph of $f(x) = (x-2)\,|x-3|$ is



Clearly from the graph, $f(x)$ increases in $\left(-\infty, \frac{5}{2}\right) \cup (3, \infty)$.

Q.9 A function $g(x)$ is defined as $g(x) = \frac{1}{4} f(2x^2 - 1) + \frac{1}{2} f(1 - x^2)$ and $f'(x)$ is an increasing function, then $g(x)$ is increasing in the interval

(A) $(-1, 1)$

(B) $\left(-\sqrt{\frac{2}{3}}, 0\right) \cup \left(\sqrt{\frac{2}{3}}, \infty\right)$

(C) $\left(-\sqrt{\frac{2}{3}}, \sqrt{\frac{2}{3}}\right)$

(D) None of these

Sol. $g'(x) = xf'(2x^2 - 1) - xf'(1 - x^2) = x\left(f'\left(2x^2-1\right) - f'\left(1-x^2\right)\right)$,

$g'(x) > 0$

If $x > 0,\ 2x^2 - 1 > 1 - x^2$ (as f' is an increasing function)

$\Rightarrow 3x^2 > 2 \Rightarrow x \in \left(-\infty, -\sqrt{\frac{2}{3}}\right) \cup \left(\sqrt{\frac{2}{3}}, \infty\right) \Rightarrow x \in \left(\sqrt{\frac{2}{3}}, \infty\right)$

If $x < 0,\ 2x^2 - 1 < 1 - x^2$

$\Rightarrow 3x^2 < 2 \Rightarrow x \in \left(-\sqrt{\frac{2}{3}}, \sqrt{\frac{2}{3}}\right) \Rightarrow x \in \left(-\sqrt{\frac{2}{3}}, 0\right)$.

Q.10 Let $f(x) = 1 - x - x^3$. Find all real values of x satisfying the inequality, $1 - f(x) - f^3(x) > f(1 - 5x)$

Sol. $f(x) = 1 - x - x^3 \Rightarrow f'(x) = -1 - 3x^2$ which is – ve $\forall\ x \in R \Rightarrow f$ is decreasing

$f[f(x)] = 1 - f(x) - f^3(x)$

$\therefore\ f[f(x)] > f(1 - 5x)]$ given

since, $f(x)$ is decreasing hence

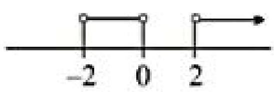
$f(x_1) > f(x_2) \Rightarrow x_1 < x_2$

$\therefore\ f(x) < 1 - 5x$

$1 - x - x^3 < 1 - 5x$

$x^3 - 4x > 0$

$x(x^2 - 4) > 0$



$\therefore\ x \in (-2, 0) \cup (2, \infty)$

Alternatively: $f'(x) = -(1 + 3x)^2 \Rightarrow f$ is decreasing

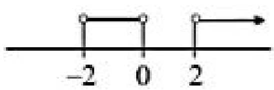
$1 - f(x) - f^3(x) > f(1 - 5x)$

$f[f(x)] > f(1 - 5x)$

but f is decreasing ; $\therefore$ $f(x) < 1 - 5x$; $1 - x - x^3 < 1 - 5x$.

$x^3 - 4x > 0$

$x(x^2 - 4) > 0$



$\therefore\ x \in (-2, 0) \cup (2, \infty)$]

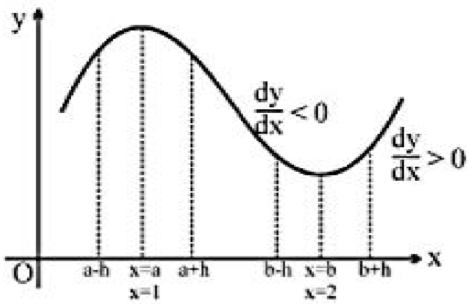
MAXIMA - MINIMA

(A) GENERAL INTRODUCTION :

The notion of optimising functions is one of the most useful application of calculus used in almost every sphere of life including geometry, business, trade, industries, economics, medicines and even at home. In this chapter we shall see how calculus defines the notion of maxima and minima and distinguishes it from the greatest and least value or global maxima and global minima of a function.

(B) DEFINITION MAXIMA & MINIMA :

A function $f(x)$ is said to have a maximum at $x = a$ if $f(a)$ is greater than every other value assumed by $f(x)$ in the immediate neighbourhood of $x = a$. Symbolically



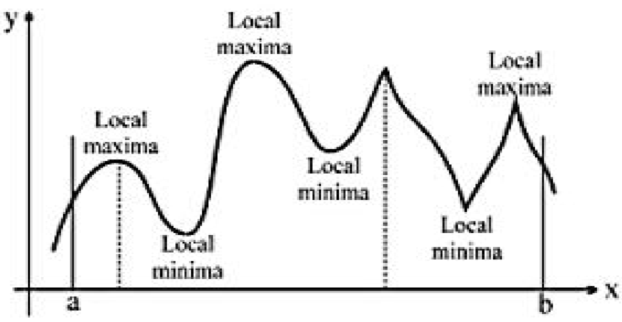
$$\left.\begin{array}{l} f(a) > f(a+h) \\ f(a) > f(a-h) \end{array}\right] \Rightarrow x = a \text{ gives maxima}$$

for a sufficiently small positive h.

Similarly, a function $f(x)$ is said to have a minimum value at $x = b$ if $f(b)$ is least than every other value assumed by $f(x)$ in the immediate neighbourhood at $x = b$. Symbolically if

$$\left.\begin{array}{l} f(b) < f(b+h) \\ f(b) < f(b-h) \end{array}\right] \Rightarrow x = b \text{ gives minima for a sufficiently small positive h.}$$

Note that :



(i) the maximum & minimum values of a function are also known as local/relative maxima or local/relative minima as these are the greatest & least values of the function relative to some neighbourhood of the point in question.

(ii) the term 'extremum' or (extremal) or 'turning value' is used both for maximum or a minimum value.

(iii) a maximum (minimum) value of a function may not be the greatest (least) value in a finite interval.

(iv) a function can have several maximum & minimum values & a minimum value may even be greater than a maximum value.

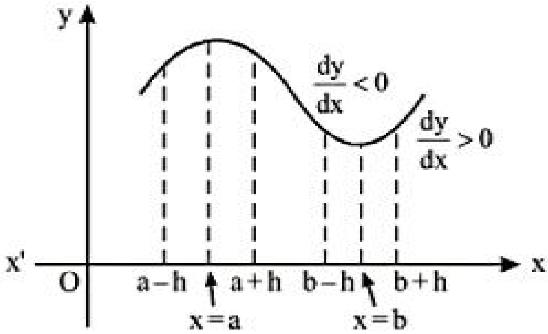
(v) maximum & minimum values of a continuous function occur alternately & between two consecutive maximum values there is a minimum value & vice versa.

Tests for local maximum/minimum, when f(x) is differentiable :

(1) First-order derivative test in Ascertaining the maxima or minima :

Consider the interval $(a-h, a)$, we find $f(x)$ is increasing $\Rightarrow \frac{dy}{dx} > 0$. Similarly, for the interval $(a, a-h)$, we find $f(x)$ is decreasing $\Rightarrow \frac{dy}{dx} < 0$. Hence, at the point $x = a$ (maxima); $\frac{dy}{dx} = 0$.

Similarly, $\frac{dy}{dx} = 0$ at $x = b$ which is the point of minima.



Hence $\frac{dy}{dx} = 0$ is the necessary condition for maxima or minima.

These points, where $\frac{dy}{dx}$ vanishes, are known as stationary points as instantaneous rate of change of function momentarily ceases at this point.

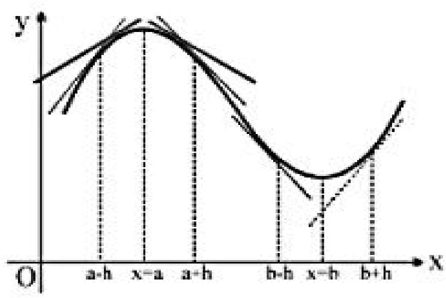
Hence, if $\left.\begin{matrix} f'(a-h) > 0 \\ f'(a+h) < 0 \end{matrix}\right] \Rightarrow x = a$ is a point of local maxima, where $f'(a) = 0$. It means that $f'(x)$ should change its sign from positive to negative.

Similarly, $\left.\begin{matrix} f'(b-h) < 0 \\ f'(b+h) < 0 \end{matrix}\right] \Rightarrow x = b$ is a point of local minima, where $f'(b) = 0$. It means that $f'(x)$ should change its sign from negative to positive.

However, if $f'(x)$ does not change sign, i.e., has the same sign in a certain complete neighbourhood of c, then $f(x)$ is either increasing or decreasing throughout this neighbourhood implying that $f(c)$ is not an extreme value of f, e.g., $f(x) = x^3$ at $x = 0$.

(2) Use of second order derivative in ascertaining the Maxima or Minima for a differentiable function:

As shown in the figure it is clear that as x increases from $a-h$ to $a+h$, the function $\frac{dy}{dx}$ continuously decreases, i.e. (+) ve for $x < a$, zero at $x = a$ and (–) ve for $x > a$. Hence $\frac{dy}{dx}$ itself is a decreasing function. Therefore $\frac{d^2y}{dx^2} < 0$ in $(a-h, a+h)$.



Hence at local maxima, $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} < 0$.

$f'(a) = 0$ and $f''(a) < 0$

Similarly at local minima, $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} > 0$.

i.e. $f'(b) = 0$ and $f''(b) > 0$

Hence if

(a) f(a) is a maximum value of the function f then $f'(a) = 0$ & $f''(a) < 0$.

(b) f(b) is a minimum value of the function f, if $f'(b) = 0$ & $f''(b) > 0$.
However, if $f''(c) = 0$ then the test fails. In this case f can still have a maxima or minima or point of inflection (neither maxima nor minima). In this case revert back to the first order derivative check for ascertaning the maxima or minima.

(3) n th Derivative Test :

It is nothing but the general version of the second derivative test. It says that if $f'(a) = f''(a) = f''(a) = \ldots\ldots f^n(a) = 0$ and $f^{n+1}(a) \neq 0$ (all derivatives of the function up to order n vanish and (n + 1)th order derivative does not vanish at x = a), then f (x) would have a local maximum or minimum at x = a iff n is odd natural number and that x = a would be a point of local maxima if $f^{n+1}(a) < 0$ and would be a point of local minima if $f^{n+1}(a) > 0$. However, if n is even, then f has neither a maxima nor a minima at x = a.

Illustration :

If $f(x) = \begin{cases} x^2, & x \le 0 \\ 2\sin x, & x > 0 \end{cases}$, *investigate the function at x = 0 for maxima/minima.*

Sol. *Analyzing the graph of f(x), we get x = 0 is a point of minima.*

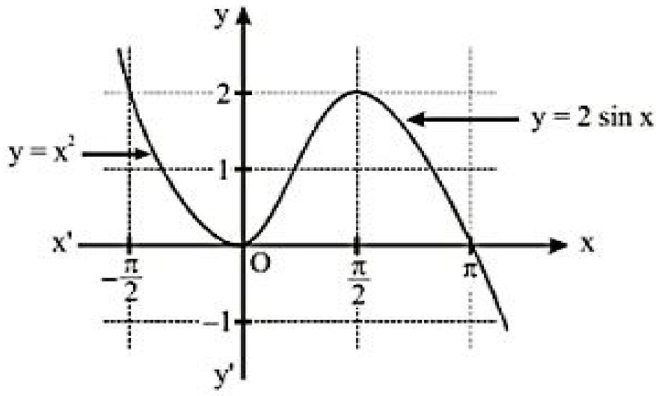


Illustration :

The function $y = \dfrac{ax+b}{(x-1)(x-4)}$ *has turning point at P(2, 1). Then find the value of a and b.*

Sol. $y = \dfrac{ax+b}{(x-1)(x-4)} = \dfrac{ax+b}{x^2-5x+4}$ *has turning point at P(2, –1)*

$\Rightarrow$ *P(2, –1) lies on the curve* $\Rightarrow$ $2a + b = 2$ ...(i)

Also $\dfrac{dy}{dx} = 0$ *at P(2, –1)*

Now $\dfrac{dy}{dx} = \dfrac{a(x^2-5x+4)-(2x-5)(ax+b)}{(x^2-5x+4)}$

At $P(2, -1)$, $\dfrac{dy}{dx} = \dfrac{-2a+2a+b}{4} = 0$

$\Rightarrow$ $b = 0$ $\Rightarrow$ $a = 1$ [from equation (i)]

Illustration :

Find the points of maxima and minima of the function

$f(x) = 12x^5 - 45x^4 + 40\,x^3 + 40.$

Check whether second derivative can be used to find the point of extrema.

Sol. We have

$f(x) = 12x^5 - 45x^4 + 40x^3 + 40$

$f\,'(x) = 60x^4 - 180x^3 + 120x^2 = 60x^2\,(x-1)\,(x-2)$

and $f\,''(x) = 60\,(4x^3 - 9x^2 + 4x)$

The critical points of $f(x)$ are

$x = 0, 1, 2$

At the critical points, we have

$f\,''(x) = 0 \Rightarrow$ more investigation required

$f\,''(1) = 60\,(4 - 9 + 4) < 0 \Rightarrow$ maxima at $x = 1$

and $f\,''(2) = 60\,(32 - 36 + 8) > 0 \Rightarrow$ minima at $x = 2$.

Hence, we can see that the nature of the critical points cannot be predicted by the use of second derivative.

Since $f\,'(x)$ does not change sign as x passes through 0, hence $x = 0$ is not an extrema.

Illustration :

Discuss the extremum of $f\,(x) = x^2 + \dfrac{1}{x^2}$.

Sol. $f\,(x) = x^2 + \dfrac{1}{x^2}$

$f\,'(x) = 2x - \dfrac{2}{x^3}$

Let $f\,'(x) = 0$ $\Rightarrow$ $x^4 = 1$ $\Rightarrow$ $x = \pm 1$

Also, $f\,''(x) = 2 + \dfrac{6}{x^4} > 0$ for all $x \neq 0$

$\Rightarrow$ Both the points $x = 1$ and $x = -1$ are the points of minima.

Illustration :

The function $f(x) = (4\sin^2 x - 1)^n\,(x^2 - x + 1)$, $n \in N$, *has a local minimum at* $x = \dfrac{\pi}{6}$, *then*

(A) n is any even number
(B) n is an odd number
(C) n is odd prime number
(D) n is any natural number

Sol. $f(x) = (4\sin^2 x - 1)^n (x^2 - x + 1)$

$x^2 - x + 1 > 0 \ \forall x \in R$

$$f\left(\frac{\pi}{6}\right) = 0 \quad \Rightarrow \quad f\left(\frac{\pi^+}{6}\right) = \lim_{x \to \frac{\pi^+}{6}} (4\sin^2 x - 1)^n (x^2 - x + 1) = \to 0^+$$

$$f\left(\frac{\pi^-}{6}\right) = \lim_{x \to \frac{\pi^-}{6}} (4\sin^2 x - 1)^n (x^2 - x + 1) \to (0^-)^n \ \ (a\ positive\ value)$$

$$f\left(\frac{\pi^-}{6}\right) > 0 \ if\ \ n\ is\ an\ even\ number.$$

When F(x) is not differentiable at x = a :

Case-I :

When f(x) is continuous at $x = a$ and $f'(a-h)$ and $f'(a+h)$ exist and are non-zero, then f(x) has a local maximum or minimum at $x = a$ if $f'(a-h)$ and $f'(a+h)$ are of opposite signs.

If $f'(a-h) > 0$ and $f'(a+h) < 0$, then $x = a$ will be a point of local maximum.

If $f'(a-h) < 0$ and $f'(a+h) > 0$, then $x = a$ will be a point of local minimum.

Case-II :

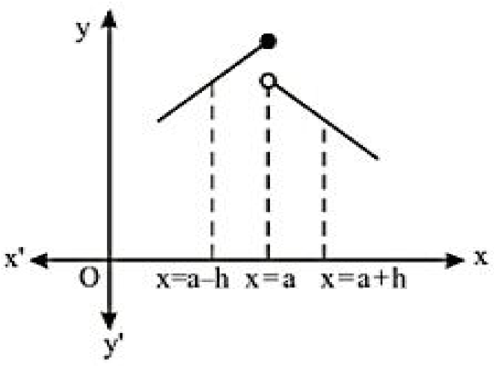
When f(x) is continuous and $f'(a-h)$ and $f'(a+h)$ exist but one of them is zero, we should infer the information about the existence of local maxima/minima from the basic definition of local maxima/minima.

Case-III :

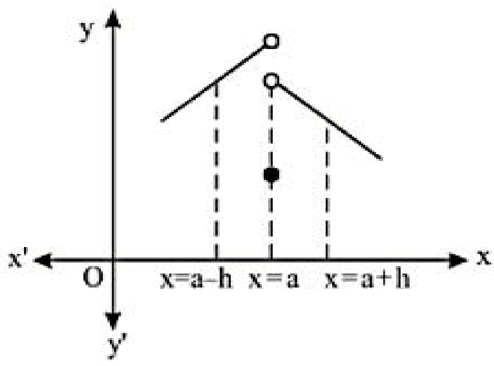
If $f(x)$ is not continuous at $x = a$ and $f'(a-h)$ and/or $f'(a+h)$ are not finite, then compare the values of $f(x)$ at the neighbouring points of $x = a$.

It is advisable to draw the graph of the function in the vicinity of the point $x = a$, because the graph would give us the clear picture about the existence of local maxima/minima at $x = a$.

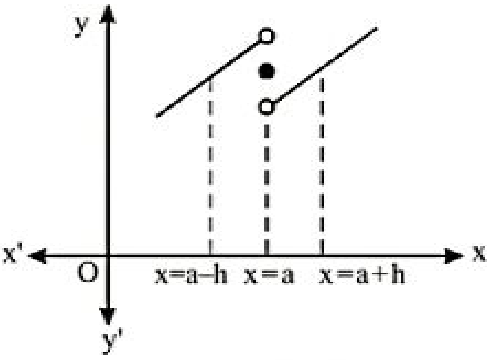
Consider the following cases :



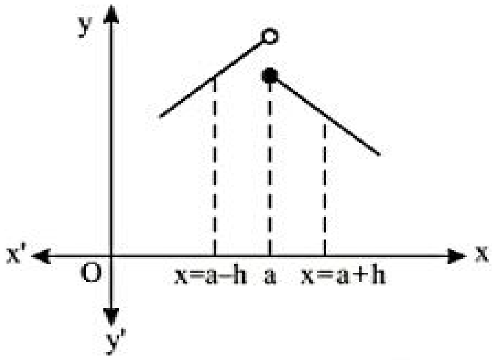
$x = a$ is the point of maxima as
$f(a) > f(a-h)$ and
$f(a) > f(a+h)$



$x = a$ is the point of minima as
$f(a) < f(a-h)$ and
$f(a) < f(a+h)$



x = a is not the point of extremum as
f (a) < f (a – h) and
f (a) > f (a + h)



x = a is not the point of extremum as
f (a) < f (a – h) and
f (a) > f (a + h)

Practice Problem

Q.1 Consider the function $f(x) = \sin^3 x + \lambda \sin^2$, $x \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$.

Find the possible values of λ such that f(x) has exactly one minima and one maxima.

Q.2 $f(x) = \begin{cases} \cos\frac{\pi x}{2}, & x > 0 \\ x + a, & x \le 0 \end{cases}$. Find the values of a if $x = 0$ is a point of maxima.

Q.3 Investigate for maxima & minima for the function, $f(x) = \int_1^x [2(t-1)(t-2)^3 + 3(t-1)^2(t-2)^2]\,dt$

Q.4 Let f(x) be a cubic polynomial which has local maximum at $x = -1$ and $f'(x)$ has a local minimum at $x = 1$. If $f(-1) = 10$ and $f(3) = -22$, then find the distance between its two horizontal tangents.

Q.5 If $f(x) = x^5 - 5x^4 + 5x^3 - 10$ has local maximum and minimum at $x = p$ and $x = q$, respectively, then (p, q) =

(A) (0, 1) (B) (1, 3) (C) (1, 0) (D) None of these

Answer key

Q.1 $\frac{-3}{2} < \lambda < \frac{3}{2}$ and $\lambda \ne 0$.

Q.2 $a \ge 1$

Q.3 Max. at $x = 1$; $f(1) = 0$, Min. at $x = 7/5$; $f(7/5) = -108/3125$

Q.4 32

Q.5 B

Concept of global maximum/minimum :

Let $y = f(x)$ be a given function with domain D. Let $[a, b] \subseteq D$. Global maximum / minimum of $f(x)$ in [a, b].

Global maximum and minimum in [a, b] would occur at critical point f(x) within [a, b] or at the endpoints of the interval.

Global maximum/minimum in [a, b] :

In order to find the global maximum and minimum of f(x) in [a, b], find the critical points of f(x) in (a, b).
Let $c_1, c_2, \ldots\ldots, c_n$ be the different critical points. Find the value of the function at these critical points.
Let $f(c_1), f(c_2), \ldots\ldots, f(c_n)$ be the values of the function at critical points.
Say, $M_1 = \max\{f(a), f(c_1), f(c_2), \ldots\ldots, f(c_n), f(b)\}$
and $M_2 = \min\{f(a), f(c_1), f(c_2), \ldots\ldots, f(c_n), f(b)\}$
Then M_1 is the greatest value of f(x) in [a, b] and M_2 is the least value of f(x) in [a, b].

Illustration :

Find the maximum value of $f(x) = \left(\frac{1}{x}\right)^x$

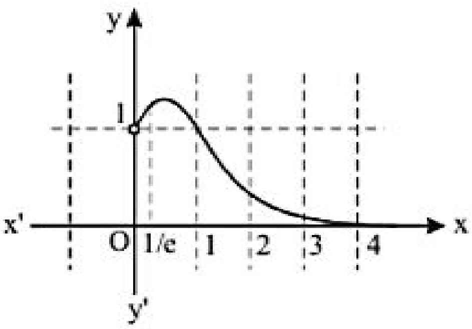
Sol. $f(x) = \left(\frac{1}{x}\right)^x \Rightarrow f'(x) = \left(\frac{1}{x}\right)^x \left(\ln\frac{1}{x} - 1\right)$

$f'(x) = 0 \Rightarrow \ln\frac{1}{x} = 1 \Rightarrow \frac{1}{x} = e \Rightarrow x = \frac{1}{e}$

Also for $x < \frac{1}{e}$, $f'(x)$ *is positive and for* $x > \frac{1}{e}$, $f'(x)$ *is negative.*

Hence, $x = \frac{1}{e}$ *is point of maxima.*

Therefore, the maximum value of function is $e^{1/e}$.



Also $\lim_{x\to 0}\left(\frac{1}{x}\right)^x = e^{\lim_{x\to 0} x\ln\left(\frac{1}{x}\right)} = e^{-\lim_{x\to 0} x\ln x} = e^0 = 1$

$\lim_{x\to 0}\left(\frac{1}{x}\right)^x = 1.$

Illustration :

Let $f(x) = 2x^3 - 9x^2 + 12x + 6$. Discuss the global maxima and minima of $f(x)$ in $[0, 2]$ and $(1, 3)$.

Sol. $f(x) = 2x^3 - 9x^2 + 12x + 6$

$\Rightarrow \quad f'(x) = 6x^2 - 18x + 12 = 6(x^2 - 3x + 2) = 6(x-1)(x-2)$

Clearly the critical point of $f(x)$ in $[0, 2]$ is $x = 1$.

Now $f(0) = 6,\ f(1) = 11,\ f(2) = 10$

Thus $x = 0$ is the point of minimum of $f(x)$ in $[0, 2]$ and $x = 1$ is the point of global maximum.

For $x \in (1, 3)$

Clearly $x = 2$ is the only critical point in $(1, 3)$

$$f(2) = 10. \quad \lim_{x\to 1^+} f(x) = 11 \text{ and } \lim_{x\to 3^-} f(x) = 15$$

Thus $x = 2$ is the point of global minimum in $(1, 3)$ and the global maximum in $(1, 3)$ does not exist.

Illustration :

Find the greatest and least values of function $f(x) = 3x^4 - 8x^3 - 18x^2 + 1$.

Sol. We have

$$f(x) = 3x^4 - 8x^3 - 18x^2 + 1$$

and $f'(x) = 12x^3 - 24x^2 - 36x = 12x(x+1)(x-3)$.

The points at which $f(x)$ may have extreme values, are the critical points $x = -1, 0, 3$

and the end points $x = \pm\infty$.

Now, $f(-1) = -6, f(0) = 1, f(3) = -134$, and $f(\pm\infty) \to +\infty$.

Hence, the least value of the function is -134 whereas the greatest value does not exist.

Illustration :

Find the greatest and least values of function $f(x) = \begin{cases} -x, & -1 \le x < 0 \\ 2-(x-1)^2, & 0 \le x \le 2 \end{cases}$.

Sol.

We have

$$f(x) = \begin{cases} -x, & -1 \le x < 0 \\ 2-(x-1)^2, & 0 \le x \le 2 \end{cases} \quad \text{and} \quad f'(x) = \begin{cases} -1, & -1 < x < 0 \\ -2x(x-1), & 0 < x < 2 \end{cases}$$

Thus, the points at which $f(x)$ may have extreme values, are the critical points $x = 0, 1$

$[f'(1) = 0$ and $f'(0) = DNE]$

and the end points $x = -1, 2$

Now, $f(-1) = 1, f(1) = 2$ and $f(2) = 1$.

Since f is discontinuous at $x = 0$, we also need to find the limiting values of $f(x)$ as $x \to 0$.

We have

$$f(0^-) \to 0, f(0^+) \to 1 \text{ and } f(0) = 1$$

the largest and the smallest among the above six values are 2 and 0 respectively.

Hence, the greatest value is 2 but the least value does not exist since the function approaches 0 but is never equal to 0.

Illustration :

Find greatest and least values of $f(x) = \dfrac{a^2}{x} + \dfrac{b^2}{1-x}$, $x \in (0, 1)$ $(a, b > 0)$.

Sol. *We have*

$$f(x) = \frac{a^2}{x} + \frac{b^2}{1-x},\quad x \in (0, 1)$$

and $$f'(x) = \frac{-a^2}{x^2} + \frac{b^2}{(1-x)^2}$$

which exists everywhere in (0, 1) and vanishes at points, given by

$$\frac{b^2}{(1-x^2)} = \frac{a^2}{x^2}$$

$$a^2 (1-x)^2 = b^2x^2$$

i.e. $a(1-x) = bx$ *i.e.* $x = \dfrac{a}{a+b}$

To find the greatest and least value, we need to check the values of f(x) at $x = 0^+, 1^-, \dfrac{a}{a+b}$.

We have $f(0^+) \to +\infty, f(1^-) \to +\infty$ *and* $f\left(\dfrac{a}{a+b}\right) = (a+b)^2$

Hence, we have

least value $= (a+b)^2$

and greatest value does not exist. ***Ans.***

Illustration :

Find greatest and least values of $f(x) = \dfrac{(a+x)(b+x)}{(c+x)}$, $x > -c$.

Sol. *We have* $f(x) = \dfrac{(a+x)(b+x)}{(c+x)}$, $x \in (-c, \infty)$

and $$f'(x) = \frac{(c+x)(2x+a+b) - [x^2+(a+b)x+ab]}{(c+x)^2}$$

$$= \frac{x^2+2cx+ac+bc-ab}{(c+x)^2},\quad x \in (-c, \infty)$$

which vanishes at points given by

$$x^2 + 2cx + ac + bc - ab = 0$$

i.e. $x = -c \pm \sqrt{c^2-(ac+bc-ab)} = -c \pm \sqrt{(a-c)(b-c)}$

Thus, the expression for f '(x) can be written as

$$f'(x) = \frac{(x-\alpha)(x-\beta)}{(c+x)^2}$$

choosing $\alpha = -c - \sqrt{(a-c)(b-c)}$ *and* $\beta = -c + \sqrt{(a-c)(b-c)}$

The critical point $x = \alpha$ *is of no interest since it does lie in the interval* $(-c, \infty)$.

Now, we have

$$f(-c^+) \to \infty, f(\infty) \to \infty$$

and $$f(\beta) = \frac{(a-c+\sqrt{(a-c)(b-c)}\,)(b-c+\sqrt{(a-c)(b-c)}\,)}{c-c+\sqrt{(a-c)(b-c)}}$$

$$= \frac{(a-c)(b-c)+(a+b-2c)\sqrt{(a-c)(b-c)}+(a-c)(b-c)}{\sqrt{(a-c)(b-c)}}$$

$$= 2\sqrt{(a-c)(b-c)} + a + b - 2c$$

$$= a - c + b - c + 2\sqrt{(a-c)(b-c)}$$

$$= \left(\sqrt{(a-c)} + \sqrt{(b-c)}\right)^2$$

Hence, we have

Least value $= \left(\sqrt{(a-c)} + \sqrt{(b-c)}\right)^2$ *and greatest value does not exist.*

Practice Problem

Q.1 Find greatest and least values of $f(x) = x - \sin 2x + \frac{1}{3}\sin 3x,\ x \in [0, \pi]$.

Q.2 Discuss the global maxima and global minima of $f(x) = \tan^{-1}x - \log_e x$ in $\left[\frac{1}{\sqrt{3}}, \sqrt{3}\right]$.

Q.3 The maximum value of $x^4 e^{-x^2}$ is

(A) e^2 (B) e^{-2} (C) $12e^{-2}$ (D) $4e^{-2}$

Q.4 Find the greatest & least value for the function ;

(i) $y = x + \sin 2x,\ 0 \le x \le 2\pi$ (ii) $y = 2\cos 2x - \cos 4x,\ 0 \le x \le \pi$

Answer key

Q.1 Least value $= -\frac{3\sqrt{3}-(\pi+2)}{6}$ and greatest value $= \frac{5\pi+2+3\sqrt{3}}{6}$

Q.2 $\frac{\pi}{3} - \ln\sqrt{3}$, $\frac{\pi}{6} - \ln\frac{1}{\sqrt{3}}$ Q.3 D

Q.4 (i) Max at $x = 2\pi$, Max value $= 2\pi$, Min. at $x = 0$, Min value $= 0$

(ii) Max at $x = \pi/6$ & also at $x = 5\pi/6$ and Max value $= 3/2$, Min at $x = \pi/2$, Min value $= -3$

PROBLEMS BASED ON MENSURATION AND GEOMETRY :

Summary-Working Rule :

(1) When possible, draw a figure to illustrate the problem & label those parts that are important in the problem. Constants & variables should be clearly distinguished.

(2) Write an equation for the quantity that is to be maximised or minimised. If this quantity is denoted by 'y', it must be expressed in terms of a single independent variable x. This may require some algebraic manipulations.

(3) If $y = f(x)$ is a quantity to be maximum or minimum, find those values of x for which $dy/dx = f'(x) = 0$.

(4) Test each values of x for which $f'(x) = 0$ to determine whether it provides a maximum or minimum or neither. The usual tests are :

(a) If d^2y/dx^2 is positive when $dy/dx = 0 \Rightarrow$ y is minimum.
If d^2y/dx^2 is negative when $dy/dx = 0 \Rightarrow$ y is maximum.
If $d^2y/dx^2 = 0$ when $dy/dx = 0$, the test fails.

(b) If $\frac{dy}{dx}$ is $\left.\begin{matrix} \text{positive} & \text{for} & x < x_0 \\ \text{zero} & \text{for} & x = x_0 \\ \text{negative} & \text{for} & x > x_0 \end{matrix}\right] \Rightarrow$ a maximum occurs at $x = x_0$.

But if dy/dx changes sign from negative to zero to positive as x advances through x_0 there is a minimum. If dy/dx does not change sign, neither a maximum nor a minimum. Such points are called INFLECTION POINTS.

(5) If the function $y = f(x)$ is defined for only a limited range of values $a \le x \le b$ then examine $x = a$ & $x = b$ for possible extreme values.

(6) If the derivative fails to exist at some point, examine this point as possible maximum or minimum.

Useful formulae of Mensuration to remember :

- Volume of a cuboid = $l$bh.
- Surface area of a cuboid = $2(l\text{b} + \text{bh} + \text{h}l)$.
- Volume of a prism = area of the base x height.
- Lateral surface of a prism = perimeter of the base x height.
- Total surface of a prism = lateral surface + 2 area of the base
(Note that lateral surfaces of a prism are all rectangles).
- Volume of a pyramid = $\frac{1}{3}$(area of the base) × (height).
- Curved surface of a pyramid = $\frac{1}{2}$ (perimeter of the base) x slant height.
(Note that slant surfaces of a pyramid are triangles).

- Volume of a cone $= \frac{1}{3}\pi r^2 h$.
- Curved surface of a cylinder $= 2\pi rh$.
- Total surface of a cylinder $= 2\pi rh + 2\pi r^2$.
- Volume of a sphere $= \frac{4}{3}\pi r^3$.
- Surface area of a sphere $= 4\pi r^2$.
- Area of a circular sector $= \frac{1}{2} r^2\theta$, when θ is in radians.

Illustration :

Find two positive numbers x and y such that $x + y = 60$ and x^3y is maximum.

Sol. $x + y = 60$

$\Rightarrow y = 60 - x$

$\Rightarrow x^3y = (60 - x)x^3$

Let $f(x) = (60 - x)\, x^3$; $x \in (0, 60)$

For maximizing $f(x)$, let us find critical points

$f'(x) = 3x^2 (60 - x) - x^3 = 0$

$f'(x) = x^2 (180 - 4x) = 0$

$\Rightarrow x = 45$ ($\because x \neq 0$)

$f'(45^+) < 0$ and $f'(45) > 0$

Hence local maxima at $x = 45$.

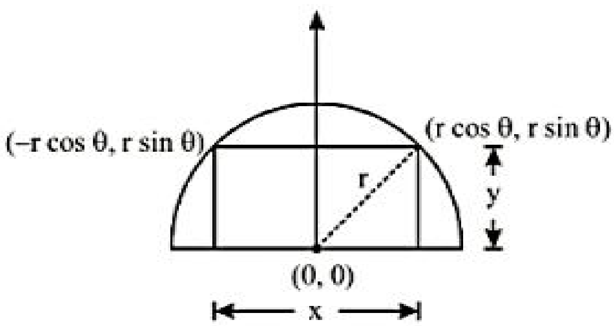
So $x = 45$ *and* $y = 15$

Illustration :

Rectangles are inscribe inside a semi-circle of radius r. Find the rectangle with maximum area.

Sol. *Let us choose co-ordinate system with origin as centre of circle*

Area, $A = xy$



$\Rightarrow \quad A = 2(r\cos\theta)(r\sin\theta),\ \theta \in \left(0, \frac{\pi}{2}\right)$

$\Rightarrow \quad A = r^2 \sin 2\theta$

A is maximum when $\sin 2\theta = 1 \quad \Rightarrow 2\theta = \frac{\pi}{2}$

$\Rightarrow \quad \theta = \frac{\pi}{4}$

$\Rightarrow$ *Sides of the rectangle are* $2r\cos\left(\frac{\pi}{4}\right) = \sqrt{2}\, r$ *and* $r\sin\left(\frac{\pi}{4}\right) = \frac{r}{\sqrt{2}}$.

Illustration :

The tangent to the parabola $y = x^2$ has been drawn so that the abscissa x_0 of the point of tangency belong to the interval (1, 2). Find x_0 for which the triangle is to be bounded by the tangent, the axis of ordinates, and the straight line $y = x_0^2$ has the greatest area.

Sol. $y = x^2,\ \dfrac{dy}{dx} = 2x$

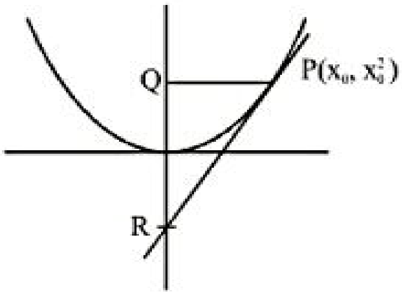
$\Rightarrow$ Equation of the tangent at (x_0, x_0^2) is $y - x_0^2 = 2x_0(x - x_0)$. It meets y-axis in $R(0, -x_0^2)$.

Q is $(0, x_0^2)$

$\Rightarrow$ Z = area of the triangle PQR

$$= \frac{1}{2} 2x_0^2 x_0 = x_0^3,\ 1 \le x_0 \le 2$$

$$\frac{dZ}{dx_0} = 3x_0^2 > 0 \text{ in } 1 \le x_0 \le 2$$



$\Rightarrow$ Z is an increasing function in [1, 2]

Hence, Z, i.e., the area of ΔPQR is greatest at $x_0 = 2$.

Illustration :

A sheet of area 40 m^2 is used to make an open tank with square base. Find the dimensions of the base such that volume of this tank is maximum.

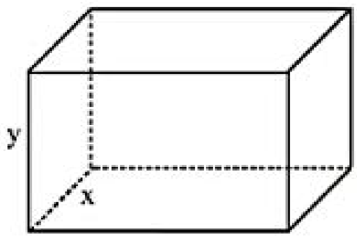
Sol. Let the length of base be x m and height be y m

Volume $V = x^2 y$

Again x and y are related to the surface area of this tank which is equal to 40 m^2.

$\Rightarrow$ $x^2 + 4xy = 40$

$$y = \frac{40 - x^2}{4x},\quad x \in \left(0, \sqrt{40}\right)$$



$$\Rightarrow \quad V(x) = x^2\left(\frac{40 - x^2}{4x}\right) = \frac{40x - x^3}{4}$$

Maximizing volume,

$$V'(x) = \frac{40 - 3x^2}{4} = 0 \Rightarrow x = \sqrt{\frac{40}{3}}\ m$$

and $\quad V''(x) = \dfrac{-3x}{2} \quad \Rightarrow V''\left(\sqrt{\dfrac{40}{3}}\right) < 0$

$\Rightarrow$ volume is maximum at $x = \sqrt{\dfrac{40}{3}}$ m.

Illustration :

If a right-circular cylinder is inscribed in a given cone. Find the dimensions of the cylinder such that its volume is maximum.

Sol. *Let x be the radius of cylinder and y be its height*

Volume $V = \pi x^2 y$

x, y can be related by using similar triangles

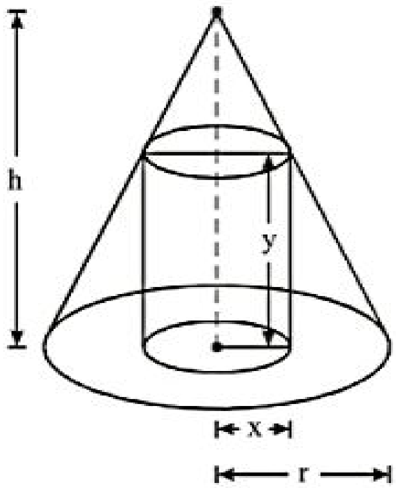
$$\frac{y}{r-x} = \frac{h}{r}$$

$$\Rightarrow \quad y = \frac{h}{r}(r-x)$$

$$\Rightarrow \quad V(x) = \pi x^2 \frac{h}{r}(r-x),\ x \in (0, r)$$

$$\Rightarrow \quad V(x) = \frac{\pi h}{r}(rx^2 - x^3)$$

$$\Rightarrow \quad V'(x) = \frac{\pi h}{r}x(2r-3x)$$



$$V'(x) = 0 \quad \Rightarrow \quad x = \frac{2r}{3}$$

$$\text{Also} \quad V''(x) = \frac{\pi h}{r}(2r-6x) \quad \Rightarrow \quad V''\left(\frac{2r}{3}\right) < 0$$

This volume is maximum when, $x = \frac{2r}{3}$ *and* $y = \frac{h}{3}$.

Practice Problem

Q.1 For a right circular cone of given total surface area (including the base) and maximum volume, find the value of the semi-vertical angle.

Q.2 If the sum of the lengths of the hypotenuse and another side of a right-angled triangle is given, show that the area of the triangle is maximum when the angle between these sides is $\pi/3$.

Q.3 Find the point (α, β) on the ellipse $4x^2 + 3y^2 = 12$, in the first quadrant, so that the area enclosed by the lines $y = x$, $y = \beta$, $x = \alpha$ and the x-axis is maximum.

Q.4 A rectangle is inscribed in an equilateral trianle of side length 2a units. The maximum are of this rectangle can be

(A) $\sqrt{3}\,a^2$ (B) $\frac{\sqrt{3}\,a^2}{4}$ (C) a^2 (D) $\frac{\sqrt{3}\,a^2}{2}$

Q.5 Tangents are drawn to $x^2+y^2=16$ from the point P(0, h). These tangents meet the x-axis at A and B. If the area of triangle PAB is minimum, then

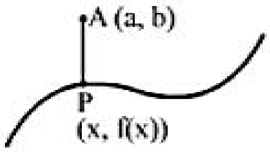
(A) $h=12\sqrt{2}$ (B) $h=6\sqrt{2}$ (C) $h=8\sqrt{2}$ (D) $h=4\sqrt{2}$

Answer key

Q.1	$\theta=\sin^{-1}(1/3)$	Q.3	$\left(\frac{3}{2},1\right)$	Q.4	D	Q.5	D

GENERAL CONCEPT :

Given a fixed point A (a, b) and a moving point P (x, f (x)) on the curve y = f (x). Then AP will be maximum or minimum if it is normal to the curve at P.



Significance of the Sign of 2nd order Derivative and Points of Inflection :

A point where the graph of function is continuous and has a tangent line and where the concavity changes is called point of inflection.

- At the point of inflection either $y''=0$ and changes sign or y'' fails to exist.
- At the point of inflection, the curve crosses its tangent at that point.
- A function can not have point of inflection and extrema at same point.

Note: If $\frac{d^2y}{dx^2}>0$ then y is concave up and if $\frac{d^2y}{dx^2}<0$ the y is concave down.

Illustration :

$f(x)=x^{1/5}$ at $x=0$ has inflection point.
y'' D.N.E. at $x=0$
Note that $f(x)$ has a vertical tangent and the curve crosses its tangent line.

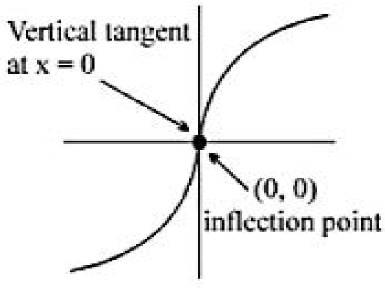


Illustration :

$f(x)=x^3$ at $x=0$ has inflection point
$y''=0$ at $x=0$ and changes sign

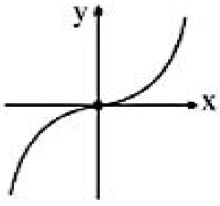


Illustration :

$f(x) = |x^2 - 1|$
has no inflection point in its domain.
as no tangent can be drawn at these points.

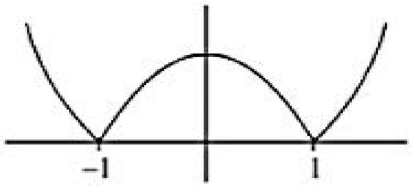


Illustration :

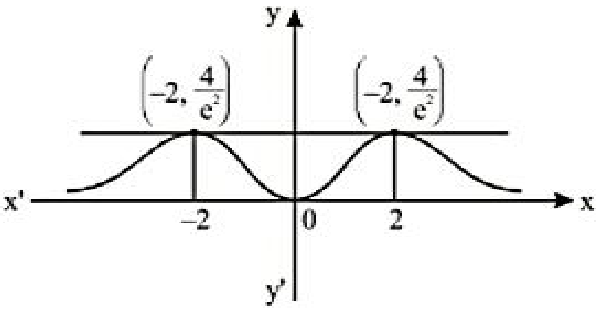
Number of points of inflection for $f(x) = x^2 e^{-|x|}$ is
(A) 1 *(B) 2* *(C) 3* *(D) 4*

Sol. *We have* $f(x) = x^2 e^{-|x|} = \begin{cases} x^2 e^{-x}, & x \ge 0 \\ x^2 e^{x}, & x < 0 \end{cases}$

$$\Rightarrow f'(x) = \begin{cases} e^{-x}(2x - x^2), & x \ge 0 \\ e^{x}(x^2 + 2x), & x < 0 \end{cases}$$

$f(x)$ increases in $(-\infty, -2) \cup (0, 2)$
and $f(x)$ decreases in $(-2, 0) \cup (2, \infty)$

$$\Rightarrow f''(x) = \begin{cases} e^{-x}(x^2 - 4x + 2), & x \ge 0 \\ e^{x}(x^2 + 4x + 2), & x < 0 \end{cases}$$

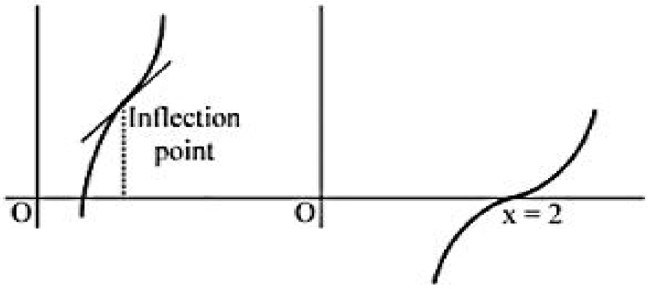


DIFFERENT GRAPHS OF THE CUBIC:

$y = ax^3 + bx^2 + cx + d$

(1) One real & two imaginary roots. (always monotonic) $\forall\ x \in R$

Condition :
$f'(x) \ge 0$ or $f'(x) \le 0$ together with either $f'(x) = 0$ has no root (i.e. $D < 0$) or $f'(x) = 0$ has a root $x = \alpha$ then $f(\alpha) = 0$.

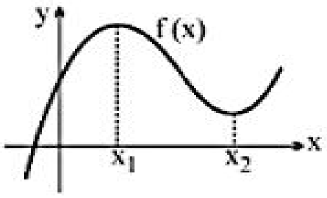


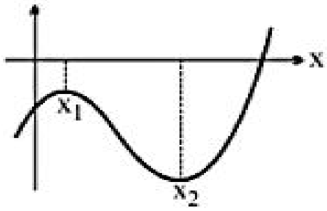
(i) either $f'(x) = 0$ has no real root
or (ii) if $f'(x) = 0$ has a root $x = \alpha$ then $f(\alpha) = 0$

e.g. $y = x^3 - 2x^2 + 5x + 4$ $\qquad$ $y = (x-2)^3$
$y' = 3x^2 - 4x + 5$ $(D < 0)$ $\qquad$ $y' = 3(x-2)^2 = 0 \Rightarrow x = 2$, also $f(2) = 0$
gives $x = 2$, $y(2) = 0$

Note: In this case if $f'(x) = 0$ has a root $x = \alpha$ and $f(\alpha) = 0$ this would mean $f(x) = 0$ has repeated roots which is dealt separately.

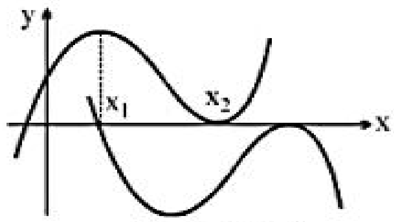
(2) Exactly one root and non monotonic.





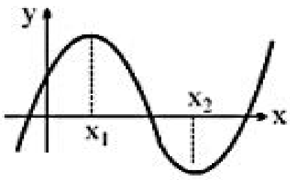
$f(x_1) \cdot f(x_2) > 0$
where x_1 & x_2 are the roots of $f'(x) = 0$

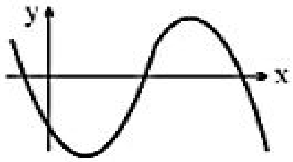
(3) Three roots < two concident / One different



$f(x_1) \cdot f(x_2) = 0$

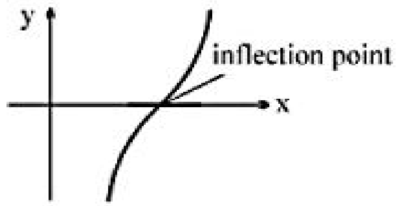
(4) All three distinct real roots





$f(x_1) \cdot f(x_2) < 0$
where x_1 & x_2 are the roots of $f'(x) = 0$

(5) All three roots concident



$f'(x) \geq 0$ or $f'(x) \leq 0$ & $f(\alpha) = 0$
where α is a root of $f'(x) = 0$
e.g. $y = (x - 1)^3$

Note :

(i) Graph of every cubic polynomial must have exactly one point of *inflection.*

(ii) In case (4) if f(a), f(b), f(c) and f(d) alternatively change sign.

Illustration :

Find the value of a if $x^3 - 3x + a = 0$ has three real distinct roots.

Sol. *Let $f(x) = x^3 - 3x + a$*
Let $f'(x) = 0$
$\Rightarrow 3x^2 - 3 = 0 \Rightarrow x = \pm 1$
For three distinct roots, $f(1)\,f(-1) < 0$
$\Rightarrow (1 - 3 + a)(-1 + 3 + a) < 0$
$\Rightarrow (a + 2)(a - 2) < 0$
$\Rightarrow -2 < a < 2$

Illustration :

Prove that three exist exactly two non-similar isosceles triangle ABC such that $\tan A + \tan B + \tan C = 100$.

Sol. *Let $A = B$, then $2A + C = 180^o$ and $2\tan A + \tan C = 100$*
Now $2A + C = 180^o \Rightarrow \tan 2A = -\tan C$ ...(i)

Also $\quad 2\tan A + \tan C = 100$

$\Rightarrow \quad 2\tan A - 100 = -\tan C \qquad\qquad ...(ii)$

From (i) and (ii), $2\tan A - 100 = \dfrac{2\tan A}{1-\tan^2 A}$

Let $\tan A = x$, then $\dfrac{2x}{1-x^2} = 2x - 100$

$\Rightarrow \quad x^3 - 50x^2 + 50 = 0$

Let $f(x) = x^3 - 50x^2 + 50$. Then $f'(x) = 3x^2 - 100x$. Thus $f'(x) = 0$ has roots $0, \dfrac{100}{3}$. Also $f(0)\, f\left(\dfrac{100}{3}\right) < 0$. Thus $f(x) = 0$ has exactly three distinct real roots. Therefore, $\tan A$ and hence A has three distinct values but one of them will be obtuse angle. Hence, there exist exactly two non similar isosceles triangles.

Illustration :

Find the set of value of m for the cubic $x^3 - \dfrac{3}{2}x^2 + \dfrac{5}{2} = \log_{1/4}(m)$ has 3 distinct solutions.

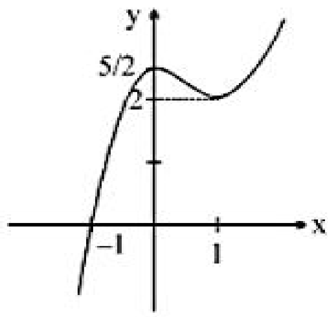
Sol. Consider $y = x^3 - \dfrac{3}{2}x^2 + \dfrac{5}{2}$

$$\frac{dy}{dx} = 3x^2 - 3x = 3x(x-1) = 0 \quad \Rightarrow \quad x = 0 \text{ or } 1$$

$$\frac{d^2y}{dx^2} = 6x - 3 \;;\; \left.\frac{d^2y}{dx^2}\right|_{x=0} = -3 \text{ i.e. } < 0 \;\Rightarrow\; \text{maximum at } x = 0$$

$$\left.\frac{d^2y}{dx^2}\right|_{x=1} = \text{is } 3 \text{ i.e.} > 0 \;\Rightarrow\; \text{mimimum}$$

Hence the graph of the cubic is now for 3 distinct roots



$$2 < \log_{1/4}(m) < \frac{5}{2}$$

$$2 < -\log_4(m) < \frac{5}{2}$$

$$-\frac{5}{2} < \log_4(m) < -2$$

$$\frac{1}{32} < m < \frac{1}{16}$$

Practice Problem

Q.1 The graph of the derivative f' of a continuous function f is shown with $f(0) = 0$

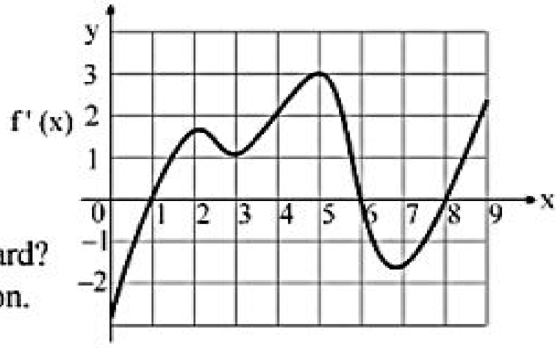
(i) On what intervals is f increasing or decreasing?

(ii) At what values of x does f have a local maximum or minimum?

(iii) On what intervals is f concave upward or downward?

(iv) State the x-coordinate(s) of the point(s) of inflection.

(v) Assuming that $f(0) = 0$, sketch a graph of f.



Q.2 The set of value(s) of a for which the function $f(x) = \frac{ax^3}{3} + (a+2)x^2 + (a-1)x + 2$ possesses a negative point inflection is

(A) $(-\infty, -2)$ (B) $\left\{-\frac{4}{5}\right\}$ (C) $(-2, 0)$ (D) empty set

Q.3 The number of values of k for which the equation $x^3 - 3x + k = 0$ has two distinct roots lying in the interval (0, 1) is

(A) three (B) two (C) infinitely many (D) zero

Q.4 For the function $f(x) = x^4(12 \log_e x - 7)$

(A) there is no point of inflection

(B) $x = e^{1/3}$ is the point of minima

(C) the graph is concave upwards in (0, 1)

(D) the graph is concave downwards in $(1, \infty)$

Answer key

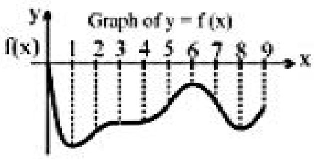
Q.1 (i) I in $(1, 6) \cup (8, 9)$ and D in $(0, 1) \cup (6, 8)$;

(ii) L.Min. at $x = 1$ and $x = 8$; L.Max. $x = 6$

(iii) CU in $(0, 2) \cup (3, 5) \cup (7, 9)$ and CD in $(2, 3) \cup (5, 7)$;

(iv) $x = 2, 3, 5, 7$

(v) One of the possible graph is



Q.2 A Q.3 D Q.4 B

Solved Examples

Q.1 Find the shortest distance between the curves $y^2 = x^3$ and $9x^2 + 9y^2 - 30y + 16 = 0$.

Sol. We have

$$9x^2 + 9y^2 - 30y + 16 = 0$$

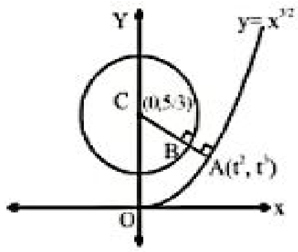
which is a circle having

$$\text{centre} \equiv \left(0, \frac{5}{3}\right) \text{ and radius} = \sqrt{\left(\frac{5}{3}\right)^2 - \frac{16}{9}} = 1.$$

Let us choose any point on the curve $y^2 = x^3$ as $A(t^2, t^3)$. If B is the point on the circle and nearest to A(see figure), then

$$AB = AC - \text{radius}$$

$$= \sqrt{(t^2-0)^2 + \left(t^3 - \frac{5}{3}\right)^2} - 1$$



Let $f(t) = t^4 + \left(t^3 - \frac{5}{3}\right)^2$

and $f'(t) = 4t^3 + 2\left(t^3 - \frac{5}{3}\right) \cdot 3t^2 = 2t^2\left[2t + 3\left(t^3 - \frac{5}{3}\right)\right]$

$$= 2t^2(3t^3 + 2t - 5) = 2t^2(t-1)(3t^2 + 3t + 5).$$

The value of 't' at which AB attains minima is given by the equation

$$f'(t) = 0$$

gives $t = 0, 1$

But $t = 0$ is not a point of extrema, since $f'(t)$ does not change sign in the neighbourhood of $t = 0$. However, $t = 1$ is a point of minima, since $f'(1^-) < 0$ and $f'(1^+) > 0$. Hence, the minimum value of AB is

$$\sqrt{1 + \left(1 - \frac{5}{3}\right)^2} - 1 \simeq 0.2. \text{ **Ans.**}$$

Q.2 Consider the function :

$$f(x) = \begin{cases} -x^3 + \dfrac{b^3 - b^2 + b - 1}{b^2 + 3b + 2}, & 0 \le x < 1 \\ 2x - 3, & 1 \le x \le 3 \end{cases}$$

Find all possible real values of b such that f(x) has the least value at $x = 1$.

Sol. We have

$$f(x) = -x^3 + g(b), \quad 0 \le x < 1$$
$$= 2x - 3, \quad 1 \le x \le 3$$

where $g(b) = \dfrac{b^3 - b^2 + b - 1}{b^2 + 3b + 2}$ (b is a constant)

and $\quad f'(x) = -3x^2, \quad 0 < x < 1$
$\qquad\qquad = 2, \qquad 1 < x < 3$

$\Rightarrow \quad$ f(x) strictly decreases in (0, 1)
strictly increases in (1, 3).

Now, we have for minima at x = 1

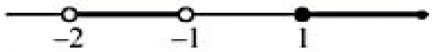
$f(1) \le f(1^-)$

i.e. $\quad 2 - 3 \le -1 + g(b)$

i.e. $\quad g(b) \ge 0$

i.e. $\quad \dfrac{(b^2+1)(b-1)}{(b+1)(b+2)} \ge 0$

The number line shown alongside, gives



$b \in (-2, -1) \cup [1, \infty)$.

Q.3 If the function

$$f(x) = (a+3)x^3 + (a-3)x^2 + 4(a-4)x + 5$$

has maxima at some $x \in R^-$ and a minima at some $x \in R^+$, find the possible values of a.

Sol. We have

$$f(x) = (a+3)\,x^3 + (a-3)\,x^2 + 4\,(a-4)\,x + 5$$

and $\quad f'(x) = 3\,(a+3)\,x^2 + 2\,(a-3)\,x + 4\,(a-4)$

According to the given condition, f'(x) must vanish at two real and distinct points say α, β such that $\alpha < 0$ and $\beta > 0$. Thus, we have $f'(x) = 3\,(a+3)\,(x-\alpha)\,(x-\beta)$

and $\quad f''(x) = 3\,(a+3)\,(x-\alpha+x-\beta)$.

According to the given condition, α is to be a maxima

i.e. $\quad f''(\alpha) < 0$

i.e. $\quad 3\,(a+3)\,(\alpha-\beta) < 0$

i.e. $\quad a + 3 > 0 \qquad\qquad [\alpha - \beta < 0]$

i.e. $\quad a > -3$

and $\quad \beta$ is to be a minima

i.e. $\quad f''(\beta) > 0$

i.e. $\quad 3\,(a+3)\,(\beta-\alpha) > 0$

i.e. $\quad a + 3 > 0 \qquad\qquad [\beta - \alpha < 0]$

i.e. $\quad a > -3$

Thus, taking together all the above results (a + 3 > 0), the graph of

$$y = 3\,(a+3)\,x^2 + 2\,(a-3)\,x + 4\,(a-4)$$

must look as shown alongside. For the curve to look like this, the neccessary and sufficient condition is

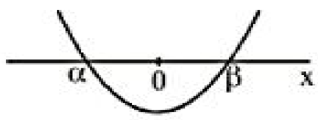
$y(0) < 0$

i.e. $\quad 4(a-4) < 0$

i.e. $\quad a < 4$

Hence, the possible values of a, are

$a \in (-3, 4)$.



[a + 3 > 0 ⇒ concavity along the +ve Y-axis]

Q.4 Find a point M on the curve $y = \frac{3}{\sqrt{2}}\, x\, \ln x, x \in (e^{-1.5}, \infty)$ such that the segment of the tangent at M intercepted between M and the Y-axis is shortest.

Sol. We have

$$y = \frac{3}{\sqrt{2}}\, x \ln x, x \in (e^{-1.5}, \infty)$$

and $$y' = \frac{3}{\sqrt{2}}(1 + \ln x), x \in (e^{-1.5}, \infty) \qquad \text{.......(1)}$$

Any point on the curve can be chosen as $M \equiv \left(h, \frac{3}{\sqrt{2}} h \ln h\right)$

Slope of the tangent at $M = \frac{3}{\sqrt{2}}(1 + \ln h)$ [using eq. (1)]

Equation of the tangent at M, is given by

$$y - \frac{3}{\sqrt{2}} h \ln h = \frac{3}{\sqrt{2}}(1 + \ln h)(x - h) \qquad \text{......(2)}$$

The tangent cuts the Y-axis at

$$N \equiv \left(0, \frac{-3h}{\sqrt{2}}\right) \qquad \text{[putting } x = 0 \text{ in eq. (2)]}$$

Length l of the tangent segment MN, is given by

$$l^2 = h^2 + \left\{\frac{3}{\sqrt{2}} h(1 + \ln h)\right\}^2 = h^2 + \frac{9}{2} h^2 (1 + \ln h)^2$$

The value of h at which l^2 attains minima, is given by the equation

$$\frac{d}{dh}(l^2) = 0$$

i.e. $$2h + 9h(1 + \ln h)^2 + 9h^2(1 + \ln h)\left(\frac{1}{h}\right) = 0$$

i.e. $$h\left[(1 + \ln h)^2 + (1 + \ln h) + \frac{2}{9}\right] = 0$$

i.e. $$1 + \ln h = \frac{-1 \pm \sqrt{1 - \frac{8}{9}}}{2} = \frac{-2}{3}, \frac{-1}{3} \qquad [h = 0 \text{ does not lie in } (e^{-1.5}, \infty)]$$

gives $h = e^{-5/3}, e^{-4/3}$.

Only $h = e^{-4/3}$ is acceptable since $e^{-5/3}$ does not lie in $(e^{-1.5}, \infty)$. Hence, the required point, is

$$M \equiv \left(e^{-4/3}, -2\sqrt{2}e^{-4/3}\right).$$

Q.5 Assuming that the petrol burnt per unit time in driving a motor boat, varies as the cube of its velocity. Find the most economical speed of the motor boat when moving against a current whose speed is cm/sec.

Sol. Let s m/sec be the speed of the motor boat w.r.t stream. Then, the rate at which petrol is burnt, is given by

$$\frac{dp}{dt} = ks^3 \qquad \text{........(1)}$$

where k is a constant of proportionality.

Now that the boat is moving against a current of c m/sec, the absolute speed of the motor boat is $(s - c)$ m/sec. Then, the rate at which distance is covered by the motor boat, is given by

$$\frac{dx}{dt} = s - c \qquad \text{........(2)}$$

Dividing equation (1) by equation (2), gives

$$\frac{dp}{dx} = \frac{ks^3}{s-c}$$

which is the amount of petrol consumed per unit distance travelled by the boat,.

Let $\quad f(s) = \dfrac{ks^3}{s-c}$

The value of s at which f(s) attains minima, is given by

$$\frac{df}{ds} = 0$$

i.e. $\quad \dfrac{(s-c)\,3s^2 - s^3}{(s-c)^2} = 0$

i.e. $\quad 2s - 3c = 0$

gives $\quad s = \dfrac{3}{2}c$

At this value of s, the petrol consumption per unit distance, is minimum and the absolute speed of the motor boat is

$$\frac{3}{2}c - c = \frac{3c}{2} \text{ m/sec.}$$

Q.6 Find the point on the curve $y = \dfrac{x}{1+x^2}$ where the tangent to the curve has the greatest slope.

Sol. We have

$$y = \frac{x}{1+x^2}$$

The slope of the tangent at any point on the curve, is given by

$$\frac{dy}{dx} = \frac{(1+x^2)1 - x(2x)}{(1+x^2)^2} = \frac{1-x^2}{(1+x^2)^2}$$

The values of x at which the slope becomes maximum or minimum, is given by

$$\frac{d^2y}{dx^2} = 0$$

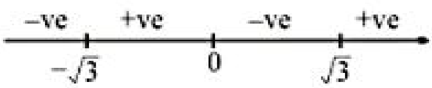
i.e. $$\frac{d}{dx}\left[\frac{1-x^2}{(1+x^2)^2}\right] = 0$$

i.e. $$\frac{(1+x^2)^2(-2x) - (1-x^2)\, 2(1+x^2)\cdot 2x}{(1+x^2)^4}$$

i.e. $$\frac{2x\,(1+x^2)\,(x^2-3)}{(1+x^2)^4} = 0$$

i.e. $x = 0, \pm\sqrt{3}$.

Now, from the sign scheme for $\frac{d^2y}{dx^2}$, we have



$\frac{d^2y}{dx^2}$ changes sign from + ve to – ve at x = 0

$\Rightarrow$ $\frac{dy}{dx}$ is maximum at x = 0

Hence, the point on the curve where slope of the tangent is maximum, is (0, 0).

Q.7 Discuss the extremum of $f(x) = a \sec x + b \operatorname{cosec} x, 0 < a < b$.

Sol. $f(x) = a \sec x + b \operatorname{cosec} x, 0 < a < b$.
$f'(x) = a \sec x \tan x - b \operatorname{cosec} x \cot x$

Let $f'(x) = 0 \Rightarrow a\frac{\sin x}{\cos^2 x} - b\frac{\cos x}{\sin^2 x} = 0$

$\Rightarrow \tan^3 x = \frac{b}{a} \Rightarrow x = \tan^{-1}\left(\frac{b}{a}\right)^{1/3}$; a, b > 0 $\Rightarrow x = \tan^{-1}\left(\frac{b}{a}\right)^{1/3} > 0$

$\Rightarrow$ x lies in either the first or third quadrant for extremum.

Case-I : $0 < x < \frac{\pi}{2}$

$$\lim_{x\to 0} (a \sec x + b \operatorname{cosec} x) \to \infty$$

$$\lim_{x\to \frac{\pi}{2}} (a \sec x + b \operatorname{cosec} x) \to \infty$$

Also $f(x)$ is +ve for this value of x.
Hence, only one point of extremum is the point of minima.

and $\quad \tan x = \left(\frac{b}{a}\right)^{1/3}$

$$\Rightarrow \quad \cos x = \frac{a^{1/3}}{\sqrt{a^{2/3}+b^{2/3}}}, \sin x = \frac{b^{1/3}}{\sqrt{a^{2/3}+b^{2/3}}}$$

$$\Rightarrow \quad \text{Minimum value of } f = \frac{a\sqrt{a^{2/3}+b^{2/3}}}{a^{1/3}} + \frac{b\sqrt{a^{2/3}+b^{2/3}}}{b^{1/3}} = (a^{2/3}+b^{2/3})^{3/2}$$

Case-II : $\quad \pi < x < \frac{3\pi}{2}$

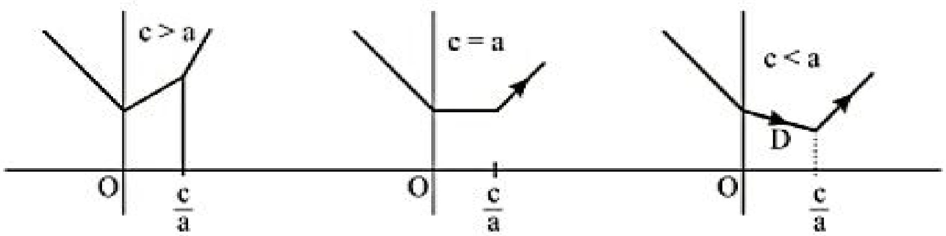
$$\lim_{x\to\pi} (a \sec x + b \operatorname{cosec} x) \to -\infty$$

$$\lim_{x\to\frac{3\pi}{2}} (a \sec x + b \operatorname{cosec} x) \to -\infty$$

Also $\quad f(x)$ is –ve for this values of x.
Hence, only one point of extremum is the point of maxima.
$\Rightarrow$ Maximum value $f_{max} = -(a^{2/3}+b^{2/3})^{3/2}$

Q.8 The function $f(x) = |ax-b| + c\,|x|\ \forall\ x \in (-\infty, \infty)$, where $a > 0, b > 0, c > 0$. Find the condition if $f(x)$ attains the minimum value only at one point.

Sol. $$f(x) = \begin{cases} b-(a+c)x, & x<0 \\ b+(c-a)x, & 0 \le x < \frac{b}{a} \\ (a+c)x+b, & x \ge \frac{b}{a} \end{cases}$$



x = 0 gives minima $\qquad$ Infinite points of minima $\qquad$ $x = \frac{c}{a}$ gives minima

Figure clearly indicates that for exactly one point of minima, $a \ne c$.

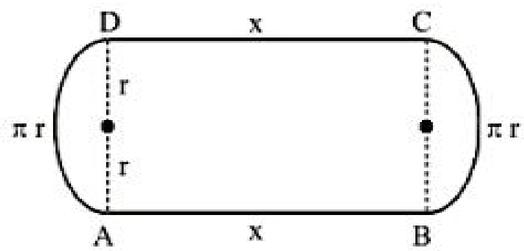
Q.9 A running track of 440 ft is to be laid out enclosing a football field, the shape of which is a rectangle with a semi-circle at each end. If the area of the rectangular portion is to be maximum, then find the lengths of its sides.

Sol. Perimeter = 440 ft.

$\Rightarrow$ $2x + \pi r + \pi r = 440$ or $2x + 2\pi r = 440$

A = Area of the rectangular portion = x 2r

$$\Rightarrow A = x\frac{(440-2x)}{\pi} = \frac{1}{\pi}(440x - 2x^2)$$



Let $\dfrac{dA}{dx} = \dfrac{1}{\pi}(440 - 4x) = 0$

$\Rightarrow$ $x = 110$ for which $\dfrac{d^2A}{dx^2} < 0$

$\Rightarrow$ A is maximum when $x = 110$

$$\Rightarrow \quad 2r = \frac{440-2x}{\pi} = \frac{440-220}{22/7} = 70$$

$\Rightarrow$ r= 35 ft and x = 110 ft

Q.10 Find the points on the curves $5x^2 - 8xy + 5y^2 = 4$ whose distance from the origin is maximum or minimum.

Sol. Let (r, θ) be the polar coordinates of any point P on the curve where r is the distance of the point from the origin.

$\Rightarrow$ $r^2[5(\cos^2\theta + \sin^2\theta) - 8\sin\theta\cos\theta] = 4$

$$\Rightarrow \quad r^2 = \frac{4}{5 - 4\sin 2\theta}$$

r^2 is maximum when $5 - 4\sin 2\theta$ is minimum $= 5 - 4 = 1$ (when $\sin 2\theta = 1$)

$\Rightarrow$ $2\theta = 90^\circ \Rightarrow \theta = 45^\circ \Rightarrow r = \pm 2,\ \theta = 45^\circ$ ......(i)

Again r^2 is minimum when $5 - 4\sin 2\theta$ is maximum

$$= 5 + 4 = 9 \text{ when } \sin 2\theta = -1 \Rightarrow 2\theta = \frac{3\pi}{2} \Rightarrow \theta = \frac{3\pi}{4}$$

$$\Rightarrow \quad r = \pm\frac{2}{3},\ \theta = \frac{3\pi}{4}$$

Hence, the points are $(r\cos\theta, r\sin\theta)$ where r and θ are given by equations (i) and (ii).

Thus, we get four points $(\sqrt{2}, \sqrt{2}), (-\sqrt{2}, -\sqrt{2}), \left(\dfrac{\sqrt{2}}{3}, -\dfrac{\sqrt{2}}{3}\right)$ and $\left(-\dfrac{\sqrt{2}}{3}, \dfrac{\sqrt{2}}{3}\right)$.

Q.11 Discuss the monotonocity of $Q(x)$, where $Q(x) = 2f\left(\frac{x^2}{2}\right) + f(6-x^2)\ \forall\ x \in R$. It is given that $f''(x) > 0\ \forall\ x \in R$. Find also the points of maxima and minima of $Q(x)$.

Sol. Given $Q(x) = 2f\left(\frac{x^2}{2}\right) + f(6-x^2)$

$$\therefore \quad Q'(x) = 2xf'\left(\frac{x^2}{2}\right) - 2xf'(6-x^2) = 2x\left\{f'\left(\frac{x^2}{2}\right) - f'(6-x^2)\right\}$$

But given that $f''(x) > 0 \Rightarrow f'(x)$ is increasing for all $x \in R$.

Case-I : Let $\frac{x^2}{2} > (6-x^2) \Rightarrow x^2 > 4$

$\therefore \quad x \in (-\infty, -2) \cup (2, \infty)$

$\Rightarrow \quad f\left(\frac{x^2}{2}\right) > f(6-x^2)$

$\Rightarrow \quad f\left(\frac{x^2}{2}\right) - f(6-x^2) > 0$

If $x > 0$, then $Q'(x) > 0 \quad \Rightarrow \quad x \in (2, \infty)$

and if $x < 0$, then $Q'(x) < 0 \quad \Rightarrow \quad x \in (-\infty, -2)$

Case-II : Let $\frac{x^2}{2} < (6-x^2) \Rightarrow x^2 < 4 \Rightarrow x \in (-2, 2)$

$\Rightarrow \quad f'\left(\frac{x^2}{2}\right) < f'(6-x^2) \Rightarrow f'\left(\frac{x^2}{2}\right) - f'(6-x^2) < 0$

If $x > 0$, then $Q'(x) < 0 \quad \Rightarrow \quad x \in (0, 2)$

and If $x < 0$, then $Q'(x) > 0 \quad \Rightarrow \quad x \in (-2, 0)$

Q.12 The largest term in the sequence $a_n = \dfrac{n^2}{n^3+200}$ is given by

(A) $\dfrac{529}{49}$ (B) $\dfrac{8}{89}$ (C) $\dfrac{49}{543}$ (D) None of these

Sol. Conisder the function $f(x) = \dfrac{x^2}{(x^3+200)}$ ......(i)

$$f'(x) = x\dfrac{(400-x^3)}{(x^3+200)^2} = 0$$

When $x = (400)^{1/3}$ $(\because\ x \neq 0)$

$x = (400)^{1/3} - h \Rightarrow f'(x) > 0$

$x = (400)^{1/3} + h \Rightarrow f'(x) < 0$

$\therefore$ $f(x)$ has maxima at $x = (400)^{1/3}$

Since $7 < (400)^{1/3} < 8$, either a_7 or a_8 is the greatest term of the sequence.

$\because$ $a_7 = \dfrac{49}{543}$ and $a_8 = \dfrac{8}{89}$ and $\dfrac{49}{543} > \dfrac{8}{89}$

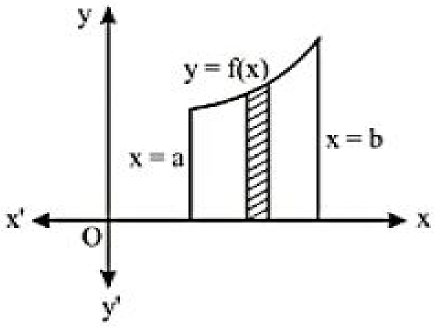
$\Rightarrow$ $a_7 = \dfrac{49}{543}$ is the greatest term.

AREA UNDER THE CURVE

DIFFERENT CASES OF BOUNDED AREA :

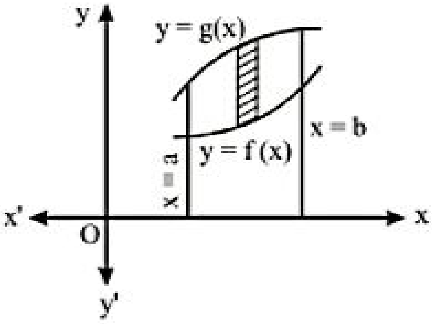
1. The area bounded by the continuous curve $y = f(x)$, the axis of x and the ordinates $x = a$ and $x = b$ (where $b > a$) is given by

$$A = \int_a^b f(x)dx = \int_a^b ydx$$



2. The area bounded by the straight line $x = a, x = b$ $(a < b)$ and the curves $y = f(x)$ and $y = g(x)$, provided $f(x) < g(x)$ (where $a \le x \le b$), is given by

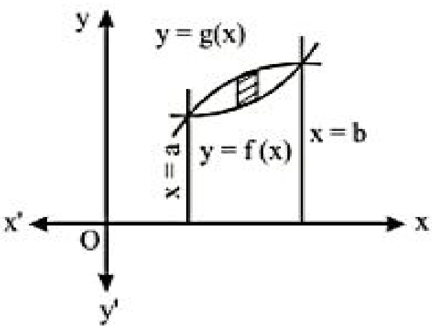
$$A = \int_a^b [g(x) - f(x)]dx$$



3. When two curves $y = f(x)$ and $y = g(x)$ intersect, the bounded area is

$$A = \int_a^b [g(x) - f(x)]dx \;\; ; \text{ where } a < b.$$

where a and b are the roots of the equation $f(x) = g(x)$.

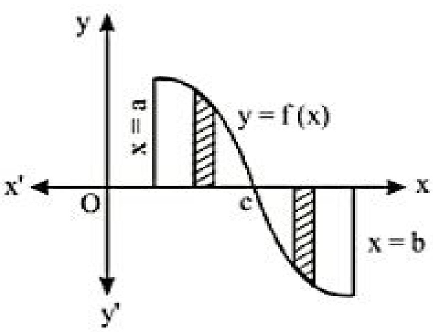


4. If some part of a curve lies below the x-axis, then its area becomes negative but area cannot be negative. Therefore, we take its modulus.

If the curves crosses the x-axis at c, then the area bounded by the curve $y = f(x)$ and ordinates $x = a$ and $x = b$

(where $b > a$) is given by $A = \left|\int_a^c f(x)dx\right| + \left|\int_c^b f(x)dx\right|$

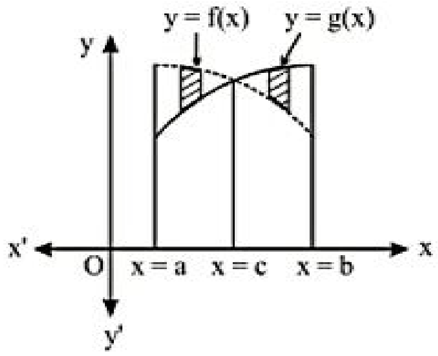
$$A = \int_a^c f(x)dx - \int_c^b f(x)dx$$



5. The area bounded by $y = f(x)$ and $y = g(x)$ (where $a \le x \le b$), when they intersect at $x = c \in (a, b)$ is given by

$$A = \int_a^b |f(x) - g(x)| dx$$

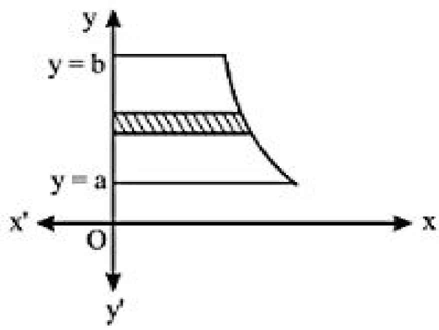
or $$\int_a^c (f(x) - g(x))dx + \int_c^b (g(x) - f(x))dx$$



DIFFERENT CASES OF BOUNDED AREA :

1. The area bounded by the continuous curve $x = f(y)$, the axis of y and the abscissa $y = a$ and $y = b$ (where $b > a$) is given by

$$A = \int_a^b f(y)dy = \int_a^b x dy$$



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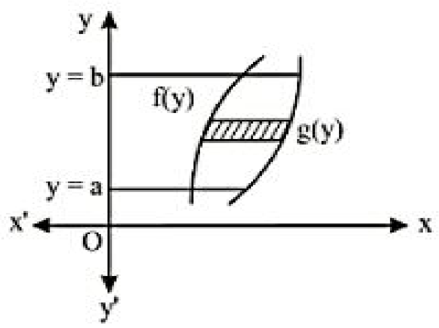
$$A = \int_a^b [g(y) - f(y)]dy$$



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$$A = \int_a^b [g(y) - f(y)]dy \text{ ; Where } a < b.$$

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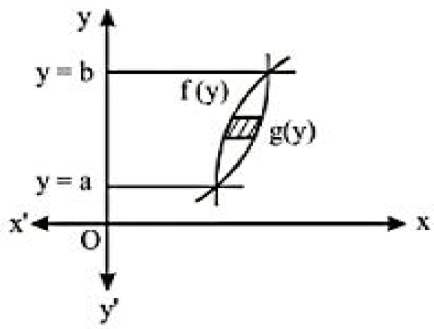

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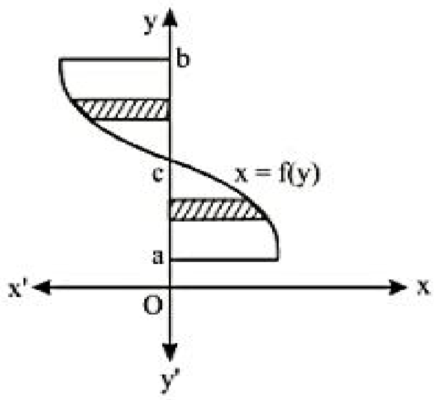
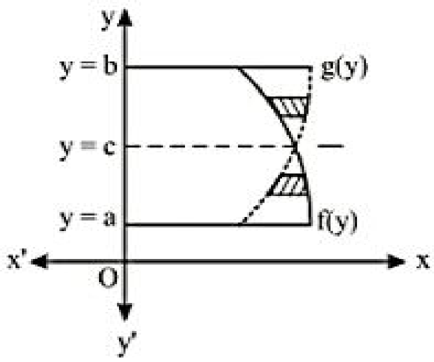


(where $b > a$) is given by $A = \left|\int_a^c f(y)dy\right| + \left|\int_c^b f(y)dy\right|$

$$= A = \int_a^c f(y)dy - \int_c^b f(y)dy$$

5. The area bounded by $x = f(y)$ and $x = g(y)$ (where $a \le y \le b$), when they intersect at $y = c \in (a, b)$ is given by



$$A = \int_a^b |f(y) - g(y)|dy$$

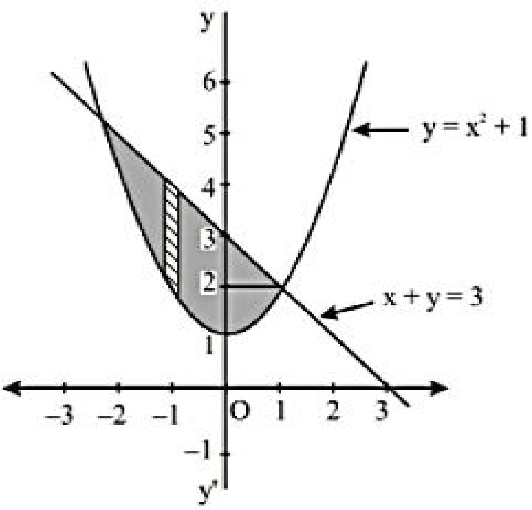
or $$\int_a^c (f(y) - g(y))dy + \int_c^b (g(y) - f(y))dy$$

Illustration :

Find the area bounded by the parabola $y = x^2 + 1$ and the straight line $x + y = 3$.

Sol. *The two curves meet at ponts where $3 - x = x^2 + 1$ i.e., $x^2 + x - 2 = 0$*
$\Rightarrow (x + 2)(x - 1) = 0 \Rightarrow x = -2, 1$

$$\therefore \text{ required area } = \int_{-2}^{1}\left[(3 - x) - (x^2 + 1)\right]dx$$



$$= \int_{-2}^{1}\left(2 - x - x^2\right)dx$$

$$= \left[2x - \frac{x^2}{2} - \frac{x^3}{3}\right]_{-2}^{1}$$

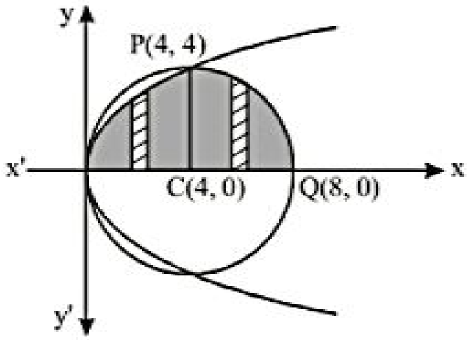
$$= \left(2 - \frac{1}{2} - \frac{1}{3}\right) - \left(-4 - \frac{4}{2} + \frac{8}{3}\right)$$

$$= \frac{9}{2} \text{ sq. units.}$$

Illustration :

Find the area, lying above x-axis and included between the circle $x^2 + y^2 = 8x$ and the parabola $y^2 = 4x$.

Sol. *Solving the curves, we get $x^2 + 4x = 8x \Rightarrow x = 0, 4$.*



$$\text{Required area} = \int_0^4 y_{parabola}\,dx + \int_4^8 y_{circle}\,dx$$

Circle is $(x-4)^2 + y^2 = 4^2$.

Area of circle in 1^{st} *quadrant* $= \frac{1}{4}\pi 4^2 = 4\pi$

$$A = 2\int_0^4 \sqrt{x}\,dx + 4\pi$$

$$= \frac{4}{3}\left[x^{3/2}\right]_0^4 + 4\pi = \frac{2}{3} \times 4\sqrt{4} + 4\pi \text{ sq. units}$$

$$= \frac{32}{3} + 4\pi \text{ sq. units}$$

Illustration :

Find the area bounded by the curve $y = (x-1)(x-2)(x-3)$ lying between the ordinates $x = 0$ and $x = 3$.

Sol. $y = (x-1)(x-2)(x-3)$

The curves will intersect the x-axis, when $y = 0$.

$\Rightarrow \quad (x-1)(x-2)(x-3) = 0$

$\Rightarrow \quad x = 1, 2, 3$

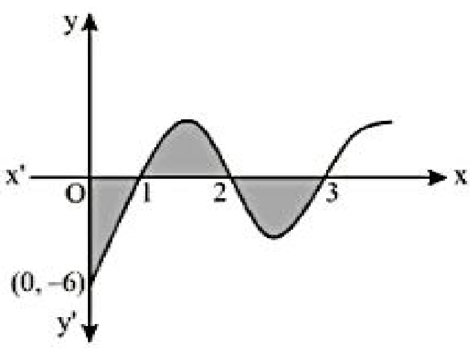
And the curve intersects the y-axis,

when $x = 0 \Rightarrow y = -6$

Thus, the graph of the given function for $0 \le x \le 3$ is as shown in figure.

Hence, the required area A = shaded area

$$= \left|\int_0^1 y\,dx\right| + \left|\int_1^2 y\,dx\right| + \left|\int_2^3 y\,dx\right|$$



Since $\int y\,dx = \int (x-1)(x-2)(x-3)dx$

$$= \int (x^3 - 6x^2 + 11x - 6)dx$$

$$= \frac{x^4}{4} - 2x^3 + \frac{11x^2}{2} - 6x$$

$\therefore$ *from equation (1)*

$$A = \left|\left[\frac{x^4}{4} - 2x^3 + \frac{11x^2}{2} - 6x\right]_0^1\right| + \left|\left[\frac{x^4}{4} - 2x^3 + \frac{11x^2}{2} - 6x\right]_1^2\right| + \left|\left[\frac{x^4}{4} - 2x^3 + \frac{11x^2}{2} - 6x\right]_2^3\right|$$

$= |-9/4| + (1/4) + |-1/4| = 11/4$ *sq. units*

Illustration :

Consider the region formed by the linex $x = 0, y = 0, x = 2, y = 2$. *Area enclosed by the curves* $y = e^x$ *and* $y = \ln x$, *within this region, is being removed. Then, find the area of the remaining region.*

Sol. *Required area = shaded region*

$$= 2\int_0^{\ln 2} (2 - e^x)\,dx$$

$$= 2[2x - e^x]_0^{\ln 2}$$

$$= 2(2\ln 2 - 1) \text{ sq. units}$$

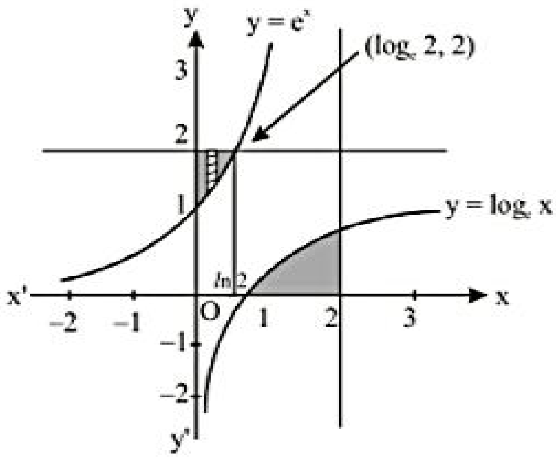


Illustration :

Find the area bounded by the curves $y = \sin x$ *and* $y = \cos x$ *between two consecutive points of their intersection.*

Sol. *Two consecutive points of intersection of* $y = \sin x$ *and* $y = \cos x$ *can be taken as* $x = \frac{\pi}{4}$ *and* $x = \frac{5\pi}{4}$

$$\therefore \text{ Required area} = \int_{\pi/4}^{5\pi/4} (\sin x - \cos x)\,dx$$

$$= [-\cos x - \sin x]_{\pi/4}^{5\pi/4}$$

$$= \frac{2}{\sqrt{2}} + \frac{2}{\sqrt{2}}$$

$$= 2\sqrt{2} \text{ sq. units}$$

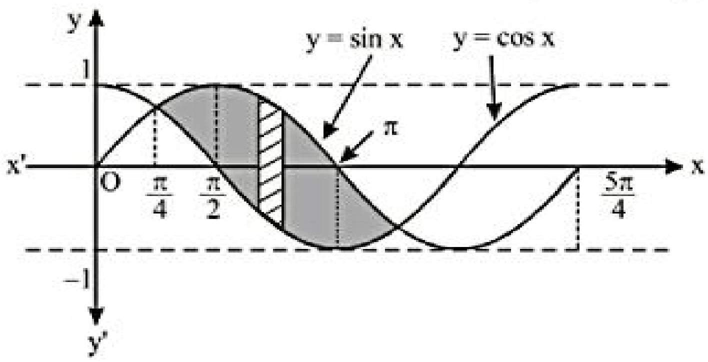


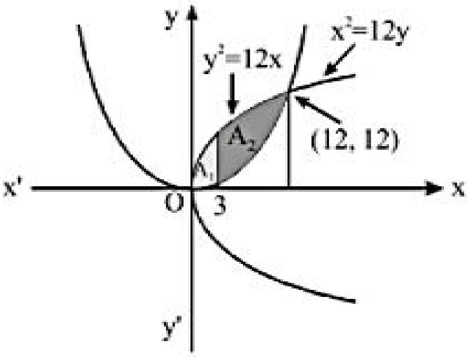
Illustration :

Find the ratio in which the area bounded by the curves $y^2 = 12x$ and $x^2 = 12y$ is divided by the line $x = 3$.

Sol. A_1 = area bounded by $y^2 = 12x$, $x^2 = 12y$ and line $x = 3$

$$= \int_0^3 \sqrt{12x}\,dx - \int_0^3 \frac{x^2}{12}\,dx$$

$$= \sqrt{12}\left|\frac{2x^{3/2}}{3}\right| - \left|\frac{x^3}{36}\right|_0^3 = \frac{45}{4} \text{ sq. units}$$



A_2 = area bounded by $y^2 = 12x$ and $x^2 = 12y$

$$= \frac{16(3)(3)}{3} = 48 \text{ sq. units}$$

$$\therefore \text{ required ratio} = \frac{\frac{45}{4}}{48 - \frac{45}{4}} = \frac{45}{147} = \frac{15}{49}$$

Illustration :

Find the area bounded by

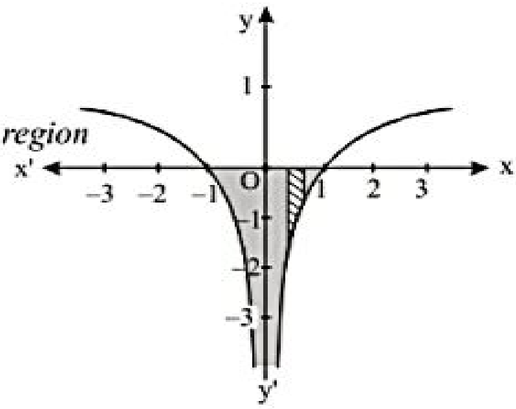
(i) $y = \log_e |x|$ and $y = 0$, *(ii) $y = |\log_e |x||$ and $y = 0$*

Sol.

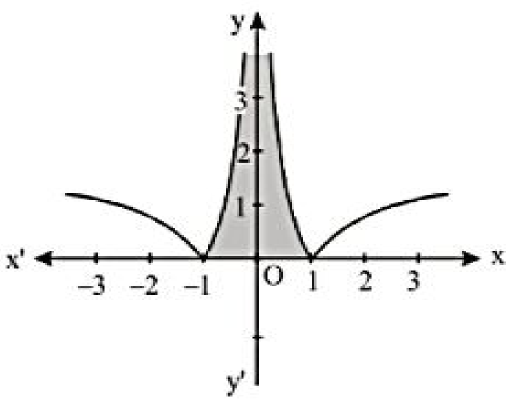
(i) $y = \log_e |x|$ and $y = 0$

From the figure, required area = area of the shaded region

$$= 2\left|\int_0^1 (\log_e x)dx\right| = 2\left|(x\log_e x - x)_0^1\right| = 2 \text{ sq. units}$$



(ii) $y = |\log_e |x||$ and $y = 0$



From the figure, required area = area of the shaded region = 1 + 1 = 2 sq. units.

Illustration :

Find the area included by the curve $y = \ln x$, x-axis and the two ordinate at $x = \frac{1}{e}$ and $x = e$.

Sol. $A = \left|\int_{1/e}^{1} \ln x\,dx\right| + \int_{1}^{e} \ln x\,dx = \left|[x(\ln x - 1)]_{1/e}^{1}\right| + [x(\ln x - 1)]_{1}^{e} = 2 - \frac{2}{e}$

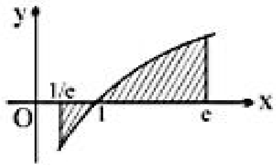
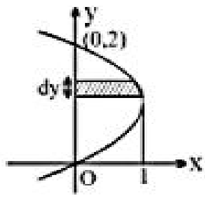


Illustration :

Find the area included by the curve $x = 2y - y^2$ and the y-axis

Sol. Let $x = 2y - y^2$ and the y-axis

$\frac{dx}{dy} = 2 - 2y = 0 \Rightarrow y = 1 \Rightarrow$ curve bends at $y = 1$;

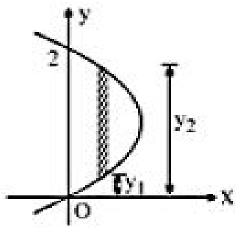


$$A = \int_0^2 x\,dy = \int_0^2 (2y - y^2)\,dy = y^2 - \frac{y^3}{3}\Bigg]_0^2 = 4 - \frac{8}{3} = \frac{4}{3} \textbf{ Ans.}$$

Alternative method :

This can also be done by taking vertical strip.

$y^2 - 2y + x = 0$



$y = \frac{2 \pm \sqrt{4 - 4x}}{2}$

$y = 1 + \sqrt{1 - x}$ $\quad (y_2)$

$y = 1 - \sqrt{1 - x}\ (y_1)$

$A = \int_0^1 y\,dx = \int_0^1 2(\sqrt{1 - x})\,dx$

Illustration :

For $b > a > 1$, the area enclosed by the curve $y = \ln x$, y axis and the straight lines $y = \ln a$ and $y = \ln b$ is

(A) $b - a$ (B) $b(\ln b - 1) - a(\ln a - 1)$

(C) $(\ln a)(b - a)$ (D) $(\ln b)(\ln a)$

Sol. *Required area* $= \int_{\ln a}^{\ln b} e^y dy = \left[e^y\right]_{\ln a}^{\ln b} = (b-a)$

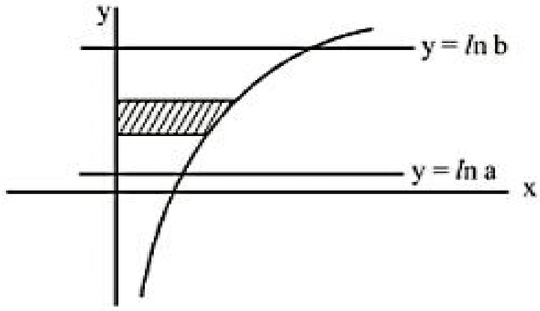


Illustration :

Find the area enclosed between y = sin x ; y = cos x and y-axis in the 1st quadrant

Sol. $A = \int_0^{\pi/4} (\cos x - \sin x)\, dx = \left[\sin x + \cos x\right]_0^{\pi/4} = \sqrt{2} - 1$

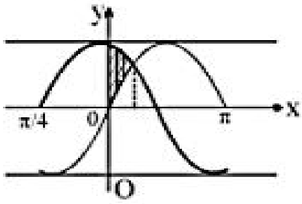
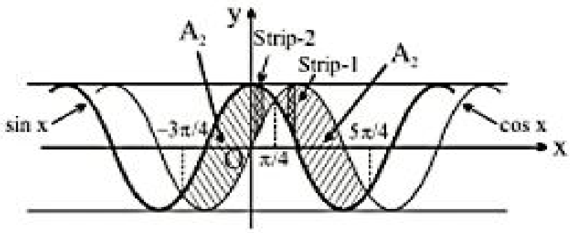


Illustration :

Curves y = sin x ; y = cos x intersect each other at infinite number of points enclosing regions of equal areas. Compute the area of one such equal region.

Sol. *(Strip-1)* $A_1 = \int_{\pi/4}^{5\pi/4} (\sin x - \cos x)dx = 2\sqrt{2}$

(Strip-2) $A_2 = \int_{-3\pi/4}^{\pi/4} (\cos x - \sin x)dx = 2\sqrt{2}$



So $A_1 = A_2 = 2\sqrt{2}$ *sq. unit*

Illustration :

Find the area enclosed by y = tan x ; y = cot x and x-axis in 1st quadrant.

Sol. $A = \int_0^{\pi/4} \tan x\, dx + \int_{\pi/4}^{\pi/2} \cot x\, dx$

$A = 2\int_0^{\frac{\pi}{4}} \tan x\, dx = 2\left[\ln|\sec x|\right]_0^{\frac{\pi}{4}}$

$= 2\ln\sqrt{2} = \ln 2$

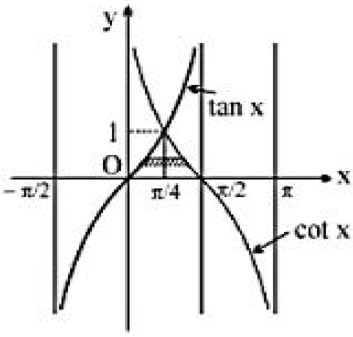
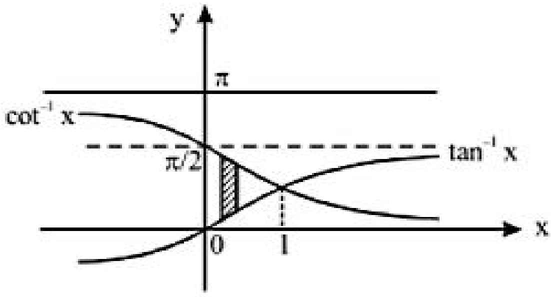


Illustration :

Compute the area enclosed between $y = \tan^{-1}x$; $y = \cot^{-1}x$ and y-axis.



Sol. $A = \int_0^1 (\cot^{-1}x - \tan^{-1}x)dx$

$$A = \int_0^{\frac{\pi}{4}} (\tan y)\,dy + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (\cot y)\,dy = ln2$$

Illustration :

Area enclosed by $y = 9 - x^2$ and coordinates axes.

Sol. $A = \int_0^3 (9 - x^2)dx = 18$

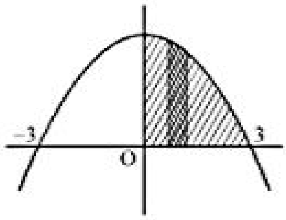


Illustration :

Compute the larger area bounded by $y = 4 + 3x - x^2$ and the coordinates axes.

Sol. $A = \int_0^4 y\,dx = \int_0^4 (4 + 3x - x^2)dx$

$$= \left[4x + \frac{3}{2}x^2 - \frac{1}{3}x^3\right]_0^4 = \frac{56}{3}$$

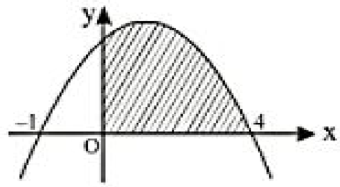


Illustration :

Find the area bounded by $y = \sin^{-1}x$, $y = \cos^{-1}x$ and x-axis.

Sol. *$y = \sin^{-1}x$, $y = \cos^{-1}x$ and the x-axis if vertical stripe is used, we get*

$$A = \int_0^{1/\sqrt{2}} \sin^{-1}x\,dx + \int_{1/\sqrt{2}}^{1} \cos^{-1}x\,dx$$

If horizontal strip is used, then

$$A = \int_0^{\pi/4} (\cos y - \sin y)dy$$

$$= [\sin y + \cos y]_0^{\pi/4}$$

$$= \left[\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} - 1\right] = \sqrt{2} - 1$$

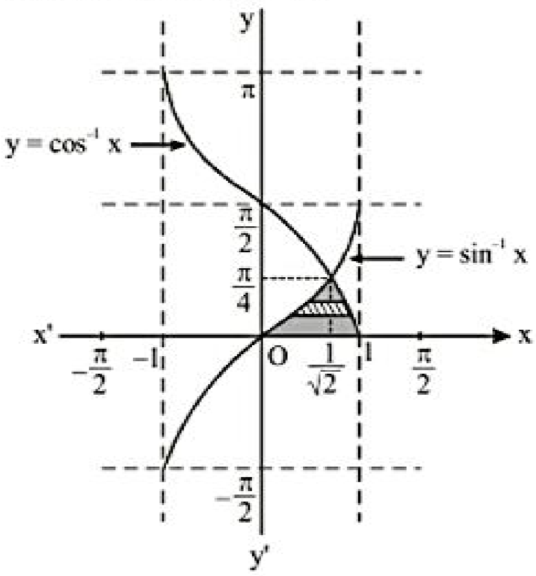
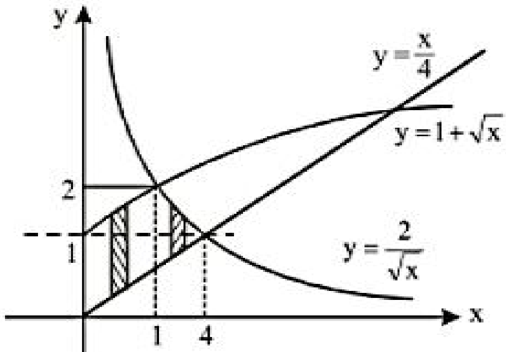


Illustration :

Find the area of the region in the 1st quadrant bounded on the left by the y-axis, below by the line $y = \frac{x}{4}$, *above left by the curve* $y = 1 + \sqrt{x}$ *and above right by the curve* $y = \frac{2}{\sqrt{x}}$

Sol. *Required area* $= \int_0^1 \left(1 + \sqrt{x} - \frac{x}{4}\right) dx + \int_1^4 \left(\frac{2}{\sqrt{x}} - \frac{x}{4}\right) dx$



$$= \left(x + \frac{2}{3}x^{3/2} - \frac{x^2}{8}\right)_0^1 + \left(4\sqrt{x} - \frac{x^2}{8}\right)_1^4$$

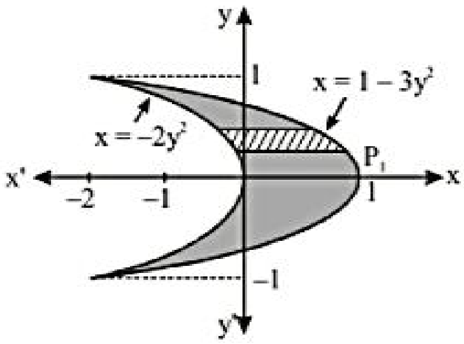
$$= \left(1 + \frac{2}{3} - \frac{1}{8}\right) + \left(4 - \frac{15}{8}\right) = \frac{5}{3} + 2 = \frac{11}{3} \text{ sq. units.}$$

Illustration :

Find the area of the figure bounded by the parabolas $x = -2y^2$, $x = 1 - 3y^2$.

Sol. *Solving the equation* $x = -2y^2$, $x = 1 - 3y^2$, *we find that ordinates of the point of intersection of the two curves as* $y_1 = -1$, $y_2 = 1$. *The points are* $(-2, -1)$ *and* $(-2, 1)$.
The required area (using horizontal strip)

$$A = 2\int_0^1 (x_1 - x_2)\, dy$$



$$= 2\int_0^1 \left[\left(1 - 3y^2\right) - \left(-2y^2\right)\right] dy$$

$$= 2\int_0^1 \left(1 - y^2\right) dy = 2\left[y - \frac{y^3}{3}\right]_0^1 = \frac{4}{3}$$

Practice Problem

Q.1 Find the area lying in the first quadrant and bounded by the curve $y = x^3$ and the line $y = 4x$.

Q.2 Find the area enclosed by the curves $x^2 = y$, $y = x + 2$ and x-axis.

Q.3 A curve is given by $y = \begin{cases} \sqrt{(4-x^2)}, & 0 \le x < 1 \\ \sqrt{(3x)}, & 1 \le x \le 3 \end{cases}$. Find the area lying between the curve and x-axis.

Q.4 Find the area of the region bounded by the limits $x = 0$, $x = \frac{\pi}{2}$ and $f(x) = \sin x$, $g(x) = \cos x$.

Q.5 Find the area bounded by the curve $y = \sin^{-1} x$ and the line $x = 0$, $|y| = \frac{\pi}{2}$.

Q.6 Find the area bounded by $y = \tan^{-1} x$, $y = \cot^{-1} x$ and y-axis is first quadrant.

Q.7 Find the area bounded by $y = \log_e x$, $y = -\log_e x$, $y = \log_e(-x)$ and $y = -\log_e(-x)$.

Q.8 Find the equation of the tangent to the parabola $x^2 = 4y$ with gradient unity. Also find the area enclosed by the curve, the tangent line and

(i) the y-axis (ii) the x-axis

Q.9 Pair of tangents are drawn from the point (3, 0) on the parabola $y = x^2$. Find the area enclosed by these tangents and the parabola.

Q.10 Compute the area included between the straight lines, $x - 3y + 5 = 0$; $x + 2y + 5 = 0$ and the circle $x^2 + y^2 = 25$.

Answer key

Q.1	4	Q.2	$\frac{5}{6}$	Q.3	$\frac{1}{6}\left(2\pi - \sqrt{3} + 36\right)$
Q.4	$2\left(\sqrt{2} - 1\right)$	Q.5	2	Q.6	$\log\sqrt{2}$
Q.7	4	Q.8	$x - y = 1$; P (2, 1) ; (i) 2/3; (ii) 1/6		
Q.9	18 sq. units	Q.10	$\frac{5}{4}$ $(5\pi + 14)$ sq. Units		

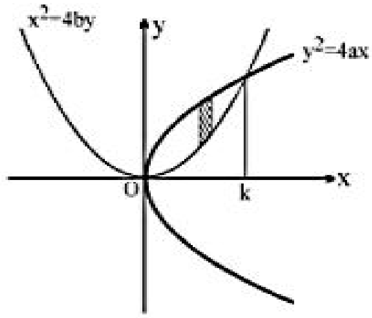
STANDARD AREAS TO BE REMEMBERED :

(1) Area bounded by the curve $y^2 = 4ax$; $x^2 = 4by$ is equal to $\frac{16\,ab}{3}$:

At point of intersection

$$\left(\frac{x^2}{4b}\right)^2 = 4ax \quad \Rightarrow \quad x^4 = 64\,ab^2\,x$$

$$\Rightarrow \quad x = 0,\ (64\,ab^2)^{1/3}$$



Let $\quad k = 4\,(ab^2)^{1/3}$

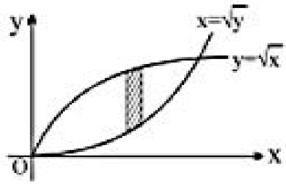
$$A = \int_0^k \left(2\sqrt{a}\sqrt{x} - \frac{x^2}{4b}\right)dx$$

$$= \left[2\sqrt{a}\,\frac{x^{\frac{3}{2}}}{\frac{3}{2}} - \frac{x^3}{12b}\right]_0^k = \frac{4\sqrt{a}}{3}\,k^{\frac{3}{2}} - \frac{k^3}{12b} = \frac{4}{3}\sqrt{a}\,8\left(ab^2\right)^{\frac{1}{2}} - \frac{64(ab^2)}{12b}$$

$$= \frac{32}{3}ab - \frac{16}{3}ab = \frac{16ab}{3}$$

Illustration :

Find the area bounded by the curve $y = \sqrt{x}$; $x = \sqrt{y}$



Sol. $a = \frac{1}{4}$; $b = \frac{1}{4}$

Required area $= \frac{16ab}{3} = \frac{16\cdot\frac{1}{4}\cdot\frac{1}{4}}{3}$

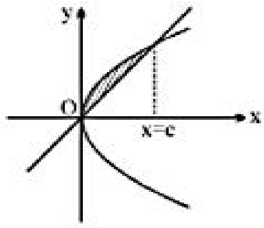
Area $= \frac{1}{3}$

(2) Area bounded by the parabola $y^2 = 4ax$ and $y = mx$ is equal to $\frac{8a^2}{3m^3}$:

$y^2 = 4ax$ and $y = mx$

At point of intersection

$$m^2x^2 = 4ax \quad \Rightarrow \quad x = 0, \ \frac{4a}{m^2}$$



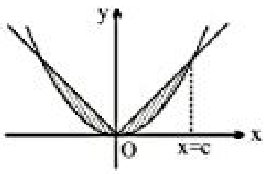
$$\text{Area} = \int_0^c \left(2\sqrt{a}\sqrt{x} - mx\right)dx \quad \text{where} \quad c = \frac{4a}{m^2}$$

$$= \left(2\sqrt{a}\,\frac{x^{\frac{3}{2}}}{\frac{3}{2}} - \frac{mx^2}{2}\right)_0^c = \frac{4\sqrt{a}}{3}c^{\frac{3}{2}} - \frac{mc^2}{2}$$

$$= \frac{4\sqrt{a}}{3}\cdot\frac{8a\sqrt{a}}{m^3} - \frac{m}{2}\cdot\frac{16a^2}{m^4} = \frac{32a^2}{3m^3} - \frac{8a^2}{m^3} = \frac{8a^2}{3m^3}$$

Illustration :

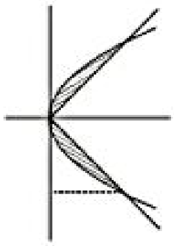
Find the area bounded by the curves $x^2 = y$; $y = |x|$.



Sol. $$Area = 2\left(\frac{8a^2}{3m^3}\right) = 16\frac{\left(\frac{1}{4}\right)^2}{3(1)^3} = \frac{1}{3}$$

Illustration :

Find the curve bounded $y^2 = x$; $x = |y|$.

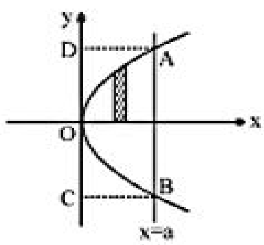


Sol. $$Area = 2\left(\frac{8a^2}{3m^3}\right) = \frac{16}{3}\cdot\frac{\left(\frac{1}{4}\right)^2}{(1)^3} = \frac{1}{3}$$

(3) Area enclosed by $y^2 = 4ax$ and its double ordinate at $x = a$:

(chord perpendicular to the axis of symmetry)

Required area = OABO



$$= 2\cdot\int_0^a \left(2\sqrt{ax}\right)dx = 4\sqrt{a}\left(\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right)_0^a$$

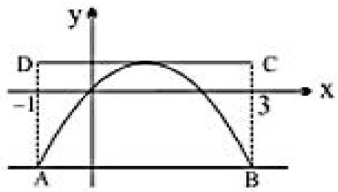
$$= \frac{8}{3}\sqrt{a}\cdot(a\sqrt{a}) = \frac{8a^2}{3}$$

Area of rectangle ABCD = $4a^2$

$$\Rightarrow \boxed{\text{Area of AOB} = \frac{2}{3}(\text{area} \square \text{ABCD})}$$

Illustration :

Find the area bounded by the curve. $y = 2x - x^2$, $y + 3 = 0$

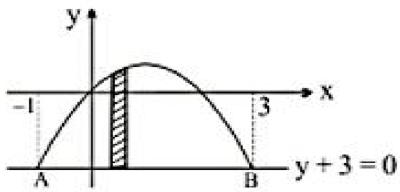


Sol. *For point of intersection of* $y = 2x - x^2$ *and* $y + 3 = 0$

Area ($ABCD) = 4 \times 4 = 16$

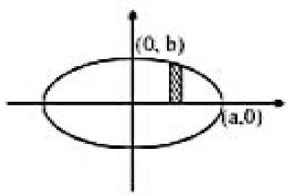
Required area $= \frac{2}{3} \times 16 = \frac{32}{3}$

Alternative method :



By integration $A = \int_{-1}^{3} [(2x - x^2) - (-3)]\, dx = \frac{32}{3}$

(4) Whole area of ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is equal to π ab :



$$A = 4\int_0^a \left(b\sqrt{1 - \frac{x^2}{a^2}}\right)dx$$

Put $\quad x = a\sin\theta$

$$A = 4\int_0^{\pi/2} ab\cos^2\theta\, d\theta = 4ab\int_0^{\pi/2}\cos^2\theta\, d\theta$$

$$= 4ab\int_0^{\pi/2}\left(\frac{1+\cos 2\theta}{2}\right)d\theta = 4ab\left(\frac{\pi}{4}\right) = \pi\, ab$$

Illustration :

Find the area of ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1.$

Sol. *Area of ellipse* $= \pi ab = \pi (4)(3) = 12\pi$

SHIFTING OF ORIGIN :

Since area remains invariant even if the coordinates axes are shifted, hence shifting of origin in many cases proves to be very convenient in computing the areas.

Illustration :

Area enclosed between the parabolas $y^2 - 2y + 4x + 5 = 0$ and $x^2 + 2x - y + 2 = 0$.

Sol. $y^2 - 2y + 1 \Rightarrow (y-1)^2 = -4(x+1)$ ... (1)

$x^2 + 2x + 1 = y - 1 \Rightarrow (x+1)^2 = (y-1)$ ... (2)

Let $y - 1 = Y$ and $x + 1 = X$

So equation $Y^2 = -4X$ and $X^2 = Y$

$$a = 1,\ b = \frac{1}{4}$$

$$\text{so required area} = \frac{16ab}{3} = \frac{16}{3} \cdot 1 \cdot \frac{1}{4} = \frac{4}{3}$$

Illustration :

Area enclosed between the ellipse $9x^2 + 4y^2 - 36x + 8y + 4 = 0$ and the line $3x + 2y - 10 = 0$ in the first quadrant.

Sol. $9x^2 + 4y^2 - 36x + 8y + 4 = 0$

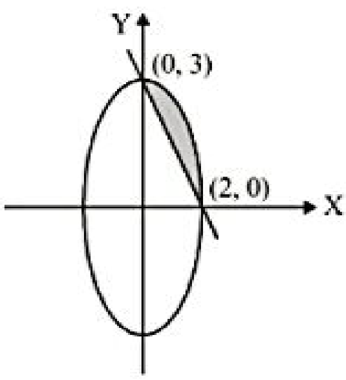
$\Rightarrow 9(x-2)^2 + 4(y+1)^2 = 36$

$$\Rightarrow \frac{(x-2)^2}{2^2} + \frac{(y+1)^2}{3^2} = 1 \qquad \text{... (1)}$$

Let $X = x - 2$ and $Y = y + 1$

So equation of ellipse will be

$$\frac{X^2}{2^2} + \frac{Y^2}{3^2} = 1$$



and equation of line $3x + 2y - 10 = 0$ ... (2)

$3(X + 2) + 2(Y - 1) - 10 = 0$

$3X + 2Y - 6 = 0$

So required area (shaded region)

$$= \frac{\pi ab}{4} - \frac{1}{2}(ab)$$

$$= \frac{\pi}{4}(2)(3) - \frac{1}{2}(2)(3) = \frac{3\pi}{2} - 3 = \frac{3(\pi - 2)}{2}$$

CURVE TRACING :

The approximate shape of a curve, the following procedure in order

(I) SYMMETRY :

(a) Symmetry about x-axis

If the equation of the curve remain unchanged by replacing y by –y then the curve is symmetrical about the x-axis.

e.g., $y^2 = 4ax$.

(b) Symmetry about y-axis

If the equation of the curve remain unchanged by replacing x by –x then the curve is symmetrical about the y-axis.

e.g., $x^2 = 4ay$

(c) Symmetry about both axes

If the equation of the curve remain unchanged by replacing x by –x and y by –y then the curve is symmetrical about the axis of 'x' as well as 'y'.

e.g., $x^2 + y^2 = a^2$

(d) Symmetry about the line y = x

If the equation of curve remains unchanged on interchaning 'x' and 'y', then the curve is symmetrical about the line y = x

e.g., $x^3 + y^3 = 3xy$.

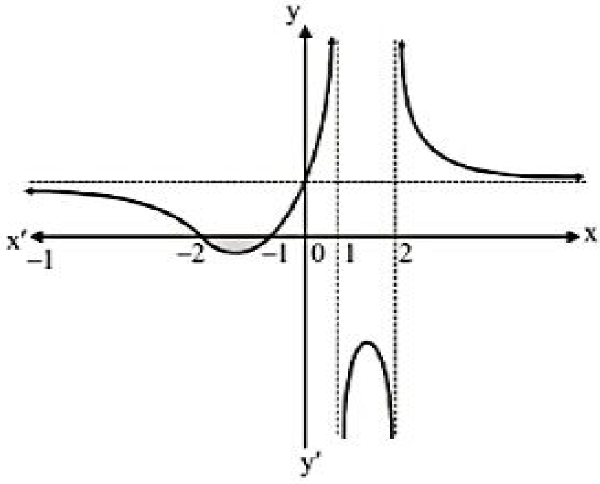
(II) Find the points where the curve crosses the x-axis and the y-axis.

(III) Find $\frac{dy}{dx}$ and examine, if possible, the intervals where f (x) is increasing or decreasing and also its stationary points.

(IV) Examine y when $x \to \infty$ or $x \to -\infty$.

Illustration :

Draw a rough sketch of the curve, $y = \dfrac{x^2+3x+2}{x^2-3x+2}$ *and find the area of the bounded region between the curve and x-axis.*



Sol. $f(x) = \dfrac{(x+1)(x+2)}{(x-1)(x-2)}$

Graph will cut x-axis $x = -1$ *and* $x = -2$.
It is discontinuous at $x = 1$ *and* $x = 2$.

$\lim\limits_{x\to\pm\infty} f(x) \to 1$, $\lim\limits_{x\to 1^-} f(x) \to +\infty$.

$\lim\limits_{x\to 1^+} f(x) \to -\infty$.

$\lim\limits_{x\to 2^-} f(x) \to -\infty$.

$\lim\limits_{x\to 2^+} f(x) \to +\infty$, $f(0) = 1$.

Now we have to find area of the shaded region. The required area

$$= \left|\int_{-2}^{-1} f(x)\,dx\right| = \left|\int_{-2}^{-1}\left(\frac{x^2+3x+2}{x^2-3x+2}\right)dx\right| = \left|\int_{-2}^{-1}\left(1+\frac{6x}{(x-1)(x-2)}\right)dx\right|$$

$$= \left|\left[x\Big|_{-2}^{-1} + 6\int_{-2}^{-1}\left(\frac{2}{x-2}-\frac{1}{x-1}\right)dx\right]\right|$$

$$= |\,1 + 6\,[2\ln|x-2| - \ln|x-1|\,]_{-2}^{-1}|$$

$$= |\,1 + 6\,[2(\ln 3 - \ln 4) - (\ln 2 - \ln 3)]| = |\,1 + 6\,[3\ln 3 - 5\ln 2]|$$

$$= 6\ln\left(\frac{32}{27}\right) - 1 \text{ sq. units}$$

Illustration :

Find the area bounded by the curves $y = -x^2 + 6x - 5$, $y = -x^2 + 4x - 3$ *and the straight line* $y = 3x - 15$.

Sol. *The given curves are*

$y = -x^2 + 6x - 5$ *or* $(x-3)^2 = -(y-4)$ ...(i)

which is a parabola with vertes at A_1 $(3, 4)$ *and axis parallel to negative y-axis. It intersects the x-axis at the point* $P(1, 0)$ *and* $Q(5, 0)$

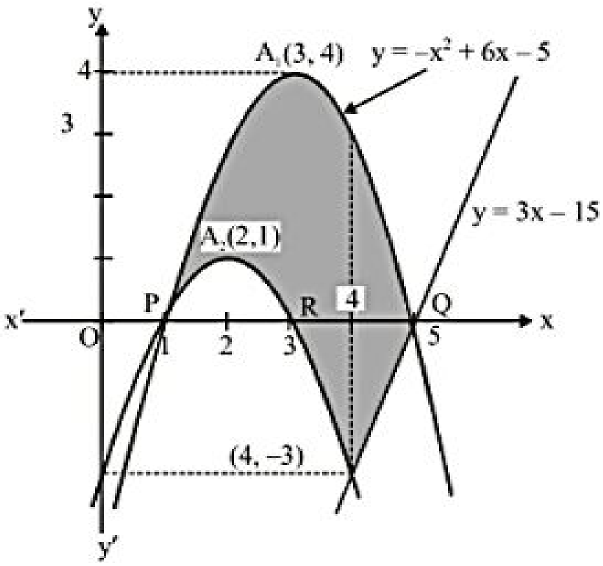
$y = -x^2 + 4x - 3$ or $(x-2)^2 = -(y-1)$ ...(ii)

which is parabola with vertex at $A_2(2, 1)$ and axis parallel to negative y-axis. It intersects the x-axis at the points P(1, 0) and R(3, 0).

And $y = 3x - 15$. ...(iii)

Solving, the points of intersection of (i), (ii) is (1, 0); (i), (iii) are (–2, –21) and (5, 0) and (ii), (iii) are (–3, –24) and (4, –3).

Thus, the required area is the shaded area in the diagram.



$$\text{Required area} = \left|\int_1^4 (y_1 - y_2)dx\right| + \left|\int_4^5 (y_1 - y_3)dx\right|$$

$$= \left|\int_1^4 [(-x^2+6x-5)-(-x^2+4x-3)]dx\right| + \left|\int_4^5 [(-x^2+6x-5)-(3x-15)]dx\right|$$

$$= \left|\int_1^4 (2x-2)dx\right| + \left|\int_4^5 (-x^2+3x+10)dx\right|$$

= 9 + 19/6 = 73/6 sq. units.

Illustration :

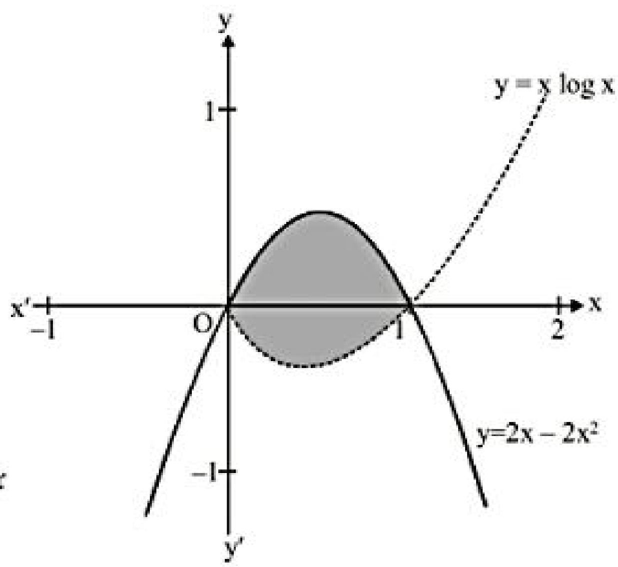
The area of the region enclosed by the curves $y = x \log x$ and $y = 2x - 2x^2$ is

(A) $\frac{7}{12}$ sq. units (B) $\frac{1}{2}$ sq. units (C) $\frac{5}{12}$ sq. units (D) None of these

Sol. *Curve tracing : $y = x \log_e x$*
Clearly, $x > 0$,
For $0 < x < 1$, $x \log_e x < 0$,
and for $x > 1$, $x \log_e x > 0$
Also $x \log_e x = 0 \Rightarrow x = 1$

Further, $\frac{dy}{dx} = 0 \Rightarrow 1 + \log_e x = 0$

$\Rightarrow x = 1/e$, which is a point of minima.



$$\text{Required area} = \int_0^1 (2x - 2x^2)dx - \int_0^1 x \log x \, dx$$

$$= \left[x^2 - \frac{2x^3}{3}\right]_0^1 - \left[\frac{x^2}{2}\log x - \frac{x^2}{4}\right]_0^1$$

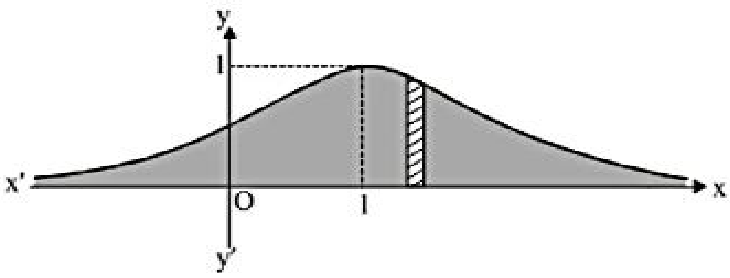
$$= \left(1 - \frac{2}{3}\right) - \left[0 - \frac{1}{4} - \frac{1}{2}\lim_{x\to 0} x^2 \log x\right] = \frac{1}{3} + \frac{1}{4} = \frac{7}{12}$$

Illustration :

Area bounded by $y = \dfrac{1}{x^2 - 2x + 2}$ *and x-axis is*

(A) 2π sq. units (B) $\dfrac{\pi}{2}$ sq. units (C) 2 sq. units (D) π sq. units

Sol. $y = \dfrac{1}{(x-1)^2 + 1}$



$$\text{Area} = 2\int_1^{\infty} \frac{1}{(x-1)^2 + 1}\,dx$$

$$= 2\left[\tan^{-1}(x-1)\right]_1^{\infty} = \pi \text{ sq. units}$$

Illustration :

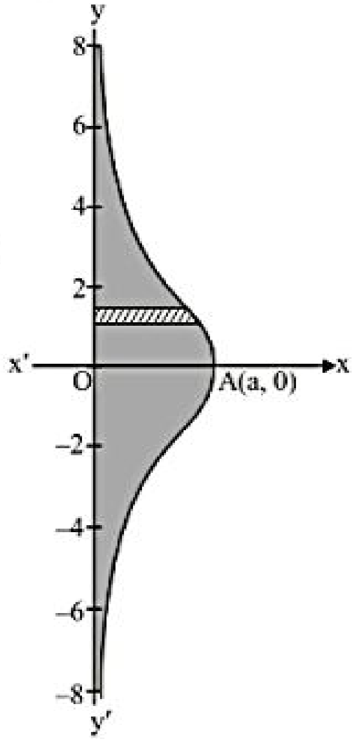
Area bounded by the curve $xy^2 = a^2(a - x)$ *and y-axis is*

(A) $\dfrac{\pi r^2}{2}$ sq. units (B) πa^2 sq. units (C) $3\pi a^2$ sq. units (D) None of these

Sol. $xy^2 = a^2(a - x)$

$$\Rightarrow x = \frac{a^3}{y^2 + a^2}$$

The given curve is symmetrical about x-axis, and meets it at (a, 0). The line x = 0, i.e., y-axis is an asymptote.



$$\text{Area} = 2\int_0^{\infty} x\,dy = 2\int_0^{\infty} \frac{a^3}{y^2 + a^2}\,dx$$

$$= 2a^3 \frac{1}{a}\left[\tan^{-1}\frac{y}{a}\right]_0^{\infty}$$

$$= 2a^2\frac{\pi}{2} = \pi a^2 \text{ sq. units.}$$

Illustration :

The area between the curve $y = 2x^4 - x^2$, *the x-axis and the ordinates of the two minima of the curve is*

(A) $\frac{11}{60}$ *sq. units* (B) $\frac{7}{120}$ *sq. units* (C) $\frac{1}{30}$ *sq. units* (D) $\frac{7}{90}$ *sq. units*

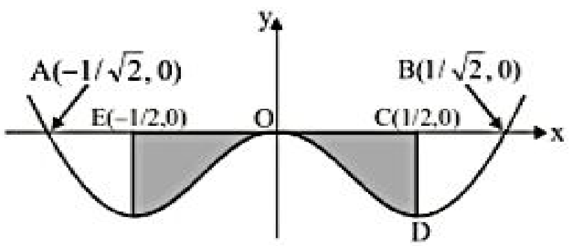
Sol. *The curve is* $y = 2x^4 - x^2 = x^2 (2x^2 - 1)$

The curve is symmetrical about of axis of y.

The curve passes through the origin and the tangent at the origin is $y = 0$, *i.e., x-axis.*

The turning points of the curve are given by

$$\frac{dy}{dx} = 8x^3 - 2x = 0 \Rightarrow 2x(4x^2 - 1) = 0$$



$\Rightarrow \quad x = 0, \pm 1/2$

Now, $\frac{d^2y}{dx^2} = 24x^2 - 2$

Obviously, $\frac{d^2y}{dx^2}$ *is +ve when* $x = \pm\frac{1}{2}$ *and –ve when* $x = 0$

$x = -1/2$ *and* $x = 1/2$

At $x = -1/2$, *min* $y = -1/8$.

The curve intersects the axes at $O(0, 0)$, $A(-1/\sqrt{2}, 0)$ *and* $B(1/\sqrt{2}, 0)$.

Thus, the graph of the curve is known in the figure

Here, $y \le 0$, *as x varies from* $x = -1/2$ *to* $x = 1/2$

$\therefore$ *The required area = 2 Area OCDO*

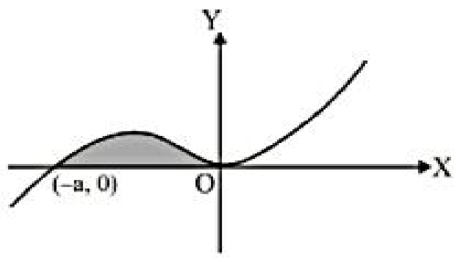
$$= 2\left|\int_0^{1/2} y\,dx\right| = 2\left|\int_0^{1/2} (2x^4 - x^2)\,dx\right| = \frac{7}{120} \text{ sq. units.}$$

Illustration :

The area bounded by the curve $a^2y = x^2 (x + a)$ *and x-axis is*

(A) $\frac{a^2}{3}$ *sq. units* (B) $\frac{a^2}{4}$ *sq. units* (C) $\frac{3a^2}{4}$ *sq. units* (D) $\frac{a^2}{12}$ *sq. units*

Sol. *The curve is* $y = \frac{x^2(x+1)}{a^2}$, *which is a cubic polynomial.*



Since $\frac{x^2(x+a)}{a^2} = 0$ *has repeated root* $x = 0$,

it touches x-axis at (0, 0) and intersects at (–a, 0).

$$\text{Required area} = \int_{-a}^{0} y\,dx = \int_{-a}^{0}\left[\frac{x^2(x+a)}{a^2}\right]dx = \frac{a^2}{12} \text{ sq. units.}$$

Illustration :

The area of the loop of the curve, $ay^2 = x^2(a-x)$ is

(A) $4a^2$ sq. units (B) $\frac{8a^2}{15}$ sq. units (C) $\frac{16a^2}{9}$ sq. units (D) None of these

Sol. $ay^2 = x^2(a-x) \Rightarrow y = \pm x\sqrt{\frac{a-x}{a}}$

Curve tracing : $y = x\sqrt{\frac{a-x}{a}}$

We must have $x \le a$

For $0 < x \le a$, $y > 0$ *and for* $x < 0$, $y < 0$

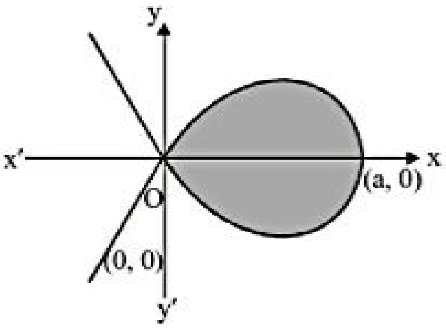
Also $y = 0 \Rightarrow x = 0, a$

Curve is symmetrical about x-axis.

When $x \to -\infty$, $y \to -\infty$

Also, it can be verified that y has only one point of maxima for $0 < x < a$.

$$\text{Area} = 2\int_0^a x\sqrt{\frac{a-x}{a}}\,dx$$



$$\sqrt{\frac{a-x}{a}} = t \Rightarrow 1 - \frac{x}{a} = t^2 \Rightarrow x = a(1-t^2)$$

$$\Rightarrow A = 2\int_1^0 a(1-t^2)t(-2at)\,dt$$

$$= 4a^2\int_0^1 (t^2 - t^4)\,dt = 4a^2\left[\frac{t^3}{3} - \frac{t^5}{5}\right]_0^1 = 4a^2\left[\frac{1}{3} - \frac{1}{5}\right] = \frac{8a^2}{15} \text{ sq. units.}$$

Illustration :

The area of the region enclosed between the curves $x = y^2 - 1$ *and* $x = |y|\sqrt{1-y^2}$ *is*

(A) 1 sq. units (B) $\frac{4}{3}$ sq. units (C) $\frac{2}{3}$ sq. units (D) 2 sq. units

Sol. $A = 2\int_0^1\left[y\sqrt{1-y^2}-(y^2-1)\right]dy$

$= 2$ sq. units

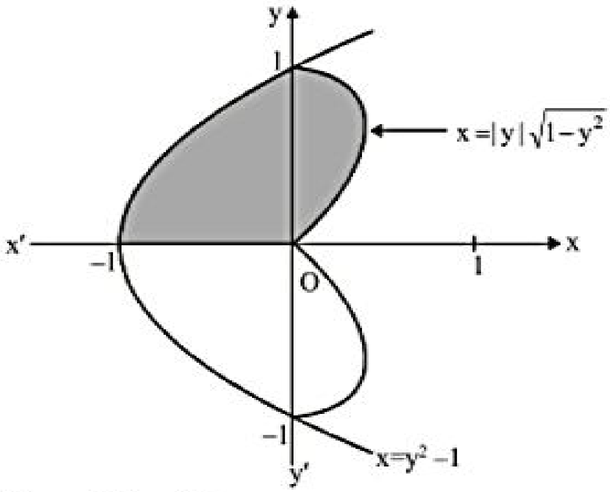


Illustration :

The area bounded by the loop of the curve $4y^2 = x^2(4-x^2)$ is

(A) $\frac{7}{3}$ sq. units (B) $\frac{8}{3}$ sq. units (C) $\frac{11}{3}$ sq. units (D) $\frac{16}{3}$ sq. units

Sol. $4y^2 = x^2(4-x^2)$

$\Rightarrow y = \pm\frac{1}{2}\sqrt{x^2(4-x^2)}$

$\Rightarrow y = \pm\frac{x}{2}\sqrt{(4-x^2)}$

$\therefore$ Area $(A) = 4\int_0^2 \frac{x}{2}\sqrt{(4-x^2)}dx$

Let $4 - x^2 = t \quad \Rightarrow \quad -2x\,dx = dt$

$\Rightarrow \quad A = \frac{-4}{4}\int_4^0 \sqrt{t}\,dt = \int_0^4 \sqrt{t}\,dt = \left[\frac{t^{3/2}}{3/2}\right]_0^4 = \frac{2}{3}\times[\sqrt{64}-0]$

$\Rightarrow \quad A = \frac{16}{3}$ sq. units.

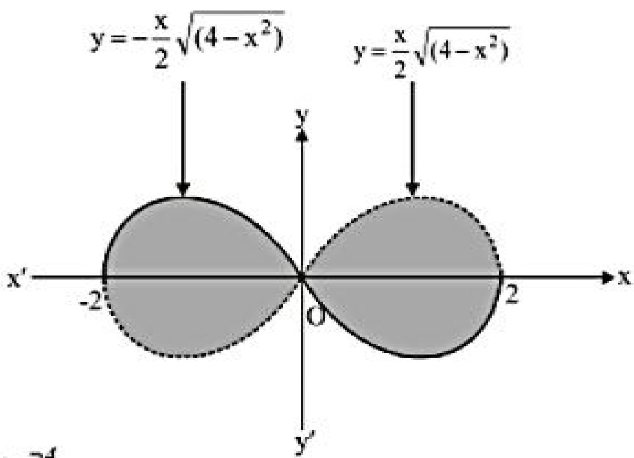


Illustration :

The area enclosed by the curves, $xy^2 = a^2(a-x)$ and $(a-x)y^2 = a^2x$ is

(A) $(\pi-2)a^2$ sq. units (B) $(4-\pi)a^2$ sq. units

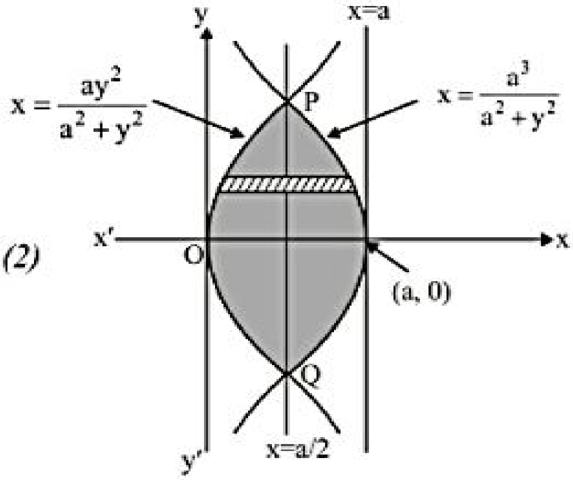
(C) $\pi\frac{a^2}{3}$ sq. units (D) None of these

Sol. The two curves are

$xy^2 = a^2(a-x) \Rightarrow x = \dfrac{a^3}{a^2+y^2}$ ...(1)

and $(a-x)y^2 = a^2x$

$\Rightarrow x = \dfrac{ay^2}{a^2+y^2} = \dfrac{ay^2+a^3-a^3}{a^2+y^2} = a - \dfrac{a^3}{a^2+y^2}$ ...(2)



Curve (1) is symmetrical about x-axis, and have y-axis as the asymptote.

Curve (2) is symmetrical about x-axis, tangent at origin as y-axis and the asymptote $x = a$.

The two curves intersect at the point $P(a/2, a)$ and $Q(a/2, -a)$.

$$\text{Required area} = 2\int_0^a\left(\frac{a^3}{a^2+y^2} - \frac{ay^2}{a^2+y^2}\right)dy = 2a\int_0^a \frac{a^2-y^2}{a^2+y^2}dy = 2a\left[2\int_0^a \frac{a^2}{a^2+y^2}dy - \int_0^a dy\right]$$

$$= 2a\left[2a\tan^{-1}\left(\frac{y}{a}\right)_0^a - a\right] = 2a\left[2a\frac{\pi}{4} - a\right] = a^2(\pi - 2)$$

Illustration :

The area bounded by the curves $y = xe^x$, $y = xe^{-x}$ and the line $x = 1$ is

(A) $\dfrac{2}{e}$ sq. units (B) $1 - \dfrac{2}{e}$ sq. units (C) $\dfrac{1}{e}$ sq. units (D) $1 - \dfrac{1}{e}$ sq. units

Sol. Curve tracing : $y = x\,e^x$

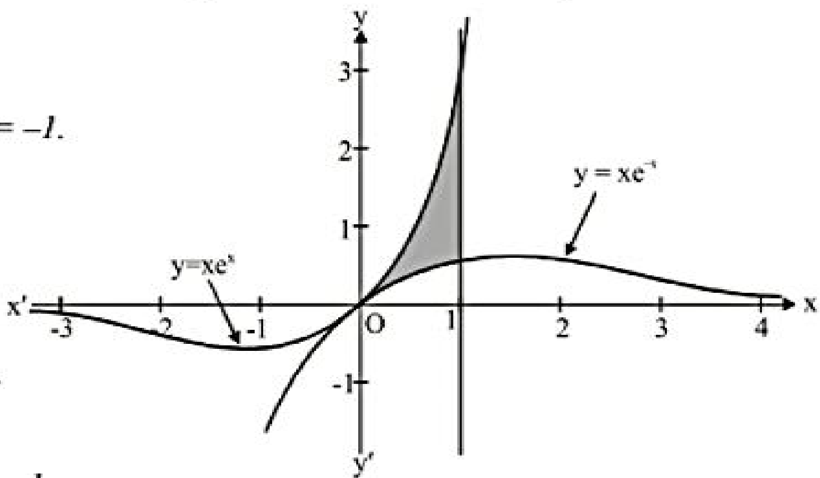
Let $\dfrac{dy}{dx} = 0 \Rightarrow e^x + xe^x = 0 \Rightarrow x = -1$.

Also, at $x = -1$,

$\dfrac{dy}{dx}$ changes sign from –ve to +ve

hence, $x = -1$ is a point of minima.

When $x \to \infty$, $y \to \infty$



Also $\lim\limits_{x\to-\infty} xe^x = \lim\limits_{x\to-\infty} \dfrac{x}{e^{-x}} = \lim\limits_{x\to-\infty} \dfrac{1}{e^{-x}} = 0$

With similary types of arguments, we can draw the graph of $y = x\,e^{-x}$.

$$Required\ area = \int_0^1 xe^x\,dx - \int_0^1 xe^{-x}\,dx \;= \left[xe^x\right]_0^1 - \int_0^1 e^x\,dx - \left(\left[-xe^{-x}\right]_0^1 + \int_0^1 e^{-x}\,dx\right)$$

$$= e - (e-1) - \left(-e^{-1} - \left(e^{-1} - 1\right)\right) = \frac{2}{e}\ sq.\ units$$

Illustration :

The area bounded by the two branches of curve $(y-x)^2 = x^3$ and the straight line $x = 1$ is

(A) $\frac{1}{5}$ sq. units (B) $\frac{3}{5}$ sq. units (C) $\frac{4}{5}$ sq. units (D) $\frac{8}{4}$ sq. units

Sol. $(y-x)^2 = x^3$, where $x \geq 0$

$\Rightarrow y - x = \pm x^{3/2}$

$\Rightarrow y = x + x^{3/2}$ ...(1)

$y = x - x^{3/2}$ ...(2)

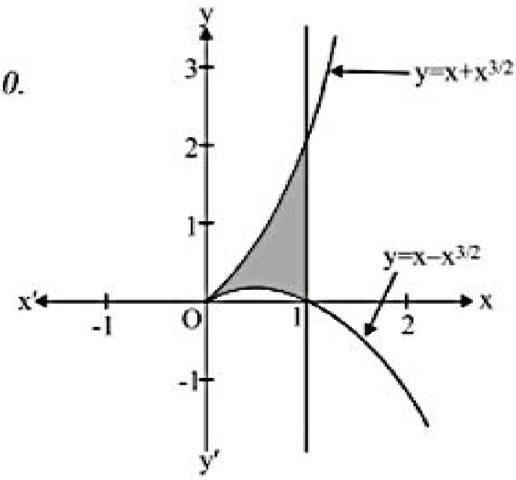
Function (1) is an increasing function.

Function (2) meets x-axis, where $x - x^{3/2} = 0$ or $x = 0, 1$.

Also, for $0 < x < 1$, $x - x^{3/2} > 0$ and for $x > 1$, $x - x^{3/2} < 0$.

When $x \to \infty$, $y \to -\infty$,

From these information, we can plot the graph as below:



$$Required\ area = \int_0^1 \left[\left(x + x^{3/2}\right) - \left(x - x^{3/2}\right)\right]dx$$

$$= 2\int_0^1 x^{3/2}\,dx = 2\left[\frac{x^{5/2}}{5/2}\right]_0^1 = \frac{4}{5}\ sq.\ units$$

Illustration :

(a) Sketch and find the area bounded by the curve $\sqrt{|x|}+\sqrt{|y|} = \sqrt{a}$ and $x^2+y^2=a^2$ (where $a > 0$).

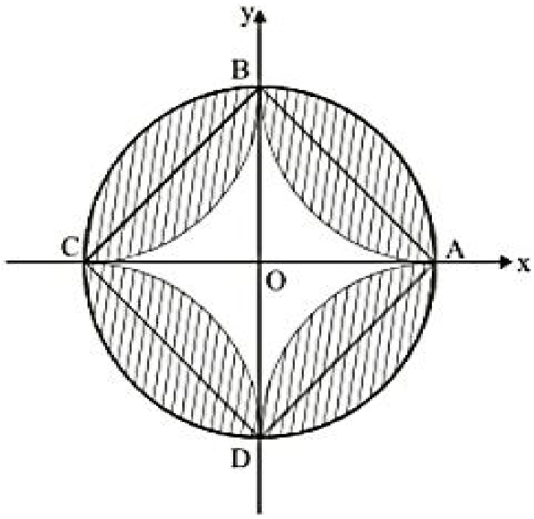
(b) If curve $|x| + |y| = a$ divides the area in two parts, then find their ratio in first quadrant only.

Sol. $\sqrt{|x|}+\sqrt{|y|}=\sqrt{a}$

$x = 0 \Rightarrow \; y = \pm a$

$y = 0 \Rightarrow \; x = \pm a$

(a) Required area



$$4\int_0^a \sqrt{a^2-x^2}\,dx - 4\int_0^a (\sqrt{a}-\sqrt{x})^2 dx$$

$$= \pi a^2 - 4\int_0^a (\sqrt{a}-\sqrt{x})^2\,dx$$

$$= \pi a^2 - 4\int_0^a [a+x-2\sqrt{a}\sqrt{x}]\,dx \quad = \pi a^2 - 4\left[a^2+\frac{a^2}{2}-2\sqrt{a}\,\frac{2}{3}a^{3/2}\right]$$

$$= \pi a^2 - \left[\frac{3a^2}{2}-\frac{4}{3}a^2\right] = \pi a^2 - 4\frac{a^2}{6} \quad = \left(\pi-\frac{2}{3}\right)a^2 \text{ sq. units.}$$

(b) Area included between curves and circle in Ist quadrant $= \dfrac{1}{4}\pi a^2 - \dfrac{1}{2}a^2 = \dfrac{(\pi-2)a^2}{4}$

Area included between $|x| + |y| = a$ and curve $\sqrt{|x|} + \sqrt{|y|} = \sqrt{a}$ in Ist quadrant

$$\frac{1}{4}\left(\pi-\frac{2}{3}\right)a^2 - \left(\frac{\pi}{4}-\frac{1}{2}\right)a^2 = \frac{a^2}{3}$$

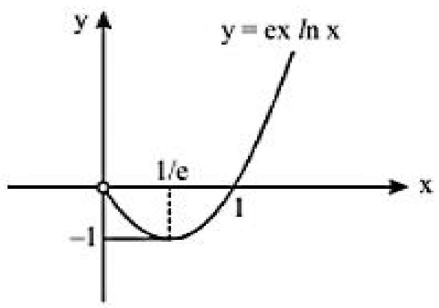
$$\text{Area ratio} = \frac{4}{3(\pi-2)}$$

Illustration :

Area enclosed between the curves $y = ex \cdot \ln x$ *and* $y = \frac{\ln x}{ex}$

Sol. $y = ex \ln x$

$$\frac{dy}{dx} = e(1 + \ln x) \quad \therefore \quad \frac{dy}{dx} = 0 \quad \Rightarrow \quad x = e^{-1}$$

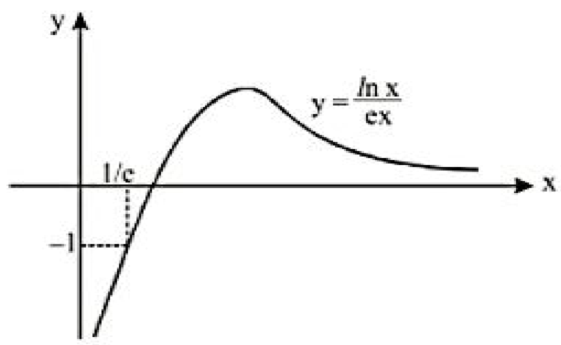


$$\frac{d^2y}{dx^2} = \frac{e}{x} \quad \therefore \quad \left.\frac{d^2y}{dx^2}\right|_{x=\frac{1}{e}} > 0$$

$$\Rightarrow \quad \text{minimum at } x = \frac{1}{e}$$

$$y = \frac{\ln x}{ex}$$

$$\frac{dy}{dx} = \frac{1}{e}\left(\frac{1 - \ln x}{x^2}\right) \quad \therefore \quad \frac{dy}{dx} = 0 \quad \text{when} \quad x = e$$



$$\left.\frac{dy}{dx}\right|_{x=e^+} < 0, \ \left.\frac{dy}{dx}\right|_{x=e^-} > 0$$

$$\Rightarrow \quad \text{at } x = e, \quad y = \frac{\ln x}{ex} \text{ has local maxima}$$

$$\text{Required area} = \int_{1/e}^{1}\left(\frac{\ln x}{ex} - ex\ln x\right)dx$$

$$= \left[\frac{(\ln x)^2}{2e} - e\left(\frac{x^2}{2}\ln x - \frac{x^2}{4}\right)\right]_{\frac{1}{e}}^{1} = \frac{e}{4} - \left[\frac{1}{2e - e} - e\left(\frac{-1}{2e^2} - \frac{1}{4e^2}\right)\right]$$

$$= \frac{e}{4} - \left[\frac{1}{2e} + \frac{3}{4e}\right] = \frac{e}{4} - \frac{5}{4e} = \left(\frac{e^2 - 20}{4e}\right)$$

Illustration :

Area enclosed by the curve $(y - \sin^{-1}x)^2 = x - x^2$.

Sol. $(y - \sin^{-1} x)^2 = x - x^2$

$y = \sin^{-1} x \pm \sqrt{x - x^2} \Rightarrow$ domain $x \in [0, 1]$

Area enclosed by the curve

$$= \int_0^1 \left(\sin^{-1} + \sqrt{x - x^2}\right) - \left(\sin^{-1} x - \sqrt{x - x^2}\right) dx = 2\int_0^1 \sqrt{x - x^2}\, dx$$

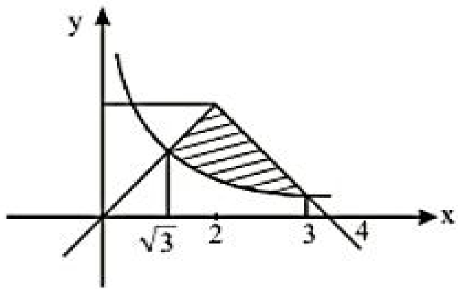
$$= 2\int_0^1 \sqrt{\frac{1}{4} - \left(x - \frac{1}{2}\right)^2} \quad = 2\left[\frac{1}{2}\left(x - \frac{1}{2}\right)\sqrt{x - x^2} + \frac{1}{2}\left(\frac{1}{4}\right)\sin^{-1}\left(\frac{x - \frac{1}{2}}{\frac{1}{2}}\right)\right]_0^1$$

$$= 2\left[\left(0 + \frac{1}{8}\frac{\pi}{2}\right) - \left(0 + \frac{1}{8}\left(-\frac{\pi}{2}\right)\right)\right] = 2\left(\frac{\pi}{16} + \frac{\pi}{16}\right) = \frac{\pi}{4}$$

Illustration :

Area of the closed figure bounded by the curves $y = 2 - |2 - x|$ and $y = \dfrac{3}{|x|}$

Sol. $y_1 = 2 - |2 - x| = \begin{cases} x; & x \le 2 \\ 4 - x; & x > 2 \end{cases}$



and $\quad y_2 = \begin{cases} \dfrac{3}{x}; & x > 0 \\ -\dfrac{3}{x}; & x < 0 \end{cases}$

so $\quad \dfrac{3}{x} = x \quad \Rightarrow \quad x = \sqrt{3}$

and $\quad \dfrac{3}{x} = 4 - x \quad \Rightarrow \quad 3 = 4x - x^2 \quad \Rightarrow \quad x^2 - 4x + 4 = 1$

$\Rightarrow \quad x - 2 = \pm 1 \quad \Rightarrow \quad x = 3$

so required area $= \displaystyle\int_{\sqrt{3}}^{2} \left(x - \frac{3}{x}\right) dx + \int_2^3 \left(4 - x - \frac{3}{x}\right) dx$

$$= \left[\frac{x^2}{2} - 3\ln x\right]_{\sqrt{3}}^{2} + \left[4x - \frac{x^2}{2} - 3\ln x\right]_2^3$$

$$= 2 - 3\ln 2 - \frac{3}{2} + 3\ln\left(\sqrt{3}\right) + \left(12 - \frac{9}{2} - 3\ln 3\right) - \left(8 - \frac{4}{2} - 3\ln 2\right)$$

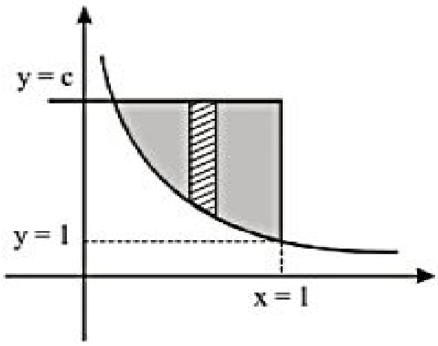
$$= 2 - 3\ ln\ 2 - \frac{3}{2} + \frac{3}{2}\ ln\ 3 + 4 - \frac{5}{2} - 3\ ln\ 3 + 3\ ln\ 2$$

$$= \frac{1}{2} + \frac{3}{2} - \frac{3}{2} ln 3 = \frac{4 - 3 ln 3}{2}$$

DETERMINATION OF PARAMETERS :

Illustration :

Find the value of c for which the area of the figure bounded by the curves $y = \frac{4}{x^2}$; $x = 1$ *and* $y = c$ *is equal to* $\frac{9}{4}$.



Sol. *So required area =* $\int_{2/\sqrt{c}}^{1} \left(c - \frac{4}{x^2}\right) dx = \left[cx + \frac{4}{x}\right]_{2/\sqrt{c}}^{1}$

$$Area = c\left(1 - \frac{2}{\sqrt{c}}\right) + 4 - 2\sqrt{c} = c - 4\sqrt{c} + 4 = \frac{9}{4}$$

$$\Rightarrow \quad (\sqrt{c} - 2)^2 = \frac{9}{4} \quad \Rightarrow \quad \sqrt{c} = 2 \pm \frac{3}{2}$$

$$\sqrt{c} = \frac{1}{2}, \frac{7}{2}$$

$$c = \frac{1}{4}, \frac{49}{4}$$

Illustration :

If the area bounded by $y = x^2 + 2x - 3$ *and the line* $y = kx + 1$ *is the least, find k and also the least area.*

Sol. x_1 *and* x_2 *are the roots of the equation*

$x^2 + 2x - 3 = kx + 1$, *or*

$x^2 + (2 - k)x - 4 = 0$

$$\Rightarrow \quad \left.\begin{matrix} x_1 + x_2 = k - 2 \\ x_1 x_2 = -4 \end{matrix}\right\}$$

$$A = \int_{x_1}^{x_2} \left[(kx + 1) - (x^2 + 2x - 3)\right] dx$$

$$= \left[(k-2)\frac{x^2}{2} - \frac{x^3}{2} + 4x\right]_{x_1}^{x_2} = \left[(k-2)\frac{x_2^2 - x_1^2}{2} - \frac{1}{3}(x_2^3 - x_1^3) + 4(x_2 - x_1)\right]$$

$$= (x_2 - x_1)\left[\frac{(k-2)^2}{2} - \frac{1}{3}\left((x_2+x_1)^2 - x_1x_2\right) + 4\right]$$

$$= \sqrt{(x_2+x_1)^2 - 4x_1x_2}\left[\frac{(k-2)^2}{2} - \frac{1}{3}\left((k-2)^2+4\right)+4\right]$$

$$= \frac{\sqrt{(k-2)^2+16}}{6}\left[\frac{1}{6}(k-2)^2 + \frac{8}{3}\right]$$

$$= \frac{\left[(k-2)^2+16\right]^{3/2}}{6}$$

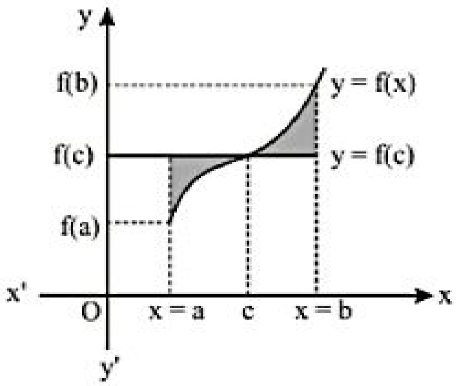
which is least when $k = 2$ and $A_{least} = 32/3$ sq. units.

VARIABLE AREA GREATEST AND LEAST VALUE :

An important concept :

If $y = f(x)$ is a monotonic function in (a, b) then the area bounded by the ordinates at $x = a$, $x = b$, $y = f(x)$ and $y = f(c)$, [where $c \in (a, b)$] is minimum when $c = \dfrac{a+b}{2}$.

Proof: $A = \displaystyle\int_a^c (f(c) - f(x))\,dx + \int_c^b (f(x) - f(c))\,dx$



$$= f(c)\,(c-a) - \int_a^c (f(x))\,dx + \int_c^b (f(x))\,dx - f(c)\,(b-c)$$

$$A = [2c - (a+b)]\, f(c) + \int_c^b (f(x))\,dx - \int_a^c (f(x))\,dx$$

Differetiating w.r.t. c,

$$\frac{dA}{dc} = \left[2c - (a+b)\right] f'(c) + 2\,f(c) + 0 - f(c) - (f(c))$$

for maxima and minima $\dfrac{dA}{dc} = 0$

$\Rightarrow \quad f'(x)[2c - (a+b)] = 0$ (as $f'(c) \neq 0$)

hence $c = \frac{a+b}{2}$

Also $c < \frac{a+b}{2}, \frac{dA}{dc} < 0$ and $c > \frac{a+b}{2}, \frac{dA}{dc} > 0$.

Hence A is minimum when $c = \frac{a+b}{2}$.

Note : Let f(x) be the bijective functon and g(x) be the inverse of it then area bounded by y = g(x), and the ordinate at x = a and x = b is same as area bounded by y = f(x) and the abscissa at y = a and y = b as f(x) and g(x) are mirror image with respect to line y = x.

Illustration :

If the area bounded by $f(x) = \frac{x^3}{3} - x^2 + a$ *and the straight lines* $x = 0\,;\, x = 2$ *and the x-axis is minimum then find the value of 'a'.*

Sol. $f(x) = \frac{x^3}{3} - x^2 + a$

$f'(x) = x^2 - 2x = x(x-2) < 0$ *(note that f(x) is monotonic in (0, 2))*

Hence for the minimum and f(x) must cross the x-axis at $\frac{0+2}{2} = 1$

Hence $f(1) = \frac{1}{3} - 1 + a = 0$

$\Rightarrow a = \frac{2}{3}$

Illustration :

The value of the parameter a for which the area of the figure bounded by the abscissa axis, the graph of the function $y = x^3 + 3x^2 + x + a$ *and the straight lines, which are parallel to the axis of ordinates and cut the abscissa axis at the point of extremum of the function, is the least, is*

(A) 2 *(B) 0* *(C) – 1* *(D) 1*

Sol. $f(x) = x^3 + 3x^2 + x + a$

$f'(x) = 3x^2 + 6x + 1 = 0$

$\Rightarrow \quad x = -1 \pm \frac{\sqrt{6}}{3}$

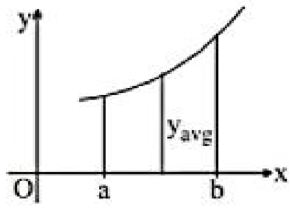
Hence, f(x) cuts the x-axis at $\frac{1}{2}\left[\left(-1+\frac{\sqrt{6}}{3}\right)+\left(-1-\frac{\sqrt{6}}{3}\right)\right]=-1.$

$f(-1) = -1 + 3 - 1 + a = 0$
$a = -1.$

AVERAGE VALUE OF A FUNCTION :

Average value of the function in y = f (x) w.r.t. x over an interval $a \le x \le b$ is defined as

$$y_{avg} = \frac{1}{b-a}\int_a^b f(x)\,dx$$



Note :

(i) Average value can be + ve, – ve or zero.

(ii) If the function is defined in $(0, \infty)$ then

$$y_{avg} = \lim_{b\to\infty} \frac{1}{b}\int_0^b f(x)\,dx \qquad \text{provided the limit exists.}$$

Root mean square value **(RMS)** is defined as

$$\rho = \left[\frac{1}{b-a}\int_a^b f^2(x)\,dx\right]^{\frac{1}{2}}$$

Illustration :

Compute the average value of $f(x) = \frac{\cos^2 x}{\sin^2 x + 4\cos^2 x}$ *in* $\left[0, \frac{\pi}{2}\right]$

Sol. *Average value of f(x)* $= \frac{\cos^2 x}{\sin^2 x + 4\cos^2 x}$

$$y_{average} = \frac{1}{\left(\frac{\pi}{2}-0\right)}\int_0^{\pi/2} \frac{\cos^2 x}{(\sin^2 x + 4\cos^2 x)}dx = \frac{2}{\pi}\int_0^{\pi/2}\frac{1}{(\tan^2 x + 4)}dx$$

$$= \frac{2}{\pi}\int_0^{\pi/2}\frac{sec^2 x}{sec^2 x(tan^2 x+4)}dx = \frac{2}{\pi}\int_0^{\pi/2}\frac{sec^2 x\, dx}{(1+tan^2 x)(4+tan^2 x)}$$

Put $t = tan\, x$ *so* $dt = sec^2 x\, dx$

$$y_{average} = \frac{2}{\pi}\int_0^{\infty}\frac{dt}{(t^2+1)(4+t^2)} = \frac{2}{3\pi}\int_0^{\infty}\left[\frac{1}{t^2+1}-\frac{1}{t^2+4}\right]dt$$

$$= \frac{2}{3\pi}\left[\tan^{-1} t - \frac{1}{2}\tan^{-1}\left(\frac{t}{2}\right)\right]_0^{\infty} = \frac{2}{3\pi}\left[\frac{\pi}{2}-\frac{\pi}{4}\right] = \frac{2}{3\pi}\,\frac{\pi}{4} = \frac{1}{6} \qquad \textbf{Ans.}$$

DETERMINATION OF FUNCTION :

The area function A(x) satisfies the differential equation $\frac{dA(x)}{dx} = f(x)$ with initial condition A(a)= 0 i.e. derivative of the area function is the function itself.

Note :

If F(x) is any integral of f(x) then ,

$A(x) = \int f(x)\,dx = [\,F(x)+c\,]$ $\qquad$ $A(a) = 0 = F(a) + c \Rightarrow c = -F(a)$

hence $A(x) = F(x) - F(a)$. Finally by taking $x = b$ we get , $A(b) = F(b) - F(a)$.

Illustration :

The area from 0 to x under a certain graph is given to be $A = \sqrt{1+3x} - 1,\ x \geq 0$;

(a) *Find the average rate of change of A w.r.t. x as x increases from 1 to 8.*

(b) *Find the instantaneous rate of change of A w.r.t. x at x = 5.*

(c) *Find the ordinate (height) y of the graph as a function of x.*

(d) *Find the average value of the ordinate (height) y, w.r.t. x as x increases from 1 to 8*

Sol. $A = \sqrt{1+3x} - 1 = \int_0^x f(x)dx$

(a) $$\left.\frac{dA}{dx}\right|_{avg} = \frac{1}{(8-1)}\int_1^8\left(\frac{dA}{dx}\right)dx$$

$$= \frac{1}{7}\left(\sqrt{1+3x}-1\right)_1^8 = \frac{1}{7}(4-1) = \frac{3}{7}$$

(b) $\left.\dfrac{dA}{dx}\right|_{x=5} = \dfrac{3}{2\sqrt{1+3x}} = \dfrac{3}{2\sqrt{1+3(5)}} = \dfrac{3}{8}$

(c) $A(x) = \displaystyle\int_0^x f(x)\,dx = \sqrt{1+3x} - 1$

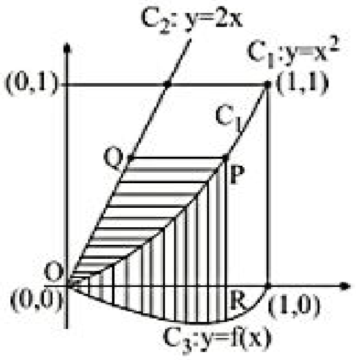
Differentiating w.r.t. x

$f(x) = \dfrac{3}{2\sqrt{1+3x}}$

(d) $y_{avg} = \dfrac{1}{(8-1)} \displaystyle\int_1^8 f(x)\,dx = \dfrac{1}{7}\left(\sqrt{1+3x}-1\right)_1^8 = \dfrac{3}{7}$

Illustration :

Let C_1 & C_2 be the graphs of the functions $y = x^2$ & $y = 2x$, $0 \le x \le 1$ respectively. Let C_3 be the graph of a function $y = f(x)$, $0 \le x \le 1$, $f(0) = 0$. For a point P on C_1, let the lines through P, parallel to the axes, meet C_2 & C_3 at Q & R respectively (see figure). If for every position of P (on C_1), the areas of the shaded regions OPQ & ORP are equal, determine the function f(x). [JEE '98, 8]



Sol. *Let $P(h, h^2)$ be a point on the curve C_1.*

$\Rightarrow$ $R(h, f(h))$

Area OPQO = Area OPRO

$$\int_0^{h^2}\left(\sqrt{y}-\frac{y}{2}\right)dy = \int_0^h\left(x^2 - f(x)\right)dx$$

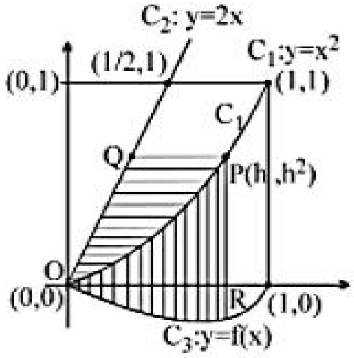
Differenting w.r.t. h

$$\left(\sqrt{h^2}-\frac{h^2}{2}\right).2h = h^2 - f(h)$$

$\Rightarrow$ $2h^2 - h^3 = h^2 - f(h)$

$\Rightarrow$ $f(h) = h^3 - h^2$

$\Rightarrow$ $f(x) = x^3 - x^2$

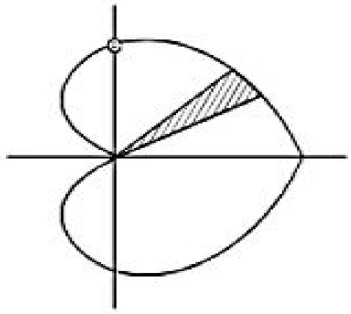


AREA ENCLOSED IN CASE ONE CURVE ARE EXPRESSED IN POLAR FORM :

$$\text{Area of any curve} = \frac{1}{2}\int r^2 d\theta$$

Illustration :

Find the area of the cardioid $r = a(1 + \cos\theta)$



Sol. $A = \dfrac{1}{2}\displaystyle\int_0^{2\pi} r^2\, d\theta = \dfrac{a^2}{2}\int_0^{2\pi} 4\cos^4\dfrac{\theta}{2}\, d\theta$ put $\dfrac{\theta}{2} = t$

$$A = a^2\int_0^{\pi} 4\cos^4 t\, dt = 8 \times \frac{3\pi a^2}{16} = \left(\frac{3\pi a^2}{2}\right)$$

AREA IN RESPECT OF CURVE REPRESENTED PARAMETRICALLY :

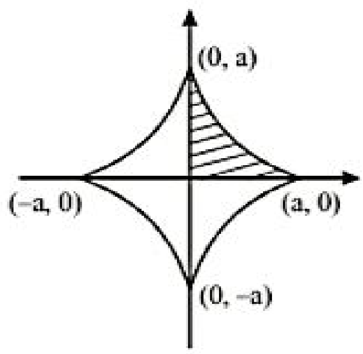
Illustration :

Find the area enclosed by the curves $x = a\sin^3 t$ and $y = a\cos^3 t$

Sol. $x^{2/3} + y^{2/3} = a^{2/3}$

$$\text{Required area} = 4\int_0^a \left(a^{2/3} - x^{2/3}\right)^{3/2} dx$$

Put $x = a\sin^3 t$; $dx = 3a\sin^2 t\cos t\, dt$



$$\text{Area} = 4\int_0^{\pi/2} \left(a^{2/3} - a^{2/3}\sin^2 t\right)^{3/2} 3a\sin^2 t\cos t\, dt$$

$$A = 12a^2\int_0^{\pi/2} \sin^2 t\cos^4 t\, dt \quad \ldots (1)$$

$$A = 12a^2\int_0^{\pi/2} \sin^4 t\cos^2 t\, dt \quad \ldots (2)$$

Adding (i) and (ii)

$$A = \frac{12a^2}{2}\int_0^{\pi/2} \cos^2 t \sin^2 t \ (\cos^2 t + \sin^2 t)\ dt$$

$$= 6a^2 \int_0^{\pi/2} \sin^2 t \cos^2 t\ dt \ = \ 6a^2 \int_0^{\pi/2} \frac{\sin^2 2t}{4} \ = \ \frac{3a^2}{2}\int_0^{\pi/2}\left(\frac{1-\cos 4t}{2}\right)dt$$

$$= \frac{3a^2}{4}\left(t - \frac{\sin 4t}{4}\right)_0^{\pi/2} \ = \ \frac{3a^2}{4}\left(\frac{\pi}{2}\right) = \frac{3\pi a^2}{8}$$

Practice Problem

Q.1 For what value of k is the area of the figure bounded by the curves $y = x^2 - 3$ and $y = kx + 2$ is the least. Determine the least area.

Q.2 Find the area enclosed by the parabola $(y-2)^2 = x - 1$ and the tangent to it at (2, 3) & x-axis.

Q.3 Area enclosed between the smaller arc of the circle $x^2 + y^2 - 2x + 4y - 11 = 0$ and the parabola $y = -x^2 + 2x + 1 - 2\sqrt{3}$.

Q.4 Find the area of the figure bounded by the parabola $y = ax^2 + 12x - 14$ and the straight line $y = 9x - 32$ if the tangent drawn to the parabola at the point $x = 3$ is known to make an angle $\pi - \tan^{-1}6$ with the x-axis.

Q.5 Find the area bounded by the curve g (x), the x-axis and the ordinate at $x = -1$ and $x = 4$ where g (x) is the inverse of the function $f(x) = \frac{x^3}{24} + \frac{x^2}{8} + \frac{13x}{12} + 1$

Q.6 $f(x) = x^3 + 3x + 2$ and g (x) is the inverse of it. Then compute the area bounded by g(x), x-axis and the ordinate at $x = -2$ and $x = 6$.

Q.7 Find the value of the parameter 'a' for which the area of the figure bounded by the abscissa axis, the graph of the function $y = x^3 + 3x^2 + x + a$, and the straight lines, which are parallel to the axis of ordinates and cut the abscissa axis at the point of extremum of the function, is the least.

Q.8 Find the average value of y^2 w.r.t. x for the curve $ay = b\sqrt{a^2 - x^2}$ between $x = 0$ & $x = a$. Also find the average value of y w.r.t. x^2 for $0 \le x \le a$.

Answer key

Q.1	$k = 0, A = \frac{20\sqrt{5}}{3}$	Q.2	0009	Q.3	$4\left(\frac{8 - 3\sqrt{3} + 2\pi}{3}\right)$
Q.4	$\frac{125}{2}$	Q.5	$\frac{16}{3}$	Q.6	$\frac{9}{2}$
Q.7	$a = -1$	Q.8	(i) $a = \frac{2b^2}{3}$, (ii) $b = \frac{2b}{3}$		

DIFFERENTIAL EQUATION

1. INTRODUCTION :

1.0 An equation that involves independent and dependent variables and at least one derivative of the dependent variable w.r.t independent variable is called a differential equation.

For example : $x\frac{dy}{dx} + y\log x = xe^x\ x^{\frac{1}{2}\log x}, (x > 0);\ \frac{d^2y}{dx^2} = \left[y + \left(\frac{dy}{dx}\right)^6\right]^{1/4}$

1.1 A differential equation is said to be **ordinary**, if the differential coefficients have reference to a single independent variable only and it is said to be **Partial** if there are two or more independent variables. We are concerned with ordinary differential equations only. While an ordinary differential equation containing two or more dependent variables with their differential coefficients w.r.t. to a single independent variable is called a **total differential equation.**

eg. $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 0$ is an ordinary diffrential equation

$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0\ ;\ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = x^2 + y$ are partial differential equation.

1.2 Order and Degree of Diffrential Equation :

The order of a differential equation is the order of the highest differential coefficient occuring in it. The degree of a differential equation which is expressed or can be expressed as a polynomial in the derivatives is the degree of the highest order derivative occuring in it, after it has been expressed in a form free from radicals & fractions so far as derivatives are concerned, thus the differential equation :

$$f(x, y)\left[\frac{d^m y}{dx^m}\right]^p + \phi(x, y)\left[\frac{d^{m-1}(y)}{dx^{m-1}}\right]^q + \ldots\ldots = 0 \text{ is order } m \ \& \text{ degree } p.$$

Illustration :

(i) $\dfrac{d^2y}{dx^2} = \left[y + \left(\dfrac{dy}{dx}\right)^6\right]^{1/4}$ (ii) $\dfrac{dy}{dx} + y = \dfrac{1}{dy/dx}$ (iii) $e^{\frac{d^3y}{dx^3}} - x\dfrac{d^2y}{dx^2} + y = 0$

(iv) $\sin^{-1}\left(\dfrac{dy}{dx}\right) = x + y$ (v) $ln\left(\dfrac{dy}{dx}\right) = ax + by$

Sol.

(i) $\dfrac{d^2y}{dx^2} = \left[y + \left(\dfrac{dy}{dx}\right)^6\right]^{1/4} \Rightarrow \left(\dfrac{d^2y}{dx^2}\right)^4 = \left[y + \left(\dfrac{dy}{dx}\right)^6\right]$

Hence order is 2 and degree is 4.

(ii) $\dfrac{dy}{dx} + y = \dfrac{1}{dy/dx} \Rightarrow \left(\dfrac{dy}{dx}\right)^2 + y\left(\dfrac{dy}{dx}\right) = 1$

Hence order is 1 and degree is 2.

(iii) $e^{\frac{d^3y}{dx^3}} - x\dfrac{d^2y}{dx^2} + y = 0$

Clearly order is 3, but degree is not defined as it cannot be written as a polynomial equation in derivatives.

(iv) $\sin^{-1}\left(\dfrac{dy}{dx}\right) = x + y \Rightarrow \dfrac{dy}{dx} = \sin(x + y)$

Hence order is 1 and degree is 1.

(v) $ln\left(\dfrac{dy}{dx}\right) = ax + by \Rightarrow \dfrac{dy}{dx} = e^{ax + by}$

Hence order is one and degree is also 1.

Practice Problem

Q.1 Find the order and degree (if defined) of the following differential equations

(i) $\dfrac{d^2y}{dx^2} = \left\{1 + \left(\dfrac{dy}{dx}\right)^4\right\}^{5/3}$ (ii) $\dfrac{d^3y}{dx^3} = x\,ln\left(\dfrac{dy}{dx}\right)$

(iii) $\left(\dfrac{d^4y}{dx^4}\right)^3 + 3\left(\dfrac{d^2y}{dx^2}\right)^6 + \sin x = 2\cos x$ (iv) $\left(\dfrac{d^3y}{dx^3}\right)^{2/3} + 4 - 3\dfrac{d^2y}{dx^2} + 5\dfrac{dy}{dx} = 0$

(v) $\rho = \dfrac{\left[1+\left(\dfrac{dy}{dx}\right)^2\right]^{3/2}}{\dfrac{d^2y}{dx^2}}$ (vi) $\dfrac{d^2y}{dx^2}+\left(\dfrac{dy}{dx}\right)^3+6y=0$

(vii) $y(x)=1+\dfrac{dy}{dx}+\dfrac{1}{1.2}\left(\dfrac{dy}{dx}\right)^2+\dfrac{1}{1.2.3}\left(\dfrac{dy}{dx}\right)^3+\ldots\ldots$ (viii) $\left(1+4\dfrac{dy}{dx}\right)^{2/3}=4\dfrac{d^2y}{dx^2}$

Answer key

Q.1	(i)	order = 2, degree = 3 ;	(ii)	order = 3, degree is not defined
	(iii)	order = 4, degree = 3 ;	(iv)	order = 3, degree = 2
	(v)	order = 2, degree = 2 ;	(vi)	order = 2, degree = 1
	(vii)	order = 1, degree = 1 ;	(viii)	order = 2, degree = 3

2. FORMATION OF DIFFERENTIAL EQUATIONS :

Consider a family of curves

$f(x, y, c_1, c_2, \ldots\ldots c_n) = 0$ ...(i)

where $c_1, c_2, \ldots\ldots, c_n$ are n independent parameters.

Equation (i) is known as an n parameter family of curves e.g. $y = mx$ is 1-parameter family of straight lines $x^2 + y^2 + ax + by = 0$ is a two-parameter family of circles.

If we differentiate equation (i) n times w.r.t x, we will get n more relations between $x, y, c_1, c_2, \ldots\ldots c_n$ and derivates of y with respect to x. By eliminating $c_1, c_2, \ldots\ldots, c_n$ from these n relations and equation (i), we get a differential equation.

Clearly order of this differential equation will be n, i.e., equal to the number of independent parameters in the family of curves.

Illustration :

From the differential equation of family of lines concurrent at the origin.

Sol. *Such lines are given by*

$y = mx$ ...(i)

$\Rightarrow \dfrac{dy}{dx} = m$

Putting the value of m is equation (i)

$$\Rightarrow \quad y = \frac{dy}{dx}x$$

$$\Rightarrow \quad xdy - ydx = 0$$

Note that the order is 1, same as number of constants.

Illustration :

From the differential equation of all concentric circle at the origin.

Sol. *Such circles are given by*

$$x^2 + y^2 = r^2$$

Differentiating w.r.t. x,

$$2x + 2y\frac{dy}{dx} = 0$$

$$\Rightarrow \quad x + y\frac{dy}{dx} = 0$$

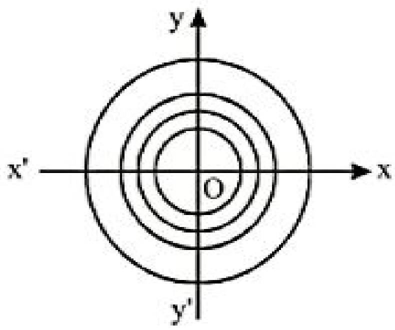


Illustration :

From the differential equation of all circles touching the x-axis at the origin and centre on y-axis.

Sol. *Such family of circles is given by*

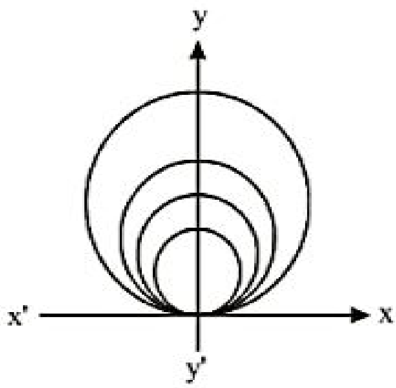
$$x^2 + (y - a)^2 = a^2$$

$$\Rightarrow \quad x^2 + y^2 - 2ay = 0 \qquad \ldots(1)$$

Differentiating,

$$2x + 2y\frac{dy}{dx} = 2a\frac{dy}{dx}$$

or $$x + y\frac{dy}{dx} = a\frac{dy}{dx}$$



substituting the value of a in equation (1)

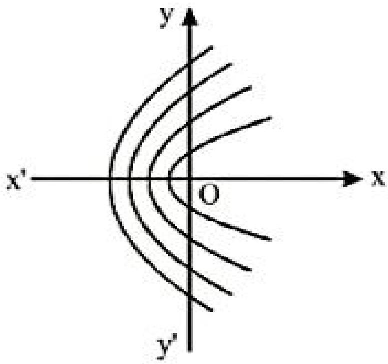
$$\Rightarrow \quad (x^2 - y^2)\frac{dy}{dx} = 2xy \qquad (order = 1 \text{ and } degree = 1)$$

From the differential equation of the family of parabolas with focus at the origin and axis of symmetry along the x-axis.

Sol. *Equation of such parabolas is* $y^2 = 4A(A + x)$

Differentiating w.r.t., we get

$$\Rightarrow \quad 2y\frac{dy}{dx} = 4A \qquad \Rightarrow \quad y\frac{dy}{dx} = 2A$$



Eliminating A from equations (2) and (1)

$$y^2 = \left(y\frac{dy}{dx}\right)^2 + 2y\frac{dy}{dx}x \quad or \quad y^2 = y^2\left(\frac{dy}{dx}\right)^2 + 2xy\frac{dy}{dx}$$

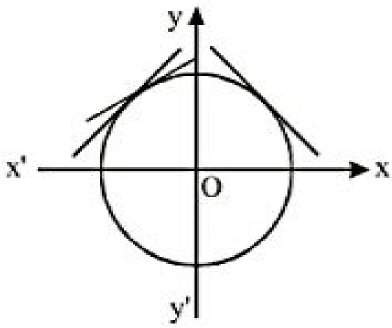
which has order 1 and degree 2.

Illustration :

Find the differential equation of family of lines situated at a constant distance p from the origin.

Sol. *All such lines are tangent to circle of radius p.*

$$y = mx + p\sqrt{1+m^2} \quad \Rightarrow \quad m = \frac{dy}{dx}$$



By eliminating m, we get

$$y = \frac{dy}{dx}x + p\sqrt{1+\left(\frac{dy}{dx}\right)^2} \Rightarrow \left(y-\frac{dy}{dx}x\right)^2 = p^2\left(1+\left(\frac{dy}{dx}\right)^2\right)$$

which has order 1 and degree 2.

Practice Problem

Q.1 Find the differential equation of all the parabola having axis parallel to x-axis.

Q.2 Find the differential equation of all ellipse whose centre is at origin and axis are co-ordinate axis.

Q.3 Consider the equation $\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} = 1$ where a and b are specified constant and λ is an arbitrary parameter. Find a differential eqation satisfied by it.

Q.4 Find the degree of the differential equation satisfying the relation

$$\sqrt{1+x^2}+\sqrt{1+y^2}=\lambda\left(x\sqrt{1+y^2}-y\sqrt{1+x^2}\right)$$

Q.5 Find the differential equation of all non-vertical lines in a plane.

Q.6 Form the differential equation

(i) $y = A + Bx + Ce^{-x}$ (ii) $y = e^{ax}\sin bx$ (iii) $y = ax\cos\left(\frac{1}{x}+b\right)$ (iv) $\sin^{-1}x+\sin^{-1}y=c$

Answer key

Q.1 $\frac{d^3y}{dx^3}=0$

Q.2 $x\left(yy''+(y')^2\right)=yy'$

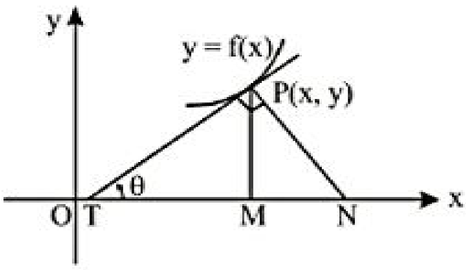
Q.3 $a^2-b^2=x^2-xy\frac{dx}{dy}-y^2+xy\frac{ydy}{dx}$

Q.4 degree 1

Q.5 $\frac{d^2y}{dx^2}=0$

Q.6 (i) $y'''+y''=0$, (ii) $y''-2ay'+(a^2+b^2)y=0$, (iii) $\sqrt{1-x^2}dy+\sqrt{1-y^2}dx=0$ (iv) $x^2y''+y=0$

3. LENGTH OF TANGENT, NORMAL SUB-TANGENT, SUB-NORMAL:



(i) Length of Tangent :

PT is defined as length of the tangent.

In ΔPMT, PT = | y cosec θ|

$$=\left|y\sqrt{(1+\cot^2\theta)}\right| \quad\Rightarrow\quad =\left|y\sqrt{\left\{1+\left(\frac{dx}{dy}\right)^2\right\}}\right|$$

$$\Rightarrow \quad \text{Length of tangent}=\left|y\sqrt{\left\{1+\left(\frac{dx}{dy}\right)^2\right\}}\right|$$

(ii) Length of Normal :

PN is defined as length of the normal.

In ΔPMN, PN = $|y \operatorname{cosec}(90^\circ - \theta)|$

$$= |y \sec\theta| \quad \Rightarrow \quad = \left| y\sqrt{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}} \right|$$

$$\Rightarrow \quad \text{Length of normal} = \left| y\sqrt{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}} \right|$$

(iii) Length of Sub-tangent :

TM is defined as sub-tangent.

$$\text{In } \Delta PTM,\ TM = |y \cot\theta| = \left|\frac{y}{\tan\theta}\right| = \left|y\frac{dx}{dy}\right|$$

$$\Rightarrow \text{Length of sub-tangent} = \left|y\frac{dx}{dy}\right|$$

(iv) Length of Sub-normal :

MN is defined as sub-normal.

$$\text{In } \Delta PMN,\ MN = |y \cot(90^\circ - \theta)| = |y \tan\theta| = \left|y\frac{dy}{dx}\right|$$

$$\Rightarrow \text{Length of sub-normal} = \left|y\frac{dy}{dx}\right|$$

4. SOLUTION OF A DIFFERENTIAL EQUATION :

Elementary types of first order & first degree differential equations.

Elementry Type of first order and first degree

- Variable Separable
  - T-1 normal
  - T-2 $\frac{dy}{dx} = f(ax + by + z)$
- Homogeneous
  - Reducible to Homogeneous
- Linear
  - Reducible to linear (Bernouli's D.E.)

4.1 Variables Separable :

If the differential equation can be expressed as; f(x)dx + g(y)dy = 0 then this is said to be variable -separable type.

A general solution of this is given by $\int f(x)\,dx + \int g(y)\,dy = c$; where c is the arbitrary constant.

Illustration :

Solve the following differential equation

(i) $\ln\frac{dy}{dx} = 3x + 4y$ with $y(0) = 0$ (ii) $x(y^2 + 1)\,dx + y(x^2 + 1)\,dy = 0$

(iii) $y'\sin x = y\ln y$; $y\left(\frac{\pi}{2}\right) = e$ (iv) $(dy/dx) = e^{x-y} + x^2 . e^{-y}$.

Sol.

(i) $\frac{dy}{dx} = e^{3x}\,e^{4y} \Rightarrow \int e^{-4y}dy = \int e^{3x}dx + C \Rightarrow \frac{-e^{4y}}{4} = \frac{e^{3x}}{3} + C \Rightarrow C = \frac{-1}{4} - \frac{1}{3} = \frac{-7}{12}$

Hence $4e^{3x} + 3e^{-4y} = 7$.

(ii) $x(y^2 + 1)\,dx + y(x^2 + 1)\,dy = 0$

$$\Rightarrow \int\frac{2x\,dx}{x^2+1} = -\int\frac{2y\,dy}{(y^2+1)} \Rightarrow \ln(x^2 + 1) = -\ln(y^2 + 1) + \ln C$$

Hence $(y^2 + 1)(x^2 + 1) =$ where C is any constant.

(iii) $\frac{dy}{dx}\sin x = y\log y \Rightarrow \int\frac{dy}{y\ln y} = \int \operatorname{cosec} x\,dx$

$\Rightarrow \ln(\ln y) = \ln(\operatorname{cosec} x - \cot x) + \ln C$

$\Rightarrow \ln y = C(\operatorname{cosec} x - \cot x)$

at $x = \frac{\pi}{2}, y = e$; so $\ln(e) = C(1 - 0) \Rightarrow C = 1$

$\Rightarrow y = e^{(\operatorname{cosec} x - \cot x)}$

(iv) $\frac{dy}{dx} = \left(e^x + x^2\right)e^{-y} \Rightarrow e^y\,dy = (e^x + x^2)\,dx \Rightarrow e^y = e^x + \frac{x^3}{3} + C$.

Illustration :

Find the foci of the conic passing through the point (1, 0) and satisfying the differential equation $(1+y^2)\,dx - xy\,dy = 0$. Find also the equation of a circle touching the conic at $\left(\sqrt{2}, 1\right)$ and passing through one of its foci.

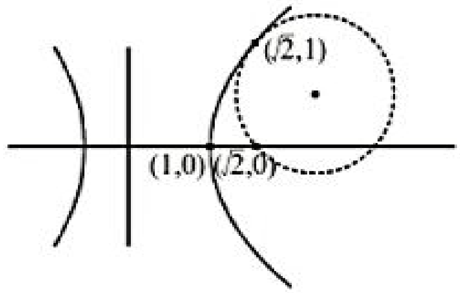
Sol. $(1 + y^2)\,dx = xy\,dy \Rightarrow \int \frac{dx}{x} = \int \frac{2y}{2(1+y^2)}\,dy$

$$\Rightarrow \ \ln x = \frac{1}{2}\ln\left(1+y^2\right) + \frac{\ln C}{2} \Rightarrow x^2 = C(1+y^2)$$

at $x = 1, y = 0 \Rightarrow 1 = C(1 + 0) \Rightarrow C = 1$ so $x^2 - y^2 = 1$

Equation of circle can be written as

$$\left(x-\sqrt{2}\right)^2 + (y-1)^2 + \lambda L = 0$$



where L is tangent of hyperbola $x^2 - y^2 = 1$ *at* $\left(\sqrt{2}, 1\right)$

i.e., $L : \sqrt{2}\,x - y = 1$

so circle $\left(x-\sqrt{2}\right)^2 + (y-1)^2 + \lambda\left(\sqrt{2}x - y - 1\right) = 0$

when this passes through focus $\left(\sqrt{2}, 0\right)$ *so* $0 + 1 + \lambda(1) = 0 \Rightarrow \lambda = -1$.

so equation of circle : $\left(x-\sqrt{2}\right)^2 + (y-1)^2 - 1\left(\sqrt{2}x - y - 1\right) = 0$

when this passes through another focus $\left(-\sqrt{2}, 0\right)$.

so $\left(-2\sqrt{2}\right)^2 + (0-1)^2 + \lambda\left(\left(\sqrt{2}\right)\left(-\sqrt{2}\right) - 0 - 1\right) = 0$

$8 + 1 + \lambda(-3) = 0 \Rightarrow \lambda = 3$

Hence equation of circle, $\left(x-\sqrt{2}\right)^2 + (y-1)^2 + 3\left(\sqrt{2}x - y - 1\right) = 0$.

Practice Problem

Q.1 Find the solution of the following differential equations

(i) $x^2 \frac{dy}{dx} = 2$ (ii) $\frac{dy}{dx} = x \log x$ (iii) $\frac{dy}{dx} = e^{y+x} + e^{y-x}$

(iv) $\frac{dy}{dx} + \frac{1}{\sqrt{1-x^2}} = 0$ (v) $\frac{dy}{dx}\tan y = \sin(x+y) + \sin(x-y)$

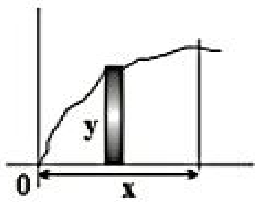
(vi) $\frac{dy}{dx} + \sqrt{\frac{1-y^2}{1-x^2}} = 0$ (vii) $\frac{dy}{dx} + \frac{1+\cos 2y}{1-\cos 2x} = 0$

Q.2 Solve $e^{\frac{dy}{dx}} = x + 1$, given that when $x = 0, y = 3$.

Q.3 Find the curve for which the segment of the tangent contained between the co-ordinate axes is bisected by the point . Curve passes through (2, 3).

Q.4 Find the $y = f(x)$ $(f(x) \geq 0$ and $f(0) = 0)$ bounding a curvilinear trapezoid with the base [0, x] if area bounded by curve, coordinate axes & var. ordinate.

Area $\propto (f(x))^{n+1}$ and $f(1) = 1$



[Hint : $\int_0^x y\,dx = K.(f(x))^{n+1}$; now differeniate both sides]

Q.5 Show that the curve passing through (1, 2) for which the segment of the tangent between P and T is bisected at its point of intersection with the y-axis is a parabola.

Answer key

Q.1 (i) $y = c - \frac{2}{x}$, (ii) $y = \frac{x^2}{2}\log x - \frac{x^2}{4} + c$, (iii) $e^{-y} = e^{-x} - e^x + c$, (iv) $y + \sin^{-1} x = c$

(v) $\sec y + 2\cos x = c$, (vi) $x\sqrt{1-y^2} + y\sqrt{1-x^2} = c$, (vii) $\tan y - \cot x = c$

Q.2 $y = (x + 1)\,ln\,(x + 1) - (x + 1) + 4$ Q.3 $xy = 6$ Q.4 $f(x) = x^{1/n}$ Q.5 $y^2 = 4x$

4.2 Differential Equation Reducible to the Separable Variable Type:

$$\frac{dy}{dx} = f(ax + by + c),\ a, b \neq 0$$

To solve this , substitute $t = ax + by + c$. Then the equation reduces to separable type in the variable t and x which can be solved.

Illustration :

Solve $\frac{dy}{dx} = (x + y)^2$.

Sol. $\frac{dy}{dx} = (x + y)^2$ ...(i)

Here the variable are not separable but by putting

$x + y = v$, *we have*

$$1 + \frac{dy}{dx} = \frac{dv}{dx}$$

$\Rightarrow$ *Equation (i) reduces to*

$$\frac{dv}{dx} = v^2 + 1 \quad or \quad \int \frac{dv}{v^2+1} = \int dx \qquad ...(ii)$$

in which variables are seperated.

Hence from equation (ii),

$\tan^{-1} v = x + c$ *or* $x + y = \tan(x + c)$, *which is a required solution.*

Illustration :

Solve $\frac{dy}{dx}\sqrt{1+x+y} = x + y - 1.$

Sol. *Putting* $\sqrt{1+x+y} = v$, *we have*

$\Rightarrow \quad x + y - 1 = v^2 - 2$

$\Rightarrow \quad 1 + \frac{dy}{dx} = 2v\frac{dv}{dx}$

Then the given equation transforms to

$$\left(2v\frac{dv}{dx} - 1\right)v = v^2 - 2$$

$\Rightarrow \quad \frac{dv}{dx} = \frac{v^2+v-2}{2v^2} \qquad \Rightarrow \qquad \int \frac{2v^2}{v^2+v-2}dv = \int dx$

$\Rightarrow \quad 2\int\left[1 + \frac{1}{3(v-1)} - \frac{4}{3(v+2)}\right]dv = \int dx \quad \Rightarrow \quad 2\left[v + \frac{1}{3}\log|v-1| - \frac{4}{3}\log|v+2|\right] = x + c$

where $v = \sqrt{1+x+y}$

4.3 Differential Equation of the Form :

$$\frac{dy}{dx} = \frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2} \qquad \text{where } b_1 + a_2 = 0$$

Following illustration will clarify the concept.

Illustration :

Solve the differential Equation $\frac{dy}{dx} = \frac{x-y}{x+y}$.

Sol. $\frac{dy}{dx} = \frac{x-y}{x+y}$

$x\,dy + y\,dy = x\,dx - y\,dx$

$x\,dy + y\,dx + y\,dy = x\,dx$

$\Rightarrow \quad d(xy) + y\,dy = x\,dx \quad \Rightarrow \quad \int d(xy) + \int y\,dy = \int x\,dx + \frac{C}{2}$.

$\Rightarrow \quad xy + \frac{y^2}{2} = \frac{x^2}{2} + \frac{C}{2} \quad \Rightarrow \quad 2xy + y^2 = x^2 + C.$

4.4 Polar Coordinates :

Sometimes transformation to the polar co–ordinates facilitates separation of variables. In this connection it is convenient to remember the following differentials. If $x = r\cos\theta$; $y = r\sin\theta$ where r and θ both are variable.

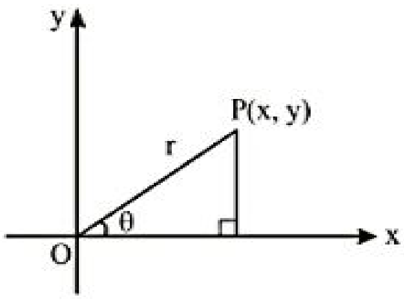
(a) (i) $x\,dx + y\,dy = r\,dr$ (ii) $x\,dy - y\,dx = r^2\,d\theta$

Proof : $x = r\cos\theta$; $y = r\sin\theta$ $\Rightarrow x^2 + y^2 = r^2$

$\Rightarrow x\,dx + y\,dy = r dr$

Also $\tan\theta = y/x \quad \Rightarrow \quad x dy - y\,dx = x^2\sec^2\theta\,d\theta$

$\Rightarrow \quad x dy - y dx = r^2\,d\theta$



(b) If $x = r\sec\theta$ & $y = r\tan\theta$ then

(i) $x\,dx - y\,dy = r\,dr$ and (ii) $x\,dy - y\,dx = r^2\sec\theta\,d\theta$.

Proof : $x = r\sec\theta$ and $y = r\tan\theta$

$\Rightarrow x^2 - y^2 = r^2 \quad \Rightarrow x dx - y dy = r dr$

$y/x = \sin\theta \quad \Rightarrow \quad x dy - y dx = x^2\cos\theta\,d\theta = r^2\sec\theta\,d\theta$

Illustration :

Solve the following differential Equation

(i) $x dx + y dy = x(x dy - y dx)$ *(ii)* $\dfrac{x + y\frac{dy}{dx}}{x\frac{dy}{dx} - y} = \sqrt{\dfrac{1 - x^2 - y^2}{x^2 + y^2}}$

(iii) $\dfrac{x dx + y dy}{\sqrt{x^2 + y^2}} = \dfrac{y dx - x dy}{x^2}$

Sol.

(i) $x\,dx + y\,dy = x\,(x\,dy - y\,dx)$

Let $x = r\cos\theta,\ y = r\sin\theta$

$\therefore\ r\,dr = r\cos\theta\,(r^2\,d\theta)$

$$\int \frac{dr}{r^2} = \int \cos\theta\,d\theta \Rightarrow \frac{-1}{r} = \sin\theta + c$$

$\Rightarrow 1 + r\sin\theta = -cr \Rightarrow (1 + r\sin\theta)^2 = c^2 r^2$

$\Rightarrow (1 + y)^2 = c^2(x^2 + y^2)$

(ii) $$\frac{x + y\dfrac{dy}{dx}}{x\dfrac{dy}{dx} - y} = \sqrt{\frac{1 - x^2 - y^2}{x^2 + y^2}} \Rightarrow \frac{x\,dx + y\,dy}{x\,dy - y\,dx} = \sqrt{\frac{1 - (x^2 + y^2)}{(x^2 + y^2)}}$$

put $x = r\cos\theta,\ y = r\sin\theta$ so $x\,dx + y\,dy = r\,dr\ ;\ x\,dy - y\,dy = r^2\,d\theta$

$$\Rightarrow \frac{r\,dr}{r^2 d\theta} = \frac{\sqrt{1 - r^2}}{r} \Rightarrow \int \frac{dr}{\sqrt{(1 - r^2)}} = \int d\theta \Rightarrow \sin^{-1}(r) = \theta + c$$

$$\Rightarrow r = \sin(\theta + c)\ ;\ \sqrt{(x^2 + y^2)} = \sin(\theta + c) \text{ where } \theta = \tan^{-1}\left(\frac{y}{x}\right)$$

(iii) $$\frac{x\,dx + y\,dy}{\sqrt{(x^2 + y^2)}} = \frac{y\,dx - x\,dy}{x^2}$$

put $x = r\cos\theta,\ y = r\sin\theta$

$x\,dx + y\,dy = r\,dr\ ;\ x\,dy - y\,dx = r^2\,d\theta$

$$\frac{r\,dr}{r} = \frac{-r^2 d\theta}{r^2\cos^2\theta} \Rightarrow \int dr = -\int \sec^2\theta\,d\theta$$

$$r = -\tan\theta + c \Rightarrow \sqrt{(x^2 + y^2)} = -\frac{y}{x} + c$$

so $$\sqrt{(x^2 + y^2)} + \frac{y}{x} = c.$$

4.5 Homogeneous Equations :

The function f(x, y) is said to be a homogeneous function of degree n if for any real number t ($\neq 0$), we have $f(tx, ty) = t^n (x, y)$. For example, $f(x, y) = ax^{2/3} + hx^{1/3} \times y^{1/3} + by^{2/3}$ is a homogeneous function of degree 2/3.

Homogeneous Differential Equation :

A differential equaiton of the form $\frac{dy}{dx} = \frac{f(x,y)}{\phi(x,y)}$, where f(x, y) and f(x, y) are homogeneous function of x and y, and of the same degree, is called Homogeneous. This equation may also be reduced to the form $\frac{dy}{dx} = g\left(\frac{x}{y}\right)$ and is solved by putting y = vx so that the dependent variable y is changed to another variable v, where v is some unknown function, the differential equation is transformed to an equation with variable separable.

Illustration :

Solve $x^2\, dy + y(x + y)\, dx = 0$.

$$\frac{dy}{dx} = -\frac{y(x+y)}{x^2} \quad or \quad \frac{dy}{dx} = -\frac{y}{x} - \frac{y^2}{x^2}$$

Sol. *Putting $y = vx$*

$$\Rightarrow \quad \frac{dy}{dx} = v + x\,\frac{dv}{dx}$$

Given equation transforms to

$$v + x\frac{dv}{dx} = -v - v^2$$

$$\Rightarrow \int \frac{dv}{v^2 + 2v} = -\int \frac{dx}{x}$$

$$\Rightarrow \frac{1}{2}\int \left[\frac{1}{v} - \frac{1}{v+2}\right] dv = -\int \frac{dx}{x}$$

$$\Rightarrow \log|v| - \log|v+2| = -2\log|x| + \log c \ (c > 0)$$

$$\Rightarrow \left|\frac{vx^2}{v+2}\right| = c$$

$$\Rightarrow \left|\frac{x^2 y}{2x+y}\right| = c \ (c > 0)$$

Solve $\left(x\sin\frac{y}{x}\right)dy=\left(y\sin\frac{y}{x}-x\right)dx$.

Sol. *Putting* $y = vx$, *we get* $dy = vdx + xdv$

$\Rightarrow x \sin v\,(vdx + xdv) = x(v \sin v - 1)\,dx$

$\Rightarrow \sin v dv + \dfrac{dx}{x} = 0$

Integrating, we get $\cos v = \ln x + c$

$\Rightarrow \cos\dfrac{y}{x} = \ln x + c$

4.6 Equations Reducible to the Homogenous Form :

Equation of the form $\dfrac{dy}{dx}=\dfrac{ax+by+c}{Ax+By+C}$ $(aB \neq Ab \text{ and } A+b\neq 0)$ can be reduced to a homogeneous form by changing the variable x, y, to X, Y by writing $x=X+h$ and $y=Y+k$; where h, k are constant to be chosen so as to make the given equation homogeneous. We have

$$\frac{dy}{dx}=\frac{d(Y+k)}{d(X+h)}=\frac{dY}{dX}$$

Hence the given equation becomes,

$$\frac{dY}{dX}=\frac{aX+bY+(ah+bk+c)}{Ah+Bk+(Ah+Bk+C)}$$

Let h and k be chosen to satisfy the relation $ah+bk+c=0$ and $Ah+Bk+C=0$.

Illustration :

Solve $x\dfrac{dy}{dx} = y + 2\sqrt{y^2-x^2}$.

Sol. *Putting,* $y = vx$ *and* $\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$

$xv + x^2\dfrac{dv}{dx} = vx + 2x\sqrt{v^2-1} \Rightarrow \dfrac{dv}{2\sqrt{v^2-1}} = \dfrac{dx}{x}$, *integrating, we get*

$\frac{1}{2}\ln\left(v+\sqrt{v^2-1}\right)=\ln(cx) \Rightarrow \frac{1}{2}\ln\left(\dfrac{y+\sqrt{y^2-x^2}}{x}\right) = \ln(cx)$

$\Rightarrow\ y + \sqrt{y^2-x^2} = c^2x^3$

Illustration :

Solve $x(dy/dx) = y(\log y - \log x + 1)$.

Sol. $x\left(\frac{dy}{dx}\right) = y\left(\log y - \log x + 1\right) \Rightarrow \frac{dy}{dx} = \frac{y}{x}\left(\log\left(\frac{y}{x}\right) + 1\right)$

Putting $y = vx$, we get $\frac{dy}{dx} = v + x\frac{dv}{dx}$

and the given equation transforms to $v + x\frac{dv}{dx} = v[\log v + 1] \Rightarrow x\frac{dv}{dx} = v\log v$

$$\Rightarrow \int \frac{dv}{v\log v} = \int \frac{dx}{x} \Rightarrow \log(\log v) = \log x + \log k,\ k > 0$$

$$\Rightarrow \log(v) = kx \Rightarrow \frac{y}{x} = e^{kx} \Rightarrow y = x\,e^{kx};\ \text{where } k > 0.$$

Illustration :

Solve $(x + y\sin(y/x))dx = x\sin(y/x)\,dy$.

Sol. $\frac{dy}{dx} = \frac{x + y\sin\left(\frac{y}{x}\right)}{x\sin\left(\frac{y}{x}\right)}$ or $\frac{dy}{dx} = \frac{1 + \left(\frac{y}{x}\right)\sin\left(\frac{y}{x}\right)}{\sin\left(\frac{y}{x}\right)}$

Put $y = vx$ then $\frac{dy}{dx} = v + x\frac{dv}{dx}$ and the given equation transforms to $v + x\frac{dv}{dx} = \operatorname{cosec}(v) + v$

$\Rightarrow \sin v\,dv = \frac{dx}{x}$ Integrating and replacing v by $\frac{y}{x}$, we get ;

$$\cos\left(\frac{y}{x}\right) + \log|x| = c,\ c \in R.$$

Illustration :

Solve $(2x - y + 1)\,dx + (2y - x + 1)\,dy$

Sol. $(2x - y + 1)\,dx = -(2y - x + 1)\,dy$

$\frac{dy}{dx} = \frac{-(2x - y + 1)}{(2y - x + 1)}$ Let $y = Y + k$ and $x = X + h$

then $\frac{dy}{dx} = \frac{dY}{dX} = \frac{-(2X - Y + 2h - k + 1)}{(2Y - X + 2k - h + 1)}$

Now, 'h' and 'k' can be chosen such that $2h - k + 1 = 0$ ......(1)
and $2k - h + 1 = 0$ ......(2)
By solving these equation $h = -1$, $k = -1$.

$$\text{So } \frac{dy}{dx} = \frac{-(2X-Y)}{(2Y-X)}.$$ *Now homogenous equation so put $Y = vX$;* $\frac{dY}{dX} = v + x\frac{dv}{dX}$

$$v + X\frac{dv}{dX} = \frac{-(2-v)}{(2v-1)} \Rightarrow X\frac{dv}{dX} = -\left(\frac{2-v}{2v-1} + v\right)$$

$$X\frac{dv}{dX} = \left(\frac{2-v+2v^2-v}{2v-1}\right) \Rightarrow \int\frac{(2v-1)}{(2v^2-2v+2)}dv = -\int\frac{dX}{X}$$

$$\Rightarrow \frac{1}{2}\int\frac{(2v-1)}{(v^2-v+1)}dv = -\int\frac{dX}{X} \Rightarrow \frac{1}{2}\ln(v^2 - v + 1) = -\ln X + \ln k$$

$$\Rightarrow \ln\left(\sqrt{(v^2-v+1)}\,X\right) = \ln k$$

$$\Rightarrow X\sqrt{(v^2-v+1)} = k \Rightarrow \sqrt{(Y^2-XY+X^2)} = k$$

$$\Rightarrow (y + 1)^2 - (y + 1)(x + 1) + (x + 1)^2 = k^2.$$

Practice Problem

Q.1 Solve the following equations

(i) $\frac{dy}{dx} = \frac{x-2y+5}{2x+3y-1}$ (ii) $\frac{dy}{dx} = \frac{x+y+1}{x+y-1}$ (iii) $\frac{dy}{dx} = \cos(x+y) - \sin(x+y)$

Q.2 Solve the following equations

(i) $\frac{dy}{dx} = \frac{xy}{x^2+y^2}$ (ii) $\frac{dy}{dx} = \frac{x}{2y-x}$ (iii) $x + y\left(\frac{dy}{dx}\right) = 2y$

(iv) $y' = \frac{x-y}{x+y}$ (v) $(2x - y + 1)dx + (2y - x + 1)dy = 0$

(vi) $xdy - ydx = (\sqrt{x^2+y^2})\,dx$ (vii) $(x + y)dx + xdy = 0$ (viii) $x(x-y)\frac{dy}{dx} = y(x+y)$

Answer key

Q.1 (i) $4xy + 3y^2 - 2y = x^2 + 10x + c$, (ii) $x + y = ce^{y-x}$ (iii) $\ln\left|1 - \tan\frac{x+y}{2}\right| + x + c = 0$

Q.2 (i) $cy^2 = e^{\frac{x^2}{y^2}}$, (ii) $(x-y)(x+2y)^2 = c$, (iii) $\log(y-x) - \frac{x}{y-x} = c$,

(iv) $y^2 + 2xy - x^2 = c$, (v) $y^2 + x^2 - xy + x + y = c$ (vi) $y + \sqrt{x^2+y^2} = cx^2$,

(vii) $x^2 + 2xy = c$, (viii) $\frac{x}{y} + \log xy = c$

4.7 Linear Differential Equations :

A differential equation is said to be linear if the dependent variable & all its differential coefficients occur in degree one only and are never multiplied together.

The nth order linear differential equation is of the form ;

$a_0(x)\frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1}y}{dx^{n-1}} + ... + a_n(x).y = \phi(x)$. Where $a_0(x)$, $a_1(x)$ ..... $a_n(x)$ are the coefficients of the differential equation.

Linear Differential Equations of First Order :

The most general form of a linear differential equations of first order is $\frac{dy}{dx} + Py = Q$, where P & Q are functions of x (Independent variable).

For solving such equations we multiply both sides by

Integrating factor = I.F. = $e^{\int P dx}$

So we get $e^{\int P dx}\left(\frac{dy}{dx} + Py\right) = Qe^{\int P dx}$

$$\Rightarrow \frac{dy}{dx}e^{\int P dx} + yPe^{\int P dx} = Qe^{\int P dx}$$

$$\Rightarrow \frac{d}{dx}\left(ye^{\int P dx}\right) = Qe^{\int P dx} \quad \left[\text{Since } \frac{d}{dx}\left(e^{\int P dx}\right) = Pe^{\int P dx}\right]$$

$$\Rightarrow \int \frac{d}{dx}\left(ye^{\int P dx}\right)dx = \int Q\left(e^{\int P dx}\right)dx$$

$$\Rightarrow ye^{\int P\,dx} = \int Qe^{\int P\,dx} + C$$

which is the required solution of the given differential equation.

In some cases a linear differential equation may be of the form $\frac{dx}{dy} + P_1x = Q_1$, where P_1 and Q_1 are functions of y alone or constants. In such a case the integrating factor is $e^{\int P_1\,dy}$, and solutions is given by

$$x\,e^{\int P_1\,dy} = \int Q_1e^{\int P_1\,dy}\,dy + C$$

Illustration :

Solve $x^2\,(dy/dx) + y = 1$.

Sol. *The given diffential equation can be written as*

$\frac{dy}{dx} + \frac{1}{x^2}y = \frac{1}{x^2}$, *which is linear*

Here $P = 1/x^2$ *and* $Q = \frac{1}{x^2}$

$$I.F. = e^{\int(1/x^2)dx} = e^{-1/x}$$

$$\Rightarrow \quad ye^{\frac{-1}{x}} = \int e^{\frac{-1}{x}} \cdot \frac{1}{x^2}\,dx = e^{\frac{-1}{x}} + C$$

$$\Rightarrow \quad y = 1 + Ce^{\frac{1}{x}}$$

Illustration :

Solve the following differential equations

(i) $\cos^2x\,\frac{dy}{dx} + y = \tan x$

(ii) $x\ln x\,\frac{dy}{dx} + y = 2\ln x$

(iii) $x\,(x^2 + 1)\,\frac{dy}{dx} = y(1 - x^2) + x^2\ln x$

(iv) $\frac{dy}{dx} = \frac{1}{x\cos y + \sin 2y}$

Sol.

(i) $\cos^2x\,\frac{dy}{dx} + y = \tan x$

$$\frac{dy}{dx} + \sec^2x\,y = \tan x\,\sec^2x$$

Integrating Factor $= e^{\int \sec^2 x\,dx} = e^{\tan x}$.

$$y\, e^{\tan x} = \int e^{\tan x} \tan x \sec^2 x\, dx \qquad \text{Let } t = \tan x \text{ so } dt = \sec^2 x\, dx$$

$$\int e^{\tan x} \tan x \sec^2 x dx = \int e^t\, t\, dt = (t-1)\, e^t$$

$$\text{so } y\, e^{\tan x} = (\tan x - 1)\, e^{\tan x} + k$$

$$y = (\tan x - 1) + k\, e^{-\tan x}$$

(ii) $$x \ln x \frac{dy}{dx} + y = 2 \ln x$$

$$\frac{dy}{dx} + \frac{y}{x \ln x} = \frac{2}{x}$$

$$\text{Integrating factor} = e^{\int \frac{dx}{x \ln x}} = e^{\ln(\ln x)} = \ln x$$

$$\text{So, } y\,(\ln x) = \int \frac{2}{x} \ln x\, dx = (\ln x)^2 + c$$

(iii) $$x\,(x^2+1) \frac{dy}{dx} = y\,(1 - x^2) + x^2 \ln x$$

$$\frac{dy}{dx} + \frac{(x^2-1)\, y}{x\,(x^2+1)} = \frac{x^2 \ln x}{x\,(x^2+1)}$$

$$\int \frac{(x^2-1)\, dx}{x(x^2+1)} = \int \frac{x}{(x^2+1)} dx - \int \frac{dx}{x\,(x^2+1)} = \frac{1}{2} \int \frac{2x}{(x^2+1)} dx - \int \left(\frac{1}{x} - \frac{2x}{2\,(x^2+1)} \right) dx$$

$$= \frac{1}{2} \ln (x^2+1) - \ln x + \frac{1}{2} \ln (x^2+1) = \ln \left(\frac{x^2+1}{x} \right)$$

$$\text{Integrating factor} = e^{\ln_e \left(\frac{x^2+1}{x} \right)} = \left(\frac{x^2+1}{x} \right)$$

$$\text{So, } y \left(\frac{x^2+1}{x} \right) = \int \left(\frac{x^2+1}{x} \right) \cdot \frac{x^2 \ln x}{x\,(x^2+1)} dx = \int \ln x\, dx$$

$$\Rightarrow y \left(\frac{x^2+1}{x} \right) = x\,(\ln x - 1) + C.$$

(iv) $$\frac{dy}{dx} = \frac{1}{x \cos y + \sin 2y} \Rightarrow \frac{dx}{dy} = x \cos y + \sin 2y$$

$$\frac{dy}{dx} - (\cos y)\, x = (\sin 2y)$$

Integrating factor $= e^{-\int (\cos y)\,dy} = e^{-\sin y}$

$x\, e^{-\sin y} = \int e^{-\sin y}\, 2 \sin y \cos y\, dy$ *Put* $\sin y = t,\ \cos y = \dfrac{dt}{dy}$

$\int e^{-\sin y}\, 2 \sin y \cos y\, dy = \int e^{-t}\,(2t)\,dt = 2\,[-t\,e^{-t} - e^{-t}] = -2\,e^{-t}\,(t+1) = -2\,e^{-\sin y}\,(\sin y + 1)$

$\therefore\ x\, e^{-\sin y} = -2\, e^{-\sin y}\,(\sin y + 1) + C$

$x = -2\,(\sin y + 1) + c\, e^{\sin y}.$

Practice Problem

Q.1 Find the intergrating factor for the following linear differential equations

(i) $\dfrac{dy}{dx} + y \tan x - \sec x = 0$ (ii) $\dfrac{dy}{dx} + \dfrac{y}{x} = x^3 - 3$

(iii) $(1 - x^2)\dfrac{dy}{dx} - xy = 1$ (iv) $(x^2+1)\dfrac{dy}{dx} + 2xy = x^2 - 1$

Q.2 Solve the following D ifferential equation

(i) $\dfrac{dy}{dx} + \dfrac{1}{x} = \dfrac{e^y}{x^2}$ (ii) $\dfrac{dy}{dx} + x \sin 2y = x^3 \cos^2 y$

(iii) $\dfrac{dy}{dx} + \dfrac{xy}{(1-x^2)} = x\sqrt{y}$ (iv) $\dfrac{dy}{dx} = \dfrac{y}{2y\, \ln y + y - x}$

Q.3 Find the solution of the following differential equations

(i) $\dfrac{dy}{dx} + \dfrac{y}{x} = y^2$ (ii) $x \log x \dfrac{dy}{dx} + y = 2 \log x$

(iii) $\dfrac{dy}{dx} + 2y \cot x = 3x^2 \operatorname{cosec}^2 x$ (iv) $x\dfrac{dy}{dx} = y + x^2$

Answer key

Q.1 (i) $\sec x$, (ii) x, (iii) $\sqrt{1-x^2}$, (iv) x^2+1

Q.2 (i) $\dfrac{e^{-x}}{x} = \dfrac{x^{-2}}{2} + C$, (ii) $\tan y\, e^{x^2} = \dfrac{1}{2} e^{x^2}\left(x^2 - 1\right) + C$, (iii) $\dfrac{\sqrt{y}}{\left(1-x^2\right)^{\frac{1}{4}}} = \dfrac{-1}{3}\left(1-x^2\right)^{\frac{3}{4}} + C$

(iv) $xy = y^2 \ln y + c$

Q.3 (i) $xy \log_e\left(\dfrac{c}{x}\right) = 1$, (ii) $y \log x = (\log x)^2 + c$, (iii) $y \sin^2 x = x^3 + c$, (iv) $y = x^2 + cx$

4.8 Equations Reducible To Linear Form (Bernoulli's Equation):

The equation $\frac{dy}{dx} + py = Q \cdot y^n$ where P & Q functions of x, is reducible to the linear form by dividing it by y^n & then substituting $y^{-n+1} = Z$. Its solution can be obtained as in the normal case.

Illustration :

Solve the following Differential Equation.

(i) $\frac{dy}{dx} = xy + x^3y^2$

(ii) $\frac{dy}{dx} - 2y\tan x + y^2 \tan^4 x = 0$

(iii) $y \sin x \frac{dy}{dx} = \cos x(\sin x - y^2)$

(iv) $\frac{dy}{dx} + \frac{1}{x}\sin 2y = x^3 \cos^2 y$.

(v) $\frac{dy}{dx} - \frac{\tan y}{x+1} = (1+x)e^x \sec y$

Sol.

(i) $\frac{dy}{dx} = xy + x^3y^2 \Rightarrow \frac{dy}{dx} - yx = x^3y^2$

$\frac{1}{y^2}\frac{dy}{dx} - \frac{x}{y} = x^3$ put $t = \frac{-1}{y}$ so $dt = \frac{dy}{y^2}$

$\frac{dt}{dx} + tx = x^3 \Rightarrow$ Integrating factor $= e^{\int x\,dx} = e^{\frac{x^2}{2}}$

$t\, e^{\frac{x^2}{2}} = \int e^{\frac{x^2}{2}} x^3\, dx = \int e^{\frac{x^2}{2}} x\, x^2\, dx$ Put $z = \frac{x^2}{2}$; $dz = x\, dx$

$\int e^{\frac{x^2}{2}} x^3\, dx = \int e^z(2z)\,dz = 2(z-1)\,e^z = (x^2-2)\, e^{\frac{x^2}{2}}$

$t\, e^{\frac{x^2}{2}} = (x^2-2)\, e^{\frac{x^2}{2}} + C \Rightarrow \frac{-e^{\frac{x^2}{2}}}{y} = (x^2-2)\, e^{\frac{x^2}{2}} + C \Rightarrow y = \frac{-e^{\frac{x^2}{2}}}{(x^2-2)e^{\frac{x^2}{2}} + C}$

(ii) $\frac{dy}{dx} - 2y\tan x + y^2\tan^4 x = 0 \Rightarrow \frac{dy}{dx} - 2y\tan x = -y^2\tan^4 x$

$\frac{-1}{y^2}\frac{dy}{dx} + \frac{2\tan x}{y} = \tan^4 x$ put $t = \frac{1}{y}$; $dt = \frac{-1}{y^2}\, dy$

$\frac{dt}{dx} + (2\tan x)\, t = \tan^4 x$

$$Integrating\ factor = e^{2\int \tan x\,dx} = e^{2\log_e \sec x} = \sec^2 x$$

$$t\sec^2 x = \int \tan^4 x \sec^2 x\,dx = \frac{\tan^5 x}{5} + c$$

$$\frac{\sec^2 x}{y} = \frac{\tan^5 x}{5} + C$$

(iii) $y\sin x \dfrac{dy}{dx} = \cos x(\sin x - y^2)$

Let $z = y^2 \quad \therefore dz = 2y\,dy$

$$\frac{1}{2}\frac{dz}{dx} + \cot x\, z = \cos x \Rightarrow \frac{dz}{dx} + 2\cot x\, z = 2\cos x$$

$$Integrating\ factor = e^{\int 2\cot x\,dx} = e^{2\ln(\sin x)} = \sin^2 x$$

$$z \cdot \sin^2 x = \int 2\cos x \sin^2 x\,dx = \frac{2\sin^3 x}{3} + C$$

$$\therefore\ y^2\sin^2 x = \frac{2\sin^3 x}{3} + C.$$

(iv) $\dfrac{dy}{dx} + \dfrac{\sin 2y}{x} = x^3\cos^2 y$

$$\Rightarrow \quad \sec^2 y\,\frac{dy}{dx} + \frac{2\tan y}{x} = x^3$$

$$z = \tan y \Rightarrow dz = \sec^2 y\,dy \Rightarrow \frac{dz}{dx} + \frac{2z}{x} + x^3$$

$$Integrating\ factor = e^{\int \frac{2dx}{x}} = e^{2ln} = x^2$$

$$\Rightarrow \quad z \cdot x^2 = \int x^5 dx = \frac{x^6}{6} + C$$

$$\Rightarrow \quad x^2\tan y = \frac{x^6}{6} + C.$$

(v) $$\frac{dy}{dx}-\frac{\tan y}{x+1}=(x+1)e^x \sec y$$

$$\cos y\frac{dy}{dx}-\frac{\sin y}{x+1}=(x+1)e^x$$

Let $z=\sin y \quad \therefore \quad dz=\cos y\, dy$

$$\frac{dz}{dx}-\frac{z}{x+1}=(x+1)e^x$$

$$\text{Integrating factor} = e^{\int\frac{-dx}{(x+1)}}=e^{-ln(x+1)}=\frac{1}{x+1}$$

$$\therefore \quad \frac{z}{(x+1)}=\int e^x dx=e^x+c$$

$$z=(e^x+c)(x+1)$$

$$\therefore \quad \sin y=(e^x+c)(x+1).$$

Note : Following exact differentials must be remembered :

(i) $x dy + y\, dx = d(xy)$

(ii) $\dfrac{x dy - y dx}{x^2}=d\left(\dfrac{y}{x}\right)$

(iii) $\dfrac{y dx - x dy}{y^2}=d\left(\dfrac{x}{y}\right)$

(iv) $\dfrac{x dy + y dx}{xy}=d(\ln xy)$

(v) $\dfrac{dx + dy}{x+y}=d(ln(x+y))$

(vi) $\dfrac{x dy - y dx}{xy}=d\left(\ln\dfrac{y}{x}\right)$

(vii) $\dfrac{y dx - x dy}{xy}=d\left(\ln\dfrac{x}{y}\right)$

(viii) $\dfrac{x dy - y dx}{x^2+y^2}=d\left(\tan^{-1}\dfrac{y}{x}\right)$

(ix) $\dfrac{y dx - x dy}{x^2+y^2}=d\left(\tan^{-1}\dfrac{x}{y}\right)$

(x) $\dfrac{x dx + y dy}{x^2+y^2}=d\left[\ln\sqrt{x^2+y^2}\right]$

(xi) $d\left(-\dfrac{1}{xy}\right)=\dfrac{x dy + y dx}{x^2y^2}$

(xii) $d\left(\dfrac{e^x}{y}\right)=\dfrac{ye^x dx - e^x dy}{y^2}$

(xiii) $d\left(\dfrac{e^y}{x}\right)=\dfrac{xe^y dy - e^y dx}{x^2}$

Illustration :

Solve $x\,dx + y\,dy = \dfrac{x\,dy - y\,dx}{x^2 + y^2}$.

Sol. *The D.E. can be written as*

$$\frac{1}{2} d(x^2 + y^2) = d\{\tan^{-1}(y/x)\}$$

Integrating, we get

$$\frac{1}{2}(x^2 + y^2) = \tan^{-1}(y/x) + c$$

Illustration :

Solve $\{(x + 1)(y/x) + \sin y\}dx + (x + \ln x + x\cos y)\,dy = 0$.

Sol. *We can re-write the differential equation as*

$$(y\,dx + x\,dy) + \left(\frac{y}{x}dx + \ln x\,dy\right) + (\sin y\,dx + x\cos y\,dy) = 0$$

$\Rightarrow d(xy) + d(y\ln x) + d(x\sin y) = 0$

Integrating both sides we have

$xy + y\ln x + x\sin y = c$

Practice Problem

Q.1 Solve the following equations

(i) $y\,dx - x\,dy + 3x^2y^2\,e^{x^3}\,dx = 0$ (ii) $y\,dx + (x + x^2y)dy = 0$

(iii) $(xy^4 + y)\,dx - x\,dy = 0$ (iv) $\dfrac{dy}{dx} = \dfrac{2xy}{x^2 - 1 - 2y}$.

Q.2 The function $y(x)$ satisfies the equation $y(x) + 2x\displaystyle\int_0^x \frac{y(u)}{1+u^2}du = 3x^2 + 2x + 1$. Show that the substitution $z(x) = \displaystyle\int_0^x \frac{y(u)}{1+u^2}du$ converts the equation into a first order linear differential equation for $z(x)$. Find its integrating factor.

Q.3 A differentiable function satisfies $f(x) = \displaystyle\int_0^x (f(t)\cos t - \cos(t - x))dt$. Which of the following hold good?

(A) $f(x)$ has a minimum value $1 - e$. (B) $f(x)$ has a maximum value $1 - e^{-1}$.

(C) $f''\left(\dfrac{\pi}{2}\right) = e$ (D) $f'(0) = 1$

Answer key

Q.1 (i) $\frac{x}{y}+e^{x^3}=C$, (ii) $\frac{-1}{xy}+\log y=C$, (iii) $3x^4y^3+4x^3=cy^3$, (iv) $\frac{x^2}{y}=\frac{1}{y}-2\log y+C$

Q.2 $y(x)=\frac{(1+x^2)^2+2x}{(1+x^2)}$ Q.3 ABC

5. PHYSICAL APPLICATION OF DIFFERENTIAL EQUATION :

5.1 Mixture Problems :

A chemical in a liquid solution (or dispersed in a gas) runs into a container holding the liquid (or the gas) with, possibly, a specified amount of the chemical dissolved as well. The mixture is kept uniform by stirring and flows out of the container at a known rate. In this process it is often important to know the concentration of the chemical in the container at any given time. The differential equation describing the process is based on the formula.

$$\begin{matrix}\textit{Rate of change}\\ \textit{of amount}\\ \textit{in container}\end{matrix} = \begin{pmatrix}\textit{rate at which}\\ \textit{chemical}\\ \textit{arrives}\end{pmatrix} - \begin{pmatrix}\textit{rate at which}\\ \textit{chemical}\\ \textit{departs}\end{pmatrix} \qquad \text{....(1)}$$

If $y(t)$ is the amount of chemical in the container at time t and $V(t)$ is the total volume of liquid in the container at time t, then the departure rate of the chemical at time t is

$$\text{Departure rate} = \frac{y(t)}{V(t)}\cdot(\text{out flow rate})$$

$$= \begin{pmatrix}\textit{concentration in}\\ \textit{container at time } t\end{pmatrix}\cdot(\text{out flow rate})$$

Accordingly, Equation (1) becomes

$$\frac{dy}{dt} = (\text{chemical's given arrival rate}) - \frac{y(t)}{V(t)}\cdot(\text{out flow rate}) \qquad \text{....(2)}$$

If, say, y is measured in grams, V in liters, and t in minutes, then unit in equation (2) are

$$\frac{\textit{grams}}{\textit{minute}} = \frac{\textit{grams}}{\textit{minute}} - \frac{\textit{grams}}{\textit{litre}}\cdot\frac{\textit{litre}}{\textit{minute}}$$

Illustration :

A tank initially contains 100 litres of brine in which 50 gms of salt dissolved. A brine containing 2 gm/litre of salt runs into the tank at the rate of 5 litre/min. The mixture is kept stirring and flows out of the tank at the rate of 4 litres/min then

(a) At what rate (gms/min) does salt enter the tank at time t.

(b) What is the volume of the brine in the tank at time t.

(c) At what rate (gms/min) does salt leave the tank at time t.

(d) form the DE of the process and solve it to find an expression for the amount of salt present at time t.

Sol.

(a) Inflow rate of brine solution = (2 gm/litre) · (5 litre/min) = 10 gm/min.

(b) Volume of the brine in the tank at time t = initial volume + (inflow – outflow) · t

$$= 100 + (5 - 4)\,t = (100 + t) \text{ litres.}$$

(c) Let $y(t)$ is the amount of salt at time t then outflow rate of salt $= \dfrac{4y}{(100+t)}$

(d) Rate of change of salt in container = (rate at which salt arrivrs) – rate of which salt leaves

$$\frac{dy}{dt} = 10 - \frac{4y}{(100+t)} \Rightarrow \frac{dy}{dx} + \frac{4y}{(100+t)} = 10$$

$$\text{Integrating factor} = e^{\int \frac{4dt}{(100+t)}} = e^{4\ln(100+t)} = (100+t)^4$$

$$y\,(100+t)^4 = 10\int (100+t)^4 dt = 2\,(100+t)^5 + C \text{ at } t = 0,\quad y = 50$$

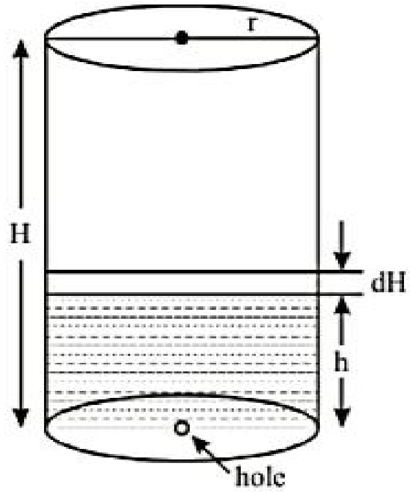
$$\Rightarrow \quad 50 \cdot (100)^4 = 2\,(100)^5 + C \Rightarrow C = -150\,(100)^4$$

$$\Rightarrow \quad y\,(100+t)^4 = 2(100+t)^5 - 150\,(100)^4 \Rightarrow y = 2\,(100+t) - \frac{150}{\left(1+\frac{t}{100}\right)^4} \quad \textbf{Ans.}$$

Illustration :

Find the time required for a cylindrical tank of radius r and height H to empty through a round hole of area 'a' at the bottom. The flow through the hole is according to the law $v(t) = k\sqrt{2gh(t)}$ where v(t) and h(t) are respectively the velocity of flow through the hole and the height of the water level above the hole at time t and g is the acceleration due to gravity.

Sol. *Let at time t the depth of water is h and radius of water surface is r.*
If in time dt the decrease of water level is dh then



$$-\pi r^2 dh = ak\sqrt{(2gh)}\,dt$$

$$\Rightarrow \quad \frac{-\pi r^2}{ak\sqrt{2g}\sqrt{h}}dh = dt \quad \Rightarrow \quad -\frac{\pi r^2}{ak\sqrt{2g}}\frac{dh}{\sqrt{h}} = dt$$

Now when t = 0, h = H and when t = t, h = 0

$$\text{then,} \quad -\frac{\pi r^2}{ak\sqrt{2g}}\int_H^0 \frac{dh}{\sqrt{h}} = \int_0^t dt$$

$$\Rightarrow \quad -\frac{\pi r^2}{ak\sqrt{2g}}\left\{2\sqrt{h}\right\}_H^0 = t$$

$$\Rightarrow \quad t = \frac{\pi r^2 2\sqrt{H}}{ak\sqrt{2g}} = \frac{\pi r^2}{ak}\sqrt{\left(\frac{2H}{g}\right)}$$

Illustration :

Suppose that a mothball loses volume by evaporation at a rate proportional to its instantaneous area. If the diameter of the ball decreases from 2 cm to 1cm in 3 months, how long will it take until the ball has practically gone?

Sol. *Let at any instance (t), radius of moth ball be 'r' and 'v' be its volume*

$$\Rightarrow \quad v = \frac{4}{3}\pi r^3 \quad \Rightarrow \quad \frac{dv}{dt} = 4\pi r^2\frac{dr}{dt}$$

Thus as per the information

$$4\pi r^2\frac{dr}{dt} = -k\,4\pi r^2, \text{ where } k \in R^+$$

$$\Rightarrow \quad \frac{dr}{dt} = -k \quad \text{or} \quad r = -kt + c \quad \text{at} \quad t = 0, r = 2cm; \; t = 3 \text{ month}, r = 1 \text{ cm}$$

$$\Rightarrow \quad c = 2, k = \frac{1}{3} \quad \Rightarrow \quad r = -\frac{1}{3}t + 2$$

now for $r \to 0$, $t \to 6$
Hence, it will take six months until the ball is practically gone.

Illustration :

A body at a temperature of 50F is placed outdoors where the temperature is 100 F. If the rate of change of the temperature of a body is proportional to the temperature difference between the body and its surrounding medium. If after 5 min. the temperature of the body is 60F, find

(a) how long it will take the body to reach a temperature of 75F and

(b) the temperature of the body after 20 min.

Sol. *Let T be the temperature of the body at time t and* $T_m = 100$ *(the temperature of the surrounding medium)*

$$\frac{dT}{dt} = -k(T - T_m) \qquad \text{or} \qquad \frac{dT}{dt} + kT = kT_m,$$ *where k is constant of proportionally.*

$$\Rightarrow \quad \frac{dT}{dt} + kT = 100k$$

This differential equation whose solution is

$$T = ce^{-kt} + 100 \qquad \ldots(i)$$

Since $T = 50$ *when* $t = 0$*, then from equation (i)* $50 = ce^{-k(0)} + 100$*, or* $c = -50$*.*

Substituting this value in equation (i), we obtain

$$T = -50e^{-kt} + 100 \qquad \ldots(ii)$$

At $t = 5$*, we are given that* $T = 60$*; hence, from equation (ii),* $60 = -50e^{-5k} + 100$*.*

Solving for k, we obtain $-40 = -50\,e^{-5k}$ *or* $k = -\frac{1}{5}\ln\frac{40}{50}$

Substituting this value in equation (ii), we obtain the temperature of the body at any time t as

$$T = -50e^{\frac{1}{5}\ln\frac{4}{5}t} + 100 \qquad \ldots(iii)$$

(a) We require t when $T = 75$*, Substituting* $T = 75$ *in equation (iii), we have*

$$75 = -50e^{\frac{1}{5}\ln\frac{4}{5}t} + 100,$$ *from which we get t*

(b) We require T when $t = 20$*. Substituing* $t = 20$ *in equation (iii) and then solving for T, we find*

$$T = -50e^{\frac{1}{5}\ln\frac{4}{5}(20)} + 100$$

5.2 Statistical Applications of Differential Equation:

Illustration :

The population of a certain country is known to increase at a rate proportional to the number of people presently living in the country. If after two years the population has doubled and after three years the population is 20,000, estimate the number of people initially living in the country.

Sol. *Let N denote the number of people living in the country at any time t, and let N_0 denote the number of people initially living in the country.*

Then, from equation $\frac{dN}{dt} - kN = 0$

which has the solution $N = ce^{kt}$ ...(i)

At $t = 0$, $N = N_0$; *hence equation (i) states that* $N_0 = ce^{k(0)}$, *or that* $c = N_0$

Thus, $N = N_0e^{kt}$ ...(ii)

At $t = 2$, $N = 2N_0$

Substituting these values into equation (ii), we have

$$2N_0 = N_0e^{2k} \text{ from which } k = \frac{1}{2}\ln 2$$

Substituting this value into equation (i) gives

$$N = N_0\, e^{t/2 \ln 2} \quad ...(iii)$$

At $t = 3$, $N = 20,000$

Substituting these values into equation (iii), we obtain

$$20,000 = N_0e^{3/2 \ln 2} \quad \Rightarrow \quad N_0 = \frac{20000}{2\sqrt{2}} \approx 7071.$$

Illustration :

What constant interest rate is required if an initial deposit placed into an account the accrues interest compounded continuously is to double its value in six years? ($\ln|2| = 0.6930$)

Sol. *The balance N(t) in the account at any time t.*

$$\frac{dN}{dt} - kN = 0, \text{ its solution is } N(t) = ce^{kt} \quad ...(i)$$

Let initial deposit be N_0.

At $t = 0$, $N(0) = N_0$, *which when substituted into equation (i) yields*

$$N_0 = ce^{k(0)} = c$$

and equation (i) becomes $N(t) = N_0e^{kt}$ .....(ii)

We seek the value of k for which $N = 2N_0$ *when* $t = 6$, *Substituting these values into equation (ii) and solving for k we find* $2N_0 = N_0e^{k(6)} \quad \Rightarrow \quad e^{6k} = 2 \quad \Rightarrow k = \frac{1}{6}\ln|2| = 0.1155$

An interest rate of 11.55 percent is required.

5.3 Geometrical Applications of Differential Equation :

We also use differential equations for finding the family of curves for which some conditiong involving the derivatives are given. For this we proceed in the following way

Equation of the tangent at a point (x, y) to the curve $y = f(x)$ is given by $Y - y = \frac{dy}{dx}(X - x)$.

At the X axis, $Y = 0$, and $X = x - \frac{y}{dy/dx}$ (intercept on X-axis)

At the Y axis, $X = 0$, and $Y = y - \frac{dy}{dx}$ (intercept on Y-axis)

Similar information can be obtained for normals by writing equations as $(Y - y)\frac{dy}{dx} + (X - x) = 0$.

Illustration :

Find the equation of the curve passing through (2, 1) which has constant sub-tangent.

Sol. *We are given that*

$$\text{sub-tangent} = \frac{y}{\frac{dy}{dx}} = (\text{constant}) \qquad = k \text{ (say)}$$

$$\Rightarrow \quad k\frac{dy}{y} = dx$$

Integrating we get, $k \ln y = x + c$

Given that curve passes through (2, 1) $\Rightarrow c = -2$

Hence the equation of such curve is $k \ln y = x - 2$.

Illustration :

Find the curve such that the intercept on the x-axis cut off between the origin and the tangent at a point is twice the abscissa and which passes through the point (1, 2).

Sol. *The equation of the tangent at any point P(x, y) is*

$$Y - y = \frac{dy}{dx}(X - x)$$

Given that intercept on X-axis (putting $Y = 0$) = 2(x-coordinate of P)

$$\Rightarrow \quad x - y\frac{dx}{dy} = 2x$$

$$\Rightarrow \quad -\frac{dy}{y} = -\frac{dx}{x}$$

Integrating we get xy = c

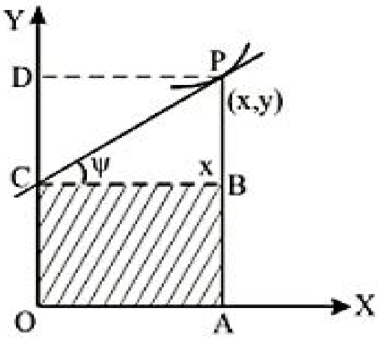
Since the curve passes through (1, 2), c = 2

Hence, the equation of the required curve is xy = 2

Illustration :

Find the equation of the curve which is such that the area of the rectangle constructed on the abscissa of any point and the intercept of the tangent at this point on y-axis is equal to 4.

Sol. *Equation of tangent P(x, y) is* $Y - y = \frac{dy}{dx}(X - x)$



$$\therefore \text{ Y-intercept} = y - x\frac{dy}{dx}$$

$$\therefore \text{ area of } OABC = \left| x\left(y - x\frac{dy}{dx}\right)\right| = 4$$

$$\Rightarrow xy - x^2\frac{dy}{dx} = \pm 4$$

$$\Rightarrow \frac{dy}{dx} - \frac{1}{x}y = \pm\frac{4}{x^2}$$

$$\therefore \text{ I.F.} = e^{-\int \frac{1}{x}dx} = e^{-\ln x} = \frac{1}{x}$$

$$\therefore \text{ the solution is } \left(\frac{y}{x}\right) = \pm 4\int \frac{1}{x^3}dx + c$$

$$\Rightarrow \frac{y}{x} = \pm\frac{2}{x^2} + c$$

Illustration :

Find the equation of the curve passing through the origin if the middle point of the segment of its normal from any point of the curve to the x-axis lies on the parabola $2y^2 = x$.

Sol. *Equation of normal at any point P(x, y) is*

$$\frac{dy}{dx}(Y - y) + (X - x) = 0$$

This meets the x-axis at $A\left(x + y\frac{dy}{dx}, 0\right)$

Mid point of AP is $\left(x + \frac{1}{2}y\frac{dy}{dx}, \frac{y}{2}\right)$ *which lies on the parabola* $2y^2 = x$.

$$\therefore \quad 2 \times \frac{y^2}{4} = x + \frac{1}{2} y \frac{dy}{dx} \text{ or } y^2 = 2x + y\frac{dy}{dx}$$

Putting $y^2 = t$, *so that* $2y\dfrac{dy}{dx} = \dfrac{dt}{dx}$,

we get $\dfrac{dt}{dx} - 2t = -4x$ *(linear)*

$I.F. = e^{-2\int dx} = e^{-2x}$

$\therefore$ *solution is*

$$t\,e^{-2x} = -4\int xe^{-2x}\,dx + c = -4\left[\frac{1}{2}xe^{-2x} - \int -\frac{1}{2}e^{-2x}\,dx\right] + c$$

$\Rightarrow \quad y^2 e^{-2x} = 2xe^{-2x} + e^{-2x} + C$

or $\quad y^2 = 2x + 1 - e^{2x}$ *is the equation of the required curve.*

5.4 Trajectories :

Suppose we are given the family of plane curves $\phi(x, y, a) = 0$, depending on a single parameter a.

A curve making at each of its points a fixed angle α with the curve of the family passing through that point is called an ***isogonal trajectory*** of that family ; if in particular $\alpha = \pi/2$, then it is called an ***orthogonal trajectory***.

To find Orthogonal trajectories :

We set up the differential equation of the given family of curves. Let it be of the form $F(x, y, y') = 0$

The differential equation of the orthogonal trajectories is of the form $F\left(x,\ y,\ -\dfrac{1}{y'}\right) = 0$

The general integral of this equation $\Phi_1(x, y, C) = 0$, gives the family of orthogonal trajectories.

Illustration :

Find the orthogonal trajectory of the following curves

(i) $\quad x^2 + y^2 = a^2$

(ii) $\quad x^2 + y^2 - 2ay = 0$

Sol.

(i) $\quad x^2 + y^2 = a^2 \quad \Rightarrow \quad 2x + 2yy' = 0 \quad \Rightarrow \quad x + yy' = 0$

For orthogonal trajectory, replace y' *by* $\dfrac{-1}{y'}$

$$x - \frac{y}{y'} = 0 \quad \Rightarrow \quad xy' - y = 0$$

$$\Rightarrow \quad x\,dy - y\,dx = 0 \qquad \Rightarrow \quad \frac{dy}{y} - \frac{dx}{x} = 0$$

$$\Rightarrow \quad \ln y - \ln x = \ln c \qquad \Rightarrow \quad y = cx$$

(ii) $x^2 + y^2 - 2ay = 0 \quad \Rightarrow \quad 2x + 2yy' - 2ay' = 0 \quad \Rightarrow \quad 2a = \left(\frac{2x + 2yy'}{y'}\right)$

$$\therefore \quad x^2 + y^2 - \left(\frac{2x + 2yy'}{y'}\right)y = 0$$

For orthogonal trajectory, replace y' by $\frac{-1}{y'}$

$$x^2 + y^2 - \frac{\left(2x - \frac{2y}{y'}\right)y}{\left(\frac{-1}{y'}\right)} = 0 \quad \Rightarrow \quad x^2 + y^2 + (2xy' - 2y)\,y = 0 \quad \Rightarrow \quad x^2 + y^2 + 2xy\,\frac{dy}{dx} - 2y^2 = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y^2 - x^2}{2xy} \quad \text{(a homogeneous equation)}$$

Put $y = vx \qquad \therefore \qquad \frac{dy}{dx} = v + x\frac{dv}{dx}$

$$\Rightarrow \quad v + x\frac{dv}{dx} = \frac{v^2 - 1}{2v} \quad \Rightarrow \quad x\frac{dv}{dx} = \frac{v^2 - 1}{2v} - v = \frac{-(v^2 + 1)}{2v} \quad \Rightarrow \quad \frac{2v}{v^2 + 1}dv = \frac{-dx}{x}$$

$$\Rightarrow \quad \ln(v^2 + 1) = -\ln x \qquad + \ln c \quad \Rightarrow \quad x^2 + y^2 = cx$$

Illustration :

Find the orthogonal trajector of $y^2 = 4ax$ (a being the parameter).

Sol. $y^2 = 4ax$

$$2y\,\frac{dy}{dx} = 4a$$

Eliminating a from equaiton (1) and (2)

$$y^2 = 2y\,\frac{dy}{dx}x$$

Replacing $\frac{dy}{dx}$ *by* $-x\,\frac{dx}{dy}$, *we get*

$$y = 2\left(-\frac{dx}{dy}\right)x$$

$2\,x\,dx + y\,dy = 0$

Integrating each term,

$$x^2 + \frac{y^2}{2} = c$$

$2x^2 + y^2 = 2c$

which is required orthogonal trajectory.

Illustration :

Find the orthogonal trajectories of $x\,y = c$.

Sol. $x\,y = c$

Differentiating w.r.t. x, we get $x\,\frac{dy}{dx} + y = 0.$

Replace $\frac{dy}{dx}$ *by* $-\frac{dx}{dy}$ *to get* $x\frac{dx}{dy} - y = 0$

Integrating $x\,dx - y\,dy = 0$

$\Rightarrow x^2 - y^2 = c$

This is the family of required orthogonal trajectories.

Practice Problem

Q.1 A storage tank contains 2000 litres of gasoline which initially has 100 gms of an additive dissolved in it. Gasoline containing 2 gms of additive per litre is pumped into the tank at a rate of 40 litre/min. The well mixed solution is pumped out at a rate of 45 litre/min. Form the DE and express the amount of additive in gasoline as a function of t.

Q.2 A tank contains 200 litres of brine in which 20 gms of salt dissolved. Brine containing 1/4 gm of salt/lit. runs into the tank at the rate of 2 litre/min. The mixture is kept stirring runs out at the same rate. Express the concentration of solution in tank in grams as a function of time t. What is the limiting value approached by the amount of salt as $t \to \infty$. Whenever was the amount of salt in solution be 20 gm.

Q.3 Find the time required for a cylindrical tank of radius 2.5m and height 3 m to empty through a round hole of 2.5 cm with a velocity 2.5 $\sqrt{h}\ ms^{-1}$, h being the depth of the water in the tank.

Q.4 If the population of country doubles in 50 years, in how many years will it triple under the assumption that the rate of increse in proportional to the number of inhabitants.

Q.5 The rate at which a substance cools in moving air is proportional to the difference between the temperatures of the substance and that of the air. If the temperature of the air is 290 K and the substance cools from 370 K to 330 K in 10 min., when will the temperature be 295 K.

Q.6 Find the orthogonal trajectory of the following curves

(i) $x^2 - \frac{1}{3}y^2 = a^2$ (ii) $y = \tan x + c$ (iii) $\cos y = ae^{-x}$ (iv) $y^2 = 4(x - a)$

Answer key

Q.1 $y = 10(400 - t) - 3900\left(1 - \frac{t}{400}\right)^9$

Q.2 (i) $y = 50 - 30 \cdot e^{-\frac{t}{100}}$, (ii) y when $t \to \infty = 50$ gms, (iii) $y > 20$ for all $t > 0$

Q.3 $8000\sqrt{3}$ s

Q.4 $50 \log_2 3$

Q.5 40 min

Q.6 (i) $xy^3 = c$, (ii) $2x + 4y + \sin 2x = k$, (iii) $\sin y = c\, e^{-x}$, (iv) $y = c\, e^{-x/2}$

VECTOR

VECTORS AND SCALARS :

The physical quantities (we deal with) are generally of two types:

Scalar Quantity:

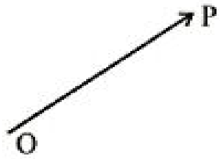
A quantity which has magnitude but no sense of direction is called scalar quantity (or scalar), e.g., mass, volume, density, speed etc.

Vector Quantity:

A quantity which has magnitude as well as a sense of direction in space and obey the laws of vector algebra is called a vector quantity, e.g., velocity, force, displacement etc.

Notation and Representation of Vectors :

Vectors are represented by $\vec{a}$, $\vec{b}$, $\vec{c}$ and their magnitude (modulus) are represented by a, b, c, or $|\vec{a}|$, $|\vec{b}|$, $|\vec{c}|$. The vectors are represented by directed line segments.



For example, line segment $\overrightarrow{OP}$ represents a vector with magnitude OP (length of line segment), arrow denotes its direction. O is initial point and P is terminal point, also called as head & tail of vector respectively.

KINDS OF VECTORS :

1. **Zero or null vector :** A vector whose magnitude is zero is called zero or null vector and it is denoted by 0 or $\vec{0}$. The initial and terminal points of the directed line segment representing zero vector are coincident and its direction is arbitrary.

2. **Unit vector :** A vector of unit magnitude is called a unit vector. A unit vector in the direction of $\vec{a}$ is denoted by $\hat{a}$. Thus

$$\hat{a} = \frac{\vec{a}}{|\vec{a}|} = \frac{\text{Vector a}}{\text{Magnitude of a}}$$

Note :

(i) $|\hat{a}| = 1$

(ii) Unit vectors parallel to x- axis, y- axis and z- axis are denoted by $\hat{i}$, $\hat{j}$ and $\hat{k}$ respectively.

(iii) Two unit vectors may not be equal unless they have the same direction.

3. **Equal Vectors :** Two vectors $\vec{a}$ and $\vec{b}$ are said to be equal , if

(a) $|\vec{a}| = |\vec{b}|$

(b) they have the same sense of direction

4. **Co-initial vectors:** Vectors having same initial point.

5. **Free vectors:** All such vectors are those which when transformed into space from one point to another point without affecting their magnitude and direction, can be considered as equal. i.e. the physical effects produced by them remains unaltered. e.g. displacement, velocity

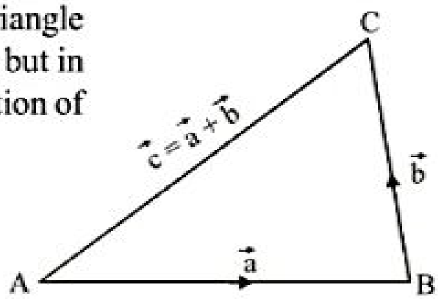
6. **Localised vectors:** e.g. force , different physical effect if line of application is changed.

Note : In mathematics we mainly deal with free vectors.

ADDITION OF VECTORS :

Triangle law of addition :

If two vectors are represented by two consecutive sides of a triangle then their sum is represented by the third side of the triangle but in opposite direction. This is known as the triangle law of addition of vectors.



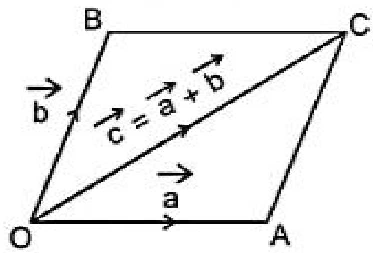
Thus, if $\overrightarrow{AB} = \vec{a}$, $\overrightarrow{BC} = \vec{b}$, and $\overrightarrow{AC} = \vec{c}$

then $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$ i.e. $\vec{a} + \vec{b} = \vec{c}$

Converse of triangle law is also true.

Parallelogram Law of Addition :

If two vectors are represented by two adjacent sides of a parallelogram, then their sum is represented by the diagonal of the parallelogram whose initial point is the same as the initial point of the given vectors. This is known as parallelogram law of addition of vectors.



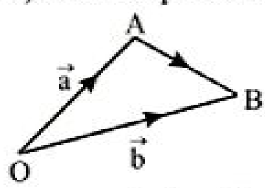
Thus if $\overrightarrow{OA} = \vec{a}$, $\overrightarrow{OB} = \vec{b}$ and $\overrightarrow{OC} = \vec{c}$

then $\overrightarrow{OA} + \overrightarrow{OB} = \overrightarrow{OC}$

i.e. $\vec{a} + \vec{b} = \vec{c}$

POSITION VECTORS :

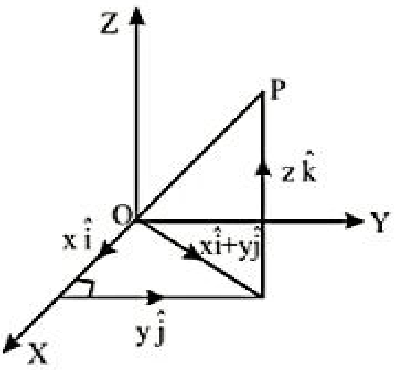
Let O be fixed point in space, then vector $\overrightarrow{OP}$ (P is any point in space) is called position vector of point P w.r.t. O. If A and B are any two point in space then



$\overrightarrow{AB}$ = p.v. of B – p.v. of A = $\overrightarrow{OB} - \overrightarrow{OA}$.

i.e. $\overrightarrow{AB} = \vec{b} - \vec{a}$

Note : Position vector of a point P (x, y, z) in terms of its cartesian coordinate is $\overrightarrow{OP} = x\hat{i} + y\hat{j} + z\hat{k}$.



MULTIPLICATION OF A VECTOR BY SCALAR :

If $\vec{a}$ is a vector & m is a scalar, then $m\vec{a}$ is a vector parallel to $\vec{a}$ whose modulus is $|m|$ times that of $\vec{a}$. This multiplication is called SCALAR MULTIPLICATION. If $\vec{a}$ & $\vec{b}$ are vectors & m, n are scalars, then :

$m(\vec{a})=(\vec{a})m=m\vec{a}$ $\qquad$ $m(n\vec{a})=n(m\vec{a})=(mn)\vec{a}$

$(m+n)\vec{a}=m\vec{a}+n\vec{a}$ $\qquad$ $m(\vec{a}+\vec{b})=m\vec{a}+m\vec{b}$

DISTANCE BETWEEN TWO POINTS :

Let A and B be two given points whose coordinate are respectively (x_1, y_1, z_1) and (x_2, y_2, z_2)

If $\vec{a}$ and $\vec{b}$ are p.v. of A and B relative to point O, then

$$\vec{a}=x_1\hat{i}+y_1\hat{j}+z_1\hat{k}$$

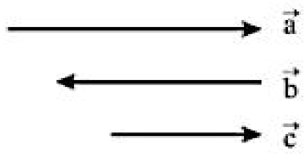
$$\vec{b}=x_2\hat{i}+y_2\hat{j}+z_2\hat{k}$$

Now $\overrightarrow{AB}=\overline{OB}-\overline{OA}=\vec{b}-\vec{a}=(x_2-x_1)\hat{i}+(y_2-y_1)\hat{j}+(z_2-z_1)\hat{k}$

Distance between the points $\vec{A}$ and $\vec{B}$ = magnitude of $\overrightarrow{AB}=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$

COLLINEAR VECTORS OR PARALLEL VECTORS :

Vectors which are parallel to the same line are called collinear vectors or parallel vectors. Such vectors have either same direction or opposite direction. If they have the same direction they are said to be like vectors, and if they have opposite directions, they are called unlike vectors.



In the diagram $\vec{a}$ and $\vec{c}$ are like vectors whereas $\vec{a}$ and $\vec{b}$ are unlike vectors.

i.e. $\quad \vec{a}=k_1\vec{c} \;\; (k_1>0), \quad \vec{a}=k_2\vec{b} \;\; (k_2<0)$

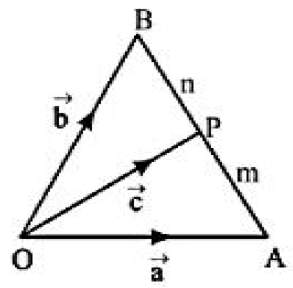
COPLANAR VECTORS :

If the directed line segments of some given vectors are parallel to the same plane then they are called coplanar vectors. It should be noted that two vectors are always coplanar but three or more vectors may or may not be coplanar.

SECTION FORMULAE :

If $\vec{a}$ and $\vec{b}$ are the position vectors of two points A and B, then the position vector $\vec{c}$ of a point P dividing AB in the ratio $m:n$ is given by

$$\vec{c}=\frac{m\vec{b}+n\vec{a}}{m+n}$$

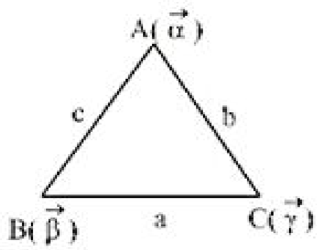


Particular Case :

1. Position vector of the mid point of AB is $\dfrac{\vec{a}+\vec{b}}{2}$

2. If the point P divides AB in the ratio m: n externally, then p.v. of P is given by $\vec{c} = \dfrac{m\vec{b}-n\vec{a}}{m-n}$

Using section Formulae we can prove that :



1. p.v. of the centroid of a triangle ABC $= \dfrac{\vec{\alpha}+\vec{\beta}+\vec{\gamma}}{3}$

(Concurrency of medians)

2. p.v. of incentre of the $\Delta = \dfrac{a\vec{\alpha}+b\vec{\beta}+c\vec{\gamma}}{a+b+c}$

(Concurrency of internal angle bisectors)

Excentres of the Δ are $\dfrac{-a\vec{\alpha}+b\vec{\beta}+c\vec{\gamma}}{-a+b+c}$; $\dfrac{a\vec{\alpha}-b\vec{\beta}+c\vec{\gamma}}{a-b+c}$ and $\dfrac{a\vec{\alpha}+b\vec{\beta}-c\vec{\gamma}}{a+b-c}$

3. p.v. of circumcentre of the $\Delta = \dfrac{\vec{\alpha}\sin 2A+\vec{\beta}\sin 2B+\vec{\gamma}\sin 2C}{\sum \sin 2A}$

(Concurrency of perpendicular bisectors of sides)

4. p.v. of orthocenter of the $\Delta = \dfrac{\vec{a}\,\tan A+\vec{b}\,\tan B+\vec{c}\,\tan C}{\sum \tan A}$

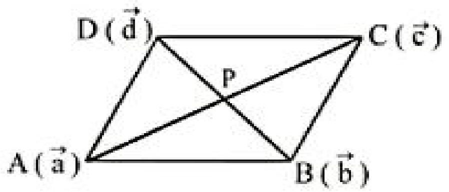
(Concurrency of altitudes)

Illustration :

ABCD is a parallelogram whose diagonals meet at P. If O is a fixed point, then $\overrightarrow{OA}+\overrightarrow{OB}+\overrightarrow{OC}+\overrightarrow{OD}$ *equals-*

(A) $\overrightarrow{OP}$ (B) $2\overrightarrow{OP}$ (C) $3\overrightarrow{OP}$ (D) $4\overrightarrow{OP}$

Sol. *Since, P bisects both the diagonal AC and BD, so*



$\therefore\ \overrightarrow{OA}+\overrightarrow{OC}=2\overrightarrow{OP}$ *and* $\overrightarrow{OB}+\overrightarrow{OD}=2\overrightarrow{OP}\Rightarrow\overrightarrow{OA}+\overrightarrow{OB}+\overrightarrow{OC}+\overrightarrow{OD}=4\overrightarrow{OP}$ *Ans.[D]*

Illustration :

If $\vec{a}$, $\vec{b}$ *are represented by the sides AB and BC of a regular hexagon ABCDEF, then vector represented by FA will be-*

(A) $\vec{a}+\vec{b}$ (B) $\vec{b}-\vec{a}$ (C) $\vec{a}-\vec{b}$ (D) $2\vec{b}-\vec{a}$

Sol. $\because\ \overrightarrow{AC}=\vec{a}+\vec{b}$

$\overrightarrow{AD}=2\,\overrightarrow{BC}=2\vec{b}$ $(\because$ *By property of hexagon AD = 2BC* $)$

$\therefore\ \overrightarrow{DC}=\overrightarrow{DA}+\overrightarrow{AC}$

$=-2\vec{b}+(\vec{a}+\vec{b})=\vec{a}-\vec{b}$

But $\overrightarrow{FA}=\overrightarrow{DC}\Rightarrow\overrightarrow{FA}=\vec{a}-\vec{b}$ *Ans.[C]*

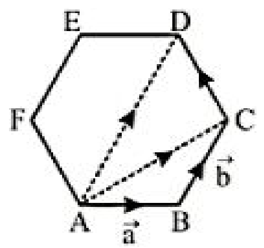


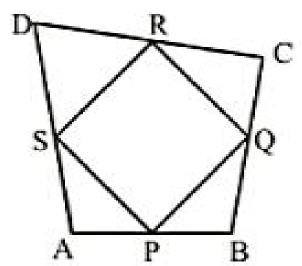
Illustration :

If the mid-points of the consecutive sides of a quadrilateral are joined, prove that the resulting quadrilateral is a parallelogram.

Sol. *Let ABCD be the given quadrilateral and P, Q, R, S be the mid-points of sides AB, BC, CD and AD respectively. Let O be origin of reference and let* $\vec{a},\vec{b},\vec{c},\vec{d}$ *be the position vectors of A, B, C and D respectively.*

Then, $\overrightarrow{OP}=\frac{1}{2}(\vec{a}+\vec{b}),\ \overrightarrow{OQ}=\frac{1}{2}(\vec{b}+\vec{c})$

$\overrightarrow{OR}=\frac{1}{2}(\vec{c}+\vec{d}),\ \overrightarrow{OS}=\frac{1}{2}(\vec{d}+\vec{a})$



Now, $\overrightarrow{PQ}=\overrightarrow{OQ}-\overrightarrow{OP}=\frac{1}{2}(\vec{b}+\vec{c}-\vec{a}-\vec{b})=\frac{1}{2}(\vec{c}-\vec{a})$

and $\overrightarrow{SR}=\overrightarrow{OR}-\overrightarrow{OS}=\frac{1}{2}(\vec{c}+\vec{d}-\vec{d}-\vec{a})=\frac{1}{2}(\vec{c}-\vec{a})$

$\therefore\ \overrightarrow{PQ}=\overrightarrow{SR}$

Thus, PQ = SR and PQ || SR i.e., two opposite sides of PQRS are equal and parallel. Hence PQRS is a parallelogram.

Practice Problem

Q.1 If G is the centroid of ΔABC, prove that $\overrightarrow{GA}+\overrightarrow{GB}+\overrightarrow{GC}=0$. Further if G_1 be the centroid of another ΔPQR, show that $\overrightarrow{AP}+\overrightarrow{BQ}+\overrightarrow{CR}=3\overrightarrow{GG_1}$.

Q.2 Show that the points $A\left(2\hat{i}-\hat{j}+\hat{k}\right), B\left(\hat{i}-3\hat{j}-5\hat{k}\right)$ and $C\left(3\hat{i}-4\hat{j}-4\hat{k}\right)$ are the vertices of a right-angled triangle.

Q.3 If $\vec{a}$ and $\vec{b}$ are non-collinear vectors and

$$\vec{A}=(x+4y)\vec{a}+(2x+y+1)\vec{b} \text{ and } \vec{B}=(y-2x+2)\vec{a}+(2x-3y-1)\vec{b}$$

find x and y such that $3\vec{A}=2\vec{B}$.

Answer key

Q.3 $x=2, y=-1$

RELATION BETWEEN TWO PARALLEL VECTORS :

1. If $\vec{a}$ and $\vec{b}$ be two parallel vectors, then there exists a non-zero scalar k such that $\vec{a}=k\vec{b}$
i.e. there exist two non- zero scalar quantities x and y so that $x\vec{a}+y\vec{b}=0$

2. If a and b be two non-zero non-parallel vectors then $x\vec{a}+y\vec{b}=0 \Rightarrow x=0$ and $y=0$

3. If $x\vec{a}+y\vec{b}=0 \Rightarrow \begin{cases} \vec{a}=0, \vec{b}=0 \\ \text{or} \\ x=0, y=0 \\ \text{or} \\ \vec{a}\parallel\vec{b} \end{cases}$

4. If $\vec{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$ and $\vec{b}=b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$ then from the property of parallel vector, we have $\vec{a}\parallel\vec{b}$

$$\Rightarrow \quad \frac{a_1}{b_1}=\frac{a_2}{b_2}=\frac{a_3}{b_3}$$

Illustration :

The value of λ when $\vec{a}=2\hat{i}-3\hat{j}+\hat{k}$ and $\vec{b}=8\hat{i}+\lambda\hat{j}+4\hat{k}$ are parallel is -
(A) 4 *(B) – 6* *(C) – 12* *(D) 1*

Sol. *Since* $\vec{a}\parallel\vec{b} \Rightarrow \frac{a_1}{b_1}=\frac{a_2}{b_2}=\frac{a_3}{b_3}$

$\therefore \frac{2}{8}=-\frac{3}{\lambda}=\frac{1}{4} \Rightarrow \lambda=-12$ *Ans. [C]*

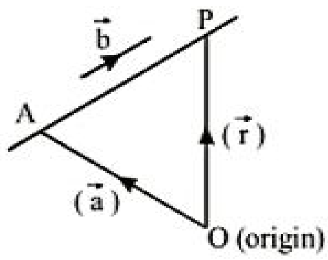
VECTOR EQUATION OF A STRAIGHT LINE :

Vector equation of a straight line passing through a given point $A(\vec{a})$ and parallel to a given vector $\vec{b}$:

Let O be the origin. Let the line pass through a given point A whose position vector is $\vec{a}$, then $\overrightarrow{OA} = \vec{a}$

Let the given line be parallel to vector $\vec{b}$

Let $\vec{r}$ be the position vector any point P on the line, then



$$\overrightarrow{OP} = \vec{r}$$

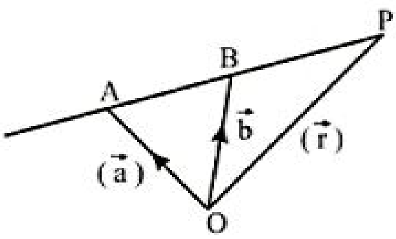
Since $\overrightarrow{AP}$ is parallel to $\vec{b}$ $\quad\therefore\quad$ $\overrightarrow{AP} = t\vec{b}$, where t is a scalar.

Now $\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AP}$ $\quad\therefore\quad$ $\boxed{\vec{r} = \vec{a} + t\vec{b}}$ ...(i)

Since for different values of t, we get different positions of point P on the line, hence (i) is the vector equation of the required straight line.

Vector equation of straight line passing through two given point $A(\vec{a})$ and $B(\vec{b})$:

Let O be the origin. Let the line pass through two given point A and B whose position vectors referred to O be $\vec{a}$ and $\vec{b}$ respectively, then



$$\overrightarrow{OA} = \vec{a} \text{ and } \overrightarrow{OB} = \vec{b}$$

$$\therefore \quad \overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \vec{b} - a$$

Clearly, the required line passes through $A(\vec{a})$

and is parallel to the vector $(\vec{b} - \vec{a})$.

Hence the vector equation of the required line is,

$$\vec{r} = \vec{a} + t(\vec{b} - \vec{a}) \quad \text{or} \quad \boxed{\vec{r} = (1-t)\vec{a} + t\vec{b}}$$

Important Note :

(i) Two lines in a plane are either intersecting or parallel conversely two intersecting or parallel lines must be in the same plane

(ii) However in space we can have two neither parallel nor intersecting lines. Such non coplanar lines are known as skew lines. If two lines are parallel and have a common point then they are coincident.

Vector equation of bisectors of angle between two straight lines :

Let OA and OB be the given straight line parallel to unit vectors $\hat{a}$ and $\hat{b}$ respectively. Take the point O as origin, and let Q be a point on the internal bisector of the angle AOB. From Q draw QR parallel to OA cutting OB at R.

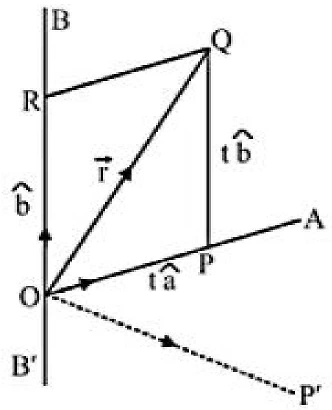
Now $\therefore$ $\angle AOQ = \angle BOQ$ (as OQ is the bisector)

and $\angle BOQ = \angle OQP$ (alternative angles)

$\therefore$ $\angle AOQ = \angle OQP$ $\therefore$ OP = QP = t (say), where t is a scalar.

$\therefore$ $\overrightarrow{OP} = t\,\hat{a}$ and $\overrightarrow{PQ} = t\,\hat{b}$

$\overrightarrow{OQ} = \vec{r}$



Let $\overrightarrow{OQ} = \overrightarrow{OP} + \overrightarrow{PQ}$

or $\vec{r} = t\hat{a} + t\hat{b}$

$\vec{r} = t\left(\hat{a} + \hat{b}\right)$

or, $\vec{r} = t\left(\dfrac{\vec{a}}{a} + \dfrac{\vec{b}}{a}\right)$

where $|\vec{a}| = a, |\vec{b}| = b$

This is the equation of internal bisector of $\angle AOB$.

Equation of external Bisector :

If OP' be the external bisector of $\angle OAB$, then OP' may be regarded as the internal bisector of the angle between the lines which are parallel to $\hat{a}$ and $-\hat{b}$. Hence its equaiton is

$\vec{r} = t(\hat{a} - \hat{b})$ or, $\vec{r} = t\left(\dfrac{\hat{a}}{a} - \dfrac{\hat{b}}{a}\right)$

Corollary : If the the lines intersect at E having position vector $\vec{\alpha}$, then the above equations becomes

$\vec{r} = \vec{\alpha} + t(\hat{a} + \hat{b})$ and $\vec{r} = \vec{\alpha} + t(\hat{a} - \hat{b})$ respectively.

Illustration :

Find the vector equation of the line through the point $2\vec{i} + \vec{j} - 3\vec{k}$ and parallel to the vector $\vec{i} + 2\vec{j} + \vec{k}$.

Sol. *Let the given point be $A(\vec{a})$ and given vector be $\vec{b}$ and O be the origin.*

Then, $\vec{a} = \overrightarrow{OA} = 2\hat{i} + \hat{j} - 3\hat{k}$ *and* $\vec{b} = \hat{i} + 2\hat{j} + \hat{k}$

Now vector equaiton of the line through A and parallel to $\vec{b}$ is

$\vec{r} = \vec{a} + t\vec{b}$, *where t is a scalar.*

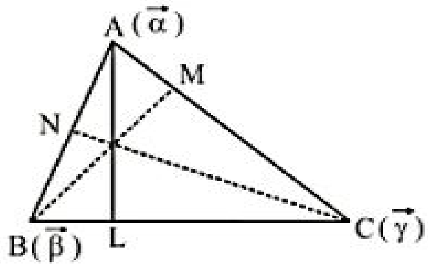
or, $\vec{r} = 2\hat{i} + \hat{j} - 3\hat{k} + t(\hat{i} + 2\hat{j} + \hat{k})$

Illustration :

Prove, by vector method that the internal bisectors of the angles of a triangle are concurrent.

Sol. *Let* $\vec{\alpha}, \vec{\beta}, \vec{\gamma}$ *be the position vectors of vertices A, B, C respectively of* ΔABC.

Let AL be the bisectors of $\angle ABC$



Then $\dfrac{BL}{LC} = \dfrac{BA}{AC} = \dfrac{c}{b}$

Thus L divides BC internally in the ratio $c : b$

$\therefore$ *P.V. of L* $= \dfrac{b\vec{\beta} + c\vec{\gamma}}{b+c}$

Now P.V. of the point which divide AL internally in the ratio $b + c : a$ *will be*

$$\frac{a\vec{\alpha} + b\vec{\beta} + c\vec{\gamma}}{a+b+c}$$

[Here we have divided AL in the ratio $b + c : a$ *because* $b + c$ *occurs in the denominator of P.V. of L and there is* $b\vec{\beta}$ *and* $c\vec{\gamma}$ *therefore, there should also be* $a\vec{\alpha}$*.]*

Similarly, we can show that the position vectors of the points which divide bisectors BM of $\angle ABC$ *and CN of* $\angle ACB$ *in the ratio* $c + a : b$ *and* $a + b : c$ *respectively will be each* $\dfrac{a\vec{\alpha} + b\vec{\beta} + c\vec{\gamma}}{a+b+c}$.

Thus the point having position vector $\dfrac{a\vec{\alpha} + b\vec{\beta} + c\vec{\gamma}}{a+b+c}$ *lies on the three internal bisectors of the angles of the triangle ABC and hence internal bisectors of angles of a triangle are concurrent and the position vector of incentre of* ΔABC *will be* $\dfrac{a\vec{\alpha} + b\vec{\beta} + c\vec{\gamma}}{a+b+c}$.

COLLINEARITY OF THREE POINTS :

1. If $\vec{a}, \vec{b}, \vec{c}$ be position vectors of three points A,B and C respectively and x, y, z be three scalars so that all are not zero, then the necessary and sufficient conditions for three points to be collinear is that

$$x\vec{a} + y\vec{b} + z\vec{c} = 0 \quad \text{where} \quad x + y + z = 0$$

2. Three points A, B and C are collinear, if $\overrightarrow{AB} = \lambda \overrightarrow{BC}$.

Illustration :

If $2\vec{a} - 3\vec{b}$, $\vec{b}$ *and* $\vec{a} - \vec{b}$ *are position vectors of three points A,B and C then they are -*

(A) Collinear *(B) Non-collinear* *(C) Can't say anything* *(D) None of these*

Sol. $\because 1\,(2\vec{a} - 3\vec{b}) + 1\,(\vec{b}) - 2\,(\vec{a} - \vec{b}) = 0$ *and* $1 + 1 - 2 = 0$

so the given vectors are collinear. *Ans.[A]*

Illustration :

If $A \equiv (2\hat{i} + 3\hat{j})$, $B \equiv (p\hat{i} + 9\hat{j})$ *and* $C \equiv (\hat{i} - \hat{j})$ *are collinear, then the value of p is-*

(A) 1/2 *(B) 3/2* *(C) 7/2* *(D) 5/2*

Sol. $\overrightarrow{AB} = (p-2)\,\hat{i} + 6\,\hat{j}$, $\overrightarrow{AC} = -\,\hat{i} - 4\,\hat{j}$

Now A,B,C are collinear $\Leftrightarrow \overrightarrow{AB} \,||\, \overrightarrow{AC} \Leftrightarrow \dfrac{p-2}{-1} = \dfrac{6}{-4} \Leftrightarrow p = 7/2$ *Ans. [C]*

PRODUCT OF VECTORS :

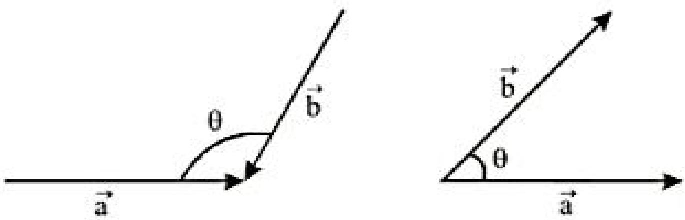
Product of two vectors is done by two methods when the product of two vectors results in a scalar quantity then it is called scalar product. It is also called as dot product because this product is represented by putting a dot.

When the product of two vectors results in a vector quantity then this product is called Vector Product. This product is represented by (x) sign so that it is also called as cross product.

Scalar or dot product of two vectors :

Definition : If $\vec{a}$ and $\vec{b}$ are two vectors and θ be the angle between their tails or heads, then their scalar product (or dot product) is defined as the number $|\vec{a}|\;|\vec{b}|\cos\theta$ where $|\vec{a}|$ and $|\vec{b}|$ are modulii of $\vec{a}$ and $\vec{b}$ respectively and $0 \le \theta \le \pi$. It is denoted by $\vec{a}\,.\,\vec{b}$. Thus

$$\vec{a}\,.\,\vec{b} = |\vec{a}|\,|\vec{b}|\cos\theta$$



Note :

(i) $\vec{a}.\vec{b} \in R$

(ii) $\vec{a}.\vec{b} \leq |\vec{a}||\vec{b}|$

(iii) $\vec{a}.\vec{b} > 0 \Rightarrow$ angle between a and $\vec{b}$ say $\theta \in \left[0, \frac{\pi}{2}\right)$.

$\vec{a}.\vec{b} < 0 \Rightarrow$ angle between $\vec{a}$ and $\vec{b}$ say $\theta \in \left(\frac{\pi}{2}, \pi\right]$.

$\vec{a}.\vec{b} = 0 \Rightarrow$ angle between $\vec{a}$ and $\vec{b}$ say $\theta = \frac{\pi}{2}$ or atleast one of $\vec{a}$ and $\vec{b}$ is zero vector.

(iv) The dot product of a zero and non- zero vector is a scalar zero i.e. $\vec{0}\cdot\vec{a} = 0$.

(v) If θ be angle between any two non zero vector $\vec{a}$ & $\vec{b}$ then $\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|\cdot|\vec{b}|}$.

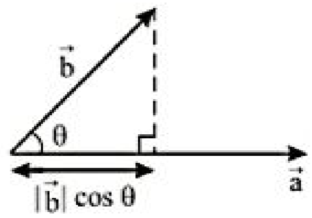
Geometrical Interpretation :

Geometrically, the scalar product of two vectors is equal to the product of the magnitude of one and the projection of second in the direction of first vector i.e. $\vec{a}.\vec{b} = |\vec{a}|\ \ (|\vec{b}|\cos\theta)$

$= |\vec{a}|$ (projection of $\vec{b}$ in the direction of $\vec{a}$)

Similarly $\vec{a}.\vec{b} = |\vec{b}|\ \ (|\vec{a}|\cos\theta)$

$= |b|$ (projection of $\vec{a}$ in the direction of $\vec{b}$)



Here projection of $\vec{b}$ on $\vec{a} = \frac{\vec{a}.\vec{b}}{|\vec{a}|}$

Projection of $\vec{a}$ on $\vec{b} = \frac{\vec{a}.\vec{b}}{|\vec{b}|}$

Illustration :

If angle between $\vec{a}$ and $\vec{b}$ is 120^o and their magnitudes are respectively 2 and $\sqrt{3}$, then a.b equals-

(A) 3 *(B) $-\sqrt{3}$* *(C) $\sqrt{3}$* *(D) -3*

Sol. *We know that $\vec{a}.\vec{b} = |\vec{a}|\ |\vec{b}|\cos\theta$*

$= 2\sqrt{3}\ \cos 120^o$

$= 2\sqrt{3}\ (-1/2) = -\sqrt{3}$ *Ans.[B]*

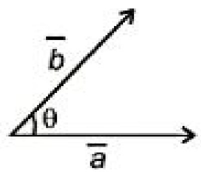
Illustration :

The projection of vector $\hat{i} + \hat{j} + \hat{k}$ on the vector $\hat{i} - \hat{j} + \hat{k}$ is-

(A) $\sqrt{3}$ *(B) $1/\sqrt{3}$* *(C) $2/\sqrt{3}$* *(D) $2\sqrt{3}$*

Sol. *Projection* $= \dfrac{(\hat{i}+\hat{j}+\hat{k}).(\hat{i}-\hat{j}+\hat{k})}{|\hat{i}-\hat{j}+\hat{k}|} = \dfrac{1-1+1}{\sqrt{1+1+1}} = \dfrac{1}{\sqrt{3}}$ *Ans.[B]*

Scalar product in particular cases :



1. $\vec{a}.\vec{b} = \vec{b}.\vec{a}$

2. $\vec{a}.(\vec{b}+\vec{c}) = \vec{a}.\vec{b} + \vec{a}.\vec{c}$

3. $(m\vec{a}).\vec{b} = m(\vec{a}.\vec{b}) = \vec{a}.(m\vec{b})$

4. If $\theta = 0 \Rightarrow \vec{a}.\vec{b} = |\vec{a}||\vec{b}|$ (like vectors)

5. If $\theta = \pi \Rightarrow \vec{a}.\vec{b} = -|\vec{a}||\vec{b}|$ (unlike vectors)

6. If $\hat{a}$ and $\hat{b}$ are unit vectors then $\hat{a}.\hat{b} = \cos\theta$ (where θ is angle between them).

7. $\vec{a}.\vec{a} = |\vec{a}|^2 \Rightarrow |\vec{a}| = \sqrt{\vec{a}.\vec{a}}$

8. If $\vec{a} \perp \vec{b} \Rightarrow \vec{a}.\vec{b} = 0$ but $\vec{a}.\vec{b} = 0 \Rightarrow \vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$ or $\vec{a} \perp \vec{b}$.

9. If $\hat{i}, \hat{j}$ and $\hat{k}$ are unit vectors along the rectangular coordinate are OX, OY and OZ then

   $\hat{i}.\hat{i} = \hat{j}.\hat{j} = \hat{k}.\hat{k} = 1$, $\hat{i}.\hat{j} = \hat{j}.\hat{k} = \hat{k}.\hat{i} = 0$

10. $\vec{a}.\vec{b} = \vec{a}.\vec{c} \not\Rightarrow \vec{b} = \vec{c}$

    Infact $\vec{a}.\vec{b} = \vec{a}.\vec{c} \Rightarrow \vec{a}.(\vec{b}-\vec{c}) = 0 \Rightarrow \vec{a} = 0$ or $\vec{b} = \vec{c}$ or $\vec{a} \perp (\vec{b}-\vec{c})$

11. $(\vec{a}.\vec{b}).\vec{c}$ is meaningless

Note : (a) $(\vec{a}.\vec{b}).\vec{b}$ is not defined

(b) $(\vec{a}+\vec{b})^2 = |\vec{a}|^2 + 2\,\vec{a}.\vec{b} + |\vec{b}|^2$

(c) $(\vec{a}-\vec{b})^2 = |\vec{a}|^2 - 2\vec{a}.\vec{b} + |\vec{b}|^2$

(d) $(\vec{a}+\vec{b}).(\vec{a}-\vec{b}) = |\vec{a}|^2 - |\vec{b}|^2$

(e) $|\vec{a}+\vec{b}| = |\vec{a}| + |\vec{b}| \Rightarrow \vec{a} \parallel \vec{b}$

(f) $|\vec{a}+\vec{b}|^2 = |\vec{a}|^2 + |\vec{b}|^2 \Rightarrow \vec{a} \perp \vec{b}$

(g) $|\vec{a}+\vec{b}| = |\vec{a}-\vec{b}| \Rightarrow \vec{a} \perp \vec{b}$

Illustration :

Determine the values of c such that for all x (real) the vectors $cx\hat{i}-6\hat{j}+3\hat{k}$ *and* $x\hat{i}+2\hat{j}+2cx\hat{k}$ *make an obtuse angle with each other.*

Sol. *If* θ *is the angle between the given vectors then* $\cos\theta=\dfrac{cx^2-12+6cx}{\sqrt{c^2x^2+45}\sqrt{x^2+4c^2x^2+4}}$

If θ *is obtuse then* $\cos\theta<0 \Rightarrow \quad cx^2+6cx-12<0\ \forall\ x\in R$
Which is possible if $c<0$ *and* $36c^2+48c<0 \Rightarrow c<0$ *and* $12c(3c+4)<0$

$\Rightarrow c<0$ *and* $-\dfrac{4}{3}<c<0$ *(but for* $c=0,\ cx^2+6cx-12<0\ \ \forall\ x$*)*

Hence $-\dfrac{4}{3}<c\le 0$.

Illustration :

If $\vec{a},\vec{b},\vec{c}$ *are unit vectors such that* $\vec{a}$ *is perpendicular to the plane of* $\vec{b}$ *and* $\vec{c}$ *then find* $|\vec{a}+\vec{b}+\vec{c}|$ *when the angle between* $\vec{b}$ *and* $\vec{c}$ *is* $\pi/3$.

Sol. *We have* $|\vec{a}|=|\vec{b}|=|\vec{c}|=1, \vec{a}.\vec{b}=\vec{a}.\vec{c}=0$ *and* $\quad \vec{b}.\vec{c}=\cos\dfrac{\pi}{3}=\dfrac{1}{2}$

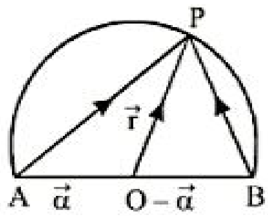
Now $|\vec{a}+\vec{b}+\vec{c}|^2=(\vec{a}+\vec{b}+\vec{c}).(\vec{a}+\vec{b}+\vec{c})=|\vec{a}|^2+|\vec{b}|^2+|\vec{c}|^2+2\vec{b}.\vec{c}\ \ (\because \vec{a}.\vec{c}=\vec{a}.\vec{b}=0)$

$$=1+1+1+2.\frac{1}{2}=4 \quad\Rightarrow\quad |\vec{a}+\vec{b}+\vec{c}|=2$$

Illustration :

Prove that the angle in a semi circle is a right angle.

Sol. *Let O be the centre of the semi circle with AOB as its diameter. Let P be a point on the semi-circle, so that* $\angle APB$ *is an angle in the semi circle. Join OP. Let O be taken as origin. Let the position vectors of A, B and P be* $\vec{\alpha}, -\vec{\alpha}$ *and* $\vec{r}$ *respectively.*



Clearly, $\quad OA=OB=OP$

Now $\quad \overrightarrow{AP}=(\vec{r}-\vec{\alpha})$ *and* $\overrightarrow{BP}=(\vec{r}+\vec{\alpha})$

$\therefore \quad \overrightarrow{AP}\cdot\overrightarrow{BP}=(\vec{r}-\vec{\alpha})\cdot(\vec{r}+\vec{\alpha})=r^2-\alpha^2=OP^2-OA^2=0$

$[\because\ OP=OA]$

$\therefore \quad AP\perp BP$ *i.e.* $\angle APB=90^o$.

Scalar product in terms of components :

Let $\vec{a}$ and $\vec{b}$ be two vectors such that $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{a} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$

Then $\vec{a}.\vec{b} = a_1b_1 + a_2b_2 + a_3b_3$

In particular

$$\vec{a}.\vec{a} = |\vec{a}|^2 = a_1^2 + a_2^2 + a_3^2$$

For any vector $\vec{a}$,

$$\vec{a} = (\vec{a}.\hat{i})\hat{i} + (\vec{a}.\hat{j})\hat{j} + (\vec{a}.\hat{k})\hat{k}$$

Illustration :

If $\vec{a} = 3\hat{i} + 2\hat{j} + \hat{k}$ and $\vec{b} = \hat{i} - 2\hat{j} + 5\hat{k}$ then find $\vec{a}.\vec{b}$.

Sol. $\vec{a}.\vec{b} = (3)(1) + (2)(-2) + (1)(5) = 3 - 4 + 5 = 4.$

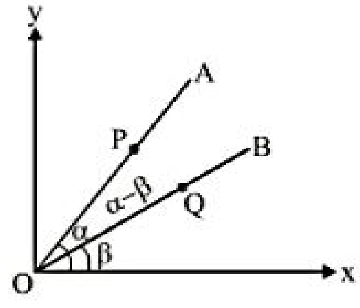
Illustration :

Prove by vector method the following formula of plane trigonometry

$$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta.$$

Sol. *Let the unit vectors along OX and OY be $\hat{i}$ and $\hat{j}$ respectively. If OA and OB be any two lines in the same plane making angles α and β respectively, with OX then, $\angle PB = \alpha - \beta$*

Again, let $\overrightarrow{OP}$ and $\overrightarrow{OQ}$ represent unit vectors along OA and OB respectively, so that their dot product is the consine of the angle between their directions.



Now, $\overrightarrow{OP}\cdot\overrightarrow{OQ} = 1 \cdot 1\cos(\alpha - \beta) = \cos(\alpha - \beta)$ ...(1)

Since $\overrightarrow{OP}$ makes an angle α with x-axis.

$\therefore$ $\overrightarrow{OP} = \cos\alpha\hat{i} + \sin\alpha\hat{j}$

Similarly, $\overrightarrow{OQ} = \cos\beta\hat{i} + \sin\beta\hat{j}$

$\therefore$ $\overrightarrow{OP}\cdot\overrightarrow{OQ} = [\cos\alpha\hat{i} + \sin\alpha\hat{j}]\cdot[\cos\beta\hat{i} + \sin\beta\hat{j}]$

$= \cos\alpha\cdot\cos\beta + \sin\alpha\cdot\sin\beta$ ...(2)

From (1) and (2), we get $\cos(\alpha - \beta) = \cos\alpha\cdot\cos\beta + \sin\alpha\cdot\sin\beta$

Illustration :

In any ΔABC, prove that $\cos C = \dfrac{a^2+b^2-c^2}{2ab}$ *with the help of vectors.*

Sol. *In* ΔABC, $\overrightarrow{BC}+\overrightarrow{CA}+\overrightarrow{AB}=\vec{0}$

$\Rightarrow \quad \overrightarrow{AB} = -(\overrightarrow{BC}+\overrightarrow{CA})$

$\Rightarrow \quad \overrightarrow{AB}\cdot\overrightarrow{AB} = (\overrightarrow{BC}+\overrightarrow{CA})\cdot(\overrightarrow{BC}+\overrightarrow{CA})$

$\Rightarrow \quad \vec{c}\cdot\vec{c} = (\vec{a}+\vec{b})\cdot(\vec{a}+\vec{b})$

$\Rightarrow \quad c^2 = a^2 + b^2 + 2\,\vec{a}\cdot\vec{b}$

$\Rightarrow \quad c^2 = a^2 + b^2 + 2ab\cos(\pi - C)$

$\Rightarrow \quad c^2 = a^2 + b^2 - 2ab\cos C$

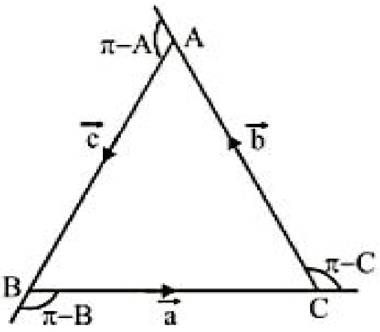


Illustration :

Prove by vector method that $(a_1b_1 + a_2b_2 + a_3b_3)^2 \le (a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2)$

Sol. *Let* $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ *and* $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$

Then $\vec{a}.\vec{b} = a_1b_1 + a_2b_2 + a_3b_3$ ... (1)

Also $|\vec{a}| = \sqrt{a_1^2+a_2^2+a_3^2}$ *and* $|\vec{b}| = \sqrt{b_1^2+b_2^2+b_3^2}$ ... (2)

If θ be the angle between the vectors $\vec{a}$ and $\vec{b}$, then

$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|} \quad \because \cos^2\theta \le 1 \quad \therefore \left(\frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}\right)^2 \le 1$$

or $\quad (\vec{a}\cdot\vec{b})^2 \le |\vec{a}|^2|\vec{b}|^2$

$\therefore \quad (a_1b_1 + a_2b_2 + a_3b_3)^2 \le (a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2)$

Angle between two vectors in terms of components :

If $\vec{a}$ and $\vec{b}$ be two vectors such that $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $b = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ and θ be the angle between them, then

$$\cos\theta = \frac{a_1b_1 + a_2b_2 + a_3b_3}{\sqrt{a_1^2+a_2^2+a_3^2}\sqrt{b_1^2+b_2^2+b_3^2}}$$

Note : If $\vec{a}$ and $\vec{b}$ are perpendicular to each other then $a_1b_1 + a_2b_2 + a_3b_3 = 0$

Illustration :

Find the angle between the vectors $4\hat{i} + \hat{j} + 3\hat{k}$ *and* $2\hat{i} + 2\hat{j} - \hat{k}$.

Sol. *Let the required angle is* θ.

$$\therefore\ \theta = \cos^{-1}\left(\frac{4.2+1.2+3(-1)}{\sqrt{16+1+9}\sqrt{4+4+1}}\right) = \cos^{-1}\left(\frac{7}{3\sqrt{26}}\right)$$

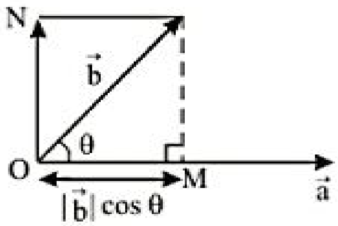
Components of $\vec{b}$ along & perpendicular to $\vec{a}$:

1. Component of $\vec{b}$ along $\vec{a} = \overrightarrow{OM}$

$$= OM\ \hat{a} = (b\cos\theta)\hat{a}$$

$$= \frac{(ab\cos\theta)}{a}\hat{a} = \frac{(\vec{a}.\vec{b})}{a^2}\hat{a}$$

$$= \frac{(\vec{a}.\vec{b})}{|a|^2}.a$$



2. Component perpendicular to $\vec{a} = \overrightarrow{ON}$

$$\because\ \vec{b} = \overrightarrow{ON} + \overrightarrow{OM}$$

$$\therefore\ \overrightarrow{ON} = \vec{b} - \overrightarrow{OM}$$

$$\overrightarrow{ON} = \vec{b} - \frac{(\vec{a}.\vec{b})}{a^2}\vec{a}.$$

Illustration :

Find the vector components of a vector $2\hat{i}+3\hat{j}+6\hat{k}$ *along and perpendicular to non-zero vector* $2\hat{i}+\hat{j}+2\hat{k}$.

Sol. *Let* $\vec{a} = 2\hat{i}+3\hat{j}+6\hat{k}$ *and* $\vec{b} = 2\hat{i}+\hat{j}+2\hat{k}$

Now vector component of $\vec{a}$ *along* $\vec{b}$

$$= \frac{\vec{a}\cdot\vec{b}}{|\vec{b}|}\vec{b} = \frac{4+3+12}{9}(2\hat{i}+\hat{j}+2\hat{k}) = \frac{19}{9}(2\hat{i}+\hat{j}+2\hat{k})$$

and vector component of $\vec{a}$ perpendicular to $\vec{b}$

$$= \vec{a} - \frac{\vec{a}\cdot\vec{b}}{|\vec{b}|^2}\vec{b} = (2\hat{i}+3\hat{j}+6\hat{k}) - \frac{19}{9}(2\hat{i}+\hat{j}+2\hat{k}) = \frac{1}{9}(-20\hat{i}+8\hat{j}+16\hat{k})$$

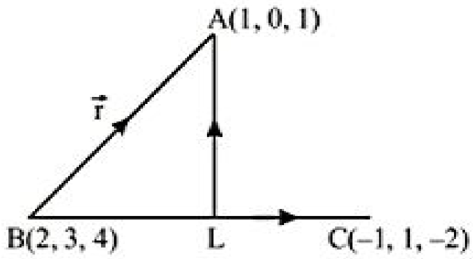
Illustration :

Find the perpendicular distance of the point A(1, 0, 1) to the line through the points B(2, 3, 4) and C(–1, 1, –2).

Sol. $\vec{r} = \overrightarrow{BA} = -\hat{i} - 3\hat{j} - 3\hat{k}$ *and* $\vec{a} = \overrightarrow{BC} = -3\hat{i} - 2\hat{j} - 6\hat{k}$

Now $\overrightarrow{BL}$ = *Projection vector of* $\vec{r}$ *on* $\vec{a}$

$$\frac{\vec{r}\cdot\vec{a}}{|\vec{a}|^2}\vec{a} = \frac{3+6+18}{49}(-3\hat{i}-2\hat{j}-6\hat{k})$$



$$= \frac{-27}{49}(3\hat{i}+2\hat{j}+6\hat{k})$$

$$\overrightarrow{LA} = \overrightarrow{LB} + \overrightarrow{BA} = \overrightarrow{BA} - \overrightarrow{BL}$$

$$= \vec{r} - \frac{\vec{r}\cdot\vec{a}}{|\vec{a}|^2}\cdot\vec{a} = (-\hat{i}-3\hat{j}-3\hat{k}) + \frac{27}{49}(3\hat{i}+2\hat{j}+6\hat{k}) = \frac{1}{49}(32\hat{i}-93\hat{j}+15\hat{k})$$

$$\therefore \quad LA = \left|\overrightarrow{LA}\right| = \frac{\sqrt{9898}}{49}$$

Linear Combination of Vectors :

A vector $\vec{r}$ is said to be a linear combination of the vectors $\vec{a}, \vec{b}, \vec{c}$......

if $\exists$ scalars x,y,z, ... such that $\vec{r} = x\vec{a} + y\vec{b} + z\vec{c} + ...$

Theorem in plane :

If $\vec{a}$ and $\vec{b}$ are two non zero non collinear vetors then any vector $\vec{r}$ coplanar with them can be expressed as a linear combination $\vec{r} = x\vec{a} + y\vec{b}$. (Explain using a sketch)

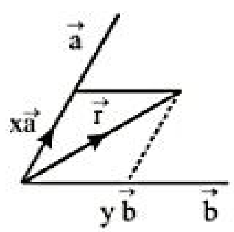
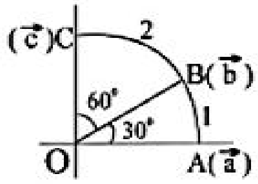


Illustration :

Arc AC of the quadrant of a circle with centre as origin and radius unity subtends a right angle at the origin. Point B divides the arc AC in the ratio 1 : 2. Express the vector $\vec{c}$ in terms of $\vec{a}$ and $\vec{b}$.



Sol. $\because\ \vec{c} = x\vec{a} + y\vec{b}$ ...(i)

Taking dot product with $\vec{a}$ in (i)

$$\vec{c}\cdot\vec{a} = \vec{a}\cdot\vec{a} + y\vec{a}\cdot\vec{b}$$

$$0 = x + y\frac{\sqrt{3}}{2} \qquad ...(ii)$$

Taking dot product with $\vec{c}$ in (i)

$$\vec{c}\cdot\vec{c} = x\,\vec{a}\cdot\vec{c} + y\,\vec{b}\cdot\vec{c}$$

$$1 = 0 + \frac{y}{2} \quad \Rightarrow \quad y = 2$$

from (ii) $x = -\sqrt{3}$ $\quad\therefore\quad$ $\vec{c} = -\sqrt{3}\,\vec{a} + 2\vec{b}$

Practice Problem

Q.1 Given that $\vec{a} = \hat{i} - \hat{j}$ and $\hat{b} = \hat{i} + 2\hat{j}$ are two vectors. Find a unit vector coplanar with $\vec{a}$ and $\vec{b}$ and perpendicular to $\vec{a}$.

Q.2 A line passes through a point with p.v. $\hat{i} - 2\hat{j} - \hat{k}$ and is parallel to the vector $\hat{i} - 2\hat{j} + 2\hat{k}$. Find the distance of a point P (5, 0, – 4) from the line.

Answer key

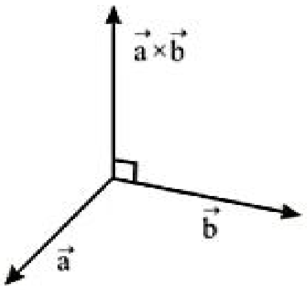
Q.1 $\pm\dfrac{\hat{i}+\hat{j}}{\sqrt{2}}$ Q.2 5

VECTOR OR CROSS PRODUCT OF TWO VECTORS :

Definition :

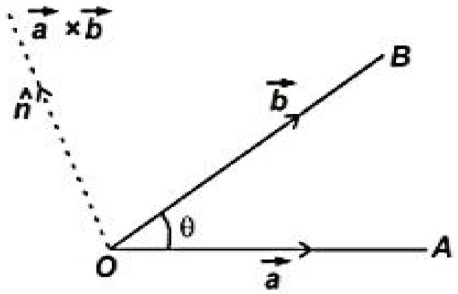
If $\vec{a}$ and $\vec{b}$ be two vectors and θ $(0 \le \theta \le \pi)$ be the angle between them, then their vector (or cross) product is defined to be a vector whose magnitude is $ab\sin\theta$ and whose direction is perpendicular to the plane of $\vec{a}$ and $\vec{b}$ such that $\vec{a}, \vec{b}$ and $\vec{a} \times \vec{b}$ form a right handed system.

$\therefore \quad \vec{a} \times \vec{b} = |\vec{a}||\vec{b}| \sin\theta \, \hat{n}$



Where $\hat{n}$ is a unit vector perpendicular to the plane of $\vec{a}$ and $\vec{b}$ such that $\vec{a}$, $\vec{b}$ and $\hat{n}$ form a right handed system.

Properties of Vector Product :



1. In general, $\vec{a} \times \vec{b} \ne \vec{b} \times \vec{a}$. In fact $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$.

2. For scalar m, $m\vec{a} \times \vec{b} = m(\vec{a} \times \vec{b}) = \vec{a} \times m\vec{b}$.

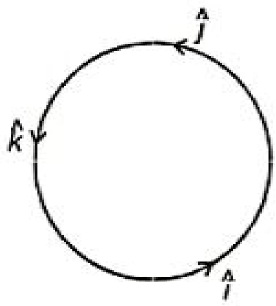
3. $\vec{a} \times (\vec{b} \pm \vec{c}) = \vec{a} \times \vec{b} \pm \vec{a} \times \vec{c}$

4. If $\vec{a} \,||\, \vec{b}$ then $\theta = 0$ or π $\Rightarrow$ $\vec{a} \times \vec{b} = \vec{0}$ (but $\vec{a} \times \vec{b} = \vec{0} \Rightarrow \vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$ or $\vec{a} \parallel \vec{b}$). In particular $\vec{a} \times \vec{a} = \vec{0}$.

5. If $\vec{a} \perp \vec{b}$ then $\vec{a} \times \vec{b} = |\vec{a}||\vec{b}|\hat{n}$ (or $|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}|$)

6. $\hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = 0$ and $\hat{i} \times \hat{j} = \hat{k}$, $\hat{j} \times \hat{k} = \hat{i}$ and

$\hat{k} \times \hat{i} = \hat{j}$ (use cyclic system)



7. Unit vector perpendicular to $\vec{a}$ and $\vec{b}$ is given by $\pm \dfrac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{b}|}$

8. If θ is angle between $\vec{a}$ and $\vec{b}$ then $\sin\theta = \dfrac{|\vec{a} \times \vec{b}|}{|\vec{a}||\vec{b}|}$

9. If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ then

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = (a_2b_3 - a_3b_2)\hat{i} + (a_3b_1 - a_1b_3)\hat{j} + (a_1b_2 - a_2b_1)\hat{k}$$

Illustration :

If $\vec{a} = 2\hat{i} + 2\hat{j} - \hat{k}$ *and* $\vec{b} = 6\hat{i} - 3\hat{j} + 2\hat{k}$ *then* $\vec{a} \times \vec{b}$ *equals*

(A) $2\hat{i} - 2\hat{j} - \hat{k}$ (B) $\hat{i} - 10\hat{j} - 18\hat{k}$ (C) $\hat{i} + \hat{j} + \hat{k}$ (D) $6\hat{i} - 3\hat{j} + 2\hat{k}$

Sol. $\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 2 & -1 \\ 6 & -3 & 2 \end{vmatrix} = \hat{i}\,(4-3) - \hat{j}\,(4+6) + \hat{k}\,(-6-12) = \hat{i} - 10\hat{j} - 18\hat{k}$ *Ans.[B]*

Illustration :

If angle between $\hat{i} - 2\hat{j} + 3\hat{k}$ *and* $2\hat{i} + \hat{j} + \hat{k}$ *is* θ *then* $\sin\theta$ *equals-*

(A) $5/\sqrt{7}$ (B) $5/21$ (C) $5/2\sqrt{7}$ (D) $3/\sqrt{14}$

Sol. *We know that* $\sin\theta = \dfrac{|\vec{a}\times\vec{b}|}{|\vec{a}|\,|\vec{b}|}$

Now $\vec{a} \times \vec{b} = -5i + 5\hat{j} + 5\hat{k}$

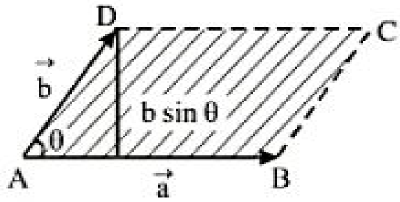
$\therefore \quad |\vec{a} \times \vec{b}| = \sqrt{(5)^2 + (5)^2 + (5)^2} = \sqrt{75} = 5\sqrt{3}$

$|\vec{a}| = \sqrt{1+4+9}, \; |\vec{b}| = \sqrt{4+1+1}$

$\therefore \quad \sin\theta = \dfrac{5\sqrt{3}}{\sqrt{1+4+9}\,\sqrt{4+1+1}} = \dfrac{5\sqrt{3}}{\sqrt{14}\,\sqrt{6}} = \dfrac{5}{\sqrt{28}} = \dfrac{5}{2\sqrt{7}}$ *Ans.[C]*

Geometrical interpretation of vector product :

The vector product of the vectors $\vec{a}$ and $\vec{b}$ represents a vector whose modulus is equal to the area of the parallelogram whose two adjacent sides are represented by $\vec{a}$ and $\vec{b}$.



Area of parallelogram = base × height = ab sin θ = $|\vec{a} \times \vec{b}|$

Area of quadraliteral if its diagonal vectors are $\vec{d}_1$ & $\vec{d}_2$ is given by $= \dfrac{1}{2}|\vec{d}_1 \times \vec{d}_2|$

Illustration :

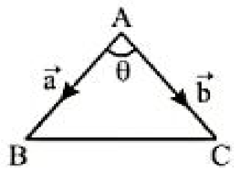
Find the area of a parallelogram whose two adjacent sides are represented by $\vec{a} = 3\hat{i} + \hat{j} + 2\hat{k}$ *and* $b = 2\hat{i} - 2\hat{j} + 4\hat{k}$.

Sol. *Area of parallelogram* $= |\vec{a} \times \vec{b}|$

$$\text{Now } \vec{a} \times b = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 1 & 2 \\ 2 & -2 & 4 \end{vmatrix} = 8\hat{i} - 8\hat{j} - 8\hat{k}$$

$$\therefore \text{ Area} = |8\hat{i} - 8\hat{j} - 8\hat{k}| = 8\sqrt{3} \text{ units}$$

Area of a triangle :



1. Area of triangle ABC $= \frac{1}{2} ab \sin\theta = \frac{1}{2}\left|\vec{a} \times \vec{b}\right| = \frac{1}{2}|\overrightarrow{AB} \times \overrightarrow{AC}|$

2. If $\vec{a}, \vec{b}, \vec{c}$ are position vectors of vertices of a ΔABC then its

Area $= \frac{1}{2}|(\vec{a} \times \vec{b}) + (\vec{b} \times \vec{c}) + (\vec{c} \times \vec{a})|$ (think !)

Illustration :

Find the area of ΔABC *if position vectors of its vertices A, B,C are* $\hat{i} + \hat{j}$, $\hat{j} + \hat{k}$ *and* $\hat{k} + \hat{i}$ *respectively.*

Sol. $\overrightarrow{AB} = (\hat{j} + \hat{k}) - (\hat{i} + \hat{j}) = \hat{k} - \hat{i}$

$\overrightarrow{AC} = (\hat{k} + \hat{i}) - (\hat{i} + \hat{j}) = \hat{k} - \hat{j}$

$$\therefore \overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} i & j & k \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{vmatrix} = \hat{i} + \hat{j} + \hat{k}$$

$$\therefore \text{ Area of } \Delta \ ABC = \frac{1}{2}|\overrightarrow{AB} \times \overrightarrow{AC}| = \frac{1}{2}\sqrt{1+1+1} = \sqrt{3}/2.$$

INTERPRETATION OF VECTOR PRODUCT AS VECTOR AREA :

1. Vector area of plane figures :

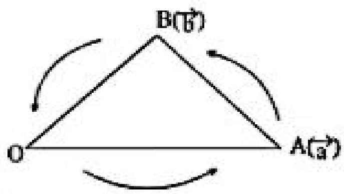
With every closed bound surface which has been described in a certain specific manner and whose boundaries do not cross, it is possible to associate a directed line segment $\vec{c}$ such that

(i) $|\vec{c}|$ = no. of units of area enclosed by the plane figure

(ii) The support of $\vec{c}$ is perpendicular to the area and

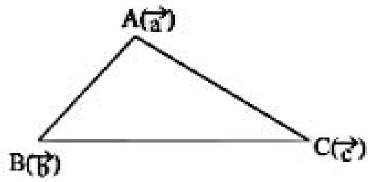
(iii) The sense of description of the boundaries and the direction of $\vec{c}$ is in accordance with the R.H. screw rule.

2. Vector area of a plane Δ (Triangle) :



Vector area of ΔOAB is $\vec{\Delta}=\frac{1}{2}(\vec{a}\times\vec{b})$

If $\vec{a},\vec{b},\vec{c}$ are the position vectors then the vector area of ΔABC is



$$\vec{\Delta}=\frac{1}{2}\left[(\vec{c}-\vec{b})\times(\vec{a}-\vec{b})\right]$$

$$\vec{\Delta}=\frac{1}{2}\left((\vec{a}\times\vec{b})+(\vec{b}\times\vec{c})\times(\vec{c}\times\vec{a})\right)$$

Note :

(i) If 3 points with position vectors $\vec{a},\vec{b}$ and $\vec{c}$ are collinear then $\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a}=0$

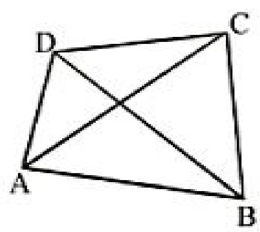
(ii) Unit vector perpendicular to the plane of the ΔABC when $\vec{a}$, $\vec{b}$, $\vec{c}$ are the p.v. of its angular point is

$\hat{n}=\pm\dfrac{\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a}}{2\Delta}$, where $\vec{a},\vec{b},\vec{c}$ are the position vectors of the angular points of the triangle ABC.

(iii) Vector Area of a quadrilateral ABCD = Vector area of ΔABC + vector area of ΔACD

$$=\frac{1}{2}\left(\overrightarrow{AB}\times\overrightarrow{AC}\right)+\frac{1}{2}\left(\overrightarrow{AC}\times\overrightarrow{AD}\right)$$

$$=\frac{1}{2}\left(\overrightarrow{AB}\times\overrightarrow{AC}-\overrightarrow{AD}\times\overrightarrow{AC}\right)=\frac{1}{2}\left(\overrightarrow{AB}-\overrightarrow{AD}\right)\times\overrightarrow{AC}$$



$$=\frac{1}{2}\left(\overrightarrow{AB}+\overrightarrow{DA}\right)\times\overrightarrow{AC}\quad=\frac{1}{2}\overrightarrow{DB}\times\overrightarrow{AC}$$

$$\therefore\ \text{Area of }\square ABCD=\frac{1}{2}\left|\overrightarrow{DB}\times\overrightarrow{AC}\right|=\frac{1}{2}\left|\overrightarrow{AC}\times\overrightarrow{BD}\right|$$

Illustration :

Using vector method, show that the points $A(2, -1, 3)$, $B(4, 3, 1)$ and $C(3, 1, 2)$ are collinear.

Sol. Let O be the origin

Given $A \equiv (2, -1, 3)$ $\quad\therefore\quad \overrightarrow{OA} = 2\hat{i} - \hat{j} + 3\hat{k}$

$B \equiv (4, 3, 1)$ $\quad\therefore\quad \overrightarrow{OB} = 4\hat{i} + 3\hat{j} + \hat{k}$

$C \equiv (3, 1, 2)$ $\quad\therefore\quad \overrightarrow{OC} = 3\hat{i} + \hat{j} + 2\hat{k}$

Now, $\overrightarrow{AB}$ = (P.V. of B) – (P.V. of A) = $\left(4\hat{i} + 3\hat{j} + \hat{k}\right) - \left(2\hat{i} - \hat{j} + 3\hat{k}\right) = \left(2\hat{i} + 4\hat{j} - 2\hat{k}\right)$

And $\overrightarrow{AC}$ = (P.V. of C) – (P.V. of A) = $\left(3\hat{i} + \hat{j} + 2\hat{k}\right) - \left(2\hat{i} - \hat{j} + 3\hat{k}\right) = \left(\hat{i} + 2\hat{j} - \hat{k}\right)$

$$\therefore\quad \overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 4 & -2 \\ 1 & 2 & -1 \end{vmatrix} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & -1 \\ 1 & 2 & -1 \end{vmatrix} = \vec{0} \qquad [\because R_2 \text{ and } R_3 \text{ are identical}]$$

Thus, $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are parallel vectors, having a common point A.
Hence, the points A, B, C are collinear.

Illustration :

AD, BE and CF are the medians of a triangle ABC intersecting in G. Show that

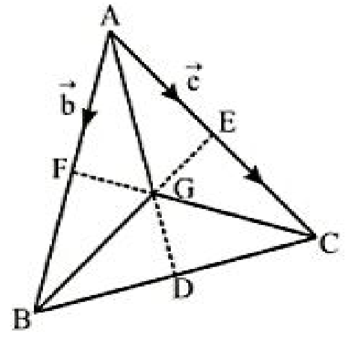
$$\Delta AGB = \Delta BGC = \Delta CGA = \frac{1}{3}\Delta ABC.$$

Sol. Let $\vec{b}, \vec{c}$ be the position vectors of B and C with respect to A as the origin of reference.

Therefore, the position vectors of D, E, F are $\frac{1}{2}(\vec{b} + \vec{c})$, $\frac{1}{2}\vec{c}$, $\frac{1}{2}\vec{b}$ respectively.

Also the position vector of the point G, the centroid, is

$$\frac{1}{3}(0 + \vec{b} + \vec{c}) = \frac{1}{3}(\vec{b} + \vec{c})$$



Therefore, area of $\Delta AGB = \frac{1}{2}\left(\overrightarrow{AB} \times \overrightarrow{AG}\right)$

$$= \frac{1}{2}\left|\vec{b} \times \frac{1}{3}\left(\vec{b} + \vec{c}\right)\right| = \frac{1}{6}\left|\vec{b} \times \vec{b} + \vec{b} \times \vec{c}\right|$$

$$= \frac{1}{6}\left|\vec{b} \times \vec{c}\right| = \frac{1}{3} \text{ area of } \Delta ABC$$

Similarly, we can show that area of $\Delta BGC = \frac{1}{3}$ area of ΔABC

and area of $\Delta CGA = \frac{1}{3}$ area of ΔABC

Practice Problem

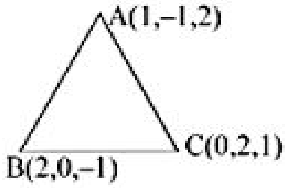
Q.1 For a non zero vector $\vec{a}$, if $\vec{a}.\vec{b} = \vec{a}.\vec{c}$ and $\vec{a}\times\vec{b} = \vec{a}\times\vec{c}$. Prove that $\vec{b}=\vec{c}$.

Q.2 Find

(i) A vector of magnitude $\sqrt{6}$ perpendicular to the plane ABC

(ii) Area of triangle ABC

(iii) Length of the altitude from A $(AB = AC = \sqrt{11})$



Q.3 Let $\overrightarrow{OA}=\vec{a}$, $\overrightarrow{OB}=10\vec{a}+2\vec{b}$ and $\overrightarrow{OC}=\vec{b}$ where O, A & C are non–collinear points. Let 'p' denote the area of the quadrilateral OABC, and let 'q' denote the area of the parallelogram with OA and OC as adjacent sides. If p = kq. Find k.

Answer key

Q.2 (i) $\pm(2\hat{i}+\hat{j}+\hat{k})$; (ii) $2\sqrt{6}$; (iii) $2\sqrt{2}$ Q.3 6

Shortest distance between 2 skew lines :

Note :

(i) 2 lines in a plane if not || must intersect and 2 lines in a plane if not intersecting must be parallel. Convertely 2 intersecting or parallel lines must be coplanar.

(ii) In space, however we come across situation when two lines neither intersect nor ||, Two such lines (like the flight paths of two planes) in space are known as skew lines or non coplanar lines.

(iii) S.D. between two such skew lines is the segment intercepted betweeen the two lines and perpendicular to both.

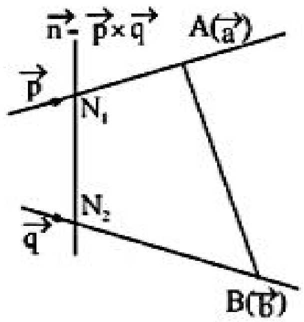
Method I: Two ways to determine the S.D.

$L_1 : \vec{r}=\vec{a}+\lambda\vec{p}$

$L_2 : \vec{r}=\vec{b}+\mu\vec{q}$

$\vec{n}=\vec{p}\times\vec{q}$

$\overrightarrow{AB}=(\vec{b}-\vec{a})$



$$\text{S.D.} = |\text{Projection of } \overrightarrow{AB} \text{ on } \vec{n}| = \left|\frac{\overrightarrow{AB}.\vec{n}}{|\vec{n}|}\right| = \left|\frac{(\vec{b}-\vec{a}).(\vec{p}\times\vec{q})}{|\vec{p}\times\vec{q}|}\right|$$

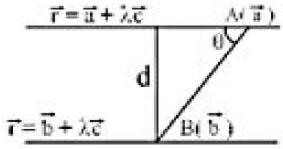
If S.D. = 0 $\Rightarrow$ lines are intersecting and hence coplanar.

Method II : p.v. of $N_1 = \vec{a} + \lambda\vec{p}$; p.v. of $N_2 = \vec{b} + \mu\vec{q}$

$$\overrightarrow{N_1N_2} = (\vec{b} - \vec{a}) + (\mu\vec{q} - \lambda\vec{p})$$

Now $\overrightarrow{N_1N_2} \cdot \vec{p} = 0$ and $\overrightarrow{N_1N_2} \cdot \vec{q} = 0$ (two linear equations to get the unique values of λ and μ.)
One p.v's of N_1 and N_2 are known we can also determine the equation to the line of shortest distance and the S.D.

Shortest Distance between two parallel lines :



$$d = |\vec{a} - \vec{b}| \sin\theta \Rightarrow \left| \frac{(\vec{a} - \vec{b}) \times \vec{c}}{|\vec{c}|} \right|$$

Illustration :

Find the shortest distance between the two lines whose vector equations are given by :

$$\vec{r} = \hat{i} + 2\hat{j} + 3\hat{k} + \lambda(2\hat{i} + 3\hat{j} + 4\hat{k}) \text{ and } \vec{r} = 2\hat{i} + 4\hat{j} + 5\hat{k} + \mu(3\hat{i} + 4\hat{j} + 5\hat{k})$$

Sol. *If the equations of the lines are $\vec{r} = \vec{a}_1 + \lambda\vec{b}_1$ and $\vec{r} = \vec{a}_2 + \lambda\vec{b}_2$, then shortest distance 'd' between them is given by*

$$d = \left| \frac{(\vec{a}_2 - \vec{a}_1) \cdot (\vec{b}_1 \times \vec{b}_2)}{|\vec{b}_1 \times \vec{b}_2|} \right| \qquad ...(i)$$

Here $\vec{a}_1 = \hat{i} + 2\hat{j} + 3\hat{k}$, $\quad \vec{b}_1 = 2\hat{i} + 3\hat{j} + 4\hat{k}$

$\vec{a}_2 = 2\hat{i} + 4\hat{j} + 5\hat{k}$, $\quad \vec{b}_2 = 3\hat{i} + 4\hat{j} + 5\hat{k}$

Now, $\vec{a}_2 - \vec{a}_1 = (2\hat{i} + 4\hat{j} + 5\hat{k}) - (\hat{i} + 2\hat{j} + 3\hat{k})$

$$= \hat{i} + 2\hat{j} + 2\hat{k} \qquad ...(ii)$$

$$\vec{b}_1 \times \vec{b}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix}$$

$$= (15 + 16)\hat{i} - (10 - 12)\hat{j} + (8 - 9)\hat{k}$$

$$= \hat{i} + 2\hat{j} - \hat{k} \qquad ...(iii)$$

$$\left|\vec{b}_1 \times \vec{b}_2\right| = \sqrt{(-1)^2 + 2^2 + (-1)^2} = \sqrt{6} \qquad ...(iv)$$

and $(\vec{a}_2 - \vec{a}_1) \cdot (\vec{b}_2 \times \vec{b}_1) = (\hat{i} + 2\hat{j} + 2\hat{k}) \cdot (-\hat{i} + 2\hat{j} - \hat{k})$

$$= 1 \times (-1) + 2 \times 2 + 2 \times (-1) = 1 \qquad ...(v)$$

Substituting the values from (iv) and (v) in (i), we get

$$d = \left|\frac{1}{\sqrt{6}}\right| = \frac{1}{\sqrt{6}}$$

Illustration :

Determine whether the following pair of lines intersect $\vec{r}=\hat{i}-\hat{j}+\lambda(2\hat{i}+\hat{k})$ *and* $\vec{r}=2\hat{i}-\hat{j}+\mu(\hat{i}+\hat{j}-\hat{k})$.

Sol. *The shortest distance between the given pair of lines is given by*

$$d=\left|\frac{(\vec{a}_2-\vec{a}_1)\cdot(\vec{b}_1\times\vec{b}_2)}{|\vec{b}_1\times\vec{b}_2|}\right|$$

The two lines will intersect if and only if $d=0$.

i.e. $(\vec{a}_2-\vec{a}_1)\cdot(\vec{b}_1\times\vec{b}_2)=0$

Here, equation of first line is $\vec{r}=\hat{i}-\hat{j}+\lambda(2\hat{i}+\hat{k})=\vec{a}_1+\lambda\vec{b}_1$

where $\vec{a}_1=\hat{i}-\hat{j}$ *and* $\vec{b}_1=2\hat{i}+\hat{k}$

Also equation of second line is $\vec{r}=2\hat{i}-\hat{j}+\mu(\hat{i}+\hat{j}-\hat{k})=\vec{a}_2+\mu\vec{b}_2$

where $\vec{a}_2=2\hat{i}-\hat{j}$ *and* $\vec{b}_2=\hat{i}+\hat{j}-\hat{k}$

Now, $\vec{b}_1\times\vec{b}_2=\begin{vmatrix}\hat{i}&\hat{j}&\hat{k}\\2&0&1\\1&1&-1\end{vmatrix}=-\hat{i}+3\hat{j}+2\hat{k}$ *and* $\vec{a}_2-\vec{a}_1=2\hat{i}-\hat{j}-(\hat{i}-\hat{j})=\hat{i}$

Since $(\vec{a}_2-\vec{a}_1)\cdot(\vec{b}_1\times\vec{b}_2)=\hat{i}\cdot(-\hat{i}+3\hat{j}+2\hat{k})=(-1)(1)+3(0)+2(0)=-1\neq 0$

Hence the given lines do not intersect

PRODUCT OF THREE OR MORE VECTORS :

Scalar triple product :

Definition : If $\vec{a},\vec{b},\vec{c}$ are three vectors, then their scalar triple product is defined as the dot product of two vectors $\vec{a}$ and $\vec{b}\times\vec{c}$. It is generally denoted by $\vec{a}.(\vec{b}\times\vec{c})$ or $[\vec{a}\ \vec{b}\ \vec{c}]$. It is read as box product of $\vec{a},\vec{b},\vec{c}$. Similarly other scalar triple products can be defined as $(\vec{b}\times\vec{c}).\vec{a}$, $(\vec{c}\times\vec{a}).\vec{b}$.

Note : Scalar triple product always results in a scalar quantity (number).

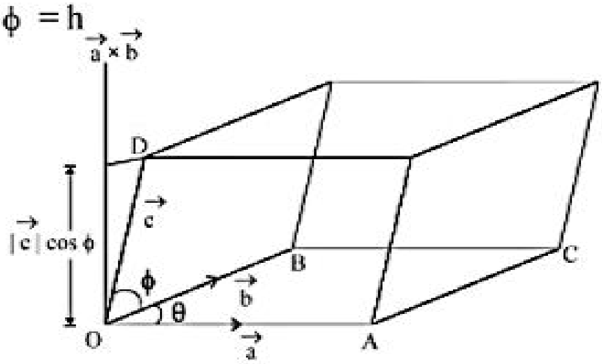
Geometrical Interpretation :

$(\vec{a}\times\vec{b}).\vec{c}=|\vec{a}||\vec{b}|\sin\theta\ \hat{n}.\vec{c}=|\vec{a}||\vec{b}||\vec{c}|\sin\theta\cos\phi$ where $\theta=\vec{a}\wedge\vec{b}$; $\phi=\vec{n}\wedge\vec{c}$

but $|\vec{a}||\vec{b}|\sin\theta$ = area of $||^{gm}$ OACB and $|\vec{c}|\cos\phi=h$

There absolute value of scalar triple product of three vectors is equal to the volume of the parallelopiped whose three coterminous edges are represented by the given vectors.

Therefore $|(\vec{a}\times\vec{b})\cdot\vec{c}|=|[\vec{a}\ \vec{b}\ \vec{c}]|$ = Volume of the parallelopiped whose coterminous edges are $\vec{a},\vec{b}$ and $\vec{c}$.



Formula for scalar Triple Product :

If $\vec{a} = a_1 i + a_2 j + a_3 k$, $\vec{b} = b_1 i + b_2 j + b_3 k$ and $\vec{c} = c_1 i + c_2 j + c_3 k$, then

$$[\vec{a}\,\vec{b}\,\vec{c}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Properties of Scalar Triple product :

1. The position of (.) and (×) can be interchanged i.e. $\vec{a}.(\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}).\vec{c}$

2. $[\vec{a}\,\vec{b}\,\vec{c}] = [\vec{b}\,\vec{c}\,\vec{a}] = [\vec{c}\,\vec{a}\,\vec{b}]$
Therefore if we don't change the cyclic order of a, b and c then the value of scalar triple product is not changed.

3. If the cyclic order of vectors is changed, then sign of scalar triple product is changed i.e.

$$\vec{a}.[\vec{b} \times \vec{c}] = -\vec{a}.(\vec{c} \times \vec{b}) \text{ or } [\vec{a}\,\vec{b}\,\vec{c}] = -[\vec{a}\,\vec{c}\,\vec{b}]$$

From (ii) and (iii) we have

$$[\vec{a}\,\vec{b}\,\vec{c}] = [\vec{b}\,\vec{c}\,\vec{a}] = [\vec{c}\,\vec{a}\,\vec{b}] = -[\vec{a}\,\vec{c}\,\vec{b}] = -[\vec{b}\,\vec{a}\,\vec{c}] = -[\vec{c}\,\vec{b}\,\vec{a}]$$

4. The scalar triple product of three vectors when two of them are equal or parallel, is zero i.e.

$[\vec{a}\,\vec{b}\,\vec{b}] = [\vec{a}\,\vec{b}\,\vec{a}] = 0$ (think !)

5. The scalar triple product of three mutually perpendicular unit vectors is ± 1 Thus

$[\hat{i}\ \hat{j}\ \hat{k}] = 1, [\hat{i}\ \hat{k}\ \hat{j}] = -1$

6. If two of the three vectors $\vec{a}, \vec{b}, \vec{c}$ are parallel then $[\vec{a}\,\vec{b}\,\vec{c}] = 0$

7. $\vec{a}, \vec{b}, \vec{c}$ are three coplanar vectors if $[\vec{a}\,\vec{b}\,\vec{c}] = 0$ i.e. the necessary and sufficient condition for three non-zero non-collinear vectors to be coplanar is

$[\vec{a}\ \vec{b}\ \vec{c}] = 0$

8. For any vectors $\vec{a}, \vec{b}, \vec{c}, d$

$[\vec{a} + \vec{b}\ \vec{c}\ d] = [\vec{a}\ \ \vec{c}\ \ d] + [\vec{b}\ \ \vec{c}\ \ d]$

9. For right handed system, $[\vec{a}\,\vec{b}\,\vec{c}] > 0$
and for left handed system, $[\vec{a}\,\vec{b}\,\vec{c}] < 0$; where $\vec{a}, \vec{b}, \vec{c}$ are non- coplanar

10. $[\vec{a}+\vec{b}, \vec{b}+\vec{c}, \vec{c}+\vec{a}] = 2[\vec{a}\,\vec{b}\,\vec{c}]$

11. $[\vec{a}-\vec{b}\ \ \vec{b}-\vec{c}\ \ \vec{c}-\vec{a}]$ is always zero.

12. $[\vec{l}\ \vec{m}\ \vec{n}][\vec{a}\,\vec{b}\,\vec{c}] = \begin{vmatrix} \vec{l}.\vec{a} & \vec{l}.\vec{b} & \vec{l}.\vec{c} \\ \vec{m}.\vec{a} & \vec{m}.\vec{b} & \vec{m}.\vec{c} \\ \vec{n}.\vec{a} & \vec{n}.\vec{b} & \vec{n}.\vec{c} \end{vmatrix}$, where $\vec{l}, \vec{m}, \vec{n}$ & $\vec{a}, \vec{b}, \vec{c}$ are non coplanar vectors.

13. $[\vec{a}\,\vec{b}\,\vec{c}]\ (\vec{p}\times\vec{q}) = \begin{vmatrix} \vec{p}.\vec{a} & \vec{q}.\vec{a} & \vec{a} \\ \vec{p}.\vec{b} & \vec{q}.\vec{b} & \vec{b} \\ \vec{p}.\vec{c} & \vec{q}.\vec{c} & \vec{c} \end{vmatrix}$

14. If $\vec{a} = a_1\ell + a_2 m + a_3 n$, $\vec{b} = b_1\ell + b_2 m + b_3 n$ and $\vec{c} = c_1\ell + c_2 m + c_3 n$, then

$$[\vec{a}\,\vec{b}\,\vec{c}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} [\ell mn]$$

Illustration :

If $\vec{a} = 2\hat{i} - 3\hat{j}$, $\vec{b} = \hat{i}+\hat{j}-\hat{k}$ and $\vec{c} = 3\hat{i}-\hat{k}$ represent three coterminous edges of a parallelopiped, then the volume of that parallelopiped is-

(A) 2 *(B) 4* *(C) 6* *(D) 10*

Sol. *Volume* $= \left|[\vec{a}\,\vec{b}\,\vec{c}]\right|$

$$= \left|\begin{vmatrix} 2 & -3 & 0 \\ 1 & 1 & -1 \\ 3 & 0 & -1 \end{vmatrix}\right| = |-2+9-3| = 4$$ *Ans.[B]*

Illustration :

For any three vectors $\vec{a}, \vec{b}, \vec{c}$ $[\vec{a}+\vec{b}, \vec{b}+\vec{c}, \vec{c}+\vec{a}]$ equals-

(A) $[\vec{a}\ \vec{b}\ \vec{c}]$ *(B) $2[\vec{a}\ \vec{b}\ \vec{c}]$* *(C) $[\vec{a}\,\vec{b}\,\vec{c}]^2$* *(D) 0*

Sol. $[\vec{a}+\vec{b}\ \ \vec{b}+\vec{c}\ \ \vec{c}\ c+\vec{a}] = (\vec{a}+\vec{b}).[(\vec{b}+\vec{c})\times(\vec{c}+\vec{a})]$

$= (\vec{a}+\vec{b}).(\vec{b}\times\vec{c} + \vec{b}\times\vec{a} + \vec{c}\times\vec{c} + \vec{c}\times\vec{a})]$

$= (\vec{a}+\vec{b}).(\vec{b}\times\vec{c} + \vec{b}\times\vec{a} + \vec{c}\times\vec{a})$ $[\because \vec{c}\times\vec{c} = 0]$

$= [\vec{a}\,\vec{b}\,\vec{c}] + [\vec{a}\,\vec{b}\,\vec{a}] + [\vec{a}\,\vec{c}\,\vec{a}] + [\vec{b}\,\vec{b}\,\vec{c}] + [\vec{b}\,\vec{b}\,\vec{a}] + [\vec{b}\,\vec{c}\,\vec{a}]$

$= [\vec{a}\,\vec{b}\,\vec{c}] + [\vec{b}\,\vec{c}\,\vec{a}] = 2[\vec{a}\,\vec{b}\,\vec{c}]$ *Ans.[B]*

Illustration :

If $\vec{a}, \vec{b}, \vec{c}$ be three non- zero vectors, then $|(\vec{a}\times\vec{b}).\ \vec{c}| = |\vec{a}|\,|\vec{b}|\,|\vec{c}|$, if-

(A) $\vec{a}.\vec{b} = \vec{b}.\vec{c} = 0$ *(B) $\vec{b}.\vec{c} = \vec{c}.\vec{a} = 0$*

(C) $\vec{c}.\vec{a} = \vec{a}.\vec{b} = 0$ *(D) $\vec{a}.\vec{b} = \vec{b}.\vec{c} = \vec{c}.\vec{a} = 0$*

Sol. $|(\vec{a}\times\vec{b}).\ c| = |\vec{a}|\,|\vec{b}|\,|\vec{c}| \Leftrightarrow |(a\times\vec{b})|\,|\vec{c}|\cos\theta = |\vec{a}|\,|\vec{b}|\,|\vec{c}|$

(where θ is the angle between $\vec{a}\times\vec{b}$ and $\vec{c}$) $\Leftrightarrow |\vec{a}|\,|\vec{b}|\,|\vec{c}|\sin\phi\cos\theta = |\vec{a}|\,|\vec{b}|\,|\vec{c}|$

(where ϕ is the angle between $\vec{a}$ and $\vec{b}$) $\Leftrightarrow \sin\phi = 1, \cos\theta = 1 \Leftrightarrow \phi = 90^o, \theta = 0^o$

$\Leftrightarrow \vec{a}.\vec{b} = 0,\ \vec{a}.\vec{c} = 0,\ \vec{b}.\vec{c} = 0$ *Ans.[D]*

Illustration :

If vectors $a\hat{i}+\hat{j}+\hat{k}$, $\hat{i}+b\hat{j}+\hat{k}$ *and* $\hat{i}+\hat{j}+c\hat{k}$ $(a \neq b \neq c \neq 1)$ *are coplanar, then* $\frac{1}{1-a}+\frac{1}{1-b}+\frac{1}{1-c}$ *equals-*

(A) 1 *(B) 0* *(C) – 1* *(D) None of these*

Sol. *Since vectors are coplanar,*

$$\therefore \begin{vmatrix} a & 1 & 1 \\ 1 & b & 1 \\ 1 & 1 & c \end{vmatrix} = 0 \Rightarrow \begin{vmatrix} a & 1 & 1 \\ 1-a & b-1 & 0 \\ 0 & 1-b & c-1 \end{vmatrix} = 0 \qquad [\text{Using } R_2 - R_1,\ R_3 - R_2]$$

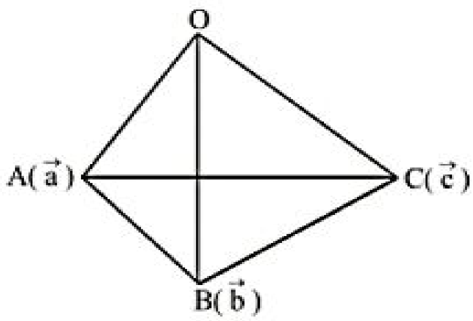
$$\Rightarrow a(b-1)(c-1) - (1-a)\ \{(c-1) - (1-b)\} = 0$$
$$\Rightarrow a(1-b)(1-c) + (1-a)(1-c) + (1-a)(1-b) = 0$$
$$\Rightarrow (a-1+1)(1-b)(1-c) + (1-a)\quad (1-c) + (1-a)(1-b) = 0$$
$$\Rightarrow (1-b)(1-c) + (1-a)(1-c) + (1-a)(1-b) = (1-a)(1-b)(1-c)$$
$$\Rightarrow \frac{1}{1-a} + \frac{1}{1-b} + \frac{1}{1-c} = 1$$

Ans.[A]

VOLUME OF TETRAHEDRON :

1. If $\vec{a}, \vec{b}, \vec{c}$ are position vectors of vertices A, B and C with respect to O, then volume of tetrahedron OABC represented by V is given by

$$V = \frac{1}{3}\ \text{Base area} \times \text{height}$$



$$\text{Base Area} = \frac{1}{2}\left|\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a}\right|$$

Let $\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a}=\vec{n}$

$$\therefore \quad \text{Base area} = \frac{1}{2}\left|\vec{n}\right|$$

Height = projection of $\vec{a}$ on $\vec{n}$

$$= \frac{\left|\vec{a}\cdot\vec{n}\right|}{\left|\vec{n}\right|} = \frac{\left|\vec{a}\cdot\left(\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a}\right)\right|}{\left|\vec{n}\right|} = \frac{\left|[\vec{a}\ \vec{b}\ \vec{c}]\right|}{\left|\vec{n}\right|}$$

$$\therefore \quad V = \frac{1}{3}\cdot\frac{1}{2}\left|\vec{n}\right|\frac{\left|[\vec{a}\ \vec{b}\ \vec{c}]\right|}{\left|\vec{n}\right|} = \frac{1}{6}\left|[\vec{a}\ \vec{b}\ \vec{c}]\right|$$

2. If $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ are position vectors of vertices A,B,C,D of a tetrahedron ABCD, then

$$\text{its volume} = \begin{cases} \frac{1}{6}\left|\left[\overrightarrow{AB} \quad \overrightarrow{AC} \quad \overrightarrow{AD}\right]\right| \\ \qquad \text{or} \\ \frac{1}{6}\left|[\vec{b}-\vec{a} \quad \vec{c}-\vec{a} \quad \vec{d}-\vec{a}]\right| \end{cases}$$

Illustration :

If the vertices of any tetrahedron be $\vec{a} = \hat{j} + 2\hat{k}$, $\vec{b} = 3\hat{i} + \hat{k}$, $\vec{c} = 4\hat{i} + 3\hat{j} + 6\hat{k}$ and $\vec{d} = 2\hat{i} + 3\hat{j} + 2\hat{k}$ then find its volume.

Sol. *Let the p.v. of the vertices A,B,C,D with respect to 0 are $\vec{a}, \vec{b}, \vec{c}$ and d respectively then*

$\overrightarrow{AB} = \vec{b} - \vec{a} = 3\hat{i} - \hat{j} - \hat{k}$,

$\overrightarrow{AC} = 4\hat{i} + 2\hat{j} + 4\hat{k}$ *and* $\overrightarrow{AD} = 2\hat{i} + 2\hat{j}$

Now volume of tetrahedron $= \frac{1}{6}\left[\overrightarrow{AB}\ \overrightarrow{AC}\ \overrightarrow{AD}\right] = \frac{1}{6}\begin{vmatrix} 3 & -1 & -1 \\ 4 & 2 & 4 \\ 2 & 2 & 0 \end{vmatrix} = -6$

$\therefore$ *Required volume = 6 units*

Practice Problem

Q.1 Find the value of p for which the vectors $(p+1)\hat{i} - 3\hat{j} + p\hat{k}$; $p\hat{i} + (p+1)\hat{j} - 3\hat{k}$ and $-3\hat{i} + p\hat{j} + (p+1)\hat{k}$ are linearly dependent/coplanar.

Q.2 Show that the lines $\vec{R} = \vec{R}_0 + t\vec{A}$ and $\vec{R} = \vec{R}_i + s\vec{B}$ intersect if $(\vec{R}_0 - \vec{R}_i)\cdot(\vec{A}\times\vec{B}) = 0$ i.e. $[\vec{R}_0\ \vec{A}\ \vec{B}] = [\vec{R}_i\ \vec{A}\ \vec{B}]$.

Q.3 If $\vec{u} = 2\hat{i} - \hat{j} + \hat{k}$; $\vec{v} = \hat{i} + \hat{j} + \hat{k}$ and $\vec{w}$ is a unit vector then find the maximum value of $[\vec{u}\ \vec{v}\ \vec{w}]$.

Answer key

Q.1 p = 1 Q.3 $[\vec{u}\ \vec{v}\ \vec{w}]_{max} = \sqrt{14}$]

VECTOR TRIPLE PRODUCT :

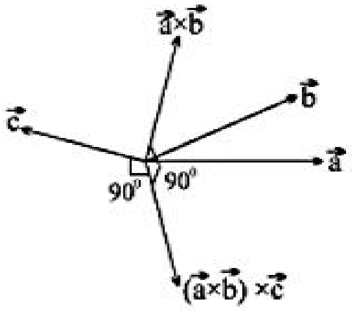
Definition : The vector triple product of three vectors $\vec{a}, \vec{b}, \vec{c}$ is defined as the vector product of two vectors $\vec{a}$ and $\vec{b}\times\vec{c}$. It is denoted by $\vec{a}\times(\vec{b}\times\vec{c})$.

$(\vec{a}\times\vec{b})\times\vec{c}$ is a vector which is coplanar with $\vec{a}$ and $\vec{b}$ and perpendicular to $\vec{c}$.

Hence $(\vec{a}\times\vec{b})\times\vec{c} = x\vec{a}+y\vec{b}$ ....(1) [linear combination of $\vec{a}$ and $\vec{b}$)

$\vec{c}\cdot(\vec{a}\times\vec{b})\times\vec{c} = x(\vec{a}.\vec{c})+y(\vec{b}.\vec{c})$

$0 = x(\vec{a}.\vec{c})+y(\vec{b}.\vec{c})$ ....(2)



$\therefore \dfrac{x}{\vec{b}.\vec{c}} = -\dfrac{y}{\vec{a}.\vec{c}} = \lambda$

$\therefore x = \lambda(\vec{b}.\vec{c})$ and $y = -\lambda(\vec{a}.\vec{c})$

Substituting the values of x and y in $(\vec{a}\times\vec{b})\times\vec{c} = \lambda(\vec{b}.\vec{c})\vec{a} - \lambda(\vec{a}.\vec{c})\vec{b}$

This is an identity and must be true for all values of $\vec{a}, \vec{b}, \vec{c}$

Put = $\vec{a}=\hat{i}$; $\vec{b}=\hat{j}$ and $\vec{c}=\hat{i}$

$(\hat{i}\times\hat{j})\times\hat{i} = \lambda(\hat{j}.\hat{i})\hat{i} - \lambda(\hat{i}.\hat{i})\hat{j}$

$\hat{j} = -\lambda\hat{j} \quad \Rightarrow \quad \lambda = -1$

hence $(\vec{a}\times\vec{b})\times\vec{c} = (\vec{a}.\vec{c})\vec{b} - (\vec{b}.\vec{c})\vec{a}$

Properties :

1. Expansion formula for vector triple product is given by

$\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}.\vec{c})\vec{b} - (\vec{a}.\vec{b})\vec{c}$

$(\vec{b}\times\vec{c})\times\vec{a} = (\vec{b}.\vec{a})\vec{c} - (\vec{c}.\vec{a})\vec{b}$.

2. $[\vec{a}\times\vec{b} \quad \vec{b}\times\vec{c} \quad \vec{c}\times\vec{a}] = [\vec{a}\,\vec{b}\,\vec{c}]^2 = \begin{vmatrix} \vec{a}.\vec{a} & \vec{a}.\vec{b} & \vec{a}.\vec{c} \\ \vec{b}.\vec{a} & \vec{b}.\vec{b} & \vec{b}.\vec{c} \\ \vec{c}.\vec{a} & \vec{c}.\vec{b} & \vec{c}.\vec{c} \end{vmatrix}$

Note that if $\vec{a}, \vec{b}, \vec{c}$ are non coplanar vectors then $\vec{a}\times\vec{b}$, $\vec{b}\times\vec{c}$ and $\vec{c}\times\vec{a}$ will also be non coplanar vectors.

3. Vector triple product is a vector quantity.

4. $\vec{a}\times(\vec{b}\times\vec{c}) \neq (\vec{a}\times\vec{b})\times\vec{c}$

6 Unit vector coplanar with $\vec{a}$ & $\vec{b}$ and perpendicular to $\vec{c}$ is $\pm\dfrac{(\vec{a}\times\vec{b})\times\vec{c}}{|(\vec{a}\times\vec{b})\times\vec{c}|}$

Illustration :

$\hat{i} \times (\hat{j} \times \hat{k}) + \hat{j} \times (\hat{k} \times \hat{i}) + \hat{k} \times (\hat{i} \times \hat{j})$ *equals-*

(A) $\hat{i}$ (B) $\hat{j}$ (C) $\hat{k}$ (D) 0

Sol. $\hat{i} \times (\hat{j} \times \hat{k}) + \hat{j} \times (\hat{k} \times \hat{i}) + \hat{k} \times (\hat{i} \times \hat{j}) \Rightarrow \hat{i} \times \hat{i} + \hat{j} \times \hat{j} + \hat{k} \times \hat{k} = 0 + 0 + 0 = 0$ *Ans.[D]*

Illustration :

If $\vec{a}, \vec{b}, \vec{c}$ *are coplanar, then show that* $\vec{a} \times \vec{b}, \vec{b} \times \vec{c}$ *and* $\vec{c} \times \vec{a}$ *are also coplanar.*

Sol. $[\vec{a}\,\vec{b}\,\vec{c}]$ *are coplanar* $\Rightarrow [\vec{a}\,\vec{b}\,\vec{c}] = 0 \Rightarrow [\vec{a}\,\vec{b}\,\vec{c}] = 0$

$\Rightarrow [\vec{a} \times \vec{b}, \vec{b} \times \vec{c}, \vec{c} \times \vec{a}] = 0$

$\Rightarrow \vec{a} \times \vec{b}, \vec{b} \times \vec{c}$ *and* $\vec{c} \times \vec{a}$ *are coplanar*

Illustration :

Let $\vec{a}, \vec{b}, \vec{c}$ *such that* $|\vec{a}| = 1, |\vec{b}| = 1$ *and* $|\vec{c}| = 2$ *and if* $\vec{a} \times (\vec{a} \times \vec{c}) + \vec{b} = 0$ *then acute angle between* $\vec{a}$ *and* $\vec{c}$ *is -*

(A) $\frac{\pi}{3}$ (B) $\frac{\pi}{4}$ (C) $\frac{\pi}{6}$ (D) None of these

Sol. *If angle between* $\vec{a}$ *and* $\vec{c}$ *is* θ *then –*

$$\vec{a}\,.\,\vec{c} = |\vec{a}|\,|\vec{c}| \cos\theta$$
$$= 1.2 \cos\theta = 2\cos\theta$$

but $\vec{a} \times (\vec{a} \times \vec{c}) + \vec{b} = 0$

$\Rightarrow (\vec{a}\,.\,\vec{c})\,\vec{a} - (\vec{a}\,.\,\vec{a})\vec{c} + \vec{b} = 0$

$\Rightarrow (2\cos\theta)\vec{a} - 1.\vec{c} = -\vec{b}$

$\Rightarrow [(2\cos\theta)\vec{a} - \vec{c}]^2 = [-\vec{b}]^2$

$\Rightarrow 4\cos^2\theta\ |\vec{a}|^2 - 2.(2\cos\theta)\ \vec{a}.\vec{c} + |\vec{c}|^2 = |\vec{b}|^2$

$\Rightarrow 4\cos^2\theta - 4\cos\theta\ (2\cos\theta) + 4 = 1 \Rightarrow 4(1 - \cos^2\theta) = 1 [\because |\vec{a}| = 1, |\vec{b}| = 1]$

$\Rightarrow \sin\theta = 1/2$

$\Rightarrow \theta = \frac{\pi}{6}$ *Ans.[C]*

Illustration :

Let $\vec{a} = 2\hat{i}+\hat{j}-2\hat{k}$ *and* $\vec{b} = \hat{i}+\hat{j}$ *if vector* $\vec{c}$ *is such that* $\vec{a}\cdot\vec{c} = |\vec{c}|, |\vec{c}-\vec{a}| = 2\sqrt{2}$ *and angle between* $(a \times b)$ *and c is the* 30^o *then* $|(\vec{a}\times\vec{b})\times\vec{c}|$ *is equal to -*

(A) $\frac{2}{3}$ (B) $\frac{3}{2}$ (C) 2 (D) 3

Sol. $|\vec{c}-\vec{a}|^2 = (\vec{c}-\vec{a}).(\vec{c}-\vec{a}) = (2\sqrt{2})^2$

$\Rightarrow \quad \vec{c}^2 + \vec{a}^2 - 2\vec{c}.\vec{a} = 8 \Rightarrow \vec{c}^2 + (4+1+4) - 2\,\vec{c}.\vec{a} = 8$

$\Rightarrow \quad \vec{c}^2 + 9 - 2|\vec{c}| = 8 \quad [\because\ \vec{a}.\vec{c} = |\vec{c}|]$

$\Rightarrow \quad \vec{c}^2 - 2|\vec{c}| + 1 = 0 \Rightarrow \vec{c}^2 - 2\vec{c} + 1 = 0$

$\Rightarrow \quad (\vec{c}-1)^2 = 0 \Rightarrow \vec{c} = 1$

$|(\vec{a}\times\vec{b})\times\vec{c}| = |\vec{a}\times\vec{b}|\,|\vec{c}|\sin 30^o = 1\times\frac{1}{2}|\vec{a}\times\vec{b}| = \frac{1}{2}|\vec{a}\times\vec{b}|$

But $\quad \vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & -2 \\ 1 & 1 & 0 \end{vmatrix} = 2\hat{i}-2\hat{j}+\hat{k}$

$\therefore |\vec{a}\times\vec{b}| = \sqrt{4+4+1} = 3$

$\therefore |(\vec{a}\times\vec{b})\times\vec{c}| = \frac{3}{2}$ *Ans.[B]*

Scalar Product of four Vector :

(I) $(\vec{a}\times\vec{b}).(\vec{c}\times\vec{d}) = (\vec{a}.\vec{c})(\vec{b}.\vec{d}) - (\vec{a}.\vec{d})(\vec{b}.\vec{c}) = \begin{vmatrix} \vec{a}.\vec{c} & \vec{a}.\vec{d} \\ \vec{b}.\vec{c} & \vec{b}.\vec{d} \end{vmatrix}$

Proof: $\underbrace{(\vec{a}\times\vec{b})}_{\vec{u}}\cdot(\vec{c}\times\vec{d}) = \vec{u}\cdot(\vec{c}\times\vec{d}) = (\vec{u}\times\vec{c})\cdot\vec{d}$ (Dot & cross are interchangeable in STP)

$$\left((\vec{a}\times\vec{b})\times\vec{c}\right)\cdot\vec{d} = \left((\vec{a}\times\vec{c})\vec{b} - (\vec{b}\times\vec{c})\vec{a}\right)\cdot\vec{d} = \left((\vec{a}\times\vec{c})(\vec{b}\times\vec{d}) - (\vec{b}\times\vec{c})(\vec{a}\times\vec{d})\right) = \begin{vmatrix} \vec{a}\cdot\vec{c} & \vec{a}\cdot\vec{d} \\ \vec{b}\cdot\vec{c} & \vec{b}\cdot\vec{d} \end{vmatrix}$$

$(\vec{a}\times\vec{b})\cdot(\vec{a}\times\vec{b}) = (\vec{a}\times\vec{b})^2 = \begin{vmatrix} \vec{a}\cdot\vec{a} & \vec{a}\cdot\vec{b} \\ \vec{a}\cdot\vec{b} & \vec{b}\cdot\vec{b} \end{vmatrix} = (\vec{a})^2(\vec{b})^2 = (\vec{a}\cdot\vec{b})^2$ which is lagrange's identity.

Illustration :

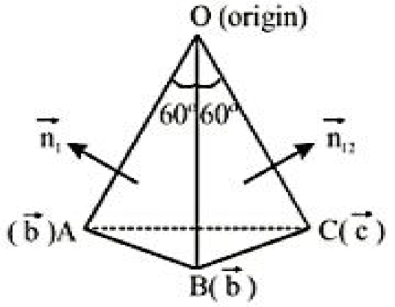
Prove that acute angle between the two plane faces of a regular tetrahedron is $\cos^{-1}\frac{1}{3}$.

Sol. *Let edge length of regular tetrahedron = 1*

$\vec{n}_1$ = *normal vector to plane OAB* = $\vec{a}\times\vec{b}$

$\vec{n}_2$ = *normal vector to plane OBC* = $\vec{b}\times\vec{c}$

$\therefore$ *acute angle between plane focus OAB & OBC is given as*



$$\cos\theta = \frac{|\vec{n}_1\vec{n}_2|}{|\vec{n}_1|\,|\vec{n}_2|} = \frac{|(\vec{a}\times\vec{b})\cdot(\vec{b}\times\vec{c})|}{|(\vec{a}\times\vec{b})|\,|(\vec{b}\times\vec{c})|} = \frac{\begin{vmatrix}\vec{a}\cdot\vec{b} & \vec{a}\cdot\vec{c}\\ \vec{b}\cdot\vec{b} & \vec{b}\cdot\vec{c}\end{vmatrix}}{\sin 60\cdot\sin 60^\circ}$$

$$= \frac{\begin{vmatrix}\cos 60^\circ & \cos 60^\circ\\ \cos 0^\circ & \cos 60^\circ\end{vmatrix}}{3/4} = \frac{\left|\frac{1}{4}-\frac{1}{2}\right|}{\frac{3}{4}} = \frac{1}{3} \Rightarrow \qquad \theta = \cos^{-1}\left(\frac{1}{3}\right)$$

Vector Product of Four Vector :

$\vec{V} = (\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d})$

$= \vec{u}\times(\vec{c}\times\vec{d}) \quad = [\vec{a}\,\vec{b}\,\vec{d}]\vec{c} - [\vec{a}\,\vec{b}\,\vec{c}]\vec{d}$ ....(1) (where $\vec{u} = \vec{a}\times\vec{b}$)

again $\vec{V} = (\vec{a}\times\vec{b})\times\underbrace{(\vec{c}\times\vec{d})}_{\vec{v}} = (\vec{a}.\vec{v})\vec{b} - (\vec{b}.\vec{v})\vec{a} = [\vec{a}\,\vec{c}\,\vec{d}]\vec{b} - [\vec{b}\,\vec{c}\,\vec{d}]\vec{a}$ ....(2)

from (1) and (2) $\quad [\vec{a}\,\vec{b}\,\vec{d}]\vec{c} - [\vec{a}\,\vec{b}\,\vec{c}]\vec{d} = [\vec{a}\,\vec{c}\,\vec{d}]\vec{b} - [\vec{b}\,\vec{c}\,\vec{d}]\vec{a}$ ....(3)

Note that $(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d}) = 0 \Rightarrow$ planes containing the vectors $\vec{a}$ & $\vec{b}$ and $\vec{c}$ & $\vec{d}$ are parallel.

|||ly $\quad (\vec{a}\times\vec{b})\cdot(\vec{c}\times\vec{d}) = 0 \Rightarrow$ the two planes are perpendicular.

(i) Equation (3) is suggestive that if $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ are four vectors no 3 three of them are coplanar then each one of them can be expressed as a linear combination of other.

(ii) If $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ are p.v.'s of four points then these four points are in the same plane if

$$[\vec{a}\,\vec{b}\,\vec{d}] - [\vec{a}\,\vec{b}\,\vec{c}] = [\vec{a}\,\vec{c}\,\vec{d}] - [\vec{b}\,\vec{c}\,\vec{d}]$$

Illustration :

If $\vec{a}, \vec{b}, \vec{c}$ are three vectors such that $\vec{a}\times\vec{b} = \vec{c},\ \vec{b}\times\vec{c} = \vec{a}$, then

(A) $|\vec{b}| = 1, |\vec{c}| = |\vec{a}|$ *(B)* $|\vec{c}| = 1, |\vec{a}| = |\vec{b}|$

(C) $|\vec{b}| = 2, |\vec{c}| = 2|\vec{a}|$ *(D)* $|\vec{a}| = 1, |\vec{b}| = |\vec{c}|$

Sol. *Given* $\vec{a}\times\vec{b} = \vec{c}$ ...(i)

$\vec{b}\times\vec{c} = \vec{a}$ ...(ii)

From (i), $\vec{c}\perp\vec{a}$ *and* $\vec{c}\perp\vec{b}$ ...(iii)

From (ii), $\vec{a}\perp\vec{b}$ *and* $\vec{a}\perp\vec{c}$ ...(iv)

From (iii) and (iv), $\vec{a}, \vec{b}, \vec{c}$ *are mutually perpedicular.*
Taking cross product of (i) with (ii), we get,

$$(\vec{a}\times\vec{b})\times(\vec{b}\times\vec{c}) = \vec{c}\times\vec{a} \quad\Rightarrow\quad [\vec{a}\ \vec{b}\ \vec{c}]\vec{b} - [\vec{a}\,\vec{b}\,\vec{b}]\vec{c} = \vec{c}\times\vec{a}$$

$$\Rightarrow \quad [\vec{a}\ \vec{b}\ \vec{c}]\vec{b} = \vec{c}\times\vec{a} \qquad [\because\ [\vec{a}\,\vec{b}\,\vec{b}] = 0\,]$$

$$\Rightarrow \quad \left|[\vec{a}\vec{b}\vec{c}]\,\vec{b}\right| = \left|\vec{c}\times\vec{a}\right| \quad\Rightarrow\quad \left|[\vec{a}\vec{b}\vec{c}]\right|\,\left|\vec{b}\right| = \left|\vec{c}\times\vec{a}\right|$$

$$\Rightarrow \quad |\vec{a}|\,|\vec{b}|\,|\vec{c}|\ |\vec{b}| = |\vec{c}|\ |\vec{a}| \quad\Rightarrow\quad |\vec{b}|^2 = 1 \quad\Rightarrow\quad |\vec{b}| = 1$$

From (i), $\vec{a}\times\vec{b} = \vec{c}$

$$\Rightarrow \quad |\vec{a}\times\vec{b}| = |\vec{c}| \quad\Rightarrow\quad |\vec{a}|\ |\vec{b}| = |\vec{c}| \qquad [\because\ \vec{a} \text{ and } \vec{b} \text{ are mutually perpendicular}\,]$$

$$\Rightarrow \quad |\vec{a}| = |\vec{c}| \qquad [\because |\vec{b}| = 1\,]$$

$\because \quad |\vec{b}| = 1$ *and* $|\vec{a}| = |\vec{c}|$ *Ans. [A]*

Practice Problem

Q.1 Prove : $\vec{d}.[\vec{a}\times\{\vec{b}\times(\vec{c}\times\vec{d})\}] = (\vec{b}.\vec{d})[\vec{a}\,\vec{c}\,\vec{d}]$

Q.2 Prove that : $(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d}) + (\vec{a}\times\vec{c})\times(\vec{d}\times\vec{b}) + (\vec{a}\times\vec{d})\times(\vec{b}\times\vec{c}) = -2[\vec{b}\,\vec{c}\,\vec{d}]\vec{a}$

Q.3 If $\hat{a}\times(\hat{b}\times\hat{c}) = \frac{1}{2}\hat{b}$ where $\hat{b}$ and $\hat{c}$ are non collinear then find the angle between $\hat{a}$ and $\hat{b}$; between $\hat{a}$ and $\hat{c}$.

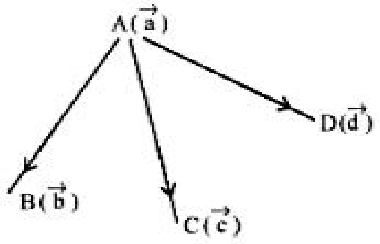
Q.4 Prove that $\hat{i}\times(\vec{a}\times\hat{i}) + \hat{j}\times(\vec{a}\times\hat{j}) + \hat{k}\times(\vec{a}\times\hat{k}) = 2\vec{a}$

Answer key

Q.3 $\pi/2$; $\pi/3$

Condition for coplanarity of four points :

4 points with pv's $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ are coplanar iff $\exists$ scalars x, y, z and t not all simultaneously zero and satisfying $x\vec{a} + y\vec{b} + z\vec{c} + t\vec{d} = 0$ where $x + y + z + t = 0$.



Case I : Let the four points A, B, C, D are in the same plane

$\Rightarrow$ the vectors $\vec{b} - \vec{a}, \vec{c} - \vec{a}$ and $\vec{d} - \vec{a}$ are in the same plane.

hence $\vec{d} - \vec{a} = l(\vec{b} - \vec{a}) + m(\vec{c} - \vec{a})$

or $\underbrace{(l + m - 1)}_{x}\vec{a} - \underbrace{l}_{y}\,\vec{b} - \underbrace{m}_{z}\,\vec{c} + \underbrace{1}_{t}\,\vec{d} = 0 \Rightarrow x\vec{a} + y\vec{b} + z\vec{c} + t\vec{d} = 0$ where, $x + y + z + t = 0$ and x, y, z, t not all simultaneous zero.

Case II : Let $x\vec{a} + y\vec{b} + z\vec{c} + t\vec{d} = 0$ where $x + y + z + t = 0$ and not all simultaneously zero

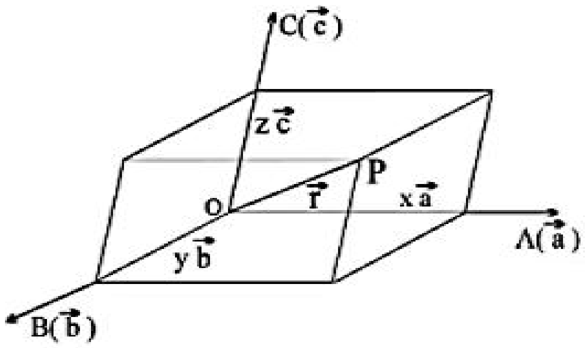
Let $t \neq 0$ $\quad (-y - z - t)\vec{a} + y\vec{b} + z\vec{c} + t\vec{d} = 0 \quad$ [putting $x = -y - z - t$]

$(\vec{d} - \vec{a})t + y(\vec{b} - \vec{a}) + z(\vec{c} - a) = 0$

$\Rightarrow \quad \vec{d} - \vec{a},\ \vec{b} - \vec{a}$ and $\vec{c} - a$ are coplanar $\quad \Rightarrow \quad$ points A, B, C, D are coplanar

Theorem in space :

If $\vec{a}, \vec{b}, \vec{c}$ are 3 non zero non coplanar vectors then any vector $\vec{r}$ can be expressed as a linear combination : $\vec{r} = x\vec{a} + y\vec{b} + z\vec{c}$ of $\vec{a}, \vec{b}, \vec{c}$



Examples :

Express the non coplanar vectors $\vec{a}, \vec{b}, \vec{c}$ in terms of $\vec{b} \times \vec{c},\ \vec{c} \times \vec{a},\ \vec{a} \times \vec{b}$.

Since $[\vec{a}\ \vec{b}\ \vec{c}]^2 = [(\vec{a} \times \vec{b})\ (\vec{b} \times \vec{c})\ (\vec{c} \times \vec{a})]$

$\therefore$ If $\vec{a}, \vec{b}, \vec{c}$ are non-coplanar

$\Rightarrow$ $\vec{a} \times \vec{b},\ \vec{b} \times \vec{c},\ \vec{c} \times \vec{a}$ are also non coplanar.

$\vec{a} = x(\vec{a} \times \vec{b}) + y(\vec{b} \times \vec{c}) + z(\vec{c} \times \vec{a})$

Taking dot product with $\vec{a}$

$$\vec{a}^2 = y[\vec{a}\,\vec{b}\,\vec{c}] \Rightarrow y = \frac{(\vec{a})^2}{[\vec{a}\,\vec{b}\,\vec{c}]}$$

Taking dot product with $\vec{b}$

$$\vec{a}\cdot\vec{b} = z[\vec{b}\,\vec{c}\,\vec{a}] \Rightarrow z = \frac{\vec{a}\,\vec{b}}{[\vec{a}\,\vec{b}\,\vec{c}]}$$

Similarly taking dot product with $\vec{c}$

$$\vec{a}\cdot\vec{c} = x[\vec{a}\,\vec{b}\,\vec{c}] \Rightarrow x = \frac{\vec{a}\cdot\vec{c}}{[\vec{a}\,\vec{b}\,\vec{c}]}$$

$$\therefore \quad \vec{a} = \frac{(\vec{a}\cdot\vec{c})(\vec{a}\times\vec{b}) + (\vec{a})^2(\vec{b}\times\vec{c}) + (\vec{a}\cdot\vec{b})(\vec{c}\times\vec{a})}{[\vec{a}\,\vec{b}\,\vec{c}]}$$

Practice Problem

Q.1 Express $\vec{b}\times\vec{c}$, $\vec{c}\times\vec{a}$, $\vec{a}\times\vec{b}$ in terms of 3 non coplanar vectors $\vec{a},\vec{b},\vec{c}$.

Q.2 Given the vector $\vec{a}$ and $\vec{b}$ orthogonal to each other find the vector $\vec{V}$ in terms of $\vec{a}$ and $\vec{b}$ satisfying $\vec{V}.\vec{a} = 0$; $\vec{V}.\vec{b} = 1$ and $[\vec{V}\ \vec{a}\ \vec{b}] = 1$

Answer key

Q.2 $\vec{V} = \frac{1}{\vec{b}^2}\vec{b} + \frac{1}{(\vec{a}\times\vec{b})^2}(\vec{a}\times\vec{b})$

Real definition of linearly independence :

If $\vec{V}_1, \vec{V}_2, \ldots\ldots\ldots \vec{V}_n$ are vectors and $\lambda_1, \lambda_2, \ldots\ldots\ldots \lambda_n$ are scalar and if the linear combination $\lambda_1\vec{V}_1 + \lambda_2\vec{V}_2 + \ldots\ldots\ldots + \lambda_n\vec{V}_n = 0$, necessarily implies $\lambda_1 = \lambda_2 = \ldots\ldots\ldots \lambda_n = 0$, we say that $\vec{V}_1, \vec{V}_2, \ldots\ldots\ldots \vec{V}_n$ are said to constitutes a linearly independent set of vectors.

Note:

(i) 2 non zero , non collinear vectors are linearly independent.

(ii) Three non zero , non coplanar vectors are linearly independent i.e. $[\vec{a}\,\vec{b}\,\vec{c}] \neq 0 \Leftrightarrow \vec{a}, \vec{b}, \vec{c}$ are linearly independent.

(iii) Four or more vectors in 3D space are always linearly dependent.

Illustration :

Show that vectors $\vec{i}-3\vec{j}+2\vec{k}$, $2\vec{i}-4\vec{j}-\vec{k}$ *and* $3\vec{i}+2\vec{j}-\vec{k}$ *are linearly independent.*

Sol. *Let* $\vec{a}=\vec{i}-3\vec{j}+2\vec{k}$, $\vec{b}=2\vec{i}-4\vec{j}-\vec{k}$ *and* $\vec{c}=3\vec{i}+2\vec{j}-\vec{k}$

$$\left[\vec{a}\ \vec{b}\ \vec{c}\right]=\begin{vmatrix}1 & -3 & 2\\ 2 & 4 & -1\\ 3 & 2 & -1\end{vmatrix}\neq 0$$

Reciprocal system of vectors :

1. If $\vec{a},\vec{b},\vec{c}$ and $\vec{a}',\vec{b}',\vec{c}'$ are 2 sets of non coplanar vectors such that $\vec{a}.\vec{a}'=\vec{b}.\vec{b}'=\vec{c}.\vec{c}'=1$ and $\vec{a}.\vec{b}'=\vec{a}.\vec{c}'=\vec{b}.\vec{a}'=\vec{b}.\vec{c}'=\vec{c}.\vec{a}'=\vec{c}.\vec{b}'=0$, then $\vec{a},\vec{b},\vec{c}$ and $\vec{a}',\vec{b}',\vec{c}'$ are said to be constitute a reciprocal system of vectors.

2. Reciprocal system of vectors exists only in case of dot product.

3. It is possible to define $\vec{a}',\vec{b}',\vec{c}'$ in terms of $\vec{a},\vec{b},\vec{c}$ as.

$$\vec{a}'=\frac{\vec{b}\times\vec{c}}{[\vec{a}\,\vec{b}\,\vec{c}]}\ ;\ \vec{b}'=\frac{\vec{c}\times\vec{a}}{[\vec{a}\,\vec{b}\,\vec{c}]}\ ;\vec{c}'=\frac{\vec{a}\times\vec{b}}{[\vec{a}\,\vec{b}\,\vec{c}]}\quad\left([\vec{a}\,\vec{b}\,\vec{c}]\neq 0\right)$$

Note: (i) $\vec{a}\times\vec{a}'+\vec{b}\times\vec{b}'+\vec{c}\times\vec{c}'=0$ i. e. $\dfrac{\vec{a}\times(\vec{b}\times\vec{c})+\vec{b}\times(\vec{c}\times\vec{a})+\vec{c}\times(\vec{a}\times\vec{b})}{[\vec{a}\,\vec{b}\,\vec{c}]}$

(ii) $(\vec{a}+\vec{b}+\vec{c}).(\vec{a}'+\vec{b}'+\vec{c}')=3$ (as $\vec{a}\cdot\vec{b}'=\vec{a}\cdot\vec{c}'=0$ etc)

(iii) If $[\vec{a}\,\vec{b}\,\vec{c}]=V$ then $[\vec{a}'\,\vec{b}'\,\vec{c}']=\dfrac{1}{V}$ $\Rightarrow$ $[\vec{a}\ \vec{b}\ \vec{c}]\,[\vec{a}'\,\vec{b}'\,\vec{c}']=1$

(iv) $\vec{a}'\times\vec{b}'+\vec{b}'\times\vec{c}'+\vec{c}'\times\vec{a}'=\dfrac{\vec{a}+\vec{b}+\vec{c}}{[\vec{a}\,\vec{b}\,\vec{c}]}$, $[\vec{a}\,\vec{b}\,\vec{c}]\neq 0$

Isolating an known vectors

Satisfying a given relationship with some known vectors:

There is no general method for solving such equations, however dot or cross with known or unknown vectors or dot with $\vec{a}\times\vec{b}$, generally isolates the unknown vector. Use of linear combination also proves to be advantageous.

Illustration :

Find vector $\vec{r}$ *if* $\vec{r}\cdot\vec{a} = m$ *and* $\vec{r}\times\vec{b} = \vec{c}$, *where* $\vec{a}.\vec{b} \neq 0$.

Sol. $\vec{r}.\vec{a} = m$ ...(i)

and $\vec{r}\times\vec{b} = \vec{c}$ ...(ii)

From (ii), $\vec{a}\times(\vec{r}\times\vec{b}) = \vec{a}\times\vec{c}$

or $(\vec{a}.\vec{b})\vec{r} - (\vec{a}.\vec{r})\vec{b} = \vec{a}\times\vec{c}$

or $(\vec{a}.\vec{b})\vec{r} = \vec{a}\times\vec{c} + (\vec{a}\cdot\vec{r})\vec{b} = \vec{a}\times\vec{c} + m\vec{b}$

$\therefore$ $$\vec{r} = \frac{1}{\vec{a}.\vec{b}}(\vec{a}\times\vec{c} + m\vec{b})$$

Illustration :

Find $\vec{r}$ *such that* $t\vec{r} + \vec{r}\times\vec{a} = \vec{b}$, *where* $\vec{a}$ & $\vec{b}$ *ae non collinear vectors.*

Sol. *Given,* $t\vec{r} + \vec{r}\times\vec{a} = \vec{b}$ ...(i)

Since $\vec{a}, \vec{b}, \vec{a}\times\vec{b}$ *are non-coplanar vectors therefore* $\vec{r}$ *can be expressed as linear combination of* $\vec{a}, \vec{b}$ *and* $\vec{a}\times\vec{b}$

Let $\vec{r} = x\vec{a} + y\vec{b} + z(\vec{a}\times\vec{b})$

Putting the value of $\vec{r}$ *in (i), we get*

$$t[x\vec{a} + y\vec{b} + z(\vec{a}\times\vec{b})] + x(\vec{a}\times\vec{a}) + y(\vec{b}\times\vec{a}) + z(\vec{a}\times\vec{b})\times\vec{a} = \vec{b}$$

$$t[x\vec{a} + y\vec{b} + z(\vec{a}\times\vec{b})] + y(\vec{b}\times\vec{a}) + z[(\vec{a}\cdot\vec{a})\vec{b} - (\vec{a}\cdot\vec{b})\vec{a}] = \vec{b}$$

$$\Rightarrow \quad [tx - z(\vec{a}\cdot\vec{b})]\vec{a} + [ty + z(\vec{a}\cdot\vec{a}) - 1]\vec{b} + (tz - y)(\vec{a}\times\vec{b}) = 0$$

Equating the coefficients of $\vec{a}$, $\vec{b}$ *and* $\vec{a}\times\vec{b}$, *we get*

$tx - z(\vec{a}\cdot\vec{b}) = 0$ ...(iii)

$ty + za^2 - 1 = 0$ ...(iv)

$tz - y = 0$ ...(v)

Solving (iii), (iv) and (v), we get

$$x = \frac{\vec{a}.\vec{b}}{t(t^2 + a^2)}, \qquad y = \frac{t}{t^2 + a^2}, \quad z = \frac{1}{t^2 + a^2}$$

Putting the values of x, y, z in (ii), we have

$$\vec{r} = \frac{1}{t^2 + a^2}\left[\frac{(\vec{a}.\vec{b})\vec{a}}{t} + t\vec{b} + \vec{a}\times\vec{b}\right]$$

Second method :

Given $\quad t\vec{r}+\vec{r}\times\vec{a}=\vec{b} \qquad ...(i)$

$\therefore \quad \vec{r}\times\vec{a}=\vec{b}-t\vec{r} \qquad ...(ii)$

Taking corss product of both sides of (i) with $\vec{a}$*, we get*

$$t(\vec{r}\times\vec{a})+(\vec{r}\times\vec{a})\times\vec{a}=\vec{b}\times\vec{a} \quad [\textit{Putting the value of } \vec{r}\times\vec{a}\,]$$

$$\Rightarrow \quad t(\vec{b}-t\vec{r})+(\vec{r}\times\vec{a})\vec{a}-a^2\vec{r}=\vec{b}\times\vec{a}$$

$$\Rightarrow \quad t\vec{b}-(t^2+a^2)\vec{r}+(\vec{r}\times\vec{a})\vec{a}=\vec{b}\times\vec{a} \qquad ...(iii)$$

In (i), taking dot product with $\vec{a}$*, we get*

$$r(\vec{r}\times a)=0=\vec{b}\times\vec{a} \qquad \therefore\ \vec{r}\times\vec{a}=\frac{\vec{a}\times\vec{b}}{t}$$

From (ii), $t\vec{b}-(t^2+a^2)\vec{r}+\dfrac{\vec{a}\times\vec{b}}{t}\vec{a}=\vec{b}\times\vec{a}$

$$\therefore \quad \vec{r}=\frac{1}{t^2+a^2}\left(\frac{\vec{a}\times\vec{b}}{t}\vec{a}+t\vec{b}+\vec{a}\times\vec{b}\right)$$

Illustration :

Solve the following simultaneous equations for $\vec{x}$ *and* $\vec{y}$ *:*

$$\vec{x}+\vec{y}=\vec{a}\,,\ \vec{x}\times\vec{y}=\vec{b} \ \text{ and } \ \vec{x}.\vec{a}=1$$

Sol. *Given* $\quad \vec{x}+\vec{y}=\vec{a} \qquad ...(i)$

$\vec{x}\times\vec{y}=\vec{b} \qquad ...(ii)$

$\vec{x}.\vec{a}=1 \qquad ...(iii)$

Putting the value of $\vec{y}$ *from (i) in (ii), we get*

$$\vec{x}\times(\vec{a}-\vec{x})=\vec{b} \quad \Rightarrow \quad \vec{x}\times\vec{a}=\vec{b} \quad \Rightarrow \quad \vec{a}\times(\vec{x}\times\vec{a})=\vec{a}\times\vec{b}$$

$$\Rightarrow \quad a^2\vec{x}-(\vec{a}.\vec{x})\vec{a}=\vec{a}\times\vec{b} \quad \Rightarrow \quad a^2\vec{x}-\vec{a}=\vec{a}\times\vec{b}$$

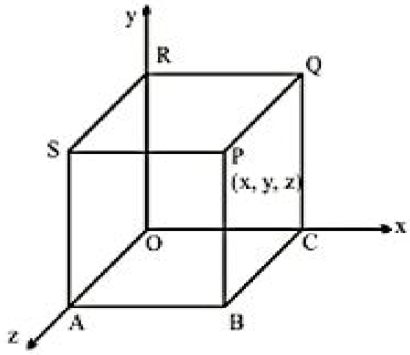
$$\therefore \quad \vec{x}=\frac{1}{\vec{a}}(\vec{a}+\vec{a}\times\vec{b}) \quad \text{and} \quad \vec{y}=\vec{a}-\vec{x}=\vec{a}-\frac{1}{a^2}(\vec{a}+\vec{a}\times\vec{b})$$

THREE DIMENSION

COORDINATES OF A POINT IN SPACE :

Consider a point P in space whose position is given by (x, y, z) where x, y, z are perpendicular distance from yz plane, zx plane and xy plane respectively.

If we assume $\hat{i}, \hat{j}, \hat{k}$ unit vectors along OX, OY, OZ respectively then the position vector of point P is $x\hat{i} + y\hat{j} + z\hat{k}$ or simply (x, y, z).



When a point lies on Co-ordinates

(i) x – axis $(\alpha, 0, 0)$

(ii) y – axis $(0, \beta, 0)$

(iii) z – axis $(0, 0, \gamma)$

(iv) XY – plane $(\alpha, \beta, 0)$

(v) XZ – plane $(\alpha, 0, \gamma)$

(vi) YZ – plane $(0, \beta, \gamma)$

Distance formulae :

Distance between the points (x_1, y_1, z_1) and (x_2, y_2, z_2) is equal to $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$.

Section formulae :

(1) Coordinates a point P which divides line joining A (x_1, y_1, z_1) and (x_2, y_2, z_2) in the ratio m : n internally is given by $\left(\frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n}, \frac{mz_2 + nz_1}{m+n}\right)$.

(2) Coordinates of a point P which divides line joining A (x_1, y_1, z_1) and B (x_2, y_2, z_2) in the ratio m : n externally is given by $\left(\frac{mx_2 - nx_1}{m-n}, \frac{my_2 - ny_1}{m-n}, \frac{mz_2 - nz_1}{m-n}\right)$.

(3) Coordinates of mid-point of line joining A (x_1, y_1, z_1) and B (x_2, y_2, z_2) is $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$.

Direction cosines :

If α, β, γ are the angles which vector $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ makes with positive direction of the x, y, z axes respectively then α, β, γ are called direction angles and their cosines $\cos\alpha, \cos\beta, \cos\gamma$ are called the direction cosines of the vector and are generally denoted l, m, n respectively.

Thus $l = \cos\alpha,\ m = \cos\beta,\ n = \cos\gamma$

Note :

(i) If a line makes α, β, γ with positive direction of x, y, z axes respectively then direction cosines of line will be $\cos\alpha, \cos\beta, \cos\gamma$ or $-\cos\alpha, -\cos\beta, -\cos\gamma$.

(ii) A unit vector along the line whose direction cosines are $\cos\alpha$, $\cos\beta$, $\cos\gamma$ can be written as $(\cos\alpha)\hat{i} + (\cos\beta)\hat{j} + (\cos\gamma)\hat{k}$.

(iii) If a vector $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ makes angles α, β, γ with positive direction of x, y, z axes respectively

then $\cos\alpha = \dfrac{\vec{a}\cdot\hat{i}}{|\vec{a}||\hat{i}|} = \dfrac{a_1}{|\vec{a}|}$, $\cos\beta = \dfrac{\vec{a}\cdot\hat{j}}{|\vec{a}||\hat{j}|} = \dfrac{a_2}{|\vec{a}|}$ and $\cos\gamma = \dfrac{\vec{a}\cdot\hat{k}}{|\vec{a}||\hat{j}|} = \dfrac{a_3}{|\vec{a}|}$

$$\therefore\ \cos^2\alpha + \cos^2\beta + \cos^2\gamma = \frac{a_1^2 + a_2^2 + a_3^2}{|\vec{a}|^2} \Rightarrow \cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

Also note that $\sin^2\alpha + \sin^2\beta + \sin^2\gamma = 2$

(iv) Direction cosines of x-axis are (1, 0, 0) or (–1, 0, 0).
Direction cosines of y-axis are (0, 1, 0) or (0, – 1, 0).
Direction cosines of z-axis are (0, 0, 1) or (0, 0, – 1).

Direction ratios :

If a, b, c are three numbers proportional to the direction cosines l, m, n of a straight line, then a, b, c are called its direction ratios. They are also called direction numbers or direction components.

Hence, we have $\dfrac{l}{a} = \dfrac{m}{b} = \dfrac{n}{c} = \lambda$ (say) $\Rightarrow l = a\lambda,\ m = b\lambda,\ n = c\lambda$

$$\because\ l^2 + m^2 + n^2 = 1 \Rightarrow (a^2 + b^2 + c^2)\lambda^2 = 1 \Rightarrow \lambda = \pm\frac{1}{\sqrt{a^2+b^2+c^2}}$$

$$\therefore\ l = \pm\frac{a}{\sqrt{a^2+b^2+c^2}},\ m = \pm\frac{b}{\sqrt{a^2+b^2+c^2}} \text{ and } n = \pm\frac{c}{\sqrt{a^2+b^2+c^2}}$$

Note :

(i) Direction ratios of a line is not unique but infinite in number but direction cosines will be for a line will be only two. (l, m, n or $-l, -m, -n$)

(ii) A vector along the line with direction ratios a, b, c can be $a\hat{i} + b\hat{j} + c\hat{k}$.

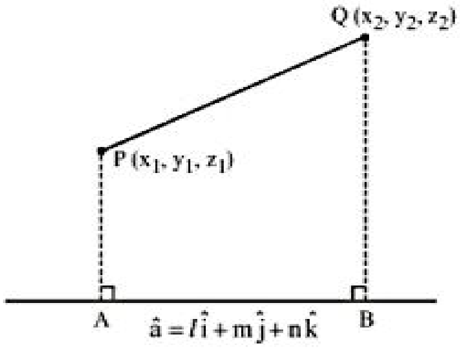
(iii) Direction ratios a line joining two points A and B are proportional to $x_2 - x_1, y_2 - y_1, z_2 - z_1$.

(iv) Projection of a Point on a Line :

Let P be a point and AB be a given line. Draw perpendicular PQ from P on AB which meets it at Q. This point Q is called projection of P on the line AB.

(v) Projection of a Line Segment Joining Two Points on a Line :

Projection of the line segment joining two points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ on another line whose direction cosines are l, m, n is $AB = |l(x_2 - x_1) + m(y_2 - y_1) + n(z_2 - z_1)|$.



Proof :

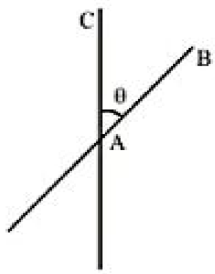
Vector $\overrightarrow{PQ} = (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}$

A unit vector along another line $\hat{a} = l\hat{i} + m\hat{j} + n\hat{k}$.

$$\therefore \text{ Projection AB} = \text{Projection of } \overrightarrow{PQ} \text{ on } \hat{a} = \frac{|\overrightarrow{PQ} \cdot \hat{a}|}{|\hat{a}|}$$

$$= |l(x_2 - x_1) + m(y_2 - y_1) + n(z_2 - z_1)|$$

Angle between two lines :



If direction ratios of two lines are a_1, b_1, c_1 and a_2, b_2, c_2 then acute angle between two lines is given by

$$\cos\theta = \frac{|a_1a_2 + b_1b_2 + c_1c_2|}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

Proof : Vector along lines can be taken as $\hat{a} = a_1\hat{i} + b_1\hat{j} + c_1\hat{k}$ and $\hat{b} = a_2\hat{i} + b_2\hat{j} + c_2\hat{k}$.

Acute angle between lines = acute angle between vectors $\vec{a}$ and $\vec{b}$.

$$\therefore \cos\theta = \frac{|\vec{a} \cdot \vec{b}|}{|\vec{a}||\vec{b}|} = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

If direction cosines of lines are l_1, m_1, n_1 and l_2, m_2, n_2 then acute angle between them is given by $\cos\theta = |l_1l_2 + m_1m_2 + n_1n_2|$.

Note :

(i) If lines are perpendiculars (i.e. vectors along them are also perpendicular) then $a_1a_2 + b_1b_2 + c_1c_2 = 0$ or $l_1l_2 + m_1m_2 + n_1n_2 = 0$.

(ii) If lines are parallel (i.e. vectors along them area also parallel) then $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$ or $\frac{l_1}{l_2} = \frac{m_1}{m_2} = \frac{n_1}{n_2}$.

Illustration :

Find the coordinates of the point which divides the line joining points (2, 3, 4) and (3, – 4, 7) in ratio 5 : 3 internally.

Sol. *Let the coordinates of the required point be (x, y, z) then* $x = \frac{2(3)+3(5)}{3+5} = \frac{21}{8}$;

$$y = \frac{3(3)-4(5)}{3+5} = \frac{-11}{8}\ ;\ z = \frac{4(3)+7(5)}{3+5} = \frac{47}{8}$$

$\therefore$ *The required point is* $\left(\frac{21}{8}, \frac{-11}{8}, \frac{47}{8}\right)$.

Illustration :

Find unit vector(s) with $\cos\alpha = \frac{1}{2}$ *and* $\cos\beta = \frac{1}{2}$, *where* α, β *are angles made by unit vector with positive direction x, y axes respectively.*

Sol. *Let unit vectors makes angle* γ *with positive z-axis.*

$\therefore$ *Unit vector will be* $(\cos\alpha)\hat{i} + (\cos\beta)\hat{j} + (\cos\gamma)\hat{k}$

$$\because\ \cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1 \Rightarrow \frac{1}{4} + \frac{1}{4} + \cos^2\gamma = 1$$

$$\Rightarrow \cos^2\gamma = \frac{1}{2} \Rightarrow \cos\gamma = \pm\frac{1}{\sqrt{2}}$$

$\therefore$ *Unit vector will be :* $\frac{\hat{i}}{2} + \frac{\hat{j}}{2} \pm \frac{\hat{k}}{\sqrt{2}}$.

Illustration :

Find the direction cosines of two lines which are connected by the relations $l - 5m + 3n = 0$ *and* $7l^2 + 5m^2 - 3n^2 = 0$.

Sol. *The given relations are* $l - 5m + 3n = 0 \Rightarrow l = 5m - 3n$ ....(1)
and $7l^2 + 5m^2 - 3n^2 = 0$ ....(2)
Putting the value of l from (1) in (2), we get
$7(5m - 3n)^2 + 5m^2 - 3n^2 = 0$

or $(2m - n)(3m - 2n) = 0 \Rightarrow \frac{m}{n} = \frac{1}{2}$ *or* $\frac{2}{3}$

When $\frac{m}{n} = \frac{1}{2}$ i.e. $n = 2m \Rightarrow l = 5m - 3n = -m$ or $\frac{l}{m} = -1$

thus $\frac{m}{n} = \frac{1}{2}$ and $\frac{l}{m} = -1$ giving $\frac{l}{-1} = \frac{m}{1} = \frac{n}{2}$

$$\text{or, } \frac{l}{-1} = \frac{m}{1} = \frac{n}{2} = \frac{\sqrt{(l^2+m^2+n^2)}}{\sqrt{\{(-1)^2+1^2+2^2\}}} = \frac{1}{\sqrt{6}}$$

So, direction cosines of one line are $\frac{-1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}$.

Again when $\frac{m}{n} = \frac{2}{3}$

$$\Rightarrow \frac{l}{m} = \frac{1}{2} \text{ giving } \frac{l}{1} = \frac{m}{2} = \frac{n}{3} = \frac{1}{\sqrt{1^2+2^2+3^2}} = \frac{1}{\sqrt{14}}$$

$\therefore$ The direction cosines of the other line are $\frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}$.

Illustration :

Find the direction ratios and direction cosines of the line joining the points A (6, – 7, – 1) and B (2, – 3, 1).

Sol. Direction ratios of AB are $(4, -4, -2) = (2, -2, -1)$

$a^2 + b^2 + c^2 = 9$

Direction cosines are $\left(\pm\frac{2}{3}, \mp\frac{2}{3}, \mp\frac{1}{3}\right)$.

Illustration :

Find direction cosines of a line perpendicular to two lines whose drs are 1, 2, 3 and – 2, 1, 4.

Sol. Vector along lines can be taken as $\vec{a} = \hat{i} + 2\hat{j} + 3\hat{k}$, $\vec{b} = -2\hat{i} + \hat{j} + 4\hat{k}$

$\therefore \quad \vec{a} \times \vec{b} = 5(\hat{i} - 2\hat{j} + \hat{k})$

$\therefore \quad$ dcs of line $= \left(\frac{1}{\sqrt{6}}, \frac{-2}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)$ or $\left(\frac{-1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, \frac{-1}{\sqrt{6}}\right)$.

Illustration :

Find the projection of the line segment joining the points (–1, 0, 3) and (2, 5, 1) on the line whose direction ratios are 6, 2, 3.

Sol. *The direction cosines ℓ, m, n of the line are given by*

$$\frac{\ell}{6} = \frac{m}{2} = \frac{n}{3} = \frac{\sqrt{\ell^2+m^2+n^2}}{\sqrt{6^2+2^2+3^2}} = \frac{1}{\sqrt{49}} = \frac{1}{7}$$

$$\therefore \quad \ell = \frac{6}{7},\ m = \frac{2}{7},\ n = \frac{3}{7}$$

The required projection is given by

$$= |\ \ell(x_2-x_1) + m(y_2-y_1) + n(z_2-z_1)\ |$$

$$= \left|\frac{6}{7}[2-(-1)+\frac{2}{7}(5-0)+\frac{3}{7}(1-3)\right|$$

$$= \left|\frac{6}{7}\times 3+\frac{2}{7}\times 5+\frac{3}{7}\times(-2)\right|$$

$$= \left|\frac{18}{7}+\frac{10}{7}-\frac{6}{7}\right| = \left|\frac{18+10-6}{7}\right| = \frac{22}{7}.$$ ***Ans.***

Practice Problem

Q.1 If points P, Q are (2, 3, 4), (1, – 2, 1), then prove that OP is perpendicular to OQ where O is (0, 0, 0).

Q.2 A line OP makes with the x-axis an angle of measure 120º and with y-axis an angle of measure 60º. Find the angle made by the line with the z-axis.

Q.3 What are the d.c's of the lines equally inclined to the axes ?

Q.4 A line makes angle $\alpha, \beta, \gamma, \delta$ with four diagonals of a cube then $\cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta$ is equal to

(A) 1 (B) $\frac{4}{3}$ (C) $\frac{3}{4}$ (D) $\frac{4}{5}$

Q.5 If a variable line in two adjacent positions has direction cosines l, m, n and $l + \delta l$, m + δm, n + δn, show that the small angle $\delta\theta$ between the two positions is given $(\delta\theta)^2 = (\delta l)^2 + (\delta m)^2 + (\delta n)^2$.

Answer key

Q.2 $\gamma = 45^\circ$ or 135° Q.3 $\left(\pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}\right)$ Q.4 B

PLANES :

Definition :

A plane is a surface such that a line joining any two points on the surface lies completely on it.

General equation of plane :

A linear equation in three variables of the type $\mathbf{ax + by + cz + d = 0}$ represents the general equation of a plane.

where a, b, c are not simultaneously zero.

Dividing by d we get $\left(\frac{a}{d}\right)x + \left(\frac{b}{d}\right)y + \left(\frac{c}{d}\right)z + 1 = 0.$

Thus equation of plane involves only three arbitrary constants. Hence in order to determine a unique plane 3 independent conditions are needed.

Note :

(i) Equation of xy plane is $z = 0$.

(ii) Equation of yz plane is $x = 0$.

(iii) Equation of zx plane is $y = 0$.

Division by Coordinate Planes :

The ratios in which the line segment PQ joining $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ is divided by coordinate planes are as follows.

(i) by yz - plane : $-\frac{x_1}{x_2}$ ratio

(ii) by zx - plane : $-\frac{y_1}{y_2}$ ratio

(iii) by xy - plane : $-\frac{z_1}{z_2}$ ratio

Illustration :

Find the ratio in which the line joining the points (3, 5, –7) and (–2,1,8) is divided by yz-plane.

Sol. *Let the line joining the points (3, 5, –7) and (–2, 1, 8) divides yz-plane in the ratio λ : 1, then coordinates of the dividing point will be*

$$\left(\frac{-2\lambda+3}{\lambda+1}, \frac{\lambda+5}{\lambda+1}, \frac{8\lambda-7}{\lambda+1}\right)$$

Now above points lies on the yz-plane, so its x-coordinate should be zero i.e.

$$\frac{-2\lambda+3}{\lambda+1} = 0 \Rightarrow \lambda = \frac{3}{2}$$

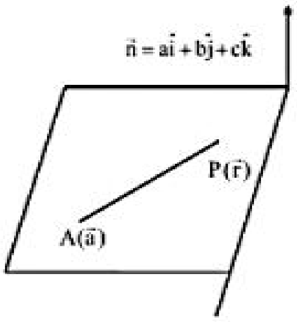
Hence yz-plane divides line joining the given points in the ratio $\frac{3}{2}:1$ *or 3 : 2.* ***Ans.***

DIFFERENT FORMS OF THE EQUATIONS OF PLANES :

1. A point in the plane and a vector normal to it is given :

Let a point $A(\vec{a})$ lies in the plane and a vector normal to it is $\vec{n} = a\hat{i} + b\hat{j} + c\hat{k}$.

$P(\vec{r})$ is a moving point whose locus is plane then for every position of vector $\overrightarrow{AP}$, vector $\vec{n}$ will be perpendicular to it.



$\therefore \quad \overrightarrow{AP} \cdot \vec{n} = 0$

$\Rightarrow \ (\vec{r} - \vec{a}) \cdot \vec{n} = 0 \ \Rightarrow \ \vec{r} \cdot \vec{n} = \vec{a} \cdot \vec{n}$

$\Rightarrow \ \vec{r} \cdot \vec{n} = d$ is general equation of plane in vector form.

It is also known as equation of plane in dot (or scalar) product form.

If $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ and $\vec{a} = x_0\hat{i} + y_0\hat{j} + z_0\hat{k}$ the equation of plane will be **$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$.**

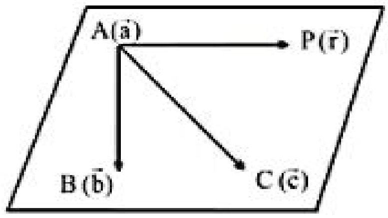
This is equation of plane containing point (x_0, y_0, z_0) and perpendicular to vector $\vec{a}\hat{i} + b\hat{j} + c\hat{k}$.

Note :

If equation of a plane is $ax + by + cz + d = 0$ then a, b, c are direction ratio of normal to the plane.

2. Plane passing through three given points :

Let three points $A(\vec{a})$, $B(\vec{b})$ and $C(\vec{c})$ lies in the plane and point $P(\vec{r})$ is moving point whose locus is plane.



$\because \ \overrightarrow{AP}, \overrightarrow{AB}$ and $\overrightarrow{AC}$ are coplanar.

$\therefore \quad \boxed{\left[\vec{r} - \vec{a} \quad \vec{b} - \vec{a} \quad \vec{c} - \vec{a}\right] = 0}$

In represents equation of plane passing through three points.

If $A = (x_1, y_1, z_1)$, $B = (x_2, y_2, z_2)$, $C = (x_3, y_3, z_3)$ and $P(x, y, z)$ then equation of plane is

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0$$

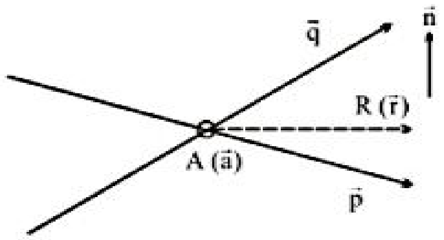
3. Plane containing two intersecting lines :

Let the equations of two lines are $\vec{r} = \vec{a} + \lambda\vec{b}$ and $\vec{r} = \vec{a} + \mu\vec{b}$.

Now, $n = \vec{p} \times \vec{q}$ is a vector perpendicular to the plane.

Hence equation of plane is $(\vec{r} - \vec{a}) \cdot (\vec{p} \times \vec{q}) = 0$

$\Rightarrow [\vec{r} - \vec{a} \quad \vec{p} \quad \vec{q}] = 0 \Rightarrow [\vec{r} \quad \vec{p} \quad \vec{q}] \Rightarrow [\vec{a} \quad \vec{p} \quad \vec{q}]$



Since vectors $\vec{r} - \vec{a}, \vec{p}, \vec{q}$ are coplanar.

Therefore $\vec{r} - \vec{a} = \lambda\vec{p} + \mu\vec{q} \Rightarrow \vec{r} = \vec{a} + \lambda\vec{p} + \mu\vec{q}$

It represents equation of a plane containing point $\vec{a}$ and parallel to two non-collinear vectors $\vec{p}$ and $\vec{q}$.
This is also known as parametric equation.

Illustration :

Express the equation of a plane $\vec{r} = \hat{i} - 2\hat{j} + \lambda\left(2\hat{i} - \hat{j} + 3\hat{k}\right) + \mu\left(3\hat{i} + 4\hat{j} - \hat{k}\right)$ in

(a) cartesian form.

(b) Scalar product form.

Sol.

(a) Clearly plane is passing through the point $\hat{i} - 2\hat{j}$ and parallel to vectors $2\hat{i} - \hat{j} + 3\hat{k}$ and $3\hat{i} + 4\hat{j} - \hat{k}$.

$\therefore$ *Equation of plane is*

$$\begin{vmatrix} x-1 & y+2 & z-0 \\ 2 & -1 & 3 \\ 3 & 4 & -1 \end{vmatrix} = 0$$

$\Rightarrow (x-1)(-11) - (y+2)(-11) + z(11) = 0$

$\Rightarrow x - 1 - y - 2 - z = 0 \Rightarrow x - y - z = 3 \Rightarrow x(1) + y(-1) + z(-1) = 3$

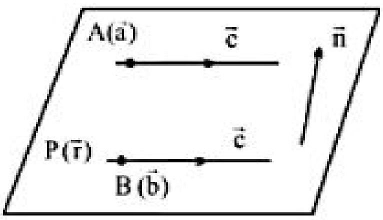
(b) Therefore equation of plane is scalar product form is $\vec{r} \cdot \left(\hat{i} - \hat{j} - \hat{k}\right) = 3$. ***Ans.***

4. Equation of plane containing two parallel lines :

Let lines be $\vec{r} = \vec{a} + \lambda\vec{b}$ and $\vec{r} = \vec{c} + \mu\vec{b}$

vector normal to plane is

$$\vec{n} = (\vec{a} - \vec{c}) \times \vec{b}$$



$\therefore$ equation of plane is

$$(\vec{r} - \vec{a}) \cdot (\vec{a} - \vec{c}) \times \vec{b} = 0$$

Alternatively : Vectors $\vec{r} - \vec{a}$, $\vec{c} - \vec{a}$ and $\vec{b}$ are coplanar

$\therefore$ equation of plane is

$$\left[\vec{r} - \vec{a} \quad \vec{c} - \vec{a} \quad \vec{b}\right] = 0.$$

Illustration :

Find the equation of the plane passing through the point (2, – 1, 3) which is the foot of the perpendicular drawn from the origin to the plane.

Sol. *The direction ratios of the normal to the plane are 2, – 1, 3.*
The equation of required plane is $2(x-2) - 1(y+1) + 3(z-3) = 0$
$\Rightarrow 2x - y + 3z - 14 = 0.$

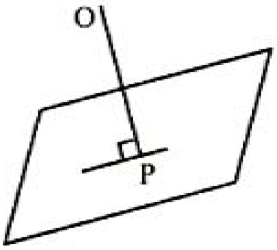


Illustration :

Find the equation of the plane through (2, 3, – 4), (1, – 1, 3) and parallel to x-axis.

Sol. *The equation of the plane passing through (2, 3, – 4) is* $a(x-2) + b(y-3) + c(z+4) = 0$ ....(1)

Since (1, – 1, 3) lies on it, we have $a + 4b - 7c = 0$ ....(2)

Since required plane is parallel to x-axis i.e. perpendicular to YZ plane i.e.

$$1 \cdot a + 0 \cdot b + 0 \cdot c = 0 \Rightarrow a = 0 \Rightarrow 4b - 7c = 0 \Rightarrow \frac{b}{7} = \frac{c}{4}$$

$\therefore$ *Equation of required plane is* $7y + 4z = 5$.

Illustration :

Two planes are given by equations $x + 2y - 3z = 0$ *and* $2x + y + z + 3 = 0$. *Find*

(i) *DC's of their normals and the acute angle between them.*

(ii) *DC's of their line of intersection.*

(iii) *Equation of the plane perpendicular to both of them through the point (2, 2, 1)*

Sol. $\vec{n}_1 = \hat{i} + 2\hat{j} - 3\hat{k}$, $\quad \vec{n}_2 = 2\hat{i} + \hat{j} + \hat{k}$

(i) $$\cos\theta = \frac{\vec{n}_1 - \vec{n}_2}{|\vec{n}_1|\,|\vec{n}_2|} = \frac{1}{2\sqrt{21}}$$

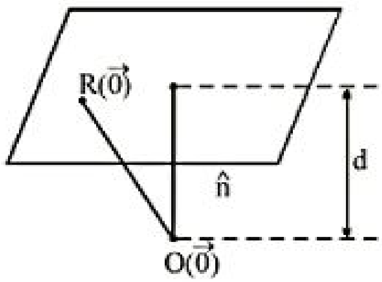
(ii) $$\vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & -3 \\ 2 & 1 & 1 \end{vmatrix} = 5\hat{i} - 7\hat{j} - 3\hat{k}$$

DC's of line of intersection of the plane $\left(\pm\frac{5}{\sqrt{83}}, \mp\frac{7}{\sqrt{83}}, \mp\frac{3}{\sqrt{83}}\right)$

(iii) $\left(\vec{r} - (2\hat{i} + 2\hat{j} + \hat{k})\right) \cdot (\vec{n}_1 \times \vec{n}_2) = 0$
$\Rightarrow \quad 5(x-2) - 7(y-2) - 3(z-1) = 0$
$\Rightarrow \quad 5x - 7y - 3z + 7 = 0$

5. Normal form of the plane :

A unit vector $\hat{n}$ normal to the plane from origin is known and perpendicular distance of the plane from the origin is d.



Projection of $\vec{r}$ on $\hat{n} = d$

$\Rightarrow \quad \vec{r}\cdot\hat{n} = d \qquad$ ....(1)

Note : d > 0, as d is distance of the plane from origin.

Cartesian form of the plane is

$$lx + my + nz = d$$

where l, m, n are dcs of normal to plane.

Illustration :

Let equation of plane be $\vec{r}\cdot(6\hat{i} - 3\hat{j} - 2\hat{k}) + 1 = 0$ then find perpendicular distance of plane from origin and also find direction cosines of this perpendicular.

Sol. *Plane is* $\vec{r}\cdot(6\hat{i} - 3\hat{j} - 2\hat{k}) = -1 \Rightarrow \vec{r}\cdot\left(\frac{-6}{7}\hat{i} + \frac{3}{7}\hat{j} + \frac{2}{7}\hat{k}\right) = \frac{1}{7}$

$\therefore$ *perpendicular distance from origin* $= \frac{1}{7}$ *and dcs of perpendicular* $= \left(\frac{-6}{7}, \frac{3}{7}, \frac{2}{7}\right)$.

Illustration :

Find the vector equation of plane which is at a distance of 8 units from the origin and which is normal to the vector $2\hat{i} + \hat{j} + 2\hat{k}$

Sol. *Here, d = 8 and* $\vec{n} = 2\hat{i} + \hat{j} + 2\hat{k}$

$$\therefore \quad \hat{n} = \frac{\vec{n}}{|\vec{n}|} = \frac{2\hat{i} + \hat{j} + 2\hat{k}}{\sqrt{2^2 + 1^2 + 2^2}} = \frac{2\hat{i} + \hat{j} + 2\hat{k}}{3}$$

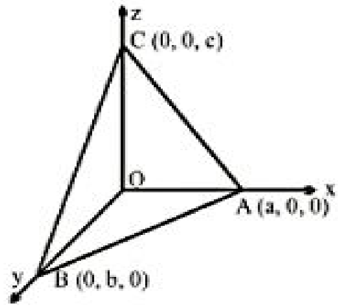
Hence, the required equation of plane is,

$$\vec{r}\,.\hat{n} = d$$

$$\Rightarrow \quad \vec{r}\left(\frac{2\hat{i} + \hat{j} + 2\hat{k}}{3}\right) = 8$$

$$or \quad \vec{r}\,.(2\hat{i} + \hat{j} + 2\hat{k}) = 24$$

6. Intercept form the plane :



Equation of plane in the intercept form is $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1.$

where $a = $ x-intercept,

$b = $ y-intercept,

$c = $ z-intercept

Proof :

Equation of plane passing through three points A (a, 0, 0), B (0, b, 0) and C (0, 0, c) will be

$$\begin{vmatrix} x-a & y-0 & z-0 \\ -a & b & 0 \\ -a & 0 & c \end{vmatrix} = 0.$$

$\Rightarrow$ $(x-a)\,bc - y\,(-ac-0) + z\,(0+ab) = 0$

$\Rightarrow$ $xbc + yac + zab = abc$

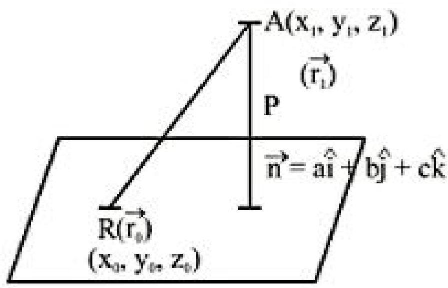
$\Rightarrow$ $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$

Note : Area of $\Delta ABC = \frac{1}{2}\left|\overrightarrow{AB}\times\overrightarrow{BC}\right| = \frac{1}{2}\left|(b\hat{j}-a\hat{i})\times(c\hat{k}-b\hat{j})\right| = \frac{1}{2}\left|bc\hat{i}+ac\hat{j}+ab\hat{k}\right|$

$$= \frac{1}{2}\sqrt{a^2b^2+b^2c^2+c^2a^2} = \sqrt{\left(\frac{ab}{2}\right)^2+\left(\frac{bc}{2}\right)^2+\left(\frac{ca}{2}\right)^2}$$

$$\therefore \text{ Area of } \Delta ABC = \sqrt{(\text{area of } \Delta OAB)^2+(\text{area of } \Delta OBC)^2+(\text{area of } \Delta OCA)^2}$$

PERPENDICULAR DISTANCE OF A POINT FROM A PLANE :



Let equation of plane is $\vec{r}\cdot\vec{n} = d$ then perpendicular distance from the point $A(\vec{r}_1)$ on the plane = projection of $\overrightarrow{RA}$ on $\vec{n}$.

$$\Rightarrow p = \frac{|(\vec{r}_1-\vec{r}_0)\cdot\vec{n}|}{|\vec{n}|} = \frac{|\vec{r}_1\cdot\vec{n}-\vec{r}_0\cdot\vec{n}|}{|\vec{n}|}$$

$$\Rightarrow p = \frac{|\vec{r}_1\cdot\vec{n}-d|}{|\vec{n}|}$$

If equation of plane is $ax+by+cz+d=0$ then perpendicular distance from point (x_1, y_1, z_1) is given by $\frac{|ax_1+by_1+cz_1+d|}{\sqrt{a^2+b^2+c^2}}$.

Note :

(i) Planes $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$ are

(a) parallel but not identical if $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2} \neq \frac{d_1}{d_2}$

(b) perpendicular if $a_1a_2 + b_1b_2 + c_1c_2 = 0$

(c) identical if $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2} = \frac{d_1}{d_2}$

(ii) The equation of a plane parallel to the plane $ax + by + cz + d$ is $ax + by + cz + k = 0$, where k is an arbitrary constant and is determined by the given condition.

(iii) Distance between two parallel planes $ax + by + cz + d_1 = 0$ and $ax + by + cz + d_2 = 0$ is equal to

$$\frac{|d_1 - d_2|}{\sqrt{a^2 + b^2 + c^2}}.$$

(iv) 3 planes $a_rx + b_ry + c_rz = d_r$ $r = 1, 2, 3$

(a) Can intersect at a point ≡ system of equations in 3 variables having unique solution.

(b) Can intersect coaxially ≡ system of equations in 3 variables having infinite solutions.

(c) May not have a common point ≡ system of equations in 3 variables having no solution.

Illustration :

A variable plane passes through a fixed point (α, β, γ) and meets the axes in A, B, C. Show that the locus of the point of intersection of the planes through A, B and C parallel to the co-ordinate planes is $\alpha x^{-1} + \beta y^{-1} + \gamma z^{-1} = 1$.

Sol. *Let the equation of the variable plane be* $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ ....(1)

where a, b, c are parameters

The plane (1) passes through the point (α, β, γ).

$\therefore \frac{\alpha}{a} + \frac{\beta}{b} + \frac{\gamma}{c} = 1$ ....(2)

The plane (1) meets the co-ordinate axes in the points A, B and C whose co-ordinates are respectively given by (a, 0, 0), (0, b, 0) and (0, 0, c). The equations of the planes through A, B and C are $x = a$, $y = b$, $z = c$ respectively ....(3)

The locus of the point of intersection of these planes is obtained by eliminating the parameters a, b, c between the equation (2), (3). Putting the values of a, b, c from (3) in (2), the required locus is given by $\frac{\alpha}{x} + \frac{\beta}{y} + \frac{\gamma}{z} = 1$ *or* $\alpha x^{-1} + \beta y^{-1} + \gamma z^{-1} = 1$.

EQUATION OF PLANES BISECTING THE ANGLES BETWEEN TWO GIVEN PLANES :

Cartesian Form :

The equation of the planes bisecting the angles between the planes $a_1 x + b_1 y + c_1 z + d_1 = 0$ and $a_2 x + b_2 y + c_2 z + d_2 = 0$ are

$$\frac{(a_1x + b_1y + c_1z + d_1)}{\sqrt{a_1^2 + b_1^2 + c_1^2}} = \pm \frac{(a_2x + b_2y + c_2z + d_2)}{\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

Note :

If angle between bisector plane and one of the plane is less than 45° then it is acute angle bisector otherwise it is obtuse angle bisector.

Vector Form :

The equation of the planes bisecting the angles between the planes $\vec{r} \cdot \vec{n}_1 = d_1$ and $\vec{r} \cdot \vec{n}_2 = d_2$ are

$$\frac{|\vec{r} \cdot \vec{n}_1 - d_1|}{|\vec{n}_1|} = \frac{|\vec{r} \cdot \vec{n}_2 - d_2|}{|\vec{n}_2|}$$

or $$\frac{|\vec{r} \cdot \vec{n}_1 - d_1|}{|\vec{n}_1|} = \pm \frac{|\vec{r} \cdot \vec{n}_2 - d_2|}{|\vec{n}_2|}$$

or $$\vec{r} \cdot (\hat{n}_1 \pm \hat{n}_2) = \frac{d_1}{|\vec{n}_1|} \pm \frac{d_2}{|\vec{n}_2|}$$

FAMILY OF PLANES :

The equation of a plane passing through the lines of intersection of $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$ is $(a_1x + b_1y + c_1z + d_1) + \lambda (a_2x + b_2y + c_2z + d_2) = 0$, where λ is a constant.

Vectorially :

Equation of a plane passing through the line of intersection of planes $\vec{r}.\vec{n}_1 = d_1$ and $\vec{r}.\vec{n}_2 = d_2$ is given by

$$(\vec{r}.\vec{n}_1 - d_1) + \lambda(\vec{r}.\vec{n}_2 - d_2) = 0$$

Illustration :

Find the equation of plane containing the line of intersection of the plane $x + y + z - 6 = 0$ and $2x + 3y + 4z + 5 = 0$ and passing through (1,1,1)

Sol. *The equation of the plane through the line of intersection of the given planes is ,*

$$(x + y + z - 6) + \lambda\,(2x + 3y + 4z + 5) = 0 \qquad ...(i)$$

If it is passes through (1, 1, 1)

$$\Rightarrow \quad (1 + 1 + 1 - 6) + \lambda\,(2 + 3 + 4 + 5) = 0 \Rightarrow \lambda = \frac{3}{14}$$

Putting $\lambda = 3/14$ in (i) we get

$$(x + y + z - 6) + \frac{3}{14}\,(2x + 3y + 4z + 5) = 0$$

$$\Rightarrow \quad 20x + 23y + 26z - 69 = 0$$

ANGLE BETWEEN TWO PLANES :

1. Vector form :

The angle between the two planes is defined as the angle between their normals.
Let θ be the angle between planes;

$\vec{r}\,.\,\vec{n}_1 = d_1$ and $\vec{r}\,.\vec{n}_2 = d_2$ is given by

$$\cos\theta = \frac{\vec{n}_1 . \vec{n}_2}{|\vec{n}_1||\vec{n}_2|}$$

2. Cartesian form :

The angle θ between the planes $a_1 x + b_1 y + c_1 z + d_1 = 0$ and $a_2 x + b_2 y + c_2 z + d_2 = 0$ is given by

$$\cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

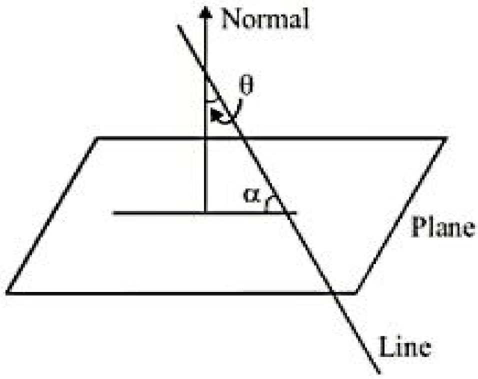
3. Two planes are perpendicular iff

$a_1a_2 + b_1b_2 + c_1c_2 = 0$ & Parallel if $\vec{n}_1 \times \vec{n}_2 = 0$

Angle between a line and a plane :

The angle between a line and a plane is the complement of the angle between the line and the normal to the plane

If α, β, γ be the direction ratios of the line and $ax + by + cz + d = 0$ be the equation of plane and θ be the angle between the line and the plane.



$$\Rightarrow \quad \cos(90° - \theta) = \frac{a\alpha + b\beta + c\gamma}{\sqrt{a^2+b^2+c^2}\sqrt{\alpha^2+\beta^2+\gamma^2}}$$

or $$\sin\theta = \frac{a\alpha + b\beta + c\gamma}{\sqrt{a^2+b^2+c^2}\sqrt{\alpha^2+\beta^2+\gamma^2}}$$

Vector form :

If θ is the angle between the line; $\vec{r} = \vec{a} + \lambda\vec{b}$ and plane $\vec{r}\cdot\vec{n} = d$

$$\Rightarrow \quad \sin\theta = \frac{\vec{b}\cdot\vec{n}}{|\vec{b}||\vec{n}|}$$

Illustration :

Find the angle between the planes $2x - y + z = 11$ and $x + y + 2z = 3$.

Sol. $$\cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{\left(a_1^2+b_1^2+c_1^2\right)\left(a_2^2+b_2^2+c_2^2\right)}}$$

$$\Rightarrow \quad \cos\theta = \frac{2\cdot 1 + (-1)\cdot 1 + 1\cdot 2}{\sqrt{2^2+(-1)^2+1^2}\sqrt{1^2+1^2+2^2}} = \frac{1}{2} \quad \Rightarrow \quad \theta = \frac{\pi}{3}.$$

Illustration :

Find the equation of plane passing through the intersection of planes $2x - 4y + 3z + 5 = 0$, $x + y + z = 6$ and parallel to straight line having direction cosines $(1, -1, -1)$.

Sol. *Equation of required plane be $(2x - 4y + 3z + 5) + \lambda(x + y - z - 6) = 0$*

i.e. $(2 + \lambda)x + (-4 + \lambda)y + z(3 - \lambda) + (5 - 6\lambda) = 0$

This plane is parallel to a straight line. So, $al + bm + cn = 0$

$1(2 + \lambda) + (-1)(-4 + \lambda) + (-1)(3 - \lambda) = 0$ *i.e.* $\lambda = -3$

$\therefore$ *Equation of required plane is* $-x - 7y + 6z + 23 = 0$.

i.e. $x + 7y - 6z - 23 = 0$.

Illustration :

Find the angle between the line $\frac{x+1}{3} = \frac{y-1}{2} = \frac{z-2}{4}$ *and the plane* $2x + y - 3z + 4 = 0$

Sol. *The given line is parallel to the vector* $\vec{b} = 3\hat{i} + 2\hat{j} + 4\hat{k}$ *and the given plane is normal to the vector*

$$\vec{n} = 2\hat{i} + \hat{j} - 3\hat{k}$$

$$\sin\theta = \frac{\vec{b}.\vec{n}}{|\vec{b}||\vec{n}|} = \frac{(3\hat{i}+2\hat{j}+4\hat{k}).(2\hat{i}+\hat{j}-3\hat{k})}{\sqrt{3^2+2^2+4^2}\sqrt{2^2+1^2+3}}$$

$$= \frac{6+2-12}{\sqrt{29}\sqrt{14}} = \frac{-4}{\sqrt{406}} \quad \therefore \quad \theta = \sin^{-1}\left(\frac{-4}{\sqrt{406}}\right)$$

Practice Problem

Q.1 The ratio in which yz-plane divides the line joining (2, 4, 5) and (3, 5, 7)
(A) – 2 : 3 (B) 2 : 3 (C) 3 : 2 (D) – 3 : 2

Q.2 The points (0, – 1, – 1), (– 4, 4, 4), (4, 5, 1) and (3, 9, 4) are
(A) collinear (B) coplanar (C) forming a square (D) none of these

Q.3 The distance of centroid from x-axis of the triangle formed by the points (2, –4, 3), (3, –1, –2) and (–2, 5, 8) is-
(A) 1 (B) 0 (C) 3 (D) $\sqrt{10}$

Q.4 The locus of a point, which moves in such a way that its distance from the origin is thrice the distance from xy-plane is -
(A) $x^2 - 8y^2 - 8z^2 = 0$ (B) $x^2 - 8y^2 + z^2 = 0$
(C) $-8x^2 + y^2 + z^2 = 0$ (D) $x^2 + y^2 - 8z^2 = 0$

Q.5 A non-zero vector $\bar{a}$ is parallel to the line of intetsection of the plane determined by the vectors $\hat{i}, \hat{i}+\hat{j}$ and the plane determined by the vectors $\hat{i}-\hat{j}, \hat{i}+\hat{k}$. The angle between $\bar{a}$ and $\hat{i}-2\hat{j}+2\hat{k}$ is.
(A) π/3 (B) π/4 (C) π/6 (D) None of these

Answer key

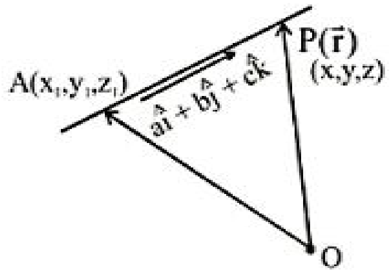
Q.1	A	Q.2	B	Q.3	C	Q.4	D	Q.5	B

STRAIGHT LINES :

Symmetric Form :

1. Equation of a straight line passing through (x_1, y_1, z_1) and having drs as a, b, c is

$$\boxed{\frac{x-x_1}{a}=\frac{y-y_1}{b}=\frac{z-z_1}{c}=\lambda}$$



Proof:

A vector parallel to line will be $a\hat{i}+b\hat{j}+c\hat{k}$.

A vector along the line can be written as

$$\overrightarrow{AP}=(x-x_1)\hat{i}+(y-y_1)\hat{j}+(z-z_1)\hat{k}$$

$\because$ vector $\overrightarrow{AP}$ is parallel to $a\hat{i}+b\hat{j}+c\hat{k}$

$\therefore$ $$\boxed{\frac{x-x_1}{a}=\frac{y-y_1}{b}=\frac{z-z_1}{c}=\lambda}$$

- Any point on this line can be taken as $(x_1+\lambda a, y_1+\lambda b, z_1+\lambda c)$.
- If dcs of line be l, m, n then its equation will $\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}=\lambda$ and any point on this line can be taken as $(x_1+\lambda a, y_1+\lambda b, z_1+\lambda c)$.

2. Equation of straight line passing through two points (x_1, y_1, z_1) and (x_2, y_2, z_2) will be

$$\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z_1}$$

Note :

(i) (a) Equation of x-axis is $\frac{x}{1}=\frac{y}{0}=\frac{z}{0}$ (or) $y=z=0$

(b) Equation of y-axis is $\frac{x}{0}=\frac{y}{1}=\frac{z}{0}$ (or) $x=z=0$

(c) Equation of z-axis $\frac{x}{0}=\frac{y}{0}=\frac{z}{1}$ (or) $x=y=0$

Here zero in denominator represents that line is perpendicular to that axis.

(ii) Line $\frac{x-2}{3}=\frac{y+1}{-2}$ and $z=2$ is written as $\frac{x-2}{3}=\frac{y+1}{-2}=\frac{z-2}{0}$

This line is perpendicular to z-axis or parallel to xy plane at a distance of 2 units.

Unsymmetrical form of straight line:

The equations $a_1x+b_1y+c_1z+d_1=0$ and $a_2x+b_2y+c_2z+d_2=0$ together represents a line in unsymmetrical form. This represent equation of line of intersection of planes $a_1x+b_1y+c_1z+d_1=0$ and $a_2x+b_2y+c_2z+d_2=0$.

Procedure to convert Unsymmetrical Form of straight line to Symmetrical Form :

Let the direction ratios of the line of intersection (AB) of two planes

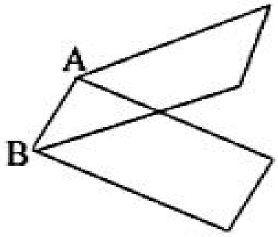
$a_1x + b_1y + c_1z + d_1 = 0$ .....(1) and $a_2x + b_2y + c_2z + d_2 = 0$ .....(2) are a, b, c

Direction ratios of normal to plane (1) are a_1, b_1, c_1 and

Direction ratios of normal to plane (2) are a_2, b_2, c_2

Line AB lies in both the planes (1) and (2)

hence normals to (1) and (2) are perpendicular to AB.



Hence $aa_1 + bb_1 + cc_1 = 0$ and $aa_2 + bb_2 + cc_2 = 0$

these two will give the proportional values of a, b, c.

Let the line AB cuts the xy plane at $(x_1, y_1, 0)$

Hence $a_1x_1 + b_1y_1 = -d_1$ and $a_2x_1 + b_2y_1 = -d_2$ This will give a point on the line AB

$\therefore$ equation of AB is $\dfrac{x - x_1}{a} = \dfrac{y - y_1}{b} = \dfrac{z - 0}{c}$.

Illustration :

Find the angle between the line $x - 2y + z = 0 = x + 2y - 2z$ and $x + 2y + z = 0 = 3x + 9y + 5z$.

Sol. *Let a_1, b_1, c_1 be the direction ratios of the line $x - 2y + z = 0$ and $x + 2y - 2z = 0$. Since it lies in both the planes, therefore, it is $\perp$ to the normals to the two planes.*

$$\therefore \quad a_1 - 2b_1 + c_1 = 0$$
$$a_1 + 2b_1 - 2c_1 = 0$$

Solving these two equations by cross-multiplication, we have

$$\frac{a_1}{4-2} = \frac{b_1}{1+2} = \frac{c_1}{2+2} \qquad or \qquad \frac{a_1}{2} = \frac{b_1}{3} = \frac{c_1}{4}$$

Let a_2, b_2, c_2 be the direction ratios of the line $x + 2y + z = 0 = 3x + 9y + 5z$. Then the discussed above

$$a_2 + 2b_2 + c_2 = 0 \qquad 3a_2 + 9b_2 + 5c_2 = 0$$

$$\Rightarrow \quad \frac{a_2}{10-9} = \frac{b_2}{3-5} = \frac{c_2}{9-6} \qquad or \qquad \frac{a_2}{1} = \frac{b_2}{-2} = \frac{c_2}{3}$$

Let θ be the angle between the given lines. Then

$$\cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

$$= \frac{(1)(2) + (-2)(3) + (3)(4)}{\sqrt{2^2 + 3^2 + 4^2}\sqrt{1^2 + (-2)^2 + (3)^2}}$$

$$= \frac{2 - 6 + 12}{\sqrt{29}\sqrt{14}} = \frac{8}{\sqrt{406}} \qquad \Rightarrow \qquad \theta = \cos^{-1}\left(\frac{8}{\sqrt{406}}\right)$$

Illustration :

Find the coordinates of the point where the line joining the points (2, – 3, 1) and (3, – 4, – 5) cuts the plane $2x + y + z = 7$.

Sol. The direction ratios of the line are $3 - 2, -4 - (-3), -5 - 1$ i.e. $1, -1, -6$

Hence equation of the line joining the given points is $\dfrac{x-2}{1} = \dfrac{y+3}{-1} = \dfrac{z-1}{-6} = r$ (say)

Coordinates of any point on this line are $(r + 2, -r - 3, -6r + 1)$

If this point lies on the given plane $2x + y + z = 7$, then

$$2(r+2) + (-r-3) + (-6r+1) = 7 \;\Rightarrow\; r = -1$$

Coordinates of the point are $(-1 + 2, -(-1) - 3, -6(-1) + 1)$ i.e. $(1, -2, 7)$.

Illustration :

Find in symmetrical form the equations of the line $3x + 2y - z - 4 = 0 = 4x + y - 2z + 3$.

Sol. The equation of the line in unsymmetrical form are $3x + 2y - z - 4 = 0$, $4x + y - 2z + 3 = 0$ .....(1)
Let l, m, n be the direction cosines of the line. Since the line is common to both the planes, it is perpendicular to the normals to both the planes. Hence, $3l + 2m - n = 0$, $4l + m - 2n = 0$

Solving these we get, $\dfrac{l}{-4+1} = \dfrac{m}{-4+6} = \dfrac{n}{3-8}$

i.e. $\dfrac{l}{-3} = \dfrac{m}{2} = \dfrac{n}{-5} = \dfrac{1}{\sqrt{(-3)^2 + 2^2 + (-5)^2}} = \dfrac{1}{\sqrt{38}}$

So, direction cosines of the lines are $\dfrac{-3}{\sqrt{38}}, \dfrac{2}{\sqrt{38}}, \dfrac{-5}{\sqrt{38}}$

Now to find the coordinates of a point on a line. Let us find out the point where it meets the plane $z = 0$. Putting $z = 0$ in the equation given by (1), we have $3x + 2y - 4 = 0$, $4x + y + 3 = 0$.
Solving these, we get $x = -2$, $y = 5$
So, one point of the line is $(-2, 5, 0)$

$\therefore$ Equation of the line in symmetrical form is $\dfrac{x+2}{\frac{-3}{\sqrt{38}}} = \dfrac{y-5}{\frac{2}{\sqrt{38}}} = \dfrac{z-0}{\frac{-5}{\sqrt{38}}}$ i.e. $\dfrac{x+2}{-3} = \dfrac{y-5}{2} = \dfrac{z}{-5}$.

Illustration :

Find the equation of the plane which contains the two parallel lines $\dfrac{x+1}{3} = \dfrac{y-2}{2} = \dfrac{z}{1}$ and $\dfrac{x-3}{3} = \dfrac{y+4}{2} = \dfrac{z-1}{1}$.

Sol. The equations of the two parallel lines are

$$\frac{x+1}{3} = \frac{y-2}{2} = \frac{z-0}{1} \qquad ....(1)$$

and $$\frac{x-3}{3} = \frac{y+4}{2} = \frac{z-1}{1} \qquad ....(2)$$

the equation of any plane through the line (1) is

$a(x + 1) + b(y - 2) + cz = 0$ ....(3)

where $3a + 2b + c = 0$ ....(4)

the line (2) will also lie on the plane (3) if the point $(3, -4, 1)$ *lies on the plane (3), and for this we have* $a(3 + 1) + b(-4 - 2) + c = 0$ *or* $4a - 6b + c = 0$ ....(5)

Solving (4) and (5), we get $\frac{a}{8} = \frac{b}{1} = \frac{c}{-26}$

Putting the values of a, b, c in (3), the required equation of the plane is $8x + y - 26z + 6 = 0$.

Illustration :

Find the shortest distance between the lines $\frac{x-3}{3} = \frac{y-8}{-1} = \frac{z-3}{1}$, $\frac{x+3}{3} = \frac{y+7}{2} = \frac{z-6}{4}$.

Also find the equation of line of shortest distance.

Sol. *Given lines are* $\frac{x-3}{3} = \frac{y-8}{-1} = \frac{z-3}{1} = r_1$ *(say)* ....(1)

$$\frac{x+3}{3} = \frac{y+7}{2} = \frac{z-6}{4} = r_2 \quad (say) \qquad ....(2)$$

any point on line (1) is $P(3r_1 + 3, 8 - r_1, r_1 + 3)$ *and on line (2) is*

$Q(-3 - 3r_2, 2r_2 - 7, 4r_2 + 6)$.

If PQ is line of shortest distance, then direction ratios of

$PQ = (3r_1 + 3) - (-3 - 3r_2), (8 - r_1) - (2r_2 - 7), (r_1 + 3) - (4r_2 + 6)$

i.e. $3r_1 + 3r_2 + 6, -r_1 - 3r_2 + 15, r_1 - 4r_2 - 3$

As PQ is perpendicular to lines (1) and (2)

$\therefore$ $3(3r_1 + 3r_2 + 6) - 1(-r_1 - 2r_2 + 15) + 1(r_1 - 4r_2 + 3) = 0$

$\Rightarrow$ $11r_1 + 7r_2 = 0$ ....(3)

and $-3(3r_1 + 3r_2 + 6) + 2(-r_1 - 2r_2 + 15) + 4(r_1 - 4r_2 + 3) = 0$

i.e. $7r_1 + 11r_2 = 0$ ....(4)

On solving equations (3) and (4), we get $r_1 = r_2 = 0$.

So, point $P(3, 8, 3)$ *and* $Q(-3, -7, 6)$

$\therefore$ *Length of shortest distance* $PQ = \sqrt{\{(-3-3)^2 + (-7-8)^2 + (6-3)^2\}} = 3\sqrt{30}$.

Direction ratios of shortest distance line is $2, 5, -1$.

$\therefore$ *Equation of shortest distance line* $\frac{x-3}{2} = \frac{y-8}{5} = \frac{z-3}{-1}$.

Illustration :

Find the distance of the point (1, – 2, 3) from the plane $x - y + z = 5$ *measured parallel to the line* $\frac{x}{2} = \frac{y}{3} = \frac{z}{-6}$.

Sol. *Here we are not to find perpendicular distance of the point from the plane but distance measured along with the given line. The method is as follow :*

The equation of the line through the point (1, – 2, 3) and parallel to given line is

$$\frac{x-1}{2} = \frac{y+2}{3} = \frac{z-3}{-6} = r \text{ (say)}$$

The coordinate of any point on it is $(2r + 1, 3r - 2, -6r + 3)$.

If this point lies in the given plane then

$$2r + 1 - (3r - 2) + (-6r + 3) = 5 \Rightarrow -7r = -1 \text{ or } r = \frac{1}{7}$$

$\therefore$ *Point of intersection is* $\left(\frac{9}{7}, \frac{-11}{7}, \frac{15}{7}\right)$.

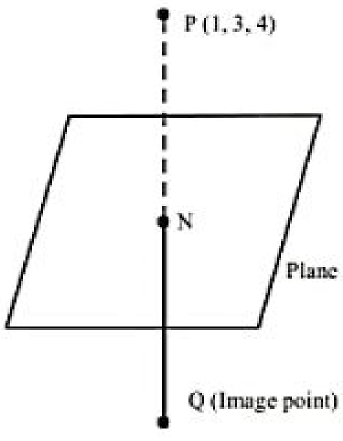
$\therefore$ *The required distance = the distance between the points (1, – 2, 3) and* $\left(\frac{9}{7}, \frac{-11}{7}, \frac{15}{7}\right)$

$$= \sqrt{\left(1-\frac{9}{7}\right)^2 + \left(-2+\frac{11}{7}\right)^2 + \left(3-\frac{15}{7}\right)^2} = \frac{\sqrt{49}}{7} = 1 \text{ Unit.}$$

Illustration :

Find the image of the point (1, 3, 4) in the plane $2x - y + z + 3 = 0$.

Sol. *As it is clear from the figure that PQ will be perpendicular to the plane and foot of this perpendicular is mid point of PQ i.e. N.*



So, direction ratios of line PQ is 2, – 1, 1

$\Rightarrow$ *Equation of line PQ* $= \frac{x-1}{2} = \frac{y-3}{-1} = \frac{z-4}{1} = r$ *(say)*

Any point on line PQ is $(2r + 1, -r + 3, r + 4)$

If this point lies on the plane, then

$$2(2r + 1) - (-r + 3) + (r + 4) + 3 = 0$$

$$\Rightarrow \quad r = -1$$

$\therefore$ *Coordinate of foot of perpendicular N = (–1, 4, 3).*

As N is middle point of PQ.

$$\therefore\ -1 = \frac{1-x_1}{2},\ 4 = \frac{3+y_1}{2},\ 3 = \frac{4+z_1}{2} \Rightarrow x_1 = -3, y_1 = 5, z_1 = 2$$

$\therefore$ *Image of point P (1, 3, 4) is the point Q (– 3, 5, 2).*

Illustration :

Determine whether each statement is true or false

(a) *Two lines parallel to a third line are parallel.*
(b) *Two lines perpendicular to a third line are parallel.*
(c) *Two planes parallel to a third plane are parallel.*
(d) *Two planes perpendicular to a third plane are parallel.*
(e) *Two lines parallel to a plane are parallel.*
(f) *Two lines perpendicular to a plane are parallel.*
(g) *Two planes parallel to a line are parallel.*
(h) *Two planes perpendicular to a line are parallel.*
(i) *Two planes either intersect or are parallel.*
(j) *Two lines either intersect or are parallel.*
(k) *A plane and a line either intersect or are parallel.*

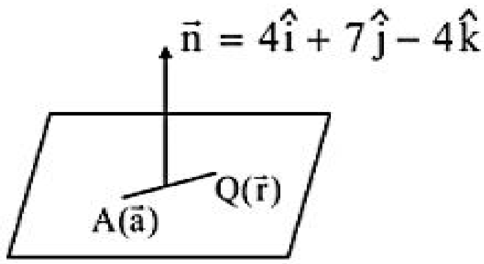
Ans. (a) True, (b) False, (c) True, (d) False, (e) False, (f) True,
(g) False, (h) True, (i) True, (j) False, (k) True

Illustration :

Find the equation of a plane passing through the point A(3, –2, 1) and perpendicular to the vector $4\vec{i} + 7\vec{j} - 4\vec{k}$. *If PM be perpendicular from the point P(1, 2, –1) to this plane, find its length.*

Sol. Let O be the origin, then $\vec{a} = \overrightarrow{OA} = 3\vec{i} - 2\vec{j} + \vec{k}$

Let $\vec{n} = 4\vec{i} + 7\vec{j} - 4\vec{k}$



Let Q(x, y, z) be any point on the plane

then, $\vec{r} = \overrightarrow{OQ} = x\vec{i} + y\vec{j} + z\vec{k}$

Now $\vec{r} - \vec{a} = (x - 3)\vec{i} + (y + 2)\vec{j} + (z - 1)\vec{k}$

Now, equation of the required plane passing point $A(\vec{a})$

and perpendicular to $\vec{n}$ is

$$(\vec{r} - \vec{a}) \cdot \vec{n} = 0$$

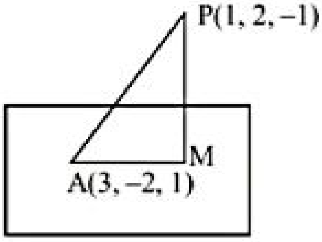
$\therefore$ $4(x - 3) + 7(y + 2) - 4(z - 1) = 0$

or, $4x - 3y - 4z + 6 = 0$. This is the required equation of the plane.

Now $PM \perp AM$

$\therefore$ Unit vector parallel to $\overrightarrow{PM}$

$$\vec{n} = \frac{\vec{n}}{|\vec{n}|} = \frac{4\vec{i} + 7\vec{j} - 4\vec{k}}{9}$$



Now PM = *length of projection of* $\overline{PA}$ *on* $\overline{PM}$

$$= \left|\overline{PA}\cdot\hat{n}\right| = \left|\left(2\vec{i} - 4\vec{j} + 2\vec{k}\right)\cdot\left(\frac{4\vec{i} + 7\vec{j} - 4\vec{k}}{9}\right)\right|$$

Let $\quad \overline{\alpha} = \overline{OP} = \overline{i} + 2\overline{j} - \overline{k}$

Second method :

Alternatively, $PM = \left|\dfrac{(\vec{\alpha} - \vec{a})\cdot\vec{n}}{n}\right|$

Here, $\vec{\alpha} - \vec{a} = \overrightarrow{AP} = -2\vec{i} + 4\vec{j} - 2\vec{k}$ *and* $\vec{n} - 4\vec{i} + 7\vec{j} - 4\vec{k}$

$$PM = \left|\frac{-8 + 28 + 8}{9}\right| = \frac{28}{9}$$

Illustration :

Find the shortest distance and the vector equation of the line of shortest distance between the lines given by $\vec{r} = 3\hat{i} + 8\hat{j} + 3\hat{k} + \lambda(3\hat{i} - \hat{j} + \hat{k})$ *and* $\vec{r} = -3\hat{i} - 7\hat{j} + 6\hat{k} + \mu(3\hat{i} + 2\hat{j} + 4\hat{k})$

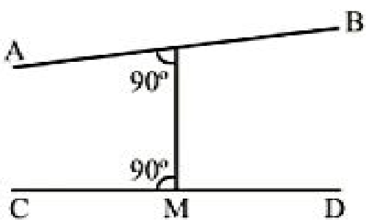
Sol. *Given lines are* $\quad \vec{r} = 3\hat{i} + 8\hat{j} + 3\hat{k} + \lambda(3\hat{i} - \hat{j} + \hat{k}) \quad$ ...(i)

and $\quad \vec{r} = -3\hat{i} - 7\hat{j} + 6\hat{k} + \mu(3\hat{i} + 2\hat{j} + 4\hat{k}) \quad$ ...(ii)

Equations of lines (i) and (ii) in cartesian from are

$$AB : \frac{x-3}{3} = \frac{y-8}{-1} = \frac{z-3}{1} = \lambda \qquad \text{...(iii)}$$

and $$CD : \frac{x-3}{-3} = \frac{y+7}{2} = \frac{z-6}{4} = \mu \qquad \text{...(iv)}$$



Let $\quad L \equiv (3\lambda + 3, -\lambda + 8, \lambda + 3)$

and $\quad M \equiv (-3\mu - 3, 2\mu - 7, 4\mu + 6)$

Direction ratios of LM are

$$3\lambda + 3\mu + 6, -\lambda - 2\mu + 15, \lambda - 4\mu - 3$$

Since $LM \perp AB$

$\therefore \quad 3(3\lambda + 3\mu + 6) - 1(-\lambda - 2\mu + 15) + 1(\lambda - 4\mu - 3) = 0$

or $\quad 11\lambda + 7\mu = 0 \qquad \ldots(v)$

Again $LM \perp CD$

$\therefore \quad -3(3\lambda + 3\mu + 6) + 2(-\lambda - 2\mu + 15) + 4(\lambda - 4\mu - 3) = 0$

or $\quad -7\lambda - 29\mu = 0 \qquad \ldots(vi)$

Solving (v) and (vi), we get $\lambda = 0, \mu = 0$

$\therefore \quad L \equiv (3, 8, 3),\ M \equiv (-3, -7, 6)$

Hence shortest distance $LM = \sqrt{(3+3)^2 + (8+7)^2 + (3-6)^2} = \sqrt{270} = 3\sqrt{30}$ *units*

Vector equation of LM is

$$\vec{r} = 3\hat{i} + 8\hat{j} + 3\hat{k} + t(6\hat{i} + 15\hat{j} - 3\hat{k})$$

Note : *Cartesian equation of LM is* $\dfrac{x-3}{6} = \dfrac{y-8}{15} = \dfrac{z-3}{-3}$

Practice Problem

Q.1 The shortest distance between the two straight lines $\dfrac{x-4/3}{2} = \dfrac{y+6/5}{3} = \dfrac{z-3/2}{4}$ and $\dfrac{5y+6}{8} = \dfrac{2z-3}{9} = \dfrac{3x-4}{5}$ is

(A) $\sqrt{29}$ (B) 3 (C) 0 (D) $6\sqrt{10}$

Q.2 A straight line passes through the point $(2, -1, -1)$. It is parallel to the plane $4x + y + z + 2 = 0$ and is perpendicular to the line $\dfrac{x}{1} = \dfrac{y}{-2} = \dfrac{z-5}{1}$. The equations of the straight line are

(A) $\dfrac{x-2}{4} = \dfrac{y+1}{1} = \dfrac{z+1}{1}$ (B) $\dfrac{x+2}{4} = \dfrac{y-1}{1} = \dfrac{z-1}{1}$

(C) $\dfrac{x-2}{-1} = \dfrac{y+1}{1} = \dfrac{z+1}{3}$ (D) $\dfrac{x+2}{-1} = \dfrac{y-1}{1} = \dfrac{z-1}{3}$

Q.3 The cosine of angle between any two diagonal of a cube is -

(A) 1/3 (B) 1/2 (C) 2/3 (D) $1/\sqrt{3}$

Q.4 The equation of the plane containing the line $\frac{x+1}{-3}=\frac{y-3}{2}=\frac{z+2}{1}$ and the point $(0, 7, -7)$ is-

(A) $x+y+z=2$ (B) $x+y+z=3$ (C) $x+y+z=0$ (D) None of these

Q.5 The distance of the point $(1, -2, 3)$ from the plane $x - y + z = 5$ measured parallel to the line $\frac{x}{2}=\frac{y}{3}=\frac{z-1}{-6}$ is-

(A) 1 (B) 2 (C) 4 (D) None of these

Q.6 The equation of the plane through the point $(2, -1, -3)$ and parallel to the lines $\frac{x-1}{3}=\frac{y+2}{2}=\frac{z}{-4}$ and $\frac{x}{2}=\frac{y-1}{-3}=\frac{z-2}{2}$ is

Answer key

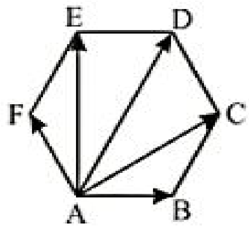
Q.1	C	Q.2	C	Q.3	A	Q.4	C	Q.5	A
Q.6	0								

Solved Examples

Q.1 If ABCDEF is a regular hexagon and $\overrightarrow{AB}+\overrightarrow{AC}+\overrightarrow{AD}+\overrightarrow{AE}+\overrightarrow{AF}=k\,\overrightarrow{AD}$, then k equals-

(A) 2 (B) 3 (C) 6 (D) 5

Sol. $\because \overrightarrow{AB}=\overrightarrow{ED}$ and $\overrightarrow{AF}=\overrightarrow{CD}$,



so $\overrightarrow{AB}+\overrightarrow{AC}+\overrightarrow{AD}+\overrightarrow{AE}+\overrightarrow{AF}$

$=\overrightarrow{ED}+\overrightarrow{AC}+\overrightarrow{AD}+\overrightarrow{AE}+\overrightarrow{CD}$

$=(\overrightarrow{AC}+\overrightarrow{CD})+(\overrightarrow{AE}+\overrightarrow{ED})+\overrightarrow{AD}$

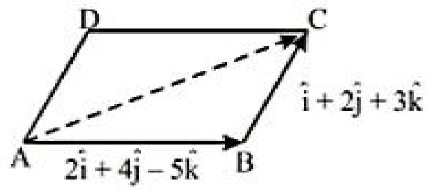
$=\overrightarrow{AD}+\overrightarrow{AD}+\overrightarrow{AD}=3\,\overrightarrow{AD}$

$\therefore$ k = 3 Ans.[B]

Q.2 The length of diagonal AC of a parallellogram ABCD whose two adjacent sides AB and AD are represented respectively by $2\hat{i}+4\hat{j}-5\hat{k}$ and $\hat{i}+2\hat{j}+3\hat{k}$ is-

(A) 3 (B) 4 (C) 5 (D) 7

Sol. $\because \overrightarrow{AC}=\overrightarrow{AB}+\overrightarrow{AD}$



$=3\hat{i}+6\hat{j}-2\hat{k}$

$\therefore$ Length of the diagonal $\overrightarrow{AC}=|\overrightarrow{AC}|$

$=\sqrt{3^2+6^2+(-2)^2}=7$ Ans.[D]

Q.3 If the middle points of sides BC, CA & AB of triangle ABC are respectively D,E,F then position vector of centre of triangle DEF, when position vector of A,B, C are respectively $\hat{i}+\hat{j}$, $\hat{j}+\hat{k}$, $\hat{k}+\hat{i}$ is-

(A) $\frac{1}{3}(\hat{i}+\hat{j}+\hat{k})$ (B) $(\hat{i}+\hat{j}+\hat{k})$ (C) $2(\hat{i}+\hat{j}+\hat{k})$ (D) $\frac{2}{3}(\hat{i}+\hat{j}+\hat{k})$

Sol. The position vector of points D,E,F are respectively

$\frac{\hat{i}+\hat{j}}{2}+\hat{k}$, $\hat{i}+\frac{\hat{k}+\hat{j}}{2}$ and $\frac{\hat{i}+\hat{k}}{2}+\hat{j}$

So, position vector of centre of ΔDEF

$=\frac{1}{3}\left[\frac{\hat{i}+\hat{j}}{2}+\hat{k}+\hat{i}+\frac{\hat{k}+\hat{j}}{2}+\frac{\hat{i}+\hat{k}}{2}+\hat{j}\right]=\frac{2}{3}\,[\hat{i}+\hat{j}+\hat{k}]$ Ans.[D]

Q.4 Let position vectors of points A,B,C and D are respectively $3\hat{i}-2\hat{j}-\hat{k}$, $2\hat{i}+3\hat{j}-4\hat{k}$, $-\hat{i}+\hat{j}+2\hat{k}$ and $4\hat{i}+5\hat{j}+\lambda\hat{k}$. If the points are coplanar, then the value of λ is-

(A) $-\frac{146}{17}$ (B) $\frac{146}{17}$ (C) 0 (D) None of these

Sol. $\overrightarrow{AB} = -\hat{i}+5\hat{j}-3\hat{k}$

$\overrightarrow{AC} = -4\hat{i}+3\hat{j}+3\hat{k}$ and $\overrightarrow{AD} = \hat{i}+7\hat{j}+(\lambda+1)\hat{k}$

If A,B,C,D are coplanar, then vectors $\overrightarrow{AB}$, $\overrightarrow{AC}$ and $\overrightarrow{AD}$ are coplanar, then

$$\left[\overrightarrow{AB}\ \overrightarrow{AC}\ \overrightarrow{AD}\right]=0 \quad \text{or} \quad \begin{vmatrix} -1 & 5 & -3 \\ -4 & 3 & 3 \\ 1 & 7 & \lambda+1 \end{vmatrix}=0$$

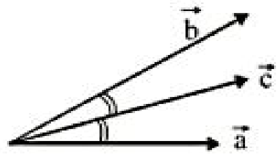
$\Rightarrow \lambda = \frac{-146}{17}$ Ans.[A]

Q.6 The vector $\vec{c}$, directed along the internal bisector of the angle between the vectors $7\hat{i}-4\hat{j}-4\hat{k}$ and $-2\hat{i}-\hat{j}+2\hat{k}$ with $|\vec{c}|=5\sqrt{6}$ is-

(A) $\frac{5}{3}\hat{i}-7\hat{j}+2\hat{k}$ (B) $\frac{5}{3}(5\hat{i}+5\hat{j}+2\hat{k})$ (C) $\frac{5}{3}(\hat{i}+7\hat{j}+2\hat{k})$ (D) None of these

Sol. Let $\vec{a}=7\hat{i}-4\hat{j}-4\hat{k}$ and $\vec{b}=-2\hat{i}-\hat{j}+2\hat{k}$

$$\vec{c}=\lambda(\hat{a}+\hat{b})=\lambda\left(\frac{7\hat{i}-4\hat{j}-4\hat{k}}{9}+\frac{-2\hat{i}-\hat{j}+2\hat{k}}{3}\right)=\lambda\left(\frac{\hat{i}-7\hat{j}+2\hat{k}}{9}\right)$$



$|\vec{c}|=5\sqrt{6} \Rightarrow \lambda=\pm 15 \Rightarrow c=\pm\frac{5}{3}(\hat{i}-7\hat{j}+2\hat{k})$ Ans.[A]

Q.7 If moduli of vectors $\vec{a},\vec{b},\vec{c}$ are 3,4 and 5 respectively and $\vec{a}$ and $\vec{b}+\vec{c}$, $\vec{b}$ and $\vec{c}+\vec{a}$, $\vec{c}$ and $\vec{a}+\vec{b}$ are perpendicular to each other, then modulus of $\vec{a}+\vec{b}+\vec{c}$ is -

(A) $5\sqrt{2}$ (B) $2\sqrt{5}$ (C) 50 (D) 20

Sol. $\because \vec{a}\perp(\vec{b}+\vec{c}) \Rightarrow \vec{a}.\vec{b}+\vec{a}.\vec{c}=0$

Similarly $\vec{b}\perp(\vec{c}+\vec{a}) \Rightarrow \vec{b}.\vec{c}+\vec{b}.\vec{a}=0$ and $\vec{c}\perp(\vec{a}+\vec{b}) \Rightarrow \vec{c}.\vec{a}+\vec{c}.\vec{b}=0$

$\therefore \vec{a}.\vec{b}+\vec{b}.\vec{c}+\vec{c}.\vec{a}=0$

Now $|\vec{a}+\vec{b}+\vec{c}|^2 = |\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2 + 2(\vec{a}.\vec{b}+\vec{b}.\vec{c}+\vec{c}.\vec{a}) = 9 + 16 + 25 = 50$

$\therefore \quad |\vec{a}+\vec{b}+\vec{c}| = 5\sqrt{2}$ Ans.[A]

Q.8 If $|\vec{a}+\vec{b}| = |\vec{a}-\vec{b}|$ then angle between a and b is -

(A) 60^o (B) 30^o (C) 90^o (D) 180^o

Sol. $|a + b| = |\vec{a}-\vec{b}|$

$\Rightarrow \quad |\vec{a}+\vec{b}|^2 = |\vec{a}-\vec{b}|^2$

$\Rightarrow \quad |\vec{a}|^2 + |\vec{b}|^2 + 2\vec{a}.\vec{b} = |\vec{a}|^2 + |\vec{b}|^2 - 2\vec{a}.\vec{b}$

$\Rightarrow \quad 4\,\vec{a}.\vec{b} = 0$

$\Rightarrow \quad \vec{a}.\vec{b} = 0 \Rightarrow \vec{a} \perp \vec{b}$ Ans.[C]

Q.9 If $\vec{a}, \vec{b}, \vec{c}$ are three vectors such that $\vec{a}+\vec{b}+\vec{c} = 0$, then-

(A) $\vec{a}\times\vec{b} = \vec{b}\times\vec{c}$ (B) $\vec{b}\times\vec{c} = \vec{c}\times\vec{a}$

(C) $\vec{a}\times\vec{b} = \vec{b}\times\vec{c} = \vec{c}\times\vec{a}$ (D) None of these

Sol. $\because \vec{a}+\vec{b}+\vec{c} = 0 \Rightarrow \vec{c} = -(\vec{a}+\vec{b})$

$\therefore \vec{b}\times\vec{c} = -\vec{b}\times(\vec{a}+\vec{b}) = -\vec{b}\times\vec{a} - \vec{b}\times\vec{b} = \vec{a}\times\vec{b}$

Similarly $\vec{c}\times\vec{a} = \vec{a}\times\vec{b}$

$\therefore \vec{a}\times\vec{b} = \vec{b}\times\vec{c} = \vec{c}\times\vec{a}$ Ans.[C]

Q.10 If $\ell\hat{i}+m\hat{j}+n\hat{k}$ is a unit vector which is perpendicular to vectors $2\hat{i}-\hat{j}+\hat{k}$ and $3\hat{i}+4\hat{j}-\hat{k}$ then $|\ell|$ is equal to-

(A) $-\frac{3}{\sqrt{155}}$ (B) $\sqrt{\frac{3}{155}}$ (C) $\frac{3}{\sqrt{155}}$ (D) None of these

Sol. Vector $2\hat{i}-\hat{j}+\hat{k}$ and $3\hat{i}+4\hat{j}-\hat{k}$

$$= \frac{(2\hat{i}-\hat{j}+\hat{k})\times(3\hat{i}+4\hat{j}-\hat{k})}{|(2\hat{i}-\hat{j}+\hat{k})\times(3\hat{i}+4\hat{j}-\hat{k})|} = \frac{\hat{i}(1-4)-\hat{j}(-2-3)+\hat{k}(8+3)}{\sqrt{9+25+121}} = \frac{-3\hat{i}+5\hat{j}+11\hat{k}}{\sqrt{155}}$$

$\therefore |\ell| = \left|\frac{-3}{\sqrt{155}}\right| = \frac{3}{\sqrt{155}}$ Ans.[C]

Q.11 The unit vector perpendicular to the plane passing through points $P(\hat{i}-\hat{j}+2\hat{k})$, $Q(2\hat{i}-\hat{k})$ and $R(2\hat{j}+\hat{k})$ is-

(A) $2\hat{i}+\hat{j}+\hat{k}$ (B) $\sqrt{6}(2\hat{i}+\hat{j}+\hat{k})$ (C) $\frac{1}{\sqrt{6}}(2\hat{i}+\hat{j}+\hat{k})$ (D) $\frac{1}{6}(2\hat{i}+\hat{j}+\hat{k})$

Sol. $\overrightarrow{PQ}=(2\hat{i}-\hat{k})-(\hat{i}-\hat{j}+2\hat{k}) = \hat{i}+\hat{j}-3\hat{k}$

$\overrightarrow{PR}=(2\hat{i}+\hat{k})-(\hat{i}-\hat{j}+2\hat{k}) = -\hat{i}+3\hat{j}-\hat{k}$

Now $\overrightarrow{PQ}\times\overrightarrow{PR} = \begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & 1 & -3\\ -1 & 3 & -1\end{vmatrix} = 8\hat{i}+4\hat{j}+4\hat{k} \Rightarrow \left|\overrightarrow{PQ}\times\overrightarrow{PR}\right| = \sqrt{64+16+16} = 4\sqrt{6}$

$\therefore$ reqd. unit vector $=\dfrac{4(2\hat{i}+\hat{j}+\hat{k})}{4\sqrt{6}} = \dfrac{1}{\sqrt{6}}(2\hat{i}+\hat{j}+\hat{k})$ Ans.[C]

Q.12 The area of parallelogram whose diagonals are respectively $3\hat{i}+\hat{j}-2\hat{k}$ and $\hat{i}-3\hat{j}+4\hat{k}$ is-

(A) $5\sqrt{2}$ (B) $5\sqrt{3}$ (C) $2\sqrt{5}$ (D) $3\sqrt{5}$

Sol. Area of parallelogram $=\frac{1}{2}|\vec{a}\times\vec{b}|$

where $\vec{a} = 3\hat{i}+\hat{j}-2\hat{k}$ and $\vec{b} = \hat{i}-3\hat{j}+4\hat{k}$

now $\vec{a}\times\vec{b} = \begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 3 & 1 & -2\\ 1 & -3 & 4\end{vmatrix} = -2\hat{i}-14\hat{j}-10\hat{k}$

$\therefore$ Area of parallelogram

$=\frac{1}{2}|-2\hat{i}-14\hat{j}-10\hat{k}| = \sqrt{1+49+25} = 5\sqrt{3}$ Ans.[B]

Q.13 If $\hat{i}-\hat{j}+2\hat{k}$, $2\hat{i}+\hat{j}-\hat{k}$ and $3\hat{i}-\hat{j}+2\hat{k}$ are position vectors of vertices of a triangle, then its area is-

(A) 26 (B) 13 (C) $2\sqrt{13}$ (D) $\sqrt{13}$

Sol. If A,B,C are given vertices, then

$\overrightarrow{AB} = \hat{i}+2\hat{j}-3\hat{k}$, $\overrightarrow{AC} = 2\hat{i}$

$\therefore\ \overrightarrow{AB}\times\overrightarrow{AC} = (\hat{i}+2\hat{j}-3\hat{k})\times 2\hat{i} = -4\hat{k}-6\hat{j} \Rightarrow \left|\overrightarrow{AB}\times\overrightarrow{AC}\right| = \sqrt{16+36} = 2\sqrt{13}$

$\therefore$ Area of $\Delta ABC = \frac{1}{2}\left|\overrightarrow{AB}\times\overrightarrow{AC}\right| = \sqrt{13}$ Ans.[D]

Q.14 If A,B,C,D are any four points, then $|\overrightarrow{AB}\times\overrightarrow{CD}+\overrightarrow{BC}\times\overrightarrow{AD}+\overrightarrow{CA}\times\overrightarrow{BD}|$ equals-

(A) Area of ΔABC (B) 2(Area of ΔABC)
(C) 3(Area of ΔABC) (D) 4 (Area of ΔABC)

Sol. Let a, b, c and d be position vectors of points A,B,C and D respectively, then

$\overrightarrow{AB}\times\overrightarrow{CD} = (\vec{b}-\vec{a})\times(\vec{d}-\vec{c}) = \vec{b}\times\vec{d}-\vec{b}\times\vec{c}-\vec{a}\times\vec{d}+\vec{a}\times\vec{c}$

Similarly

$\overrightarrow{BC}\times\overrightarrow{AD} = \vec{c}\times\vec{d}-\vec{c}\times\vec{a}-\vec{b}\times\vec{d}+\vec{b}\times\vec{a}$

$\overrightarrow{CA}\times\overrightarrow{BD} = \vec{a}\times\vec{d}-\vec{a}\times\vec{b}-\vec{c}\times\vec{d}+\vec{c}\times\vec{b}$

Therefore given expression

$= |2(\vec{b}\times\vec{a}-\vec{b}\times\vec{c}+\vec{a}\times\vec{c})| = 2\,|(\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a})|$

$= 4$ (Area of ΔABC) Ans.[D]

Q. 15 a,b,c,d are the position vectors of four coplanar points A, B, C and D respectively. If $(\vec{a}-\vec{d}).(\vec{b}-\vec{c}) = 0 = (\vec{b}-\vec{d}).(\vec{c}-\vec{a})$, then for the ΔABC,D is-

(A) incentre (B) orthocentre (C) circumcentre (D) centroid

Sol. $(\vec{b}-\vec{d}).(\vec{c}-\vec{a}) = 0 \Rightarrow (\vec{a}-\vec{d}) \perp (\vec{b}-\vec{c}) \Rightarrow \overrightarrow{AD} \perp \overrightarrow{BC}$

Similarly $(\vec{b}-\vec{d}).(\vec{c}-\vec{a}) = 0 \Rightarrow \overrightarrow{BD} \perp \overrightarrow{AC}$

$\therefore$ D is the orthocentre of ΔABC. Ans.[B]

Q.16 For any vector $\vec{a}$, $\vec{u} = \hat{i}\times(\vec{a}\times\hat{i})+\hat{j}\times(\vec{a}\times\hat{j})+\hat{k}\times(\vec{a}\times\hat{k})$ equals-

(A) $2\vec{a}$ (B) $-2\vec{a}$ (C) $\vec{a}$ (D) $-\vec{a}$

Sol. $\vec{u} = (\hat{i}.\hat{i})\,\vec{a}-(\hat{i}.\vec{a})\,\hat{i}+(\hat{j}.\hat{j})\,\vec{a}-(\hat{j}.\vec{a})\,\hat{j}+(\hat{k}.\hat{k})\,\vec{a}-(\hat{k}.\vec{a})\,\hat{k}$

$= \vec{a}-a_1\hat{i}+\vec{a}-a_2\hat{j}+\vec{a}-a_3\hat{k}$ $[\vec{a} = a_1 i + a_2\hat{j}+a_3\hat{k}\text{ (say)}]$

$\therefore\ u = 3\vec{a}-\vec{a} = 2\vec{a}$ Ans.[A]

Q.17 Let $\vec{a} = 2\hat{i} - \hat{j} + \hat{k}$, $\vec{b} = \hat{i} + 2\hat{j} - \hat{k}$ and $\vec{c} = \hat{i} + \hat{j} = 2\hat{k}$ be three vectors. A vector in the plane of $\vec{b}$ and $\vec{c}$ whose projection on a is $\sqrt{2/3}$ will be-

(A) $2\hat{i} + 3\hat{j} - 3\hat{k}$ (B) $2\hat{i} + 3\hat{j} + 3\hat{k}$ (C) $-2\hat{i} - \hat{j} + 5\hat{k}$ (D) $2\hat{i} + \hat{j} + 5\hat{k}$

Sol. Let the required vector $\vec{r} = \vec{b} + t\vec{c}$

$\Rightarrow r = (1 + t)\ \hat{i} + (2 + t)\hat{j} - (1 + 2t)\ \hat{k}$

Also projection of $\vec{r}$ on $\vec{a} = \sqrt{2/3}$

$$\Rightarrow \frac{\vec{r}.\vec{a}}{|\vec{a}|} = \sqrt{2/3} \Rightarrow \frac{2(1+t)-(2+t)-(1+2t)}{\sqrt{6}} = \sqrt{\frac{2}{3}} \Rightarrow -t-1=2 \Rightarrow t=-3$$

$\therefore\ \vec{r} = -2\hat{i} - \hat{j} + 5\hat{k}$ Ans.[C]

PROBABILITY

RANDOM EXPERIMENTS :

In our day to day life, we perform many activities which have a fixed result no matter any number of times they are repeated. For example given any triangle, without knowing the three angles, we can definitely say that the sum of measure of angles is 180°.

We also perform many experimental activities, where the result may not be same, when they are repeated under identical conditions. For example, when a coin is tossed it may turn up a head or a tail, but we are not sure which one of these results will actually be obtained. Such experiments are called random experiments.

An experiment is called **random experiment** if it satisfies the following to conditions :

(i) It has more than one possible outcome.

(ii) It is not possible to predict the outcome in advance.

Example :

(i) Tossing a coin is a random experiment.

(ii) Throwing a dice is a random experiment.

(iii) Drawing a card from a well shuffled deck of 52 playing card is also a random experiment.

OUT COMES AND SAMPLE SPACE :

A possible result of a random experiment is called its **outcome**.
Consider the experiment of rolling a die. The outcomes of this experiment are 1, 2, 3, 4, 5 or 6, if we are interested in the number of dots on the upper face of the die.

The set of outcomes {1, 2, 3, 4, 5, 6} is called the sample space of the experiment.
Thus, the set of all possible outcomes of a random experiment is called the **sample space** associated with the experiment. Sample space is denoted by the symbol S.

Each element of the sample space is called a **sample point**. In other words, each outcome of the random experiment is also called sample point.

Illustration :

Two coins (a one rupee coin and a two rupee coin) are tossed once. Find a sample space.

Sol. *Clearly the coins are distinguishable in the sense that we can speak of the first coin and the second coin Since either coin can turn up Head (H) or Tail (T), the possible outcomes may be*
Heads on both coins = (H, H) = HH
Head on first coin and Tail on the other = (H, T) = HT
Tail on first coin and Head on the other = (T, H) = TH
Tail on both coins = (T, T) = TT
Thus, the sample space is S = {HH, HT, TH, TT}

Illustration :

When a coin is tossed twice if head appears in the second throw then a dice is thrown. Write down the sample space of the experiment.

Sol. *When a coin is tossed two times then possible outcomes are {(TT), (HT), (TH), (HH)}*
If head appears in the second throw then dice is thrown.
∴ All possible outcomes of the experiment are
S = {(TT), (HT), (TH1), (TH2), (TH3), (TH4), (TH5), (TH6), (HH1),(HH2), (HH3), (HH4), (HH5), (HH6)}

EVENT :

Consider the experiment of tossing a coin two times. An associated sample space is

S = {HH, HT, TH, TT}

Now suppose that we are intersected in those outcomes which correspond to the occurrence of exactly one head. We find that HT and TH are the only elements of S corresponding to the occurrence of this happening (event). These two elements form the set E = {HT, TH}.

We know that the set E is a subset of the sample space S. Similarly, we find the following correspondence between events and subsets of S.

Description of events	Corresponding subset of 'S'
Number of tail is exactly 2	A = {TT}
Number of tails is atleast one	B = {HT, TH, TT}
Number of heads is atmost one	C = {HT, TH, TT}
Second toss is not head	D = {HT, TT}
Number of tails is atmost two	S = {HH, HT, TH, TT}
Number of tails is more than two	ϕ

Definition :

Any subset E of a sample space S is called an event.

Note : The maximum number of events which can be associated with an experiment is 2^n, where n is the number of elements in the sample space.

i.e., $^nC_0 + {}^nC_1 + {}^nC_2 + ... + {}^nC_n = 2^n$

Illustration :

In throwing a pair of dice write down two possible events.

E_1 = sum of the numbers appear on both the dice is 7.

E_2 = The sum of the numbers appear on both the dice is divisible by 3.

Sol. *E_1 = {(6, 1), (5, 2), (4, 3), (3, 4), (2, 5), (1, 6)}*

E_2 = {(2, 1), (1, 2), (5, 1), (4, 2), (3, 3), (2, 4) (1, 5), (6, 3), (5, 4), (4,5), (3, 6), (6, 6)}

Occurrence of an event :

Consider the experiment of throwing a die. Let E denotes the event " a number less than 4 appears". If actually '1' had appeared on the die then we say that event E has occurred. As a matter of fact if outcomes are 2 or 3, we say that even E has occurred.

Thus, the event E of a sample space S is said to have occurred if the outcome ω of the experiment is such that $\omega \in E$. If the outcome ω is such that $\omega \notin E$, we say that the event E has not occurred.

Impossible and Sure Events :

The empty set ϕ and the sample space S describe events. In fact ϕ is called an **impossible event** and S, i.e., the whole sample space is called the **sure event**.

To understand these let us consider the experiment of rolling a die. The associated sample space is

$$S = \{1, 2, 3, 4, 5, 6\}$$

Let E be the event" the number appears on the die is a multiple of 7".

Clearly no outcome satisfies the condition given in the event, i.e., no element of the sample space ensure the occurrence of the event E. Thus, we say that the empty set only correspond to the event E. In other words we can say that it is impossible to have a multiple of 7 on the upper face of the die. Thus, the event $E = \phi$ is an impossible event.

Now let us take up another event F " the number turns up is odd or even". Clearly $F = \{1, 2, 3, 4, 5, 6\} = S$, i.e., all outcomes of the experiment ensure the occurrence of the event F. Thus, the event $F = S$ is a sure event.

Simple Event :

If an event E has only one sample point of a sample space, it is called a simple (or elementary) event.

In a sample space containing n distinct elements, there are exactly n simple events.

$S = \{HH, HT, TH, TT\}$

There are four simple events corresponding of this sample space. These are

$E_1 = \{HH\}, E_2 = \{HT\}, E_3 = \{TH\}$ and $E_4 = \{TT\}$.

Compound Event :

If an event has more than one sample point, it is called a compound event.

For example, in the experiment of "tossing a coin thrice" the events

E : 'Exactly one head appeared'

F : 'Atleast one head appeared'

G : 'Atmost one head appeared' etc.

are all compound events. The subsets of associated with these events are

$E = \{HTT, THT, TTH\}$

$F = \{HTT, THT, TTH, HHT, HTH, THH, HHH\}$

$G = \{TTT, THT, HTT, TTH\}$

Each of the above subsets contain more than one sample point, hence they are all compound events.

ALGEBRA OF EVENTS :

In the Chapter on Sets, we have studied about different ways of combining two or more sets, viz, union, intersection, difference, complement of a set etc. Like-wise we can combine two or more events by using the analogous set notations.

Let A, B, C be events associated with an experiment whose sample space is S.

Complementary Event :

For every event A, there corresponds another event A' or $\overline{A}$ called the complementary event to A. It is also called the event 'not A'.

For example, take the experiment 'of tossing three coins'. An associated sample space is

$$S = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$$

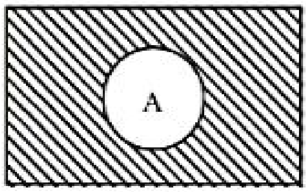
Let A = {HTH, HHT, THH} be the event 'only one tail appears'.

Clearly for the outcome HTT, the event A has not occurred. But we may say that the event 'not A' has occurred. Thus, with every outcome which is not in A, we say that 'not A' occurs.

Thus the complementary event 'not A' to the event A is

$$A' = \{HHH, HTT, THT, TTH, TTT\}$$

or $$A' = \{\omega : \omega \in S \text{ and } \omega \notin A\} = S - A$$

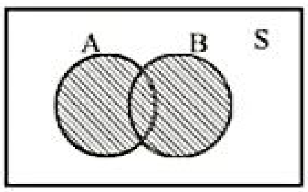


A′ or $\overline{A}$

The Event 'A or B' :

Recall that union of two sets A and B denoted by $A \cup B$ contains all those elements which are either in A or in B or in both.

When the sets A and B are two events associated with a sample space, then '$A \cup B$' is the event 'either A or B or both'. This event '$A \cup B$' is also called 'A or B'.



Therefore Event 'A or B' = $A \cup B = \{\omega : \omega \in A \text{ or } \omega \in B\}$

$A \cup B$

The Event 'A and B' :

We know that intersection of two sets $A \cap B$ is the set of those elements which are common to both A and B. i.e., which belong to both 'A and B'.

If A and B are two events, then the set $A \cap B$ denotes the event 'A and B'.

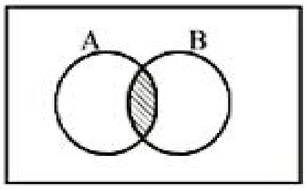
Thus, $A \cap B = \{\omega : \omega \in A \text{ and } \omega \in B\}$

For example, in the experiment of 'throwing a die twice'
Let A be the event 'score on the first thrown is six' and
B is the even 'sum of two scores is atleast 11' then

$A = \{(6, 1), (6, 2), (6, 3), (6, 4), (6, 5)\ (6, 6)\}$

and $B = \{(5, 6), (6, 5), (6, 6)\}$

so $A \cap B = \{(6, 5), (6, 6)\}$



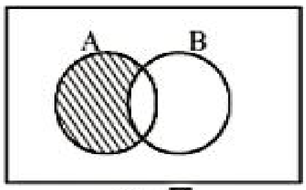
$A \cap B$

Note that the set $A \cap B = \{(6, 5), (6, 6)\}$ may represent the vent ' the score on the first throw is six and the sum of the scores is atleast 11.'

The Event 'A but not B' :

We know that A – B is the set of all those elements which are in A but not in B. Therefore, the set A – B may denote the event 'A but not B'. We know that

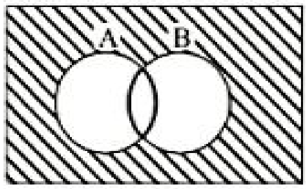
$A - B = A \cap B'$



$A \cup \overline{B}$

The Event 'neither A nor B' :

The set of the elements which are neither in set A nor in set B. i.e. $S - (A \cup B)$ and which is denoted on $\overline{A} \cap \overline{B}$.



$\overline{A} \cap \overline{B}$

Illustration :

Consider the experiment of rolling a die. Let A be the event 'getting a prime number', B be the event 'getting an odd number'. Write the sets representing the events

(i) A or B (ii) A and B (iii) A but not B (iv) 'not A'.

Sol. *Here* $S = \{1, 2, 3, 4, 5, 6\}, A = \{2, 3, 5\}$

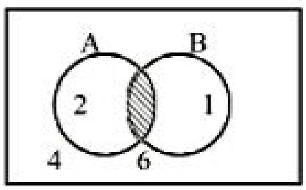
Obviously

(i) 'A or B' $= A \cup B = \{1, 2, 3, 5\}$

(ii) 'A and B' $= A \cap B = \{3, 5\}$

(iii) 'A but not B' $= A - B = \{2\}$

(iv) 'not A' $= A' = \{1, 4, 6\}$



Note : $\left.\begin{array}{l}(i)\ \overline{A \cup B} = \overline{A} \cap \overline{B} \\ (ii)\ \overline{A \cap B} = \overline{A} \cup \overline{B}\end{array}\right\}$ *De Morgan's Law*

THREE MOST IMPORTANT EVENTS :

(1) Equally Likely Events :

Events are said to be equally likely when no particular event preference to occur in relation to the other event.

Example :

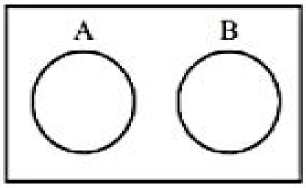
(i) The outcomes as result throwing a die are equally likely, as no particular face is more likely to occur as compared to other faces. That is why we normally write as fair die or unbiased die.

(ii) The outcomes as result of drawing a card from a well shuffled pack of 52 playing cards are equally likely to occur. Each card is as likely to be withdrawn as any other card.

(iii) How ever getting of a total of 7 is not as equally likely as getting of a total of 12 when a pair of dice are rolled once. It is also to be noted that it is 6 times more likely to get a total of 7 than to get a total of 12 in a single throw with the pair of dice.

(2) Mutually Exclusive / Disjoint / Incompatible Events :

Two events A and B are said to be mutually exclusive events if their simultaneous occurrence is impossible, i.e. both the events can not occur together.



$A \cap B = \phi$

Example :

(i) In throwing a fair die, to events A and B are such that

A : getting an odd number

B : getting an even number

then A & B are mutually exclusive events.

(ii) In drawing a card from a well shuffled pack of 52 playing card two events A and B are such that

A : getting an ace

B : getting a red card

then A and B are not mutually exclusive events.

(3) Exhaustive Events :

If $E_1, E_2, ..., E_n$ are n events associated with an experiment whose sample space is S and if

$$E_1 \cup E_2 \cup E_3 \cup ......\cup E_n = \bigcup_{i=1}^{n} E_i = S$$

then $E_1, E_2, ..., E_n$ are called exhaustive events. In other words, events $E_1, E_2, ..., E_n$ are said to be exhaustive if atleast one of them necessarily occurs whenever the experiment is performed.

Further, if $E_i \cap E_j = \phi$ for $i \neq j$ i.e., events E_i and E_j are pairwise disjoint and $\bigcup_{i=1}^{n} E_i = S$, then events $E_1, E_2, \ldots, E_n$ are called mutually exclusive and exhaustive events.

Example :

Consider the experiment of throwing a die. We have

$S = \{1, 2, 3, 4, 5, 6\}$. Let us define the following events

A : 'a number less than 4 appears'.

B : 'a number greater than 2 but less than 5 appears'

and C : 'a number greater than 4 appears'.

Then $A = \{1, 2, 3\}$, $B = \{3, 4\}$ and $C = \{5, 6\}$. We observe that

$A \cup B \cup C = \{1, 2, 3\} \cup \{3, 4\} \cup \{5, 6\} = S.$

Such events A, B and C are called exhaustive events.

Practice Problem

Q.1 Suppose 3 bulbs are selected at random from a lot. Each bulb is tested and classified as defective (D) or non-defective (N). Write the sample space of this experiment.

Q.2 A coin is tossed. If the out come is a head, a die is thrown. If the die shows up and even number, the die is thrown again. What is the sample space for the experiment?

Q.3 A die is thrown. Describe the following events :

(i)	A : a number less than 7	(ii)	B : a number greater than 7
(iii)	C : a multiple of 3	(iv)	D : a number less than 4
(v)	E : an even number greater than 4	(vi)	F : a number not less than 3

Also find $A \cup B, A \cap B, B \cup C, E \cap F, D \cap E, A - C, D - E, E \cap F', F'$

Q.4 Two dice are thrown. The events A, B and C are as follows :

A : getting an even number on the first die.

B : getting an odd number on the first die.

C : getting the sum of the numbers on the dice ≤ 5.

Describe the events

(i)	A'	(ii)	not B	(iii)	A or B
(iv)	A and B	(v)	A but not C	(vi)	B or C
(vii)	B and C	(viii)	$A \cap B' \cap C'$		

Q.5 Refer to question 4 above, state true or false; (give reason for your answer)

(i) A and B are mutually exclusive

(ii) A and B are mutually exclusive and exhaustive

(iii) $A = B'$

(iv) A and C are mutually exclusive

(v) A and B' are mutually exclusive

(vi) A', B', C are mutually exclusive and exhaustive.

Answer key

Q.1 DDD, DDN, DND, NDD, DNN, NDN, NND, NNN

Q.2 T, H1, H3, H5, H21, H22, H23, H24, H25, H26, H41, H42, H43, H44, H45, H46, H61, H62, H63, H64, H65, H66

Q.3 (i) $\{1, 2, 3, 4, 5, 6\}$ (ii) ϕ (iii) $\{3, 6\}$ (iv) $\{1, 2, 3\}$ (v) $\{6\}$

(vi) $\{3, 4, 5, 6\}$, $A \cup B = \{1, 2, 3, 4, 5, 6\}$, $B \cup C = \{3, 6\}$, $E \cap F = \{6\}$, $D \cap E = \phi$,

$A - C = \{1, 2, 4, 5\}$, $D - E = \{1, 2, 3\}$, $E \cap F' = \phi$, $F' = \{1, 2\}$

Q.4 $A = \{(2, 1), (2, 2), (2, 3), (2, 4), (2, 5), (2, 6), (4, 1), (4, 2), (4, 3), (4, 4), (4, 5), (4, 6), (6, 1), (6, 2), (6, 3), (6, 4), (6, 5), (6, 6)\}$

$B = \{(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6), (3, 1), (3, 2), (3, 3), (3, 4), (3, 5), (3, 6), (5, 1), (5, 2), (5, 3), (5, 4), (5, 5), (5, 6)\}$

$C = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (4, 1)\}$

(i) $A' = \{(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6), (3, 1), (3, 2), (3, 3), (3, 4), (3, 5), (3, 6), (5, 1), (5, 2), (5, 3), (5, 4), (5, 5), (5, 6)\} = B$

(ii) $B' = \{(2, 1), (2, 2), (2, 3), (2, 4), (2, 5), (2, 6), (4, 1), (4, 2), (4, 3), (4, 4), (4, 5), (4, 6), (6, 1), (6, 2), (6, 3), (6, 4), (6, 4), (6, 6)\} = A$

(iii) $A \cup B = \{(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6), (3, 1), (3, 2), (3, 3), (3, 4), (3, 5), (3, 6), (5, 1), (5, 2), (5, 3), (5, 4), (5, 5), (5, 6), (2, 1), (2, 2), (2, 3), (2, 5), (2, 6), (4, 1), (4, 2), (4, 3), (4, 4), (4, 5), (4, 6), (6, 1), (6, 2), (6, 3), (6, 4), (6, 5), (6, 6)\} = S$

(iv) $A \cap B = \phi$

(v) $A - C = \{(2, 4), (2, 5), (2, 6), (4, 2), (4, 3), (4, 4), (4, 5), (4, 6), (6, 1), (6, 2), (6, 3), (6, 4), (6, 5), (6, 6)\}$

(vi) $B \cup C = \{(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3), (3, 4), (3, 5), (3, 6), (4, 1), (5, 1), (5, 2), (5, 3), (5, 4), (5, 5), (5, 6)\}$

(vii) $B \cap C = \{(1, 1), (1, 2), (1, 3), (1, 4), (3, 1), (3, 2)\}$

(viii) $A \cap B' \cap C' = \{(2, 4), (2, 5), (2, 6), (4, 2), (4, 3), (4, 4), (4, 5), (4, 6), (6, 1), (6, 2), (6, 3), (6, 4), (6, 5), (6, 6)\}$

Q.5 (i) True ; (ii) True ; (iii) True ; (iv) False ; (v) False ; (vi) False

CLASSICAL (A PRIORI) DEFINITION OF PROBABILITY :

If an experiment results in a total of $(m+n)$ outcomes which are equally likely and mutually exclusive with one another and if 'm' outcomes are favourable to an event 'A' while 'n' are unfavourable, then the probability of occurrence of the event 'A', denoted by P(A), is defined by

$$\frac{m}{m+n} = \frac{\text{number of favourable outcomes}}{\text{total number of outcomes}}$$

i.e. $P(A) = \dfrac{m}{m+n}$.

Note that $P(\bar{A})$ or $P(A')$ or $P(A^C)$, i.e. probability of non-occurrence of $A = \dfrac{n}{m+n} = 1 - P(A)$

In the above we shall denote the number of out comes favourable to the event A by n(A) and the total number of out comes in the sample space S by n(S).

$\therefore \quad P(A) = \dfrac{n(A)}{n(S)}$.

If $\quad P(A) = 0$ Event is impossible

$\quad P(A) = 1$ Event is sure

$\quad P(A) \not> 1$ and $P(A) \not< 0$

Note :

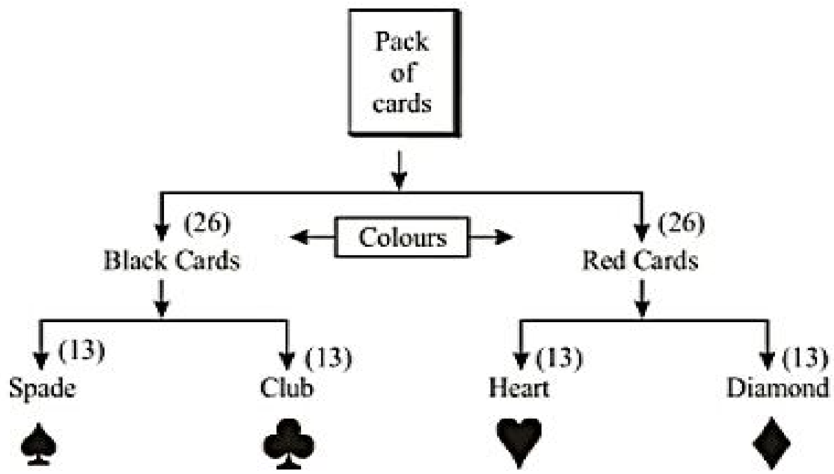
(i) More is the probability of an event, more are chances of its happening.

(ii) $P(\phi) = 0$ & $P(S) = 1$ i.e. nothing outside sample space can occur.

Designation of Cards :

Colours : There are two colours. Red & Block

Suits : There are four (4) suits (types).
Each suit contains 13 cards



Recognition of Cards :

	K King	Q Queen	J Jack	A Ace
♥	1	1	1	1
♦	1	1	1	1
♠	1	1	1	1
♣	1	1	1	1
	4	4	4	4

(i) Face Cards :

Face cards contain 12 cards all of K, Q and J having designed a figure of a person.
i.e., Face cards = 4 + 4 + 4 = 12,

(ii) Honours Cards :

It contains all face cards and also a card marked A.
i.e. Honours cards = (4 + 4 + 4) + 4 = 16 cards.

(iii) Knave Cards :

(10, J, Q) = 4 + 4 + 4 = 12 cards

Illustration :

An old man while dialing a seven digit telephone number, after having dialed the first five digits, suddenly forgets the last two. But he remembered that the last two digits were different. On this assumption he randomly dials the last two digits. What is the probability that the correct telephone number is dialed.

Sol. *Note that total number of ways in which the last two digits (different) can be dialed is $10 \times 9 = 90$. Out of these 90 EL/ME/ and exhaustive outcomes only one of them favours happening of the event "correct telephone is dialed". Hence $P(E) = \frac{1}{90}$.*

What the probability would have been if he did not even remember the last two digits were different:
Here $n(S) = 10 \times 10 = 100$

Hence $P(E) = \frac{1}{100}$.

Illustration :

4 Apples and 3 Oranges are randomly placed in a line. Find the chances that the extreme fruits are both oranges.

Sol. $n(S) = \frac{7!}{4!3!}$; $n(A) = \frac{5!}{4!} \Rightarrow P = \frac{5!}{4!} \cdot \frac{4!3!}{7!} = \frac{1}{7}$

Note whether fruits the same species are different or alike that probability of the required event remains the same.

Illustration :

Two natural are randomly selected from the set of first 20 natural numbers. Find the probability that (A) their sum is odd (B) sum is even (C) selected pair is twin prime.

Sol. $S = \{1, 2, 3, ..., 19, 20\}$; $n(S) = {}^{20}C_2$

$n(A) = {}^{10}C_1 . {}^{10}C_1 = 100 \Rightarrow P(A) = \frac{100}{190} = \frac{10}{19}$ *(sum odd $\Rightarrow$ one odd and one even)*

$n(B) = {}^{10}C_2 + {}^{10}C_2 = 2.{}^{10}C_2 = 90 \Rightarrow P(B) = \frac{90}{190} = \frac{9}{19}$

(sum even $\Rightarrow$ both odd or both even)

$n(C) = \{(3, 5), (5, 7), (11, 13), (17, 19)\} \Rightarrow P(C) = \frac{4}{190} = \frac{2}{95}$

Illustration :

What is the chance that the fourth power of an integer chosen randomly ends in the digit six.

Sol. *Any integer randomly selected can end in 0, 1, 2, 3, 4, 5, 6, 7, 8 and 9. These are EL/ME and Exhaustive cases. Out of these 10 case only four cases, when the integer ends in 2, 4, 6 and 8 favours happening of the required event. Hence*

$$P \text{ (required event)} = \frac{4}{10} = 40\%$$

It will be incorrect to think this problem as :

4th power of an integer can end in 0, 1, 5 and 6. Hence the probability = $\frac{1}{4}$ which is wrong. Note that four events are ME and exhaustive but not equally likely hence the definition of probability can not be based on them. In factor 4th power of an integer.

ending in '0' is favoured by only 1 case {0}
ending in '1' is favoured by only 4 cases {1, 3, 7, 9}
ending in '5' is favoured by only 1 case {5}
ending in '6' is favoured by only 4 cases {2, 4, 6, 8}

$$\Rightarrow \quad P(0) = \frac{1}{10}; P(1) = \frac{4}{10}; P(5) = \frac{1}{10}; P(6) = \frac{4}{10}$$

Illustration :

Pair of dice has been rolled/thrown/cast once. Find the probability that atleast one of the dice shows up the face one.

Sol. *There are four ME and Exhaustive cases*

E_1 : 1^{st} *dice only shows up the face one.*

E_2 : 2^{nd} *dice only shows up the face one.*

E_3 : *both dice shows up the face one.*

E_4 : *None of the dice shows up the face one.*

Out of these, first 3 cases favours happening of the required event. Hence

$$P\text{ (required event) } = 1 - P(E_4) = 1 - \frac{5 \times 5}{36} = \frac{11}{36}$$

Note that E_1, E_2, E_3, E_4 *are not equally likely.*

Illustration :

A leap year is selected at random. Find the probability that it has

(A) 53 Sundays *(B) 53 Sundays and Mondays* *(C) 53 Sundays or 53 Mondays*

Sol. *Leap year means which is divisible by 4 if it not a century year. If it is a century year it must be divisible by 400 as well. A leap year has 366 days out of this 364 days are consumed for 52 weeks i.e. 52 times*

S, M, T, W, Th, F and Sat. For remaining 2 days of the leap year can begin with SM, MT, TW, W Th., Th. F, F Sat and Sat Sun.

$$\Rightarrow P(A) = \frac{2}{7}; P(B) = \frac{1}{7}; P(C) = \frac{3}{7}$$

Illustration :

A card is drawn randomly from a well shuffled pack of 52 cards. The probability that the drawn card is "neither a heart nor a face card".

Sol. *Note that there are 22 cards which either H or Face cards (All K, Q and J) hence*

$$P(\text{either a H or Face card}) = \frac{22}{52} = \frac{11}{26}$$

$$\therefore \quad P(\text{neither a H nor FC}) = 1 - \frac{11}{26} = \frac{15}{26}$$

It is to be noted that

$P(\text{not } A \text{ or } \overline{A} \text{ or } A^c) = 1 - P(A)$

Note that A and A^c *makes an event a sure event and probability of a sure event is one.*

ODDS IN FAVOUR AND ODDS AGAINST OF AN EVENT :

If an experiment has (m + n) as a total number of outcomes which are equally likely, mutually exclusive and exhaustive, and if 'm' outcomes are in favour of an event 'A' and n outcomes are not in favour of that event A means n outcomes are in against of event A then we can say –

$$\text{Odds in favour of event A} = \frac{m}{n} = \frac{\text{No. of outcomes which are in favour of event A}}{\text{No. of outcomes which are not in favour of event A}}$$

$$\text{Odds in against of event A} = \frac{n}{m} = \frac{\text{No. of outcomes which are not in favour of event A}}{\text{No. of outcomes which are in favour of event A}}$$

Note : If $P(A) = \frac{a}{b}$ then

(i) odds in favour of event A = a : b – a.

(ii) odds against of event = b – a: a.

Illustration :

5 different marbles are placed in 5 different boxes randomly. Find the odds in favour that exactly two boxes remain empty. Given each box an hold any number of marbles.

Sol. *$n(S) = 5^5$; For computing favourable outcomes.*

2 boxes which are remain empty, can be selected in ${}^5C_2$ ways and 5 marbles can be placed in the remaining 3 boxes in groups of 221 or 311 in $3! \times \left[\frac{5!}{2!\,2!\,2!} + \frac{5!}{3!\,2!}\right] = 150$ *ways*

$$\therefore \quad P(E) = {}^5C_2 \times \frac{150}{5^5} = \frac{12}{25}$$

Hence, odds in favour of event E = 12 : 13 ***Ans.***

Practice Problem

Q.1 If three cards are drawn from a well shuffled pack of 52 cards randomly. What is the probability that is has

(i) all three Kings?

(ii) one King and two Queens?

(iii) all three of same colour?

(iv) all three of different suits?

(v) all three of same denomination?

(vi) at least one King?

Q.2 The first twelve letters of the alphabet are written down at random. What is the probability that there are exactly 4 letters between A and B?

Q.3 If n biscuits are distributed at random among N beggars. Find the probability that a particular beggar receives r (< n) biscuits.

Q.4 If k is chosen at random from the interval [0, 5]. Find the probability of the equation $x^2 + kx + \frac{1}{4}(k+2) = 0$ to have real roots.

Q.5 The chance of an event happening is the square of the chance of a second event but the odds against the first are the cube of the odds against the second. Find the chance of each.

Answer key

Q.1 (i) $\frac{^4C_3}{^{52}C_3}$; (ii) $\frac{^4C_1 \times {}^4C_2}{^{52}C_3}$; (iii) $\frac{^2C_1 \times {}^{26}C_3}{^{52}C_3}$; (iv) $\frac{^4C_3 \times {}^{13}C_1 \times {}^{13}C_1 \times {}^{13}C_1}{^{52}C_3}$;

(v) $\frac{^{13}C_1 \times {}^4C_3}{^{52}C_3}$; (vi) $\frac{^4C_1 \times {}^{48}C_2 + {}^4C_2 \times {}^{48}C_1 + {}^4C_3 \times {}^{48}C_0}{^{52}C_3}$

Q.2 $\frac{7}{66}$ Q.3 $\frac{^nC_r(1)^r.(N-1)^{n-r}}{N^n}$ Q.4 $\frac{3}{5}$ Q.5 1/9

DEPENDENT AND INDEPENDENT EVENTS :

Independent events – Events A and B are said to be independent if occurrences or non occurrence of one does not affect the probability of occurrence or non-occurrence of the other.

(i) Two people holding a normal dice and the other a coin, throw them once, then getting a 6 on normal dice and getting a head on the coin are the examples of events which are independent.

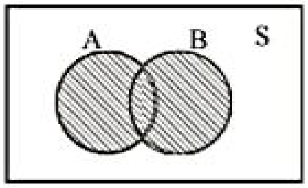
(ii) From an urn containing 2R, 3G and 4W balls, a ball is drawn its colour is noted, the ball is replaced in the urn and another ball is drawn. Getting a red and a red ball on both the occasion are the examples of events which are independent.

(iii) Similar example can be given in playing cards 'getting an ace' and 'an ace' in two successive draws from a well shuffled pack of 52 cards when the first drawn card is replaced in the pack before the second is drawn. If it is not replaced the events become dependent or contingent.

Note : Dependent/Independent events come from two different experiments while mutually exclusive events come form the same experiment.

ADDITION THEOREM ON PROBABILITY :

If A and B are two events associated with an experiment then $P(A \cup B)$ is called the sum of the probabilities of all the sample points in $A \cup B$ or probability of occurrence of atleast one of the events from A and B and the expression for $P(A \cup B)$ is called the addition theorem on probability



From the Venn diagram it is clear that

P(Occurence atleast one of the events from A and B)

P(A or B or both)

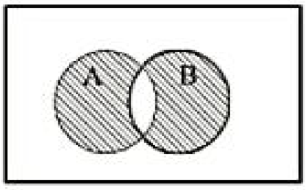
or

P(A + B)

$$\Rightarrow P(A \cup B) = \begin{cases} = P(A) + P(B) - P(A \cap B) \\ = P(A) + P(\overline{A} \cap B) \\ = P(B) + (A \cap \overline{B}) \\ = P(A \cap \overline{B}) + P(A\,B) + P(\overline{A} \cap B) \\ = 1 - P(\overline{A} \cap \overline{B}) \\ = 1 - P(\overline{A \cap B}) \end{cases}$$

P(occurrence of exactly one of the events) or P(A or B but not both)

$$\begin{cases} P(A \cap \overline{B}) + P(\overline{A} \cap B) \\ P(A) + P(B) - P(A \cap B) \end{cases}$$



Note :

(i) If A and B are mutually exclusive events then –

$P(A \cup B) = P(A) + P(B) \qquad \{\because P(A \cap B) = 0\}$

(ii) If A and B are exhaustive events then $P(A \cup B) = 1$

(iii) $P(A \cup B) = 1 - P(\overline{A \cup B})$

Illustration :

Two students Anil and Ashima appeared in an examination. The probability that Anil will qualify the examination is 0.05 and that Ashima will qualify the examination is 0.10. The probability that both will qualify the examination is 0.02. Find the probability that

(a) Both Anil and Ashima will not qualify the examination

(b) Atleast one of them will not qualify the examination and

(c) Only one of them will qualify the examination.

Sol. *Let E and F denote the events that Anil and Ashima will qualify the examination, respectively. Given that*

$$P(E) = 0.05,\ P(F) = 0.10 \quad \text{and} \quad P(E \cap F) = 0.02$$

Then

(a) The event 'both Anil and Ashima will not qualify the examination' may be expressed as $E' \cap F'$. Since, E' is not E, i.e., Anil will not qualify the examination and F' is 'not F, i.e., Ashima will not qualify the examination.

Also $E' \cap F' = (E \cup F)'$ *(By Demorgan's Law)*

Now $P(E \cup F) = P(E) + P(F) - P(E \cap F)$

or $P(E \cup F) = 0.05 + 0.10 - 0.02 = 0.13$

Therefore $P(E' \cap F') = P(E \cup F)' = 1 - P(E \cup F) = 1 - 0.13 = 0.87$

(b) *P(atleast one of them will not qualify)*

$= 1 -$ *P(both of them will qualify)*

$= 1 - 0.02 = 0.98$

(c) *The event only one of them will qualify the examination is same as the event either (Anil will qualify, and Ashima will not qualify) or (Anil will not qualify and Ashima will qualify) i.e.,* $E \cap F'$ *or* $E' \cap F$, *where* $E \cap F'$ *and* $E' \cap F$ *are mutually exclusive.*

Therefore, P(only one of them will qualify) $= P(E \cap F'$ *or* $E' \cap F)$

$= P(E \cap F') + P(E' \cap F) = P(E) - P(E \cap F) + P(F) - P(E \cap F)$

$= 0.05 - 0.02 + 0.10 - 0.02 = 0.11$

Illustration :

A and B are any two events such that $P(A) = 0.3$, $P(B) = 0.1$ *and* $P(A \cap B) = 0.16$. *Find the probability that exactly one of the events happens.*

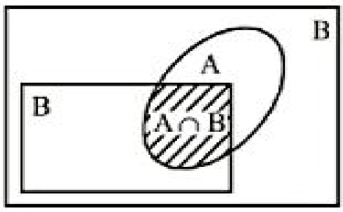
Sol. *Exactly one of the events happens* $= P(A \cap B')$ *or* $P(A' \cap B)$

$P(A \cap B') + P(A' \cap B) = P(A) + P(B) - 2P(A \cap B)$

$= 0.3 + 0.1 - 2 \times 0.16 = 0.08$

CONDITIONAL PROBABILITY :

Let A and B be any two events associated with a random experiment. The probability of occurrence of event A when the event B has already occurred is called the conditional probability of A when B is given and is denoted as P(A/B). The conditional probability P(A/B) is meaningful only when $P(B) \neq 0$, i.e., when B is not an impossible event.



By definition,

$$P\left(\frac{A}{B}\right) = \text{Probability of occurrence of event A when the event B as already occurred}$$

$$= \frac{\text{Number of cases favourable to B which are also favourable to A}}{\text{Number of cases favourable to B}}$$

$$\therefore \quad P\left(\frac{A}{B}\right) = \frac{\text{Number of cases favourable to } A \cap B}{\text{Number of cases favourable to B}}$$

$$\text{Also,}\quad P\left(\frac{A}{B}\right) = \frac{\dfrac{\text{Number of cases favourable to } A\cap B}{\text{Number of cases in the sample space}}}{\dfrac{\text{Number of cases favourable to B}}{\text{Number of cases in the sample space}}}$$

$$\therefore \quad P\left(\frac{A}{B}\right) = \frac{P(A\cap B)}{P(B)},\ \text{provided } P(B)\neq 0.$$

Similarly, we have

$$P\left(\frac{B}{A}\right) = \frac{P(A\cap B)}{P(A)},\ \text{provided } P(A)\neq 0.$$

Illustration :

Roll a fair die twice. Let A be the event that the sum of the two rolls equals six, and let B be the event that the same number comes up twice. What is P(A/B)?

(A) 1/6 *(B) 5/36* *(C) 1/5* *(D) none*

Sol. $A = \{(1, 5), (4, 4), (3, 3), (4, 2), (5, 1)\}$

$B = \{(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6)\}$

$$\therefore P\left(\frac{A}{B}\right) = \frac{P(A\cap B)}{P(B)} = \frac{n(A\cap B)}{n(B)} = \frac{1}{6}$$

Illustration :

In a class, 30% of the students failed in Physics, 25% failed in Mathematics and 15% failed in both Physics and Mathematics. If a student is selected at random failed in Mathematics, find the probability that he failed in Physics also.

Sol. *Let A be the event "failed in Physics" and B be the event "failed in Mathematics". We want to find* $P\left(\frac{A}{B}\right)$*. It is given that*

$$P(A) = \frac{30}{100} \quad \text{and} \quad P(B) = \frac{25}{100}$$

Also $P(A\cap B) = \dfrac{15}{100}$

Therefore $$P\left(\frac{A}{B}\right) = \frac{P(A\cap B)}{P(B)} = \frac{15/100}{25/100} = \frac{15}{25} = \frac{3}{5}$$

Illustration :

Let A and B be two events such that P(A) = 0.3, P(B) = 0.6 and $P\left(\frac{B}{A}\right) = 0.5$. *Then* $P\left(\frac{\bar{A}}{\bar{B}}\right)$ *equals*

(A) $\frac{3}{4}$ (B) $\frac{5}{8}$ (C) $\frac{9}{40}$ (D) $\frac{1}{4}$

Sol. $P(A \cap B) = P(A)\, P\left(\frac{B}{A}\right) = (0.3)\,(0.5) = 0.15$

Now $P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.3 + 0.6 - 0.15 = 0.75$

Also $P\left(\frac{\bar{A}}{\bar{B}}\right) = \frac{P(\bar{A} \cap \bar{B})}{P(\bar{B})} = \frac{1 - P(A \cup B)}{1 - P(B)} = \frac{1 - 0.75}{1 - 0.6} = \frac{0.25}{0.4} = \frac{250}{400} = \frac{5}{8}$

MULTIPLICATION THEOREM ON PROBABILITY :

Let A and B be two events associated with a sample space S. Clearly, the set $A \cap B$ denotes the event that both A and B have occurred. In other words, $A \cap B$ denotes the simultaneous occurrence of the events E and F. The event $A \cap B$ is also written as AB.

We know that the conditional probability of event A given that B has occurred is denoted by $P(A\,|\,B)$ and is given by

$$P(A\,|\,B) = \frac{P(A \cap B)}{P(B)},\ P(B) \neq 0$$

From this result, we can write

$$P(A \cap B) = P(B) \cdot P(A\,|\,B) \qquad \ldots\text{(i)}$$

Also, we know that

$$P(B\,|\,A) = \frac{P(A \cap B)}{P(A)},\ P(A) \neq 0$$

or $$P(B\,|\,A) = \frac{P(A \cap B)}{P(A)} \quad (\text{since } A \cap B = B \cap A)$$

Thus, $$P(A \cap B) = P(A) \cdot P(B\,|\,A) \qquad \ldots\text{(ii)}$$

Combining (i) and (ii), we find that

$$P(A \cap B) = P(A)\,P(B\,|\,A) = P(B)\,P(A\,|\,B) \text{ provided } P(A) \neq 0 \text{ and } P(B) \neq 0$$

The above result is known as the multiplication rule of probability.

Note : If A & B are independent events then $P\left(\frac{A}{B}\right) = P(A)$ and $P\left(\frac{B}{A}\right) = P(B)$ and in this case multiplication theorem $P(A \cap B) = P(A) \cdot P(B)$.

Theorem-I :

Let A and B be events associated with a random experiment. If A and B are independent, then show that the events (i) $\overline{A}$, B (ii) A, $\overline{B}$ (iii) $\overline{A}, \overline{B}$ are also independent.

Proof : The events A and B are independent.

$\therefore \quad P(A \cap B) = P(A)\,P(B) \qquad \text{......(i)}$

(i) $\quad (A \cap B) \cap (\overline{A} \cap B) = (A \cap \overline{A}) \cap (B \cap B) = \phi \cap B = \phi$

and $\quad (A \cap B) \cup (\overline{A} \cap B) = (A \cup \overline{A}) \cap B = S \cap B = B$

$\therefore$ The events $A \cap B$ and $\overline{A} \cap B$ are mutually exclusive and their union is B.

$\therefore$ By addition theorem, we have $P(B) = P(A \cap B) + P(\overline{A} \cap B) \qquad \text{......(i)}$

$$\Rightarrow P(\overline{A} \cap B) = P(B) - P(A \cap B) = P(B) - P(A)\,P(B)$$

$$= \left(1 - P(A)\right) P(B) = P(\overline{A})\,P(B) \qquad \text{(Using (i))}$$

$\therefore \quad P(\overline{A} \cap B) = P(\overline{A})\,P(B)$ i.e., $\overline{A}$ and B are independent.

(ii) $\quad (A \cap B) \cap (A \cap \overline{B}) = (A \cap A) \cap (B \cap \overline{B}) = A \cap \phi = \phi$

and $\quad (A \cap B) \cup (A \cap \overline{B}) = A \cap (B \cup \overline{B}) = A \cap S = A$

$\therefore$ The events $A \cap B$ and $A \cap \overline{B}$ are mutually exclusive and their union is A.

$\therefore$ By addition theorem, we have $P(A) = P(A \cap B) + P(A \cap \overline{B}) \qquad \text{......(i)}$

$$\Rightarrow P(A \cap \overline{B}) = P(A) - P(A \cap B) = P(A) - P(A)\,P(B)$$

$$= P(A)\left(1 - P(B)\right) = P(A)\,P(\overline{B}) \qquad \text{(Using (i))}$$

$\therefore \quad P(A \cap \overline{B}) = P(A)\,P(\overline{B})$ i.e., A and $\overline{B}$ are independent.

(iii) $\quad (\overline{A} \cap B) \cap (\overline{A} \cap \overline{B}) = (\overline{A} \cap \overline{A}) \cap (B \cap \overline{B}) = \overline{A} \cap \phi = \phi$

and $\quad (\overline{A} \cap B) \cup (\overline{A} \cap \overline{B}) = \overline{A} \cap (B \cup \overline{B}) = \overline{A} \cap S = \overline{A}$

$\therefore$ The events $\overline{A} \cap B$ and $\overline{A} \cap (B \cap \overline{B})$ are mutually exclusive and their union is $\overline{A}$.

$\therefore$ By addition theorem, we have $P(\overline{A}) = P(\overline{A} \cap B) + P(\overline{A} \cap \overline{B}) \qquad \text{......(i)}$

$$\Rightarrow P(\overline{A} \cap \overline{B}) = P(\overline{A}) - P(\overline{A} \cap B) = P(\overline{A}) - P(\overline{A})\,P(B)$$

$$= P(\overline{A})\left(1 - P(B)\right) = P(\overline{A})\,P(\overline{B}) \qquad \text{(Using (i))}$$

$\therefore \quad P(\overline{A} \cap \overline{B}) = P(\overline{A})\,P(\overline{B})$ i.e., $\overline{A}$ and $\overline{B}$ are independent.

Illustration :

A die is thrown. If E is the event 'the number appearing is a multiple of 3' and F be the event 'the number appearing is even' then find whether E and F are independent?

Sol. We know that the sample space is $S = \{1, 2, 3, 4, 5, 6\}$

Now $E = \{3, 6\}$, $F = \{2, 4, 6\}$ and $E \cap F = \{6\}$

Then $P(E) = \frac{2}{6} = \frac{1}{3}$, $P(F) = \frac{3}{6} = \frac{1}{2}$ and $P(E \cap F) = \frac{1}{6}$

Clearly $P(E \cap F) = P(E) \cdot P(F)$

Hence E and F are independent events.

Illustration :

Three coins are tossed simultaneously. Consider the event E 'three heads or three tails', F 'at least two heads' and G 'at most two heads'. Of the pairs (E, F), (E, G) and (F, G), which are independent? which are dependent?

Sol. The sample space of the experiment is given by

$S = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$

Clearly $E = \{HHH, TTT\}$, $F = \{HHH, HHT, HTH, THH\}$

and $G = \{HHT, HTH, THH, HTT, THT, TTH, TTT\}$

Also $E \cap F = \{HHH\}$, $E \cap G = \{TTT\}$, $F \cap G = \{HHT, HTH, THH\}$

Therefore $P(E) = \frac{2}{8} = \frac{1}{4}$, $P(F) = \frac{4}{8} = \frac{1}{2}$, $P(G) = \frac{7}{8}$

and $P(E \cap F) = \frac{1}{8}$, $P(E \cap G) = \frac{1}{8}$, $P(F \cap G) = \frac{3}{8}$

Also $P(E) \cdot P(F) = \frac{1}{4} \times \frac{1}{2} = \frac{1}{8}$, $P(E) \cdot P(G) = \frac{1}{4} \times \frac{7}{8} = \frac{7}{32}$

and $P(F) \cdot P(G) = \frac{1}{2} \times \frac{7}{8} = \frac{7}{16}$

Thus $P(E \cap F) = P(E) \cdot P(F)$

$P(E \cap G) \neq P(E) \cdot P(G)$ and $P(F \cap G) \neq P(F) \cdot P(G)$

Hence, the events (E and F) are independent, and the events (E and G) and (F and G) are dependent.

Illustration :

A pair of fair dice is thrown. Find the probability that either of the dice shows 2 if the sum is 6.

Sol. *The sample space of the experiment "throwing a pair of fair dice" consists of 36(= 6 × 6) ordered pair (a, b), where a and b can be any integers from 1 to 6. Let A be the event "2 appears on either of the dice" and B be the event "sum is 6". We want to find* $P\left(\frac{A}{B}\right)$. *Note that*

$A = [(2, b) \mid 1 \le b \le 6] \cup [(a, 2) \mid 1 \le a \le 6]$ and $B = [(1, 5), (2, 4), (3, 3), (4, 2), (5, 1)]$

Also, $A \cap B = [(2, 4), (4, 2)]$

Therefore $P(B) = \frac{5}{36}$ *and* $P(A \cap B) = \frac{2}{36}$

So $$P\left(\frac{A}{B}\right) = \frac{P(A \cap B)}{P(B)} = \frac{2/36}{5/36} = \frac{2}{5}$$

Illustration :

A jar contains 10 white balls and 6 blue balls, all are of equal size. Two balls are drawn without replacement. Find the probability that the second ball is white if it is known that the first is white.

Sol. *Let* E_1 *be the event " the first ball drawn is white" and* E_2 *be the event" the second ball drawn is white again. Then*

$$P(E_1) = \frac{10}{16}$$

since 10 out of 10 + 6 balls are white. But, after one ball is chosen, there remain 9 white balls and 6 blue balls. Therefore the required probability is

$$P\left(\frac{E_2}{E_1}\right) = \frac{P(E_1 \cap E_2)}{P(E_1)} = \frac{\frac{10}{16} \cdot \frac{9}{15}}{\frac{10}{16}} = \frac{9}{15} = \frac{3}{5}$$

Illustration :

There are four machines and it is known that exactly two of them are faulty. They are tested one by one, in a random order till both the faulty machines are identified. The probability that only two tests are needed is

(A) $\frac{1}{3}$ (B) $\frac{1}{6}$ (C) $\frac{1}{2}$ (D) $\frac{1}{4}$

Sol. *The procedure ends in first two tests if either both are faulty or both are good. Therefore the probability is*

$$= P(G \cap G) + P(F \cap F) = P(G) \cdot P\left(\frac{G}{G}\right) + P(F) \cdot P\left(\frac{F}{F}\right) = \frac{2}{4} \cdot \frac{1}{3} + \frac{1}{4} \cdot \frac{1}{3} = \frac{1}{3} \qquad \textit{Ans.}$$

Practice Problem

Q.1 If A and B are any two events with $P(A)=3/8$; $P(B) = 1/2$ and $P(A \cap B) = 1/4$. Find
(i) $P(A \cup B)$ (ii) $P(A^C)$ and $P(B^C)$ (iii) $P(A^C \cap B^C)$ (iv) $P(A^C \cup B^C)$
(v) $P(A \cap B^C)$ (vi) $P(B \cap A^C)$

Q.2 A problem in mathematics is given to 2 children who solve it independently. If probability of A solving it is 1/2 and probability of B solving it is 2/3. Find the probability that the problem is solved.

Q.3 A pair of dice is rolled until a total of 5 or 7 is obtained. Find the probability that the total of 5 comes before a total of 7

Q.4 A box contains 5 tubes, 2 of them defective and 3 good one. Tubes are tested by one-by-one till the 2 defective tubes are discovered. What is the probability that the testing procedure comes to an end at the end of
(i) second testing (ii) 3^{rd} testing

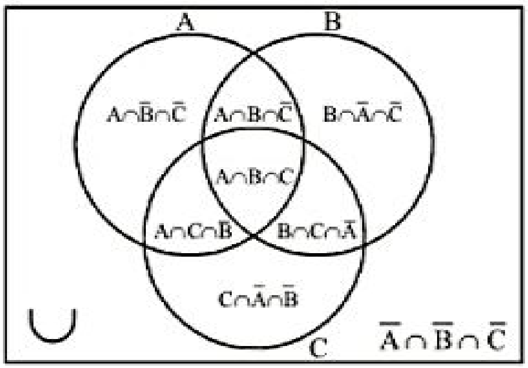
Q.5 In the following experiment, we roll a fair die 5 times
(i) What is the probability of the sequence "1, 2, 3, 4, 5".
(ii) What is the probability that the sequence starts with a "1"
(iii) What is the probability that the number "2" appears exactly twice.
(iv) Let E be the event that we find the sequence "1, 2, 3, 4, 5" and let F be the event that the sequence starts with a "1".
What are the probabilities P(E/F) and P(F/E)

Answer key

Q.1 (i) 5/8 ; (ii) 5/8 & 1/2 ; (iii) 3/8 ; (iv) 3/4 ; (v) 1/8 ; (vi) 1/4
Q.2 5/6 Q.3 2/5 Q.4 (i) 1/10 ; (ii) 3/10

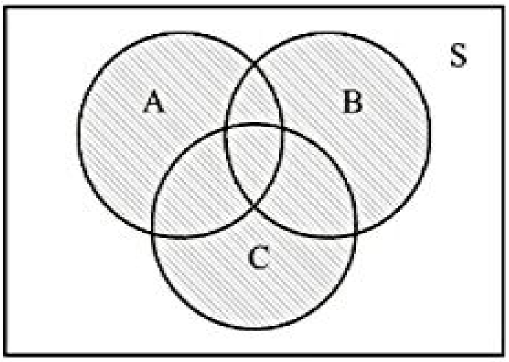
Q.5 (i) $\frac{1}{6^5}$; (ii) $\frac{1 \cdot 6^4}{6^5} = \frac{1}{6}$; (iii) $\frac{{}^5C_2 \cdot 1 \cdot 5^3}{6^5} = \frac{2}{3}\left(\frac{5}{6}\right)^4$; (iv) $P(E/F) = \frac{1}{6^4}$; $P(F/E) = 1$

THREE EVENTS ASSOCIATED WITH AN EXPERIMENTAL PERFORMANCE :



The addition theorem can be extended when three events are associated with the experiment.
If A, B and C are three events then
$P(A \cup B \cup C)$ denotes the sum of probabilities of all the sample points in $(A \cup B \cup C)$ or probability of occurrence of atleast one of the events.

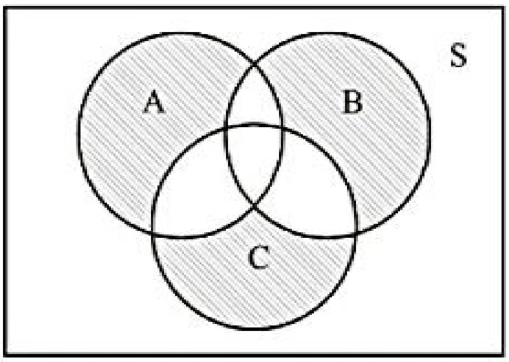
(i) $P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C)$



$(A \cup B \cup C)$

(ii) P (occurrence of exactly one of the events) =

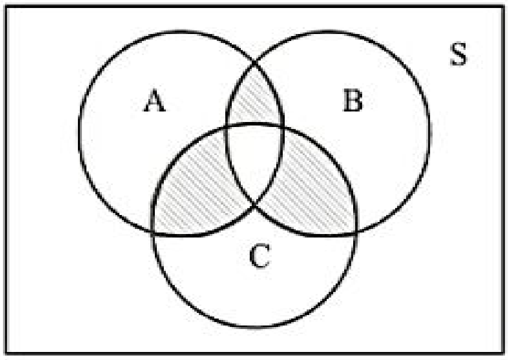
$$P(A) + P(B) + P(C) - 2[P(A \cap B) + P(B \cap C) + P(C \cap A)] + 3P(A \cap B \cap C)$$



$(A \cap \bar{B} \cap \bar{C}) \cup (\bar{A} \cap B \cap \bar{C}) \cup (\bar{A} \cap \bar{B} \cap C)$

(iii) P (occurrence of exactly two of the events) =

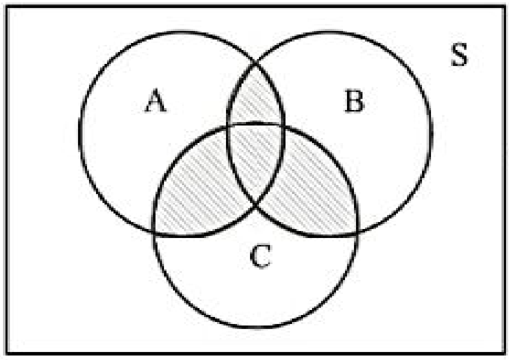
$$P(A \cap B) + P(B \cap C) + P(C \cap A) - 3P(A \cap B \cap C)$$



$$(A \cap B \cap \bar{C}) \cup (\bar{A} \cap B \cap C) \cup (A \cap \bar{B} \cap C)$$

(iv) P (occurrence of atleast two of the events) =

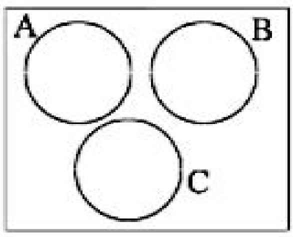
$$P(A \cap B) + P(B \cap C) + P(C \cap A) - 2P(A \cap B \cap C)$$

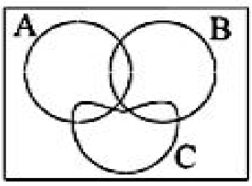


$$(A \cap B \cap \bar{C}) \cup (A \cap \bar{B} \cap C) \cup (\bar{A} \cap B \cap C) \cup (A \cap B \cap C)$$

Note :

(a) If A, B, C are three pair wise mutually exclusive $\Rightarrow$ they are mutually exclusive however if A, B, C are mutually exclusive $\not\Rightarrow$ they are pair wise mutually exclusive





ME $\not\Rightarrow$ pair wise ME   Pair wise ME $\Rightarrow$ ME

(b) However , if A,B,C are pair wise independent $\not\Rightarrow$ they are independent. Infact for 3 events A, B and C to be independent they must be

(i) pair wise (ii) mutually independent , mathematically

$$P(A \cap B) = P(A) \,.\, P(B) \,;\, P(B \cap C) = P(B).P(C) \,;\, P(C \cap A) = P(C).P(A)$$

and $P(A \cap B \cap C) = P(A).P(B).P(C)$

for n independent events, the total number of conditions would be

$${}^nC_2 + {}^nC_3 + ... + {}^nC_n = 2^n - n - 1$$

Illustration :

A, B and C are three newspapers from a city. 25% of the population reads A, 20% reads B, 15% reads C, 16% reads both A and B, 10% reads both B and C, 8% reads both A and C and 4% reads all the three. Find the percentage of the population who read atleast one of A, B and C.

Sol. *We are given that*

$$P(A) = \frac{25}{100},\ P(B) = \frac{20}{100},\ P(C) = \frac{15}{100}$$

$$P(A \cap B) = \frac{16}{100},\ P(B \cap C) = \frac{10}{100},\ P(C \cap A) = \frac{8}{100} \quad \text{and} \quad P(A \cap B \cap C) = \frac{4}{100}$$

We have to find $P(A \cap B \cap C)$. *We can use the formula*

$$P(A \cap B \cap C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C)$$

$$= \frac{1}{100}(25 + 20 + 15 - 16 - 10 - 8 + 4) = \frac{30}{100}$$

Thus 30% of the people read atleast one of the newspapers.

Illustration :

Let A, B and C be three events such that

$p = P$ *(exactly one of A or B) = P(exactly one of B or C) =P(exactly one of C or A)*

and $P(A, B, C \text{ simultaneously}) = p^2$

where $0 < p < \frac{1}{2}$. *Then P (at least one of A, B or C) is equal to*

(A) $\frac{3p+2p^2}{2}$ (B) $\frac{2p+3p^2}{2}$ (C) $\frac{2p+3p^2}{4}$ (D) $\frac{3p+2p^2}{4}$

Sol. *Exactly one of A or B means*

So $P(A) + P(B) - 2P(A \cap B) = p$ ...(i)

Similarly P (exactly one of B or C)

$P(B) + P(B \cap C) - 2P(B \cap C) = p$ ...(ii)

and $P(C) + P(A) - P(C \cap A) = p$ ...(iii)

Adding equation (i) – (iii), we have

$$2[P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A)] = 3p \quad ...(iv)$$

Now P(atleast one A, B or C) is given by [see part (3), theorem 7.2]

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C)$$

$$= \frac{3p}{2} + p^2 \qquad [\text{Equation (iv) and } P(A \cap B \cap C) = p^2]$$

$$= \frac{3p + 2p^2}{2}$$

BINOMIAL PROBABILITY :

Let an experiment has n-independent trials, and each of the trial has two possible outcomes

(i) success (ii) failure

If probability of getting success, $P(S) = p$ and probability getting failure, $P(F) = q$ such that $p + q = 1$.

Then, $P(r \text{ successes}) = {}^nC_r\, p^r q^{n-r}$

Proof :

Consider the compound event where r successes are in succession and $(n - r)$ failures are in succession.

$$P\left(\underbrace{SSS......S}_{r}\ \underbrace{FFF......F}_{(n-r)}\right) = \underbrace{P(S).P(S)......P(S)}_{r \text{ times}}\ \underbrace{P(F).P(F)......P(F)}_{(n-r)\text{times}} = P^r . q^{n-r}$$

But these r successes and $(n - r)$ failures can be arranged in $\dfrac{n!}{r!(n-r)!} = {}^nC_r$ ways and in each arrangement the probability will be $p^r . q^{n-r}$

Hence total pr. $= P(r) = {}^nC_r\, p^r q^{n-r}$ ......(1)

Recurrence relation

$$p(r + 1) = {}^nC_{r+1}\, p^{r+1} . q^{n-r-1}$$

$$\therefore \quad \frac{P(r+1)}{P(r)} = \frac{{}^nC_{r+1}}{{}^nC_r}\frac{p}{q} = \frac{n-r}{r+1}\frac{p}{1-p}$$

$$\therefore \quad P(r + 1) = \frac{n-r}{r+1}\cdot\frac{p}{1-p}P(r) \qquad ......(2)$$

Equation (2) is used for completely the probabilities of P(1); P(2); P(3); ...... etc. once P(0) is determined.

Illustration :

A pair of dice is thrown 6 times, getting a doublet is considered a success. Compute the probability of

(i) *no success* *(ii)* *exactly one success*

(iii) *at least one success* *(iv)* *at most one success*

Sol. *Total sample spaces are = 36*

In which six doublets then

$$p = \frac{3}{36} = \frac{1}{6}; \quad q = 1 - \frac{1}{6} = \frac{5}{6}$$

(i) *No success for r = 0*

$$\therefore \quad p(0) = {}^6C_0\left(\frac{1}{6}\right)^0\left(\frac{5}{6}\right)^6 = \left(\frac{5}{6}\right)^6$$

(ii) *Exactly one success for r = 1*

$$\therefore \quad p(1) = {}^6C_1\left(\frac{1}{6}\right)^1\left(\frac{5}{6}\right)^5 = \left(\frac{5}{6}\right)^5$$

(iii) *For at least are success for r = 1, 2, 3, 4, 5, 6.*

$$\therefore \quad \sum_{r=1}^{6} {}^6C_r p^r q^{6-r} = {}^6C_1\left(\frac{1}{6}\right)\left(\frac{5}{6}\right)^5 + {}^6C_2\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^4 + {}^6C_3\left(\frac{1}{6}\right)^3\left(\frac{5}{6}\right)^3 + {}^6C_4\left(\frac{1}{6}\right)^4\left(\frac{5}{6}\right)^2$$

$$+ {}^6C_5\left(\frac{1}{6}\right)^5\left(\frac{5}{6}\right)^1 + {}^6C_6\left(\frac{1}{6}\right)^6\left(\frac{5}{6}\right)^0$$

(iv) *For at most one success for r = 0, 1*

$$\sum_{r=0}^{1} {}^6C_r\left(\frac{1}{6}\right)^6\left(\frac{5}{6}\right)^{6-r} = {}^6C_0\left(\frac{1}{6}\right)^0\left(\frac{5}{6}\right)^6 + {}^6C_1\left(\frac{1}{6}\right)\left(\frac{5}{6}\right)^5$$

Illustration :

In a hurdle race a man has to clear 9 hurdles. Probability that he clears a hurdle 2/3 and the probability that he knocks down the hurdle is 1/3. Find the probability that he knocks down fewer than 2 hurdles.

Sol. *For probability that he knocks down fewer than two hurdles for r = 0, 1*

where $p = \frac{1}{3}, \quad q = \frac{2}{3}$

$$\therefore \quad \sum_{r=0}^{1} {}^9C_r\left(\frac{1}{3}\right)^r\left(\frac{2}{3}\right)^{9-r} = {}^9C_0\left(\frac{1}{3}\right)^0\left(\frac{2}{3}\right)^9 + {}^9C_1\left(\frac{1}{3}\right)^1\left(\frac{2}{3}\right)^8$$

Illustration :

A drunkard takes a step forward or backward. The probability that he takes a step forward is 0.4. Find the probability that at the end of 11 steps he is one step away from the starting point.

Sol. *At the end of 11 steps he is one step away from the starting point by two ways*

(i) *Man has taken 6 steps forward and 5 steps backward*

(ii) *Man has taken 6 steps backward and 6 steps forward*

here p = *probability of forward* $= \frac{2}{5}$

q = *probability of backward* $= \frac{3}{5}$

$$\therefore \quad \text{Probability} = {}^{11}C_6\left(\frac{2}{5}\right)^6\left(\frac{3}{5}\right)^5 + {}^{11}C_5\left(\frac{2}{5}\right)^5\left(\frac{3}{5}\right)^6$$

Practice Problem

Q.1 A die is thrown 7 times. What is the chance that an odd number turns up (i) exactly 4 times (ii) at least 4 times?

Q.2 A fair coin is tossed a fixed number of times. If the probability of getting 7 heads is equal to getting 9 heads, the probability of getting 2 heads is ,

(A) $\frac{15}{2^5}$ (B) $\frac{2}{15}$ (C) $\frac{15}{2^{13}}$ (D) None of these

Q.3 A coin is twice as likely to land heads as tails. In a sequence of five independent trials, find the probability that the third head occurs on the fifth toss.

Q.4 A fair coin is flipped n times. Let E be the event "a head is obtained on the first flip", and let F_k be the event "exactly k heads are obtained". For which one of the following pairs (n, k) are E and F_k independent?
(A) (12, 4) (B) (20, 10) (C) (40, 10) (D) (100, 51)

Answer key

Q.1 $\frac{35}{128}, \frac{1}{2}$ Q.2 C Q.3 $\frac{16}{81}$ Q.4 B

TOTAL PROBABILITY THEOREM :

Let $E_1, E_2, \ldots\ldots, E_n$ be n mutually exclusive and exhaustive events, with non-zero probabilities, of a random experiment. If A be any arbitrary event of the sample space of the above random experiment with $P(A) > 0$, then

$$P(A) = P(E_1)\, P\left(\frac{A}{E_1}\right) + P(E_2)\, P\left(\frac{A}{E_2}\right) + \ldots\ldots + P(E_n)\, P\left(\frac{A}{E_n}\right).$$

Proof : Let S be the sample space of the random experiment.

Since $E_1, E_2, \ldots\ldots, E_n$ are exhaustive, we have $S = E_1 \cup E_2 \cup \ldots\ldots \cup E_n$.

Now $A = S \cap A = (E_1 \cup E_2 \cup \ldots\ldots \cup E_n) \cap A$

$\Rightarrow$ $A = (E_1 \cap A) \cup (E_2 \cap A) \cup \ldots\ldots \cup (E_n \cap A)$ ......(i)

Since $E_1, E_2, \ldots\ldots, E_n$ are mutually exclusive, we have $E_i \cap E_j = \phi$ for $i \neq j$

Now $(E_i \cap A) \cap (E_j \cap A) = (E_i \cap E_j) \cap A = \phi \cap A = \phi$

$\therefore$ $E_1 \cap A,\ E_2 \cap A,\ \ldots\ldots, E_n \cap A$ are also mutually exclusive.

By using addition theorem, (i) implies

$$P(A) = P(E_1 \cap A) + P(E_2 \cap A) + \ldots\ldots + P(E_n \cap A)$$

$$\Rightarrow \quad P(A) = P(E_1)\, P\left(\frac{A}{E_1}\right) + P(E_2)\, P\left(\frac{A}{E_2}\right) + \ldots\ldots + P(E_n)\, P\left(\frac{A}{E_n}\right).$$

Remark : In practical problems, it is found convenient to write as follows :

$$P(A) = P(E_1A \text{ or } E_2A \text{ or } \ldots\ldots E_nA) = P(E_1A) + P(E_2A) + \ldots\ldots + P(E_nA)$$

$$P(A) = P(E_1)\, P\left(\frac{A}{E_1}\right) + P(E_2)\left(\frac{A}{E_2}\right).$$

Illustration :

A box contains three coins, one coin is fair, one coin is two-headed, and one coin is weighted so that the probability of head appearing is 1/3. A coin is selected at random and tossed. Find the probability that (i) head (ii) tail appears.

Sol. *Let E_1, E_2 and E_3 be the events of selecting at random first coin, second coin and third coin respectively.*

$$\therefore \quad P(E_1) = \frac{1}{3},\ P(E_2) = \frac{1}{3} \text{ and } P(E_3) = \frac{1}{3}$$

Let H and T be events of getting head and tail respectively.

$$\therefore \quad P\left(\frac{H}{E_1}\right) = \frac{1}{2},\ P\left(\frac{T}{E_1}\right) = \frac{1}{2} \qquad (\because \text{ First coin is fair})$$

$$P\left(\frac{H}{E_2}\right) = 1,\ P\left(\frac{T}{E_2}\right) = 0 \qquad (\because \text{ Second coin is two-headed})$$

(i) $P(\text{getting head}) = P(H) = P(E_1H \text{ or } E_2H \text{ or } E_3H)$

$$= P(E_1H) + P(E_2H) + P(E_3H)$$

$$= P(E_1)\, P\left(\frac{H}{E_1}\right) + P(E_2)\, P\left(\frac{H}{E_2}\right) + P(E_3)\, P\left(\frac{H}{E_3}\right)$$

$$= \frac{1}{3} \times \frac{1}{2} + \frac{1}{3} \times 1 + \frac{1}{3} \times \frac{1}{3} = \frac{11}{18}$$

(ii) $P(\text{getting tail}) = P(T)$

$$= P(E_1T \text{ or } E_3T) = P(E_1T) + P(E_3T) = P(E_1)\, P\left(\frac{T}{E_1}\right) + P(E_3)\, P\left(\frac{T}{E_3}\right)$$

$$= \frac{1}{3} \times \frac{1}{2} + \frac{1}{3} \times \frac{2}{3} = \frac{7}{18}$$

Illustration :

There are two bags. The first bag contains 5 white and 3 black balls and the second bag contains 3 white and 5 black balls. Two balls are drawn at random from the first bag and are put into the second bag, without nothing their colours. Then two balls are drawn from the second bag. Find the probability that the balls drawn are white and black.

Sol.

5 White 3 Black	3 White 5 Black
Bag-I	Bag-II

Let E_1, E_2 and E_3 be the events of transferring 2 white, 1 white and 1 black, 2 black balls respectively from the fist bag to the second bag.

$$\therefore \quad P(E_1) = \frac{{}^5C_2}{{}^8C_2} = \frac{10}{28} = \frac{5}{14}$$

$$P(E_2) = \frac{{}^5C_1 \times {}^8C_1}{{}^8C_2} = \frac{5 \times 3}{28} = \frac{15}{28}$$

$$P(E_3) = \frac{{}^3C_2}{{}^8C_2} = \frac{3}{28}$$

Let A be the event of drawing one white and one black ball from the second bag.

$$P(A) = P(E_1A \text{ or } E_2A \text{ or } E_3A)$$

$$= P(E_1A) + P(E_2A) + P(E_3A)$$

$$= P(E_1)\, P\left(\frac{A}{E_1}\right) + P(E_2)\, P\left(\frac{A}{E_2}\right) + P(E_3)\, P\left(\frac{A}{E_3}\right)$$

$$= \frac{5}{14} \times \frac{{}^5C_1 \times {}^5C_1}{{}^{10}C_2} + \frac{15}{28} \times \frac{{}^4C_1 \times {}^6C_1}{{}^{10}C_2} + \frac{3}{28} \times \frac{{}^3C_1 \times {}^7C_1}{{}^{10}C_2}$$

$$= \frac{5}{14} \times \frac{5}{9} + \frac{15}{28} \times \frac{8}{15} + \frac{3}{28} \times \frac{7}{15} = \frac{673}{12600}$$

Illustration :

Two machines A and B produce respectively 60% and 40% of the total numbers of items of a factory. The percentages of defective output of these machines are respectively 2% and 5%. If an item is selected at random, what is the probability that the item is (i) defective (ii) non-defective?

Sol. Let E_1, E_2 be the events of drawing an item produced by machine A and machine B respectively. Let A be the event of selecting a defective item.

$\therefore$ $\bar{A}$ represent the event of selecting a non-defective item.

We have

$P(E_1) = 60\%$; $P(E_2) = 40\%$

$P\left(\frac{A}{E_1}\right)$ = Probability that an item produced A is defective

$= 2\%$

$P\left(\frac{A}{E_2}\right)$ = Probability that an item produced by B is defective

$= 5\%$

(i) P(selected item is defective)

$= P(A) = P(E_1 A \quad or \quad E_2 A) = P(E_1 A) + P(E_2 A)$

$$= P(E_1)\, P\left(\frac{A}{E_1}\right) + P(E_2)\left(\frac{A}{E_2}\right)$$

$= (60\%)\ (2\%) + (40\%)\ (5\%)$

$$= \frac{60}{100} \times \frac{2}{100} + \frac{40}{100} \times \frac{5}{100} = \frac{320}{1000} = 0.032$$

(ii) P(selected item is non-defective)

$= P(\bar{A}) = P(E_1 \bar{A} \quad or \quad E_2 \bar{A}) = P(E_1 \bar{A}) + P(E_2 \bar{A})$

$$= P(E_1)\, P\left(\frac{\bar{A}}{E_1}\right) + P(E_2)\, P\left(\frac{\bar{A}}{E_2}\right)$$

$= (60\%)\ (98\%) + (40\%)\ (95\%)$

$$= \frac{60}{100} \times \frac{98}{100} + \frac{40}{100} \times \frac{95}{100} = \frac{9680}{10000} = 0.968$$

BAYE'S THEOREM :

If an event A can occur only with one of the n pair wise mutually exclusive and exhaustive events $B_1, B_2, ... B_n$ & if the conditional probabilities of the events.

$P(A/B_1), P(A/B_2) ....P(A/B_n)$ are known then, $P(B_i/A) = \dfrac{P(B_i)P(A/B_i)}{\sum_{i=1}^{n} P(B_i)P(A/B_i)}$

Proof :

The event A occurs with one of the 'n' mutually exclusive e exhaustive events $B_1, B_2, B_3, \ldots\ldots, B_n$

$$A = AB_1 + AB_2 + AB_3 + \ldots\ldots + AB_n$$

$$P(A) = P(AB_1) + P(AB_2) + \ldots\ldots + P(AB_n) = \sum_{i=1}^{n} P(AB_i)$$

Note :

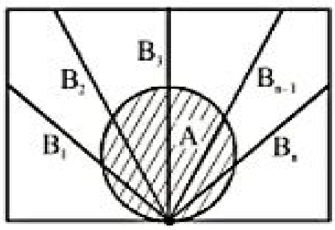
A = event what we have,

B_i = event what we want,

$B_1, B_2, \ldots\ldots, B_n$ are alternative events.

Now,

$$P(AB_i) = P(A) \cdot P\left(\frac{B_i}{A}\right) = P(B_i) \cdot P\left(\frac{A}{B_i}\right)$$



$$P\left(\frac{B_i}{A}\right) = \frac{P(B_i)P\left(\frac{A}{B_i}\right)}{P(A)} = \frac{P(B_i) \cdot P\left(\frac{A}{B_i}\right)}{\sum_{i=1}^{n} P(AB_i)} = \frac{P(B_i) \cdot P\left(\frac{A}{B_i}\right)}{\sum_{i=1}^{n} P(B_i) \cdot P\left(\frac{A}{B_i}\right)}$$

Illustration :

Bag A contains 3 white and 2 black balls. Bag B contains 2 white and 2 black balls. One ball is drawn at random from A and transferred to B. One ball is selected at random from B and is found to be white. The probability that the transferred ball is white is

(A) $\frac{8}{13}$ *(B)* $\frac{5}{13}$ *(C)* $\frac{4}{13}$ *(D)* $\frac{9}{13}$

Sol. *Let E_1 and E_2 denote the events of the transferred ball being white and black, respectively. W denotes the drawn ball from B is white. By hypothesis,*

$$P(E_1) = \frac{{}^3C_1}{{}^5C_1} = \frac{3}{5} \quad , \quad P(E_2) = \frac{{}^2C_1}{{}^5C_1} = \frac{2}{5}$$

$$P\left(\frac{W}{E_1}\right) = \frac{{}^3C_1}{{}^5C_1} = \frac{3}{5}, \quad P\left(\frac{W}{E_2}\right) = \frac{{}^2C_1}{{}^5C_1} = \frac{2}{5}$$

By Boyes' theorem

$$P\left(\frac{E_1}{W}\right) = \frac{P(E_1)P\left(\frac{W}{E_1}\right)}{P(E_1)P\left(\frac{W}{E_1}\right) + P(E_2)P\left(\frac{W}{E_2}\right)} = \frac{\frac{3}{5}\times\frac{3}{5}}{\frac{3}{5}\times\frac{3}{5}+\frac{2}{5}\times\frac{2}{5}} = \frac{9}{13}$$

Illustration :

A letter is to come from either LONDON or CLIFTON. The postal mark on the letter legibly shows consecutive letters "ON". The probability that the letter has come from LONDON is

(A) $\frac{12}{17}$ (B) $\frac{13}{17}$ (C) $\frac{5}{17}$ (D) $\frac{4}{17}$

Sol. *Let the events be defined as*

E_1 : *Letter coming from LONDON*

E_2 : *Letter coming from CLIFTON*

E_3 : *Two consecutive letters ON.*

The word LONDON contains 5 types of consecutive letters (LO, ON, ND, DO, ON) of which there are two ON's. The word CLIFTON contains 6 types of consecutive letters (CL, LI, IF, FT, TO, ON) of which there is one "ON". Now

$$P(E_1) = \frac{1}{2} = P(E_2) \quad \Rightarrow \quad P\left(\frac{E_3}{E_2}\right) = \frac{2}{5} \quad \text{and} \quad P\left(\frac{E_3}{E_2}\right) = \frac{1}{6}$$

By Boyes' theorem

$$P\left(\frac{E_1}{E_3}\right) = \frac{\frac{1}{2}\times\frac{2}{5}}{\frac{1}{2}\times\frac{2}{5}+\frac{1}{2}\times\frac{1}{6}} = \frac{12}{17}$$

Illustration :

In a factory which manufactures bolts, machines A, B and C manufacture respectively 25%, 35% and 40% of the bolts. Of their outputs, 5, 4 and 2 percent are respectively defective bolt. A bolts is drawn at random from the product and is found to be defective. What is the probability that it is manufactured by the machine B?

Sol. *Let events B_1, B_2, B_3 be the following*

B_1 : *the bolt is manufactured by machine A*

B_2 : *the bolt is manufactured by machine B*

B_3 : *the bolt is manufactured by machine C*

Clearly, B_1, B_2, B_3 are mutually exclusive and exhaustive events and hence, they represent a partition of the sample space.

Let the event E be 'the bolt is defective'.

The event E occurs with B_1 or with B_2 or with B_3. Given that,

$$P(B_1) = 25\% = 0.25,\ \ P(B_2) = 0.35\ \text{ and } P(B_3) = 0.40$$

Again $P(E|B_1)$ = Probability that the bolt drawn is defective given that it is manufactured by machine

$$A = 5\% = 0.05$$

Similarly, $P(E|B_2) = 0.04,\ \ P(E \mid B_3) = 0.02$

Hence, by Bayes' Theorem, we have

$$P(B_2 | E) = \frac{P(B_2)\ P(E|B_2)}{P(B_1)\ P(E|B_1) + P(B_2)\ P(E|B_2) + P(B_3)\ P(E|B_3)}$$

$$= \frac{0.35 \times 0.04}{0.25 \times 0.05 \times 0.35 \times 0.04 + 0.40 \times 0.02} = \frac{0.0140}{0.0345} = \frac{28}{69}$$

Illustration :

In a test, an examinee either guesses or copies or knows the answer for a multiple choice question having FOUR choices of which exactly one is correct. The probability that he makes a guess is 1/3 and the probability for copying is 1/6. The probability that his answer is correct, given that he copied it is 1/8. The probability that he knew the answer, given that his answer is correct is

(A) $\frac{5}{29}$ (B) $\frac{9}{29}$ (C) $\frac{24}{29}$ (D) $\frac{20}{29}$

Sol. *Let the events be defined as*

E_1 : *Guessing*

E_2 : *Copying*

E_3 : *Knowing*

E : *Correct answer*

By hypothesis,

$$P(E_1) = \frac{1}{3},\ P(E_2) = \frac{1}{6},\ P(E_3) = 1 - \frac{1}{3} - \frac{1}{6} = \frac{1}{2}$$

$$P\left(\frac{E}{E_1}\right) = \frac{1}{4} \quad \textit{(out of four choices only one is correct)}$$

$$P\left(\frac{E}{E_2}\right) = \frac{1}{8}$$

$$P\left(\frac{E}{E_3}\right) = 1$$

Therefore by Bayes' theorem

$$P\left(\frac{E_3}{E}\right)=\frac{P(E_3)P\left(\frac{E}{E_3}\right)}{P(E_1)P\left(\frac{E}{E_1}\right)+P(E_2)P\left(\frac{E}{E_2}\right)+P(E_3)P\left(\frac{E}{E_3}\right)}=\frac{\frac{1}{2}\times 1}{\frac{1}{3}\times\frac{1}{4}+\frac{1}{6}\times\frac{1}{8}+\frac{1}{2}\times 1}=\frac{24}{29}$$

Illustration :

A doctor is to visit a patient. From the past experience, it is known that the probabilities that he will come by train, bus scooter or by other means of transport are respectively $\frac{3}{10}$, $\frac{1}{5}$, $\frac{1}{10}$ and $\frac{2}{5}$. The probabilities that he will be late are $\frac{1}{4}$, $\frac{1}{3}$ and $\frac{1}{12}$, if he comes by train, bus and scooter respectively, but if he comes by other means of transport, then he will not be late. When he arrives, he is late. What is the probability that he comes by train?

Sol. *Let E be the event that the doctor visits the patient late and let T_1, T_2, T_3, T_4 be the events that the doctor comes by train, bus scooter, and other means of transport respectively.*

Then $P(T_1)=\frac{3}{10}$, $P(T_2)=\frac{1}{5}$, $P(T_3)=\frac{1}{10}$ *and* $P(T_4)=\frac{2}{5}$ *(given)*

$P(E|T_1)$ = *Probability that the doctor arriving late comes by train* $=\frac{1}{4}$

Similarly, $P(E|T_2)=\frac{1}{3}$, $P(E|T_3)=\frac{1}{12}$ *and* $P(E|T_4)=0$, *since he is not late if he comes by other means by other means of transport.*

Therefore, by Bayes' Theorem, we have

$P(T_1|E)$ = *Probability that the doctor arriving late comes by train*

$$=\frac{P(T_1)\,P(E|T_1)}{P(T_1)\,P(E|T_1)+P(T_2)\,P(E|T_2)+P(T_3)\,P(E|T_3)+P(T_4)\,P(E|T_4)}$$

$$=\frac{\frac{3}{10}\times\frac{1}{4}}{\frac{3}{10}\times\frac{1}{4}+\frac{1}{5}\times\frac{1}{3}+\frac{1}{10}\times\frac{1}{12}+\frac{2}{5}\times 0}=\frac{3}{40}\times\frac{120}{8}=\frac{1}{2}$$

Hence, the required probability is $\frac{1}{2}$.

Illustration :

Suppose that the reliability of a HIV test is specified as follows :

Of people having HIV, 90% of the test detect the disease but 10% go undetected. Of people free of HIV, 99% of the test are judged HIV –ive but 1% are diagnosed as showing HIV +ve. From a large population of which only 0.1% have HIV, one person is selected at random, given the HIV test, and the pathologist reports him/her as HIV +ve. What is the probability that the person actually has HIV?

Sol. *Let E denote the event that the person selected is actually having HIV and A the event that the person's HIV test is diagnosed as +ve. We need to find P(E|A). Also E' denotes the event that the person selected is actually not having HIV.*

Clearly, {E, E'} is a partition of the sample space of all people in the population. We are given that

$$P(E) = 0.1\% = \frac{0.1}{100} = 0.001$$

$$P(E') = 1 - P(E) = 0.999$$

P(A|E) = P (Person tested as HIV +ve given that he/she is actually having HIV)

$$= 90\% = \frac{90}{100} = 0.9$$

and *P(A|E') = P(Person tested as HIV +ve given that he/she is actually not having HIV)*

$$= 1\% = \frac{1}{100} = 0.01$$

Now, by Bayes' theorem

$$P(E|A) = \frac{P(E)\,P(A|E)}{P(E)\,P(A|E) + P(E')\,P(A|E')} = \frac{0.001 \times 0.9}{0.001 \times 0.9 + 0.999 \times 0.01} = \frac{90}{1089}$$

Thus, the probability that a person selected at random is actually having HIV given that he/she is tested HIV +ve is $\frac{90}{1089}$.

Illustration :

A bag contains 4 balls of unknown colours. A ball is drawn at random from it and is found to be white. The probability that all the balls in the bag are white is

(A) 4/5 *(B) 1/5* *(C) 3/5* *(D) 2/5*

Sol. *Let* W_j $(j = 1, 2, 3, 4)$ *denote 1, 2, 3 and 4 white balls are in the bag. Let W be the ball drawn is white. Then* $P(W_1) = P(W_2) = P(W_3) = P(W_4) = \frac{1}{4}$

$$P\left(\frac{W}{W_1}\right) = \frac{1}{4},\quad P\left(\frac{W}{W_2}\right) = \frac{2}{4},\quad P\left(\frac{W}{W_3}\right) = \frac{3}{4},\quad P\left(\frac{W}{W_4}\right) = 1$$

Therefore by Bayes' theorem

$$P\left(\frac{W}{W_4}\right)=\frac{P(W_4)P\left(\frac{W}{W_4}\right)}{\sum_{j=1}^{4}P(W_j)\,P\left(\frac{W}{W_j}\right)}=\frac{\frac{1}{4}\times 1}{\frac{1}{4}\left(\frac{1}{4}+\frac{2}{4}+\frac{3}{4}+\frac{4}{4}\right)}=\frac{4}{10}=\frac{2}{5}$$

PROBABILITIES THROUGH STATISTICAL (STOCHASTIC) TREE DIAGRAM:

Illustration :

A : box contains three coins A, B and C

A : Normal coin; B : Double Headed (DH) coin ; C : a weighted coin so that $P(H)=\frac{1}{3}$

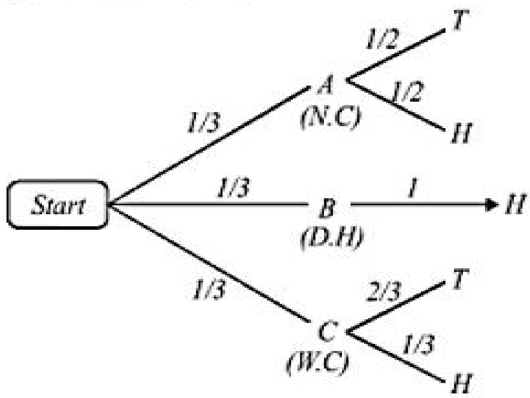
A coin is randomly selected & tossed

(A) Find the probability that head appears.

(B) If head appear find the probability that it is a normal coin P(A/H)

(C) Find the probability that tail appears.

(D) If tail appears, find the probability that it is a weighted coin P(C/T)



Sol. (A) $P(H)=\frac{1}{3}\cdot\frac{1}{2}+\frac{1}{3}\cdot 1+\frac{1}{3}\cdot\frac{1}{3}=\frac{11}{18}$

(B) $P\left(\frac{A}{H}\right)=\frac{P(A\cap B)}{P(H)}=\frac{\frac{1}{3}\cdot\frac{1}{2}}{\frac{11}{18}}=\frac{3}{11}$

(C) $P(T)=\frac{1}{3}\cdot\frac{1}{2}+\frac{1}{3}\cdot 0+\frac{1}{3}\cdot\frac{2}{3}=\frac{7}{18}$

or $1-P(H)=1-\frac{11}{18}=\frac{7}{18}$

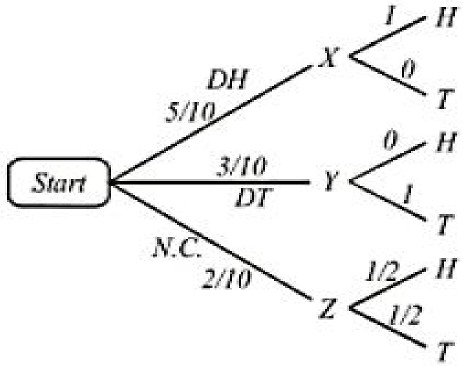
(D) $P\left(\frac{C}{T}\right)=\frac{P(C\cap T)}{P(T)}=\frac{\frac{1}{3}\cdot\frac{2}{3}}{\frac{7}{18}}=\frac{4}{7}$

Illustration :

A box contains 10 coins
- *5 coins DH denoted by say X*
- *3 coins DT denoted by say Y*
- *2 coins normal denoted by Z*

A coin is dream at random from the box and tossed, fall headwise. Find the probability that it was a normal coin.

Sol. $P\left(\frac{Z}{H}\right) = \frac{P(H \cap Z)}{P(H)}$



$$P(H) = \frac{5}{10}\cdot 1 + \frac{2-1}{10-2} = \frac{6}{10} = \frac{\frac{2}{10}\cdot\frac{1}{2}}{\frac{6}{10}} = \frac{1}{6}$$

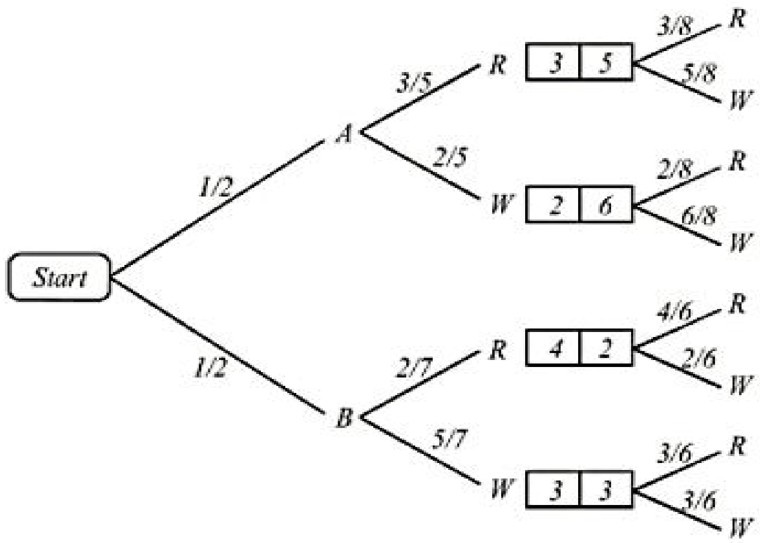
Illustration :

Let the contents of the two boxes A and B with respect to number of R and W marbles is as given below:

Bag	R	W
A	3	2
B	2	5

A bag is selected at random; a marble is drawn and put into the other box; then a marble is drawn from the second box. Find the probability the both marbles drawn the of same colour.

Sol.



$$P(E) = \frac{1}{2}\cdot\frac{3}{5}\cdot\frac{3}{8} + \frac{1}{2}\cdot\frac{2}{5}\cdot\frac{6}{8} + \frac{1}{2}\cdot\frac{2}{7}\cdot\frac{4}{6} + \frac{1}{2}\cdot\frac{5}{7}\cdot\frac{3}{6} = \frac{901}{1680}$$

Practice Problem

Q.1 Three political parties namely the Congress, the B.J.P. and the Janta Dal are contesting for a state legislative assembly elections. The state does not have a common entrance test after the 12^{th} standard, for the admissions to the medical or engineering colleges. The probabilities of these parties winning the elections are $\frac{1}{3}, \frac{4}{9}$ & $\frac{2}{9}$ respectively. If the Congress comes to the power, the probability of its introducing the common entrance test for the state is 0.6 and the corresponding probabilities for the B.J.P. and Janta Dal are 0.7 and 0.5 respectively. Find the probability that the common entrance test is introduced.

Q.2 Three persons Mr. Iyyengar, Dr. Singh and Prof. Mukherjee and competing for the post of the principal of a degree college exclusively meant for boys. Their chances are, respectively, 0.5, 0.3 and 0.2. If Mr. Iyyengar is selected, he will introduce co-education with probability 0.5 while the probabilities are 0.7 and 0.6 with regard to Dr. Singh and Prof. Mukherjee, respectively. Co-education is introduced in the college. The probability that Dr. Singh is selected as principal is

(A) $\frac{31}{58}$ (B) $\frac{21}{58}$ (C) $\frac{27}{58}$ (D) $\frac{37}{58}$

Q.3 In a city 60% are males and 40% are females. Suppose 50% of males and 30% of females have colour blindness. One is selected at random and is found to be colour blind. The probability that the selected person is male is

(A) $\frac{10}{19}$ (B) $\frac{12}{87}$ (C) $\frac{9}{19}$ (D) $\frac{5}{7}$

Q.4 A person has three coins A, B and C in his pocket out of which A is a fair coin. The probability of B showing head is $\frac{2}{3}$ and that of C is $\frac{1}{3}$. He selected one of the coins at random and tossed it three times and observed 2 heads and 1 tail. The probability that the selected coin is A is

(A) $\frac{7}{25}$ (B) $\frac{18}{25}$ (C) $\frac{9}{25}$ (D) $\frac{16}{25}$

Q.5 A bag contains 6R and 4W balls. 4 balls are drawn one by one without replacement and were found to be at least two white. Find the probability that the next draw of a ball from this bag will give a white ball.

Q.6 Box A contains nine cards numbered 1 through 9, and box B contains five cards numbered 1 through 5. A box is chosen at random and a card drawn; if the card shows an even number, another card is drawn from the same box. If the card shows an odd number, a card is drawn from the other box ;

(i) What is the probability that both cards show even numbers?

(ii) If both cards show even numbers, what is the probability that they come from box A?

(iii) What is the probability that both cards show odd numbers?

Answer key

Q.1	$\frac{28}{45}$	Q.2	B	Q.3	D	Q.4	C	Q.5	$\frac{34}{115}$

Q.6 (i) 2/15 ; (ii) 5/8 ; (iii) 1/3

MATHEMATICAL EXPECTATION (PRACTICAL USE OF PROBABILITY IN DAY TO DAY LIFE):

It is worthwhile indicating that if 'P' represents a person's chance of success in any venture and 'M 'the sum of money which he will receive in case of success, then the sum of money denoted by 'P·M' is called his expectation.

Illustration :

Two players of equal skill A and B are playing a game. They leave off playing (due to some force majeure conditions) when A wants 3 points and B wants 2 to win. If the prize money is Rs.16000/-. How can the referee divide the money in a fair way.

Sol. *Since, A wins if he scores 3 points before B scores 2.*

Probability of A's scoring a point = Probability of B's scoring at point = $\frac{1}{2}$

Hence, required probability that A succeeds

$$= \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2} + \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2} + \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2} + \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2} = \frac{5}{16}$$

Probability that B succeeds $= 1 - \frac{5}{16} = \frac{11}{16}$

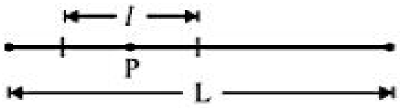
$$\therefore \quad A\text{'s expectation} = \frac{5}{16} \times 16000 = 5000$$

$$B\text{'s expectation} = \frac{11}{16} \times 16000 = 11000$$

GEOMETRICAL PROBABILITY (CONTINUOUS SAMPLE SPACE) :

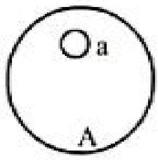
(1) One-dimensional Probability :

$$P = \frac{\text{favourable length}}{\text{total length}}$$



(2) Two-dimensional Probability :

$$P = \frac{\text{favourable area}}{\text{total area}}$$



(3) Three-dimensional Probability :

$$P = \frac{\text{favourable volume}}{\text{total volume}}$$

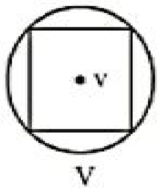
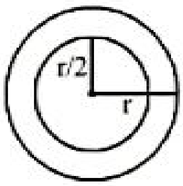


Illustration :

A point is taken inside a circle of radius find the probability that the point is closer to the centre as a circumference.

Sol. $n(s) = \pi r^2$



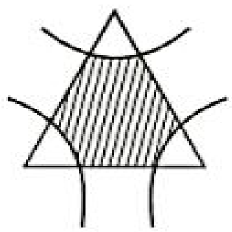
$$n(A) = \pi\left(\frac{r}{2}\right)^2$$

$$P = \frac{\pi\left(\frac{r}{2}\right)^2}{\pi r^2} = \frac{1}{4} \qquad \textit{Ans.}$$

Illustration :

A point is selected random inside a equilateral triangle whose length of side is 3. Find the probability that its distance from any corner is greater than 1.

Sol. Area of sector $= \dfrac{r^2\theta}{2}$



$$n(s) = \frac{\sqrt{3}}{4}\cdot 9$$

$$n(A) = \frac{\sqrt{3}}{4}\cdot 9 - 3\,\frac{1\cdot 1}{2}\cdot\frac{\pi}{3} \qquad \therefore\ P(A) = \frac{\frac{\sqrt{3}}{4}\cdot 9 - \frac{\pi}{2}}{\frac{\sqrt{3}}{4}\cdot 9} = 1 - \frac{2\pi}{9\sqrt{3}}$$

Illustration :

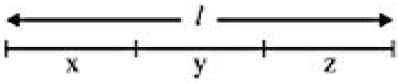
A sphere of radius r is circumscribed about a cube. Find the probability that a point lies in the sphere but out side the cube.

Sol. $$P = \frac{\text{fav. volume}}{\text{total volume}} = \frac{\frac{4\pi}{3}r^3 - \left(\frac{2r}{\sqrt{3}}\right)^3}{\frac{4}{3}\pi r^3} = 1 - \frac{2}{\sqrt{3}\pi} \quad \textbf{Ans.}$$

Illustration :

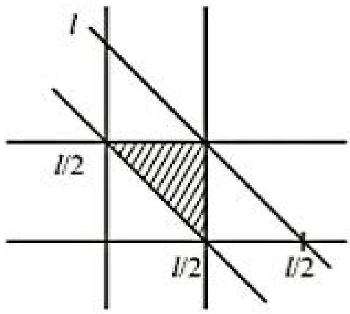
A stick of length l is broken into three parts find the probability that these three parts from a triangle.

Sol.



$x > 0 \quad y > 0, \; l-(x+y) > 0 \quad z = l-(x+y)$

For a triangle sum of two sides should be greater than third side



$$x + y > l - (x+y) \qquad x + y > \frac{l}{2} \qquad \ldots\ldots(i)$$

$$x + l - (x+y) > y \qquad l > 2y \qquad y < \frac{l}{2} \qquad \ldots\ldots(ii)$$

Similarly $\quad x < \frac{l}{2} \qquad \ldots\ldots\ldots(iii)$

So required probability $$= \frac{\frac{l}{2}\times\frac{l}{2}\times\frac{l}{2}}{\frac{1}{2}\times l\times l} = \frac{1}{4}. \quad \textbf{Ans.}$$

COINCIDENCE TESTIMONY :

If p_1 and p_2 are the probabilities of speaking the truth of two independent witnesses A and B who give the same statement

$$\text{P (their combined statement is true)} = P(H_1 / H_1 \cup H_2) = \frac{p_1 p_2}{p_1 p_2 + (1-p_1)(1-p_2)}.$$

where H_1 means both speaks the truth and H_2 means both speaks false.

In this case it has been assumed that we have no knowledge of the event except the statement made by A and B.

However if p is the probability of the happening of the event before their statement then

$$\text{P (their combined statement is true)} = \frac{p p_1 p_2}{p p_1 p_2 + (1-p)(1-p_1)(1-p_2)}.$$

Here it has been assumed that the statement given by all the independent witnesses can be given in two ways only, so that if all the witnesses tell falsehoods they agree in telling the same falsehood.

If this is not the case and c is the chance of their coincidence testimony then the

Probability that the statement is true = $P p_1 p_2$

Probability that the statement is false = $(1-p).c\,(1-p_1)(1-p_2)$

However chance of coincidence testimony is taken only if the joint statement is not contradicted by any witness.

Illustration :

A speaks the truth 3 out of 4 times, and B 5 out of 6 times. What is the probability that they will contradict each other in stating the same fact.

Sol. $P(A) = \frac{3}{4};\ P(B) = \frac{5}{6}$

$$P(contradict) = \frac{3}{4}\cdot\frac{1}{6}+\frac{5}{6}\cdot\frac{1}{4}=\frac{8}{24}=\frac{1}{3}$$

Illustration :

A speaks truth 3 times out of 4, and B 7 times out of 10. They both assert that a white ball has been drawn from a bag containing 6 balls of different colours; find the probability of the truth of their assertion. P(A) = 3/4; P(B)=7/10

Sol. *There are 2 hypothesis (i) their coincidence testimony is true*

(ii) it is false

H_1 : white ball is actually drawn & both speaks the truth

$$P(H_1) = \frac{1}{6}\cdot\frac{3}{4}\cdot\frac{7}{10}$$

H_2 : (white has not been withdrawn) and (their statement coincides) and they both speaks false

$$P(H_2) = \frac{5}{6}\times\frac{1}{5}\times\frac{1}{5}\times\frac{1}{4}\times\frac{3}{10}$$

Let E : their assertion is true

$$\therefore \quad P(E) = P\left(\frac{H_1}{H_1\cup H_2}\right) = \frac{\frac{1}{6}\cdot\frac{3}{4}\cdot\frac{7}{10}}{\frac{1}{6}\cdot\frac{3}{4}\cdot\frac{7}{10}+\frac{5}{6}\left(\frac{1}{25}\right)\cdot\frac{1}{4}\cdot\frac{3}{10}} = \frac{35}{36}$$

Practice Problem

Q.1 10 witnesses each of whom the probability of speaking the truth as 5/6, assert that certain event took place. If the probability of this event before their statement is $\frac{1}{1+5^9}$, find the probability of the truth of their assertion.

Q.2 A man is know to speak the truth 3 out if 4 times. He throws a die and reports that it is a six. The probability that it is actually a six is

(A) $\frac{3}{8}$ (B) $\frac{1}{5}$ (C) $\frac{3}{4}$ (D) None of these

Q.3 2 hunters A and B shot at a bear simultaneously. The bear was shot dead with only one hole in its body. Probability of A shooting the bear is 0.8 and that of B shooting the bear is 0.4. The hide was sold for Rs 280/-. divide this money in a fair way.

Q.4 Two points are picked at random on the unit circle $x^2 + y^2 = 1$. The probability that the chord joining the two points has length at least 1, is

(A) 4/9 (B) 1/3 (C) 1/6 (D) 2/3

Q.5 A circle of radius 'a' is inscribed in a square of side 2a. Find the probability that a point chosen at random is inside the square but outside the circle.

Q.6 A cloth of length 10 meter is to be randomly distributed among three brothers. Find the probability that no one gets more than 4 meter of cloth.

Q.7 A starts from a town 'X' any time between 1PM to 4PM walks towards the town 'Y' on a straight road with a speed of 4km/hr and B starts from 'Y' at any time between 1 PM to 4PM and walks towards 'X' at 4 km/hr. Assuming all times of starting all equally likely, find the odds in favour of their meeting on the way.

Answer key

Q.1 5/6 Q.2 A

Q.3 A's share = $\frac{0.48}{0.56} \times 280 = 240/-$; B's share = $\frac{0.08}{0.56} \times 280 = 40/-$

Q.4 D Q.5 $P(E) = 1 - p/4$ Q.6 $\frac{1}{25}$ Q.7 5 : 4

RANDOM VARIABLE :

Random variables are of two types :

(i) Discreate random variable.

(ii) Continuous random variables.

(i) A random variable is called a discrete random variable if it can take only finitely many values. For example, in the experiment of drawing three cards from a pack of playing cards, the random variable "number of kings drawn" is a discrete random variable taking value either 0 or 1 or 2 and 3.

(ii) A random variable is called a continuous random variable if it can take any value between certain limits. For example, height, weight of students in a class are continuous random variables.

Probability Distribution of a discrete random variable :

Let x be a discrete random variable assuming values $x_1, x_2, x_3, \ldots\ldots, x_n$ corresponding to the various outcomes of a random experiment. If the probability of occurrence of $x = x_i$ is $P(x_i) = p_i$, $1 \le i \le n$ such that $p_1 + p_2 + p_3 + \ldots\ldots + p_n = 1$, then the function, $P(x_i) = P_i$, $1 \le i \le n$ is called the probability function of the random variable x and the set $\{P(x_1), P(x_2), P(x_3), \ldots\ldots, P(x_n)\}$ is called the probability distribution of x.

Illustration :

Three balls are drawn are by one without replacement from a bag containing 5 white and 4 red balls. Find the probability distribution of the number of red balls drawn.

5 White
4 Red

Sol. *Let x denote the discrete random variable "number of red balls".*

$\therefore$ The possible values of x are 0, 1, 2, 3.

Let R_i be the event of drawing a red ball from the bag in the ith draw, i = 1, 2, 3.

$$P(x=0) = P(\bar{R}_1\bar{R}_2\bar{R}_3) = P(\bar{R}_1)\, P\left(\frac{\bar{R}_2}{\bar{R}_1}\right) P\left(\frac{\bar{R}_3}{\bar{R}_1\bar{R}_2}\right) = \frac{5}{9}\times\frac{4}{8}\times\frac{3}{7} = \frac{60}{504} = \frac{5}{42}$$

$$P(x=1) = P\left(R_1\bar{R}_2\bar{R}_3 \quad \text{or} \quad \bar{R}_1R_2\bar{R}_3 \quad \text{or} \quad \bar{R}_1\bar{R}_2R_3\right)$$

$$= P(R_1)\, P\left(\frac{\bar{R}_2}{R_1}\right) P\left(\frac{\bar{R}_3}{R_1\bar{R}_2}\right) + \frac{P(\bar{R}_1)}{P\left(\frac{R_2}{\bar{R}_1}\right) P\left(\frac{\bar{R}_3}{\bar{R}_1R_2}\right)} + P(\bar{R}_1)\, P\left(\frac{\bar{R}_2}{\bar{R}_1}\right) P\left(\frac{R_3}{\bar{R}_1\bar{R}_2}\right)$$

$$= \frac{4}{9}\times\frac{5}{8}\times\frac{4}{7} + \frac{5}{9}\times\frac{4}{8}\times\frac{4}{7} + \frac{5}{9}\times\frac{4}{8}\times\frac{4}{7} = \frac{240}{504} = \frac{10}{21}$$

$$P(x=2) = P\left(R_1R_2\overline{R}_3 \text{ or } R_1\overline{R}_2R_3 \text{ or } \overline{R}_1R_2R_3\right)$$

$$= P(R_1)\,P\left(\frac{R_2}{R_1}\right) P\left(\frac{\overline{R}_3}{R_1R_2}\right) + P(R_1)\,P\left(\frac{\overline{R}_2}{R_1}\right) P\left(\frac{R_3}{R_1\overline{R}_2}\right) + P(\overline{R}_1)P\left(\frac{R_2}{\overline{R}_1}\right) P\left(\frac{R_3}{\overline{R}_1R_2}\right)$$

$$= \frac{4}{9}\times\frac{3}{8}\times\frac{5}{7}+\frac{4}{9}\times\frac{5}{8}\times\frac{3}{7}+\frac{5}{9}\times\frac{4}{8}\times\frac{3}{7}=\frac{180}{504}=\frac{5}{14}$$

$$P(x=3) = P(R_1R_2R_3) = R(R_1)\,P\left(\frac{R_2}{R_1}\right) P\left(\frac{R_3}{R_1R_2}\right) = \frac{4}{9}\times\frac{3}{8}\times\frac{2}{7}=\frac{24}{504}=\frac{1}{21}$$

The required Probability distribution is

x	0	1	2	3
$P(x)$	$\frac{5}{42}$	$\frac{10}{21}$	$\frac{5}{14}$	$\frac{1}{21}$

MEAN AND VARIANCE OF A PROBABILITY DISTRIBUTION :

(1) Mean :-

If a random variable X assumes the values $x_1, x_2, \ldots\ldots, x_n$ with probabilities $p_1, p_2, \ldots\ldots, p_n$ respectively then the mean of X is defined by

Rendom variable (x_i)	Probability (p_i)	p_ix_i
x_1	p_1	p_1x_1
x_2	p_2	p_2x_2
$\vdots$	$\vdots$	$\vdots$
x_n	p_n	p_nx_n

Then mean $(\mu) = \dfrac{\sum_{i=1}^{n} p_ix_i}{\sum_{i=1}^{n} p_i} = \sum_{i=1}^{n} x_i\,p_i$ $\qquad \left\{\sum_{i=1}^{n} p_i = 1\right\}$

(2) Variance :

$$\sigma^2 = \sum_{i=1}^{n} p_i(x_i-\mu)^2 = \sum_{i=1}^{n} p_i\left(x_i^2-2\mu x_i+\mu^2\right) = \sum_{i=1}^{n}\left(p_ix_i^2-2\mu p_ix_i+p_i\mu^2\right)$$

$$= \sum_{i=1}^{n} p_ix_i^2 - 2\mu\sum_{i=1}^{n} p_ix_i + \mu^2\sum_{i=1}^{n} p_i = \sum_{i=1}^{n} p_ix_i^2 - \mu^2$$

(3) Standard Deviation :

$SD = +\sqrt{\sigma^2}$

Illustration :

Two bad eggs are accidently mixed with 10 good eggs 3 eggs are drawn simultaneously from the basket. Find the mean and variance of the number of bad eggs drawn.

Sol. $\frac{x=0}{P(0)} = \frac{^{10}C_3}{^{12}C_3} = \frac{10 \cdot 9 \cdot 8}{12 \cdot 11 \cdot 10} = \frac{6}{11}$

$x = 1$

$P(1) = \frac{^2C_1 \times ^{10}C_2}{^{12}C_3} = \frac{9}{22}$

$x = 2$

$P(2) = \frac{^2C_2 \cdot ^{10}C_1}{^{12}C_3} = \frac{10 \cdot 6}{12 \cdot 11 \cdot 10} = \frac{1}{22}$

x_i	p_i	$p_i x_i$
0	$\frac{6}{11}$	0
1	$\frac{9}{22}$	$\frac{9}{22}$
2	$\frac{1}{22}$	$\frac{2}{22}$

$\mu = \sum p_i x_i = \frac{11}{22} = \frac{1}{2}$

$\therefore \sum p_i x_i^2 = 0 + \frac{9}{22} + \frac{4}{22}$

$\sigma^2 = \sum p_i x_i^2 - \mu^2 = \frac{13}{22} - \frac{1}{4} = \frac{15}{44}$ ***Ans.***

BINOMIAL PROBABILITY DISTRIBUTION :

Let an experiment has n independent trials and each of the trial has two possible out comes i.e. success or failure.

If getting number of successes in the experiment is a random variable then probability of getting exactly r-successes is -

$$P(x = r) = {}^nC_r\, p^r \cdot q^{n-r}$$

where p = probability of getting success

and q = probability of getting failure

x_i	p_i	$p_i x_i$	$p_i x_i^2$
0	${}^nC_0 p^0 q^n$	$0 \times {}^nC_0 p^0 q^n$	$0^2 \cdot {}^nC_0 p^0 q^n$
1	${}^nC_1 p^1 q^{n-1}$	$1 \times {}^nC_1 p^1 q^{n-1}$	$1^2 \cdot {}^nC_1 p^1 q^{n-1}$
2	${}^nC_2 p^2 q^{n-2}$	$\vdots$	$\vdots$
3	${}^nC_3 p^3 q^{n-3}$	$\vdots$	$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
r	${}^nC_r p^r q^{n-r}$	$r \times {}^nC_r p^r \cdot q^{n-r}$	$r^2 \cdot {}^nC_r p^r q^{n-r}$

Mean of BPD of a random variable

$$\mu = \sum p_i x_i = \sum_{r=0}^{n} r \cdot {}^nC_r \cdot p^r \cdot q^{n-r} = \sum_{r=0}^{n} r \cdot \frac{n}{r} \cdot {}^{n-1}C_{r-1} \cdot p^r \cdot q^{n-r} = p \cdot n \sum_{r=1}^{n} {}^{n-1}C_{r-1} \cdot p^{r-1} q^{n-r}$$

$$= np\, [{}^{n-1}C_0 \cdot p^0\, q^{n-1} + {}^{n-1}C_1 \cdot p^1 q^{n-2} + \ldots\ldots + {}^{n-1}C_{n-1}\, p^{n-1}\, q^0]$$

$$= np\, (p+q)^{n-1} = np$$

Variance of BPD of a random variable :

$$\sigma^2 = \sum p_i x_i^2 - \mu^2$$

$$\sum p_i x_i^2 = \sum_{n=0}^{n} r^2 \cdot {}^nC_r \cdot p^r\, q^{n-r} = \sum_{r=0}^{n} r^r \cdot \frac{n}{r} \cdot {}^{n-1}C_{r-1} \cdot p^r \cdot q^{n-r}$$

$$= \sum_{r=0}^{n} r^2 \cdot \frac{n}{r} \cdot {}^{n-1}C_{r-1} \cdot p^r \cdot q^{n-r} = \sum_{r=0}^{n} (r-1+1) \cdot {}^{n-1}C_{r-1} \cdot p^r \cdot q^{n-r}$$

$$= \left[\sum_{r=0}^{n} (r-1) \cdot {}^{n-1}C_{r-1} \cdot p^r \cdot q^{n-r} + \sum_{r=0}^{n} {}^{n-1}C_{r-1} \cdot p^r \cdot q^{n-r} \right]$$

$$= p^2\, n \cdot (n-1) \sum_{n=r}^{n} {}^{n-2}C_{r-2} \cdot p^r \cdot q^{n-r} + pn \sum_{r=1}^{n} {}^{n-1}C_{r-1} \cdot p^r \cdot q^{n-r}$$

$$= p^2 \cdot n(n-1)\, (p+q)^{n-2} + pn \cdot (p+q)^{n-1}$$

$$= p^2 \cdot n(n-1) + pn$$

$$\therefore \quad \sigma^2 = p^2 n^2 - p^2 n + pn - n^2 p^2 \qquad \{\because\ \mu = np\}$$

$$\sigma^2 = pn(1-p) = npq$$ **Ans.**

Standard deviation of BPD of a random variable :

Positive value of square root of variance is called standrard deviation.

$$SD = +\sqrt{\sigma^2} = \sqrt{npq}$$

Illustration :

A pair of dice is thrown 5 times if getting a doublet is considered as a success then find the mean & variance of the successes.

Sol. *Here* $n = 5$ *and favorable sample space are* $(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6)$

$\therefore\ p = 1/6$ *and* $q = 5/6$

Mean $= \mu = np = 5/6$

Variance $= \sigma^2 = npq = 25/36$

Illustration :

If difference between mean & variance of a BPD is 1 and difference between squares is 11 then find the probability of getting exactly 3 successes.

Sol. *Given*

$$np - npq = 1 \qquad \ldots\ldots(i)$$

$$n^2p^2 - n^2p^2q^2 = 11 \qquad \ldots\ldots(ii)$$

$$\therefore\quad np + npq = 11$$

$$npq = 5 \qquad q = 5/6 \qquad p = 1/6 \qquad n = 36$$

$$\therefore \text{ Required probability} = {}^{36}C_3\left(\frac{1}{6}\right)^3\left(\frac{5}{6}\right)^{33}$$

Illustration :

If X follows a binomial distribution with mean 3 and variance $\left(\frac{3}{2}\right)$, *find*

(i) $P(X \geq 1)$ *(ii)* $P(X \leq 5)$

Sol. *We know that mean* $= np$ *and variance* $= npq$

$$\therefore\quad np = 3 \text{ and } npq = \frac{3}{2} \Rightarrow 3q = \frac{3}{2} \Rightarrow q = \frac{1}{2}$$

$$\therefore\quad p = (1 - q) = \left(1 - \frac{1}{2}\right) = \frac{1}{2}$$

$$\text{Now, } np = 3 \text{ and } p = \frac{1}{2} \Rightarrow n \times \frac{1}{2} = 3 \Rightarrow n = 6$$

So, the binomial distribution is given by

$$P(X=r) = {}^nC_r \cdot p^r \cdot q^{(n-r)} = {}^6C_r \cdot \left(\frac{1}{2}\right)^r \left(\frac{1}{2}\right)^{(6-r)} = {}^6C_r \left(\frac{1}{2}\right)^6$$

(i) $P(X \geq 1) = 1 - P(X = 0)$

$$= 1 - {}^6C_0 \cdot \left(\frac{1}{2}\right)^6 = \left(1 - \frac{1}{64}\right) = \frac{63}{64}$$

(ii) $P(X \leq 5) = 1 - P(X = 6)$

$$= 1 - {}^6C_6 \left(\frac{1}{2}\right)^6 = \left(1 - \frac{1}{64}\right) = \frac{63}{64}$$

Illustration :

If the sum of the mean and variance of a binomial distribution for 5 trials is 1.8, find the distribution.

Sol. We know that

mean = up and variance = npq

It is being given that $n = 5$ and mean + variance = 1.8

$\therefore$ $np + npq = 1.8$, where $n = 5$

$\Leftrightarrow 5p + 5pq = 1.8$

$\Leftrightarrow p + p(1-p) = 0.36$ $\quad [\because q = (1-p)]$

$\Leftrightarrow p^2 - 2p + 0.36 = 0$

$\Leftrightarrow 100\,p^2 - 200p + 36 = 0$

$\Leftrightarrow 25p^2 - 50p + 9 = 0$

$\Leftrightarrow 25p^2 - 45p - 5p + 9 = 0$

$\Leftrightarrow 5p\,(5p - 9) - (5p - 9) = 0$

$\Leftrightarrow (5p - 9)\,(5p - 1) = 0$

$\Leftrightarrow p = \frac{1}{5} = 0.2$ $\quad [\because$ p cannot exceed 1$]$

Thus, $n = 5$, $p = 0.2$, and $q = (1-p) = (1-0.2) = 0.8$

Let X denote the binomial variate. Then, the required distribution is

$$P(X=r) = {}^nC_r \cdot p^r \cdot q^{(n-r)} = {}^5C_r \cdot (0.2)^r \cdot (0.8)^{(5-r)} \quad \text{where } r = 0, 1, 2, 3, 4, 5$$

Practice Problem

Q.1 A pair of fair dice is thrown. Let X be the random variable which denotes the minimum of the two numbers which appear. Find the probability distribution, mean and variance of X.

Q.2 If the mean and SD of a binomial variate X are 9 and 3/2 respectively. Find the probability that X takes a value greater than one.

Q3 A pair of dice is thrown 4 times. If getting a total of 9 in a single throw is considered as a success then find the mean and variance of the successes.

Q4 The sum of the product of the mean and variance of a binomial distribution are 24 and 128 respectively. Find the distribution.

Q.5 A fair coin is tossed four times. Let Y denotes the random variable denoting the "longest string of heads" occurring. Find the probability distribution, mean, variance and S.D. of Y.

Answer key

Q.1 $\mu = 2.5$; var(X) = 2.1

Q.2 $P(X>1) = 1 - P(X = 0 \text{ or } 1)$

Q3 Mean = 4/9 ; Variance = 32/81

Q4 ${}^{32}C_r \cdot \left(\frac{1}{2}\right)^{32}$

Q.5 $\mu = 1.7$; $\sigma^2 = 0.9$; SD = 0.95

Solved Examples

Q.1 If three coins are tossed randomly then the probability of getting

(i) all three tails (ii) atleast one head

(ii) one head and two tails (iv) exactly two tails

Sol. For a single coin,

$S_1 = \{H, T\}, n(S_1) = 2$

For two coins,

$S_2 = \{H, H\}, (H, T), (T, H), (T, T), n(S_2) = 2^2 = 4$

For three coins,

$S \equiv S_2 \times S_2 = \{H, T\} \times \{(H, H), (H, T), (T, H), (T, T)\}$

$\equiv \{(H, H, H), (H, H, T), (H, T, H), (T, H, H), (H, T, T), (T, H, T)\}, (T, T, H), (T, T, T),$

$\therefore \quad n(S_2) = 2^3 = 8$

(i) Let E_1 = Event of getting all three tails $\{(T, T, T)\}$

$n(E_1) = 1$

$$\therefore \quad P(E_1) = \frac{n(E_1)}{n(S)} = \frac{\text{Fav. case}}{\text{Total case}} = \frac{1}{8}$$

(ii) Let E_2 = Event of getting at least one head.

$E_2 \equiv \{(H, H, H), (H, H, T), (H, T, H), (T, H, H), (H, T, T), (T, H, T), (T, T, H)\}$

$n(E_2) = 7$

$$\therefore \quad P(E_2) = \frac{n(E_2)}{n(S)} = \frac{7}{8}$$

(iii) Let E_3 = Event of getting one head and two tails.

$E_3 \equiv \{(H, T, T), (T, H, T), (T, T, H)\}$

$n(E_3) = 3$

$$\therefore \quad P(E_3) = \frac{n(E_3)}{n(S)} = \frac{3}{8}$$

(iv) Let E_4 = Event of getting exactly two tails.

$E_2 \equiv \{(H, T, T), (T, H, T), (T, T, H)\}$

$n(E_4) = 3$

$$\therefore \quad P(E_4) = \frac{n(E_4)}{n(S)} = \frac{3}{8}.$$

Q.2 If three tair and unbaised dice are rolled the ludo board at once. Find the probability that

(i) numbers shown are equal. (ii) numbers shown are (totally) different

(ii) sum of numbers is 10 (iv) sum of numbers is 15

Sol. Here,

$n(S) = 6^3 = 216$

E_1 = Event to show equal number on each.

$E_1 = \{(1, 1, 1), (2, 2, 2), (3, 3, 3), (4, 4, 4), (5, 5, 5), (6, 6, 6)\}$

$n(E_1) = 6$

$$P(E_1) = \frac{n(E_1)}{n(S)} = \frac{6}{216} = \frac{1}{36}$$

(ii) E_2 = Event to show different number on each.

$n(E_2) = {}^6P_3$ = to arrange any three different face out of six faces.

$$\therefore \quad P(E_2) = \frac{n(E_2)}{n(S)} = \frac{{}^6P_3}{216} = \frac{5}{9}$$

(iii) E_3 = Events to have sum of all three dice appeared as 10.

$$E_3 \rightarrow \left.\begin{array}{l} (1, 3, 6) \rightarrow 3! \ = 6 \\ (1, 4, 5) \rightarrow 3! \ = 6 \\ (2, 2, 6) \rightarrow 3!/2 = 3 \\ (2, 3, 5) \rightarrow 3! \ = 6 \\ (2, 4, 4) \rightarrow 3/2 = 3 \end{array}\right\} n(E_3) = 27$$
$$(3, 3, 4) \rightarrow 3/2! = 3$$

$$P(E_3) = \frac{n(E_3)}{n(S)} = \frac{27}{216}$$

Q.3 A bag contains 20 identical balls of which 8 are black and 12 are blue. Three balls are taken out at random from the bag one after the other without replacement. Find the probability that all the three balls drawn are blue.

Sol. The probability that the first ball drawn is blue is $\frac{12}{20}$,

since there are 12 blue balls among 20 balls in the bag. If the first ball is blue, then the probability that the second ball drawn is blue is $\frac{11}{19}$, since 11 of the remaining 19 are blue.

Similarly, if the first two balls drawn are blue, then the probability that the third ball drawn is blue is $\frac{10}{18}$.

The required probability is $\frac{12}{20} \cdot \frac{11}{19} \cdot \frac{10}{18} = \frac{11}{57}$

Note : If the drawn ball is replaced every time, then the probability is $\left(\frac{12}{20}\right)^3 = \left(\frac{3}{5}\right)^3$

Q.4 A and B are two events of a random experiment. If $P(A \cap B) = \frac{1}{4}$ and $P(\bar{A}) = \frac{5}{8}$, then $P(A \cap \bar{B})$ is equal to

(A) $\frac{1}{8}$ (B) $\frac{1}{4}$ (C) $\frac{1}{3}$ (D) $\frac{3}{8}$

Sol. We have $P(A \cap \bar{B}) = P(A - B) = P(A) - P(A \cap B)$

$$= \left(1 - \frac{5}{8}\right) - \frac{1}{4} = \frac{3}{8} - \frac{1}{4} = \frac{1}{8}$$

Q.5 A class contains 20 boys and 20 girls of which half the boys and half the girls have cat eyes. If one student is selected from the class, the probability that either the student is a boy or has cat eyes is

(A) $\frac{1}{2}$ (B) $\frac{3}{4}$ (C) $\frac{3}{8}$ (D) $\frac{2}{3}$

Sol. Let A be the event of a boy and B the event of having cat eyes. So

$$P(A) = \frac{20}{40} = \frac{1}{2} \quad \text{and} \quad P(B) = \frac{20}{40} = \frac{1}{2}$$

Now $P(A \cap B) = \frac{10}{40} = \frac{1}{4}$

Therefore $P(A \cup B) = P(A) + P(B) - P(A \cup B)$

$$= \frac{1}{2} + \frac{1}{2} - \frac{1}{4} = \frac{3}{4}$$

Q.6 An urn contains 10 black and 5 white balls. Two balls are drwn from the urn one after the other without replacement. What is the probability that both drawn balls are black?

Sol. Let E and F denote respectively the events that first and second ball drawn are black. We have to find $P(E \cap F)$ or $P(EF)$.

Now $$P(E) = P(\text{black ball in first draw}) = \frac{10}{15}$$

Also given that the first ball drawn is black, i.e., event E has occurred, now there are 9 black balls and five white balls left in the urn. Therefore, the probability that the second ball drawn is black, given that the ball in the first draw is black, is nothing but the conditional probability of F given that E has occurred.

i.e. $$P(F|E) = \frac{9}{14}$$

By multiplication cube fo probability, we have

$$P(E \cap F) = P(E) \cdot P\left(\frac{F}{E}\right) = \frac{10}{15} \cdot \frac{9}{14} = \frac{3}{7}$$

Q.7 Three cards are drawn successively, without replacement from a pack of 52 well shuffled cards. What is the probability that first two cards are kings and the third card drawn is an ace?

Sol. Let K denote the event that the card drawn is king and A be the event that the card drawn is an ace. Clearly, we have to find P(KKA)

Now $$P(K) = \frac{4}{52}$$

Also, $P(K|K)$ is the probability of second king with the condition that one king has already been drawn. Now there are three kings in $(52 - 1) = 51$ cards.

Therefore $$P(K|K) = \frac{3}{51}$$

Lastly, $P(A|KK)$ is the probability of third drawn card to be an ace, with the condition that two kings have already been drawn. Now there are four aces in left 50 cards.

Therefore $$P(A|KK) = \frac{4}{50}$$

By multiplication law of probability, we have

$$P(KKA) = P(K) \quad P(K|K) \quad P(A|KK)$$

$$= \frac{4}{52} \times \frac{3}{51} \times \frac{4}{50} = \frac{2}{5525}$$

Q.8 The probability that a candidate securing admission in IIT through entrance test is $\frac{1}{10}$. Seven candidates are selected at random from a centre. The probability that two will get admission in IIT through entrance test is

(A) $20(0.1)^2(0.9)^5$ (B) $15(0.1)^2(0.9)^5$ (C) $21(0.1)^2(0.9)^5$ (D) $2(0.1)^2(0.9)^5$

Sol. Let $p = \text{Probability of success} = \frac{1}{10} = 0.1$

$q = \text{Probability of failure} = 1 - 0.1 = 0.9$

Therefore $P(X = 2) = \text{Probability of 2 success and 5 failures}$

$$= {}^7C_2\,(0.1)^2\,(0.9)^5$$

$$= 21\,(0.1)^2\,(0.9)^5$$

Q.9 A book writer writes a good block with probability $\frac{1}{2}$. If it is a good book, the probability that it will be published is $\frac{2}{3}$, otherwise it is $\frac{1}{4}$. If he writes 2 books, the probability that at least one book will be published is

(A) $\frac{407}{576}$ (B) $\frac{411}{576}$ (C) $\frac{405}{576}$ (D) $\frac{307}{576}$

Sol. Let G = Event of good book

G' = Event of not a good book

E = Event of publication

Then $E = (G \cup G') \cap E = (G \cap E) \cup (G' \cap E)$

Now $P\left(\frac{E}{G}\right) = \frac{2}{3}$, $P\left(\frac{E}{G'}\right) = \frac{1}{4}$, $P(G) = = P(G')$

Therefore $P(E) = \frac{1}{2} \times \frac{2}{3} + \frac{1}{2} \times \frac{1}{4} = \frac{11}{24}$

Further, X denotes the number of books published. Then P(at least one book will be published)

$$= P(X = 1) + P(X = 2) = {}^2C_1\left(\frac{11}{24}\right)\left(\frac{13}{24}\right) + {}^2C_2\left(\frac{11}{24}\right)^2\left(\frac{13}{24}\right)^0 = 2 \times \frac{11}{24} \times \frac{13}{24} + \left(\frac{11}{24}\right)^2 = \frac{407}{576}$$

Q.10 $P_1, P_2 ......, P_8$ are equally strong players (i.e., for each of them the probability of win or lose is 1/2) are participating in tennis singles tournament. If the probability of P_1 losing to eventual winner of the tournament is m/n then $n - m$ is equal to ______

Sol. Let X be the eventual winner. P_1 may lose to X in I, II and III rounds.

$P(P_1 \text{ to lose in I}) = P(P_1 \text{ pairing with X and losing}) = \frac{1}{4} \times \frac{1}{2}$

since $P(P_1 \text{ pairing with X}) = \frac{1}{4}$ as there are four pairs.

$P(P_1 \text{ to lose in II}) = P(P_1 \text{ wins in I})\ P(P_1 \text{ paring with X in II})\ P(P_1 \text{ losing})$

$$= \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}$$

$P(P_1 \text{ to lose in III}) = P(P_1 \text{ wining I and II})\ P(P_1 \text{ loosing}) = \left(\frac{1}{2} \times \frac{1}{2}\right) \times \frac{1}{2} = \frac{1}{8}$

Therefore probability of P_1 losing to X is $\frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{3}{8}$

Hence $n - m = 8 - 3 = 5$

Q.11 The odds against an event A is 2 : 3 and odds in favour of another event B is 1 : 2. If A and B are independent and $P(A \cup B) = \frac{m}{n}$, then $|m - n|$ is _______ . Here m and n do not have proper common divisor.

Sol. We have

$$P(A \cup B) = P(A) + P(B) - (A \cap B) = \frac{3}{5} + \frac{1}{3} - \frac{3}{5} \times \frac{1}{3} = \frac{9 + 5 - 3}{15} = \frac{11}{15}$$

Therefore $m = 11$ and $n = 15$. So $|m - n| = 4$.

Q.12 A natural number is selected at random from the first 100 natural numbers. Let A, B and C denote the events of selection of even number, a multiple of 3 and a multiple of 5, respectively. Then

(A) $P(A \cap B) = \frac{4}{25}$ (B) $P(B \cap C) = \frac{3}{50}$

(C) $P(C \cap A) = \frac{1}{10}$ (D) $P(A \cup B \cup C) = \frac{37}{50}$

Sol. We have

Number of even numbers ≤ 100 is equal to 50.

Number of multiples of $3 \le 100$ is 33.

Number of multiples of $5 \le 100$ is 20.

Number of common multiples of 2 and 3 is 16.

Number of common multiples of 3 and 5 is 16.

Number of common multiples of 2 and 5 is 10.

Number of common multiples of 2, 3 and 5 is 3.

Now $P(A) = \frac{50}{100}$, $P(B) = \frac{33}{100}$, $P(C) = \frac{20}{100}$

$$P(A \cap B) = \frac{16}{100},\ P(B \cap C) = \frac{6}{100},\ P(C \cap A) = \frac{10}{100}$$

Also

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C)$$

$$= \frac{50}{100} + \frac{33}{100} + \frac{20}{100} - \frac{16}{100} - \frac{6}{100} - \frac{10}{100} + \frac{3}{100} = \frac{106-32}{100} = \frac{74}{100} = \frac{37}{50}$$

Hence all (A), (B), (C) and (D) are correct.

Q.13 Boxes B_1, B_2, B_3 contain different coloured balls as given in table. The prbabilities of selecting boxes are, respectively, $\frac{1}{6}$, $\frac{1}{2}$ and $\frac{1}{3}$. One of the boxes is chosen at random and a ball is drawn from it. If the probability of the drawn ball is black is $\frac{23}{90}$ then the value of n is equal to ______.

Table : Integer answer type question 5(n is a positive integer)

	White	Black	Red
B_1	2	n	2
B_2	3	2	4
B_3	4	3	2

Sol. Let B be denote the event of drawing a black ball. Then

$$B = (B_1 \cup B_2 \cup B_3) \cap B = (B_1 \cap B) \cup (B_2 \cap B) \cup (B_3 \cap B)$$

Therefore $$P(B_1) = P(B_1)\left(\frac{B}{B_1}\right) + P(B_2)\,P\left(\frac{B}{B_2}\right) + P(B_3)\,P\left(\frac{B}{B_3}\right)$$

$$= \frac{1}{6} \times \frac{n}{n+4} + \frac{1}{2} \times \frac{2}{9} + \frac{1}{3} \times \frac{3}{9} = \frac{n}{6(n+4)} + \frac{2}{9} = \frac{3n+4(n+4)}{18(n+4)}$$

By hypothesis $P(B) = \frac{23}{90}$

Therefore $\frac{3n + 4(n+4)}{18(n+4)} = \frac{23}{90}$

$35n + 80 = 23\,(n + 4)$

$12n = 12$

$n = 1$ **Ans.**

Q.14 A and B are two independent events whose probabilities are, respectively, $\frac{1}{n}$ and $\frac{1}{(n+1)}$. If the probability of $A \cap B$ is $\frac{1}{12}$, then n equals _____ .

Sol. A and B are independent events. This implies $P(A \cap B) = P(A)\,P(B)$

Therefore $\frac{1}{12} = P(A)\,P(B) = \frac{1}{n(n+1)}$ which gives $n = 3$ **Ans.**

Q.15 A number x is selected from the set of first 9 natural numbers (i.e., $x = 1, 2, 3, \ldots\ldots, 9$). If the probability that $f(f(x)) = x$ where $f(x) = x^2 - 3x + 3$ is $\frac{m}{9}$, then m is equal to _____

Sol. Clearly all the solutions of $f(x) = x$ are also solutions of $f(f(x)) = x$. First, we solve $f(x) = x$

$f(x) = 0 \Rightarrow x^2 - 3x + 3 = x \Rightarrow x^2 - 4x + 3 = 0 \Rightarrow (x-1)(x-3) = 0 \Rightarrow x = 1, 3$

Therefore $x = 1, 3$ are also solutions of $f(f(x)) = x$. We want to seek if there are any more solutions of $f(f(x)) = x$ other than 1 and 3.

$$f(f(x)) = x \Rightarrow f(x^2 - 3x + 3) = x \Rightarrow (x^2 - 3x + 3)^2 - 3(x^2 - 3x + 3) + 3 = 0$$

$$\Rightarrow x^4 - 6x^3 + 12x^2 - 9x + 3 = 0 \Rightarrow (x^2 - 4x + 3)(x^2 - 2x + 1) = 0$$

$$\Rightarrow (x-1)(x-3)(x-1)^2 = y \Rightarrow x = 1, 3$$

In this case we have no additional solutions. Therefore the probability that x satisfies equation $f(f(x)) = x$ is $\frac{2}{9}$. Therefore $m = 2$. Ans.

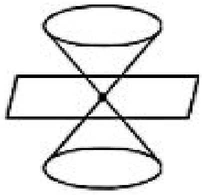
CONIC SECTION

Point, pair of straight lines, circle, parabola, ellipse and hyperbola are called conic section because they can be obtained when a cone (or double cone) is cut by a plane.

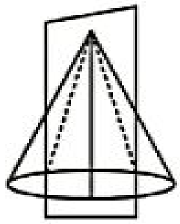
The mathematicians associated with the study of conics were Euclid, Aristarchus and Apollonius. Most of the objects around us and in space have shape of conic-sections. Hence study of these becomes a very important tool for present knowledge and further exploration.

Section of right circular cone by different planes :

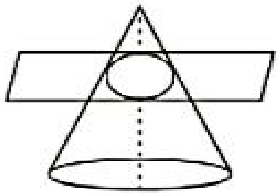
(1) When a double right circular cone is cut by a plane parallel to base at the common vertex, the cutting profile is a point.



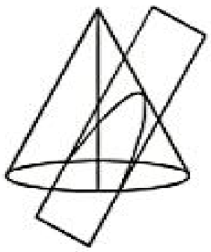
(2) When a right circular cone is cut by any plane through its vertex, the cutting profile is a pair of straight lines through its vertex



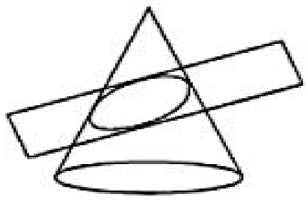
(3) When a right circular cone is cut by a plane parallel to its base the cutting profile is a circle.



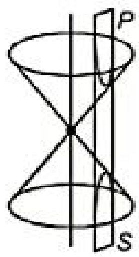
(4) When a right circular cone is cut by a plane parallel to a generator of cone, the cutting profile is a parabola.



(5) When a right circular cone is cut by a plane which is neither parallel to any generator / axis nor parallel to base, the cutting profile is an ellipse.



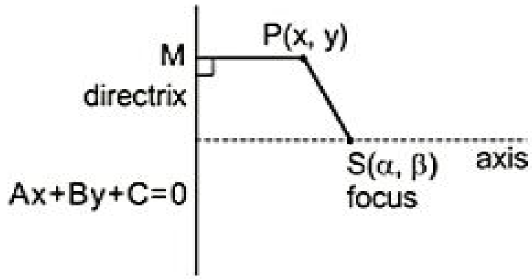
(6) When a double right circular cone is cut by plane, parallel to its common axis, the cut profile is hyperbola



Hence a point, a pair of intersecting straight lines, circle, parabola, ellipse and hyperbola, all are conic-

The conic section is the locus of a point which moves such that the ratio of its distance from a fixed point (focus) to perpendicular distance from a fixed straight line (directrix) is always constant (e). Here e is called eccentricity of conic i.e.,

$$\frac{PS}{PM} = e$$



A line through focus and perpendicular to directrix is called - axis. The vertex of conic is that point where the curve intersects its axis.

$$\frac{PS}{PM} = e \Rightarrow PS^2 = e^2PM^2$$

$$\Rightarrow \ (x-\alpha)^2 + (y-\beta)^2 = e^2\left(\frac{Ax+By+C}{\sqrt{A^2+B^2}}\right)^2$$

Simplification shall lead to the equation of the form $ax^2 + by^2 + 2hxy + 2gx + 2fy + c = 0$

Distinguishing various conics :

The nature of the conic section depends upon the position of the focus S w.r.t. the directrix & also upon the value of the eccentricity e. Two different cases arise.

Case-I : When The Focus Lies On The Directrix (De-generated conic) :

In this case $\Delta \equiv abc + 2fgh - af^2 - bg^2 - ch^2 = 0$ & the general equation of a conic represents a pair of straight lines if

$e > 1$ i.e. $h^2 > ab$ the lines will be real & distinct, intersecting at S.
$e = 1$ i.e. $h^2 = ab$ the lines will coincident.
$e < 1$ i.e. $h^2 < ab$ the lines will be imaginary.

Case-II : When The Focus Does Not Lie on the Directrix (Non de-generated conic) :

In this case $\Delta \equiv abc + 2fgh - af^2 - bg^2 - ch^2 \neq 0$ and conic represent

a parabola	an ellipse	a hyperbola	rectangular hyperbola	Circle
$e = 1$ $h^2 = ab$	$0 < e < 1$ $h^2 < ab$	$e > 1$ $h^2 > ab$	$e = \sqrt{2}$ $h^2 > ab;\ a + b = 0$	$e = 0$ $h = 0,\ a = b$

Note :

(i) For pair of straight lines $e \to \infty$

(ii) All second degree terms in parabola form a perfect square

Definition of various terms related to a conic :

(1) **Focus :** The fixed point is called a focus of the conic.

(2) **Directrix :** The fixed line is called a directrix of the conic.

(3) **Axis :** The line passing through the focus and perpendicular to the directrix is called the axis of the conic.

(4) **Vertex :** The points of intersection of the conic and the axis are called vertices of the conic.

(5) **Centre :** The point which bisects every chord of the conic passing through it, is called the centre of the conic.

(6) **Latus-rectum :** The latus-rectum of a conic is the chord passing through the focus and perpendicular to the axis.

(7) **Double ordinate :** A chord which is perpendicular to the axis of parabola or parallel to its directrix.

Illustration :

What conic does $13x^2 - 18xy + 37y^2 + 2x + 14y - 2 = 0$ represent ?

Sol. *Compare the given equation with*

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

$\therefore$ $a = 13, h = -9, b = 37, g = 1, f = 7, c = -2$

then $\Delta = abc + 2fgh - af^2 - bg^2 - ch^2$

$= (13)(37)(-2) + 2(7)(1)(-9) - 13(7)^2 - 37(1)^2 + 2(-9)^2$

$= -962 - 126 - 637 - 37 + 162$

$= -1600 \neq 0$

and also $h^2 = (-9)^2 = 81$ *and* $ab = 13 \times 37 = 481$

Here $h^2 - ab < 0$

So we have $h^2 - ab < 0$ *and* $\Delta \neq 0$. *Hence the given equation represents an ellipse.*

Illustration :

For what value of λ the equation of conic $2xy + 4x - 6y + \lambda = 0$ represents two real intersecting straight lines? if $\lambda = 17$ then this equation represents ?

Sol. *Comparing the given equation of conic with*

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

$\therefore$ $a = 0, b = 0, h = 1, g = 2, f = -3, c = \lambda$

For two intersecting real lines

$$h^2 - ab \geq 0 \text{ and } \Delta = 0$$

here $\Delta = abc + 2fgh - af^2 - bg^2 - ch^2$

$= 0 + 2 \times (-3) \times 2 \times 1 - 0 - 0 - \lambda (1)^2$

$= -12 - \lambda = 0$

$\therefore$ $\lambda = -12$

and $h^2 - ab = 1$

hence for $\lambda = -12$ above equation always represent real intersecting lines.

if $\lambda = 17$ *then* $\Delta \neq 0$ *and* $h^2 - ab > 0$

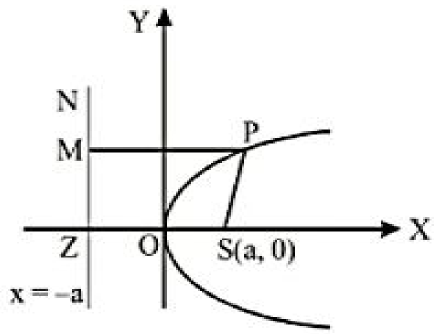
so we have $\Delta \neq 0$ and $h^2 - ab > 0$. Hence the given equation represents a Hyperbola.

PARABOLA

A parabola is the locus of a point which moves in a plane such that its distance from a fixed point (focus) is equal to its perpendicular distance from a fixed straight line (directrix). Eccentricity of parabola is 1.

Standard Equation of a parabola :

Let S be the focus and ZN is the directrix of the parabola.
From S, draw SZ perpendicular to the directrix.
Let O be the middle point of ZS. Take O as the origin and OS as x-axis and OY perpendicular to OS as the y-axis.
Let ZS = 2a, then ZO = OS = a
Now, $S \equiv (a, 0)$ and the equation of ZN is $x = -a$ or $x + a = 0$.
Let P(x, y) be any point on the parabola.
$\therefore$ PS = PM (by definition of parabola).



$$\Rightarrow \quad \sqrt{(x-a)^2+(y-0)^2} = \frac{|x+a|}{\sqrt{1^2+0}}$$

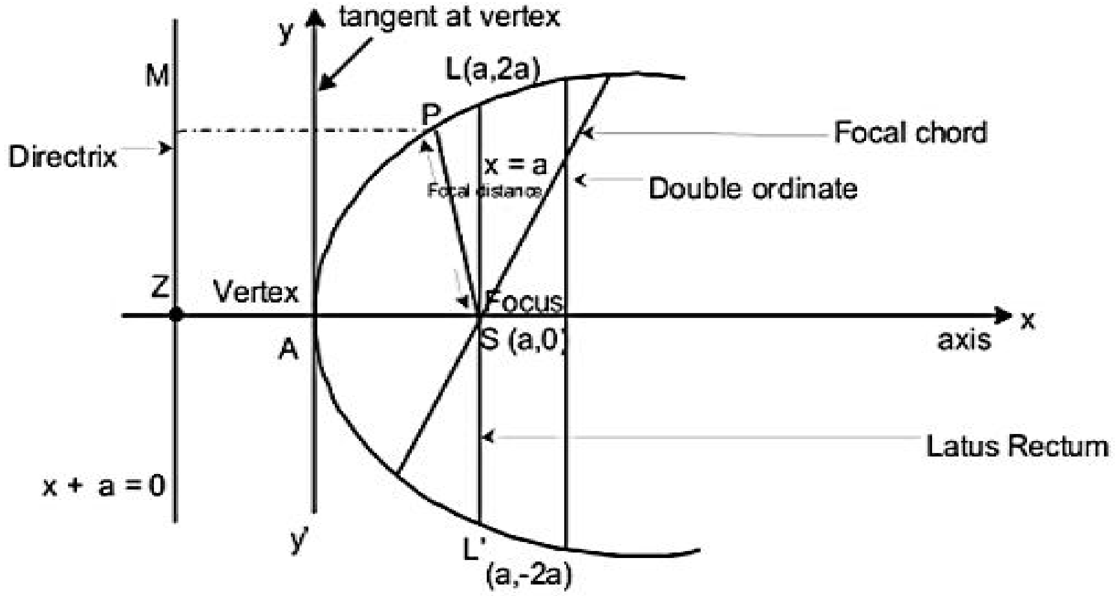
$$\Rightarrow \quad \sqrt{(x-a)^2+y^2} = |x+a|$$

or $\quad (x-a)^2 + y^2 = (x+a)^2$

or $\quad x^2 - 2ax + a^2 + y^2 = x^2 + 2xa + a^2$

or $\quad y^2 = 4ax \quad$ which is the required equation.

Terms related to Parabola :



(1) **Axis :** A straight line passes through the focus and perpendicular to the directrix is called the axis of parabola. For the parabola $y^2 = 4ax$, x-axis is the axis.
Since equation has even power of y therefore the parabola is symmetric about x-axis i.e. about its axis.

(2) **Vertex :** The point of intersection of a parabola and its axis is called the vertex of the Parabola. For the parabola $y^2 = 4ax$, O(0, 0) is the vertex.
The vertex is the middle point of the focus and the point of intersection of axis and directrix.

(3) **Focal Distance :** The distance of any point P (x, y) on the parabola from the focus is called the focal length (distance) of point P.
The focal distance of P = the perpendicular distance of the point P from the directrix.

(4) **Double Ordinate :** The chord which is perpendicular to the axis of Parabola or parallel to Directrix is called double ordinate of the Parabola.

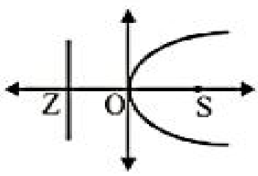
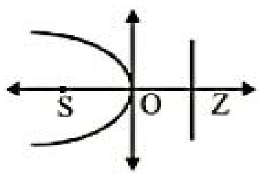
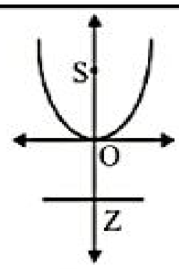
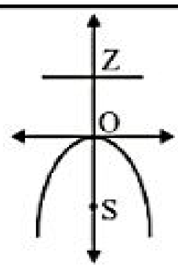
(5) **Focal Chord :** Any chord of the parabola passing through the focus is called Focal chord.

(6) **Latus Rectum :** If a double ordinate passes through the focus of parabola then it is called as latus rectum. The extremities of the latus rectum are L (a, 2a) and L'(a, –2a). Since LS = L'S = 2a, therefore length of the latus rectum LL' = 4a.

(7) **Parametric Equation of Parabola :** The parametric equation of Parabola $y^2 = 4ax$ are $x = at^2$, $y = 2at$. Hence any point on this parabola is $(at^2, 2\,at)$ which is also called as 't' point.

Note:
(i) The length of the latus rectum = 2 × perpendicular distance of focus from the directrix.
(ii) If $y^2 = lx$ then length of the latus rectum = l.
(iii) Two parabolas are said to be equal if they have same latus rectum.
(iv) The ends of a double ordinate of a parabola can be taken as $(at^2, 2\,at)$ and $(at^2, -2at)$.
(v) Parabola has no centre, but circle, ellipse, hyperbola have centre.

Other Standard parabola :

Equation of parabola	$y^2 = 4ax$	$y^2 = -4ax$	$x^2 = 4ay$	$x^2 = -4ay$
(a) Graphs				
(b) Eccentricity	e = 1	e = 1	e = 1	e = 1
(c) Focus	S(a, 0)	S(–a, 0)	S(0, a)	S(0, –a)
(d) Equation of directrix	$x + a = 0$	$x - a = 0$	$y + a = 0$	$y - a = 0$
(e) Equation of axis	$y = 0$	$y = 0$	$x = 0$	$x = 0$
(f) Vertex	O(0, 0)	O(0, 0)	O(0, 0)	O(0, 0)
(g) Extremities of latusrectum	(a, ±2a)	(–a, ±2a)	(±2a, a)	(±2a, –a)
(h) Length of latusrectum	4a	4a	4a	4a
(i) Equation of tangent at vertex	$x = 0$	$x = 0$	$y = 0$	$y = 0$
(j) Parametric coordinates of any point on parabola	$P(at^2, 2at)$	$P(-at^2, 2at)$	$P(2at, at^2)$	$P(2at, -at^2)$

REDUCTION TO GENERALIZED EQUATION OF PARABOLA :

If the equation of a parabola is either in the form $x = \ell y^2 + my + n$ or $y = \ell x^2 + mx + n$ then it can be reduced into generalised form. For this we change the given equation into the following forms-
$(y - k)^2 = 4a(x - h)$ or $(x - h)^2 = 4a(y - k)$

And then we compare from the standard equation of parabola to find all its parameters.

(A) When the equation of parabola is :

$$(y - k)^2 = 4a(x - h) \qquad \ldots(i)$$

Equation (i) is of the form $Y^2 = 4aX$
where $Y = y - k$ and $X = x - h$

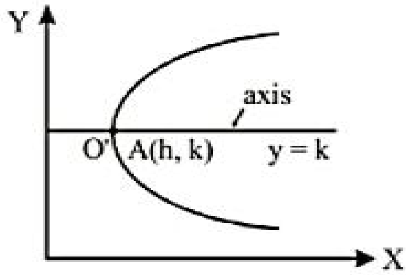
(1) Axis of parabola is $Y = 0$, i.e., $y - k = 0 \Rightarrow y = k$

(2) Coordinates of vertex of parabola are given by

$X = 0$ and $Y = 0$

i.e. $x - h = 0$ and $y - k = 0$

$\therefore$ Vertex is (h, k)



(3) Tangent at the vertex to parabola (i) is given by
$X = 0$, i.e., $x - h = 0$
Therefore, tangent at the vertex is $x = h$.

(4) Coordinates of focus of parabola are given by

$X = a$ and $Y = 0$

i.e. by $x - h = a$ and $y - k = 0$

$\therefore$ Focus of parabola is $(a + h, k)$.

(5) Equation of directrix of parabola is

$X = -a$

i.e., $x - h = -a$

Therefore, directrix of parabola is $x = h - a$

(6) Length of latus rectum of parabola is $|4a|$.

(7) Coordinates of ends of latus rectum of parabola are given by

$X = a$ & $Y = \pm 2a$

i.e., by $x - h = a$, $y - k = \pm 2a$ i.e. coordinate of latus rectum is $(a + h, k \pm 2a)$.

(8) Parametric equation is $x = h + at^2$ and $y = k + 2at$.

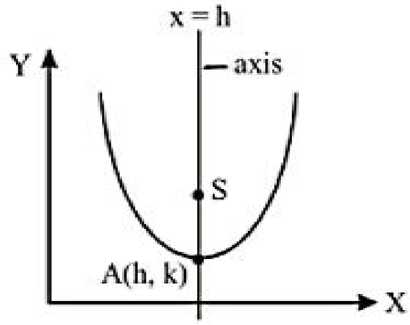
(B) When the equation of parabola is :

$$(x-h)^2 = 4a(y-k) \qquad \ldots(i)$$

Equation (i) is of the form $X^2 = 4aY$

where $X = x - h$ and $Y = y - k$

(1) Axis of parabola is $X = 0$, i.e., $x - h = 0$



(2) Coordinates of vertex of parabola is given by

$X = 0$ and $Y = 0$

i.e., by $x - h = 0$ and $y - k = 0$

$\therefore$ $x = h$ and $y = k$

Hence vertex of parabola is (h, k)

(3) Equation of tangent at the vertex to parabola is

$Y = 0$ i.e., $y - k = 0$

or $y = k$

(4) Coordinates of focus of parabola are given by

$X = 0$ and $Y = a$

i.e., by $x - h = 0$ and $y - k = a$

$\therefore$ Focus of parabola is $(h, k + a)$.

(5) Equation of directrix of parabola (i) is given by

$Y = -a$ or $y - k = -a$ or $y = k - a$

(6) Length of latus rectum of parabola is $|4a|$.

(7) Coordinates of ends of latus rectum of parabola are given by

$Y = a,\quad X = \pm 2a$

i.e., $y - k = a,\quad X - h = \pm 2a$

$\therefore$ Ends of latus rectum are $(h \pm 2a, k + a)$

(8) Parametric equation is $x = h + 2at$ and $y = k + at^2$.

Equation of Parabola	**Vertex**	**Axis**	**Focus**	**Directrix**	**Parametric equation**
$(y-k)^2 = 4a(x-h)$	(h,k)	$y = k$	$(h + a, k)$	$x + a - h = 0$	$x = h + at^2, y = k + 2at$
$(x-h)^2 = 4a(y-k)$	(h, k)	$x = h$	$(h, a + k)$	$y + a - k = 0$	$x = h + 2at, y = k + at^2$

Note :

(i) For the parabola $y = Ax^2 + Bx + C$, the length of latus rectum is $\dfrac{1}{|A|}$ and axis is parallel to y-axis. If A is positive then it is concave up parbaola, if A is negative then it is concave down parabola.

(ii) For the parabola $x = Ay^2 + By + C$, the length of latus rectum is $\dfrac{1}{|A|}$ and axis is parallel to x-axis. If A is positive then it is opening right and if A is negative then it is opening left parabola.

Illustration :

Find the equation of the parabola whose focus is at (– 1, – 2) and the directrix is the line $x - 2y + 3 = 0$.

Sol. Let $P(x, y)$ be any point on the parabola whose focus is $S(-1, -2)$ and the directrix $x - 2y + 3 = 0$. Draw PM perpendicular to directrix $x - 2y + 3 = 0$. Then by definition,

$$SP = PM$$

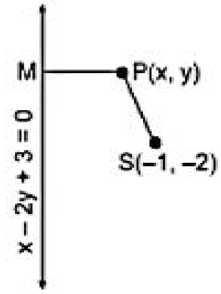
$$\Rightarrow \quad SP^2 = PM^2$$

$$\Rightarrow \quad (x+1)^2 + (y+2)^2 = \left(\frac{x-2y+3}{\sqrt{1+4}}\right)^2$$

$$\Rightarrow \quad 5\,[(x+1)^2 + (y+2)^2] = (x - 2y + 3)^2$$

$$\Rightarrow \quad 5(x^2 + y^2 + 2x + 4y + 5) = (x^2 + 4y^2 + 9 - 4xy + 6x - 12y)$$

$$\Rightarrow \quad 4x^2 + y^2 + 4xy + 4x + 32y + 16 = 0$$



This is the equation of the required parabola.

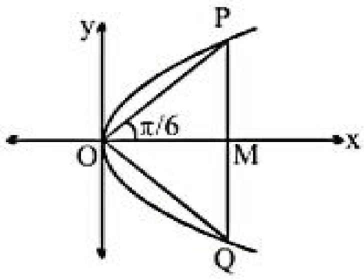
Illustration :

An equilateral triangle is inscribed in the parabola $y^2 = 4ax$, such that one vertex of this triangle coincides with the vertex of the parabola. Then find the side length of this triangle.

Sol. If ΔOPQ is an equilateral triangle then $OP = OQ = PQ = k$

$$\Rightarrow \quad \angle POM = \frac{\pi}{6}$$

$$\therefore \quad OM = k\cos\frac{\pi}{6} \text{ and } PM = k\sin\frac{\pi}{6}$$



$$\therefore \quad P\left(\frac{\sqrt{3}}{2}k, \frac{k}{2}\right) \text{ lie on the parabola } y^2 = 4ax$$

$$\Rightarrow \quad \left(\frac{k}{2}\right)^2 = 4a \cdot \frac{\sqrt{3}}{2}k \Rightarrow k = 8\sqrt{3}\,a. \qquad \textbf{Ans.}$$

Illustration :

The parametric equation of a parabola is $x = t^2 + 1,\ y = 2t + 1$. Then find the coordinate of vertex and length of latus -rectum.

Sol. Eliminate t from parametric equation, we get equation of parabola. Hence

$$x = \left(\frac{y-1}{2}\right)^2 + 1 \text{ or } (y-1)^2 = 4(x-1)$$

$\therefore$ vertex is $(1, 1)$ and length of latus rectum = 4.

Illustration :

Find the parametric equation of the parabola $(x-1)^2 = -12\,(y-2)$

Sol. $\because \quad 4a = -12 \quad \Rightarrow \quad a = -3,$
parametric equation is $y - 2 = -3t^2$
$x - 1 = -6t \quad \Rightarrow \quad x = 1 - 6t, \quad y = 2 - 3t^2$

Illustration :

The parametric equation of the curve $(y-2)^2 = 12\,(x-4)$ *are-*
(A) $6t,\ 3t^2$ (B) $2 + 3t,\ 4 + t^2$ (C) $4 + 3t^2, 2 + 6t$ (D) None of these

Sol. *Here* $a = 3$
$x - 4 = at^2 \Rightarrow x = 4 + 3.t^2 = 4 + 3t^2$ ***Ans. [C]***
$y - 2 = 2at \Rightarrow y = 2 + 2.3t = 2 + 6t$

Illustration :

Find the equation of parabola, whose axis is parallel to y-axis and which passes through points $(0, 2), (-1, 0)$ *and* $(1, 6)$.

Sol. *General equation of such parabola is* $y = Ax^2 + Bx + C$ .....(i)
Since it passes throught $(0, 2)\ (-1, 0)$ *and* $(1, 6)$, *then we have*
$C = 2$ ....(ii)
$A - B + C = 0$ ......(iii)
$A + B + C = 6$ ......(iv)
from (ii), (iii) and (iv),
$A = 1, B = 3, C = 2$
$\therefore$ *equation of parabola is* $y = x^2 + 3x + 2$. ***Ans.***

Illustration :

Find the value of λ *if the equation* $\sqrt{(x-1)^2+(y-2)^2} = \sqrt{\lambda}\ (x + y + 3)$ *represent parabola. Find its focus, directrix, axis and equation of vertex.*

Sol. $(x-1)^2 + (y-2)^2 = \lambda\,(x + y + 3)^2$ *are given equation*

$$(x-1)^2 + (y-2)^2 = 2\lambda\left(\frac{x+y+3}{\sqrt{2}}\right)^2$$

$\Rightarrow$ *above equation represents parabola if* $2\lambda = 1 \Rightarrow \lambda = \frac{1}{2}$
In this case focus is $(1, 2)$ *and directrix is* $x + y + 3 = 0$
Line perpendicular to directrix is $x - y + k = 0$ *and passes through* $(1, 2)$ *gives axis of parabola*
$\Rightarrow \quad 1 - 2 + k = 0 \Rightarrow k = 1$
$\therefore$ *axis is* $x - y + 1 = 0$
Now axis and directrix meet at $(-2, -1)$ *(called foot of directrix)*
Thus vertex is the mid point of foot of directix and the focus.

i.e. *vertex* $\left(\frac{1-2}{2}, \frac{2-1}{2}\right)$ *i.e.,* $\left(-\frac{1}{2}, \frac{1}{2}\right)$. ***Ans.***

Illustration :

If the length of a focal chord of the parabola $y^2 = 4ax$ at a distance b from the vertex is c, then show that $b^2c = 4a^3$.

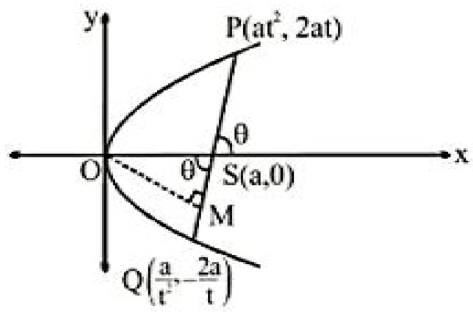
Sol. $OM = b$

focal length $PQ = 4a\ \text{cosec}^2\theta$ ....(i)

From right angled triangle OMS,

$$\sin\theta = \frac{OM}{OS} = \frac{b}{a}$$

$\therefore$ $\text{cosec}\ \theta = \dfrac{a}{b}$ ....(ii)



from (i) and (ii)

$$PQ = c = 4a \cdot \left(\frac{a}{b}\right)^2$$

$\Rightarrow$ $b^2c = 4a^3$. ***Ans.***

Illustration :

AP is perpendicular to PB, where A is the vertex of parabola $y^2 = 4x$ and P is on the parabola. If B lie on the axis of parabola, then find the locus of centroid of ΔPAB

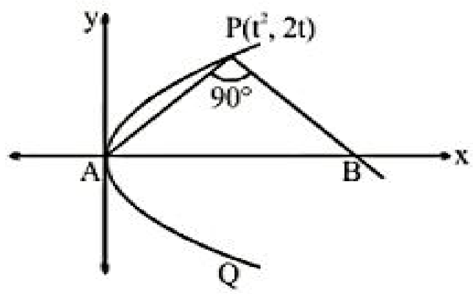
Sol. Slope of $AP = \dfrac{2}{t}$

$\therefore$ Slope of $BP = -\dfrac{t}{2}$

$\therefore$ equation of line PB is $y - 2t = -\dfrac{t}{2}(x - t^2)$

$\therefore$ point B is $(t^2 + 4, 0)$

Let centroid of ΔPAB is (h, k)



$\therefore$ $h = \dfrac{t^2 + t^2 + 4}{3}$ and $k = \dfrac{2t}{3}$

Now eliminate t from above two equation, we get

$$3h - 4 = 2\left(\frac{3k}{2}\right)^2$$

$\therefore$ locus of centroid is $3x - 4 = \dfrac{9y^2}{2}$. ***Ans.***

Illustration :

Find the equation of the parabola whose vertex is (–3, 0) and directrix is x + 5 = 0.

Sol. *A line passing through the vertex (–3, 0) and perpendicular to directrix x + 5 = 0 is x-axis which is the axis of the parabola by definition. Let focus of the parabola is (a, 0). Since vertex, is the middle point of Z(–5, 0) and focus S, therefore*

$$-3 = \frac{(a-5)}{2} \Rightarrow a = -1$$

$\therefore$ Focus = $(-1, 0)$

Thus the equation to the parabola is

$$(x+1)^2 + y^2 = (x+5)^2$$

$$\Rightarrow \quad y^2 = 8(x+3)$$

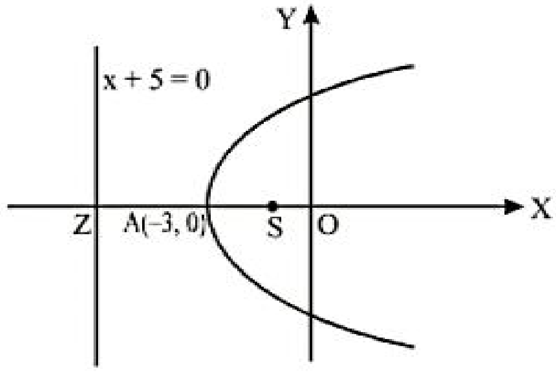


Illustration :

Find the equation of directrix and axis of the parabola $4y^2 - 6x - 4y = 5$.

Sol. Here $4y^2 - 4y = 6x + 5$

$$\Rightarrow \quad 4\left(y - \frac{1}{2}\right)^2 = 6(x+1) \quad \Rightarrow \quad \left(y - \frac{1}{2}\right)^2 = \frac{3}{2}(x+1)$$

Put $y - \frac{1}{2} = Y$ & $x + 1 = X$

The equation in standard form $Y^2 = \frac{3}{2}X$

$$4a = \frac{3}{2} \Rightarrow a = \frac{3}{8} \text{ \& length of latus rectum} = \frac{3}{2}$$

Directrix, $X + a = 0$

$$\Rightarrow \quad x + 1 + \frac{3}{8} = 0 \quad \Rightarrow \quad 8x + 11 = 0$$

Axis is $Y = 0 \Rightarrow y - \frac{1}{2} = 0 \quad \Rightarrow \quad 2y - 1 = 0.$

Parametric equation is $x + 1 = \frac{3}{8}t^2$ & $y - \frac{1}{2} = 2 \cdot \frac{3}{8} \cdot t$

i.e. $x = -1 + \frac{3}{8}t^2,\ y = \frac{1}{2} + \frac{3}{4}t.$

Illustration :

Find the vertex, axis, focus, directrix, latusrectum of the parabola, also draw their rough sketches.

$$4y^2 + 12x - 20y + 67 = 0$$

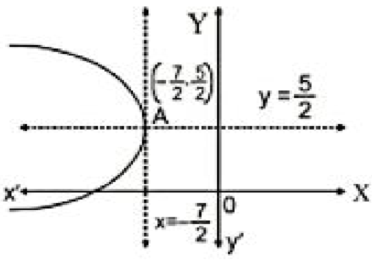
Sol. *The given equation is*

$$4y^2 + 12x - 20y + 67 = 0 \quad \Rightarrow \quad y^2 + 3x - 5y + \frac{67}{4} = 0$$

$$\Rightarrow \quad y^2 - 5y = -3x - \frac{67}{4} \quad \Rightarrow \quad y^2 - 5y + \left(\frac{5}{2}\right)^2 = -3x - \frac{67}{4} + \left(\frac{5}{2}\right)^2$$

$$\Rightarrow \quad \left(y - \frac{5}{2}\right)^2 = -3x - \frac{42}{4} \quad \Rightarrow \quad \left(y - \frac{5}{2}\right)^2 = -3\left(x + \frac{7}{2}\right) \qquad ....(i)$$

Let $X = x + \frac{7}{2}$ & $Y = y - \frac{5}{2}$ ....(ii)



Using these relations, equation (i) reduces to

$$Y^2 = -3X \qquad ....(iii)$$

This is of the form $Y^2 = -4aX$. *On comparing, we get* $4a = 3 \Rightarrow a = 3/4$.

The coordinates of the vertex are $(X = 0, Y = 0)$

i.e. the coordinates of the vertex are $\left(-\frac{7}{2}, \frac{5}{2}\right)$

The equation of the axis of the parabola is $Y = 0$.

i.e. the equation of the axis is $y = \frac{5}{2}$ *[Putting* $Y = 0$ *in (ii)]*

The coordinates of the focus are $(X = -a, Y = 0)$

$$\Rightarrow \quad x + \frac{7}{2} = -\frac{3}{4} \ \ \& \ \ y - \frac{5}{2} = 0$$

i.e. the coordinates of the focus are $(-17/4, 5/2)$

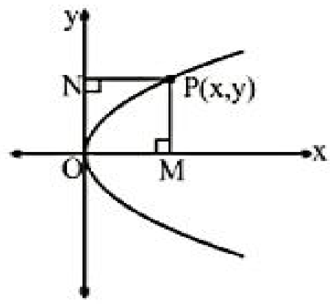
The equation of the directrix is $X = a$ *i.e.* $X = \frac{3}{4}$.

i.e. the equation of the directrix is $x = -\frac{11}{4}$

The length of the latusrectum of the given parabola is $4a = 3$.

Equation of parabola with respect to two perpendicular lines :

Let P(x, y) is any point on the parabola then equation of parabola $y^2 = 4ax$ is consider as

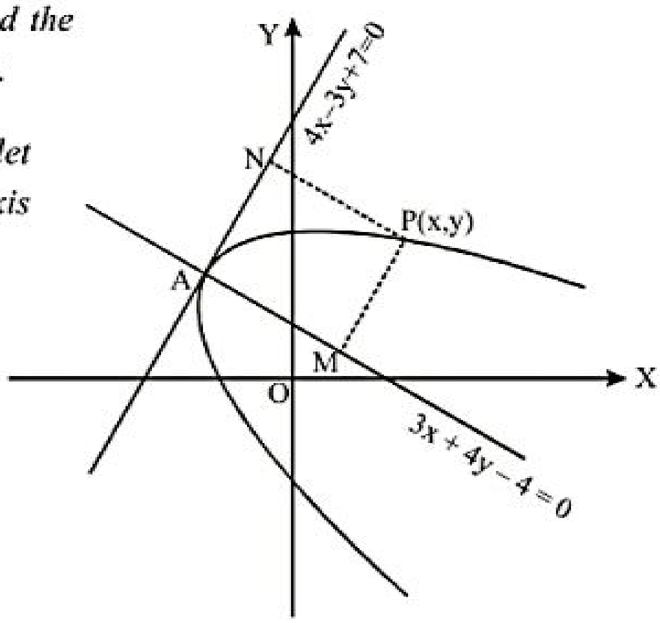


$$(PM)^2 = 4a(PN)$$

i.e. (The distance of P from its axis)^2 = (latus-rectum) × (The distance of P from the tangen at its vertex) where P is any point on the parabola.

Illustration :

Find the equation of the parabola whose latus rectum is 4 units, axis is the line $3x + 4y - 4 = 0$ and the tangent at the vertex is the line $4x - 3y + 7 = 0$.



Sol. *Let P (x, y) be any point on the parabola and let PM and PN are perpendiculars from P on the axis and tangent at the vertex respectively then*

$$(PM)^2 = (\text{latus re-ctum})\ (PN)$$

$$\Rightarrow \left(\frac{3x+4y-4}{\sqrt{3^2+4^2}}\right)^2 = 4\left(\frac{4x-3y+7}{\sqrt{4^2+(-3)^2}}\right)$$

$$\Rightarrow (3x + 4y - 4)^2 = 20\ (4x - 3y + 7)$$

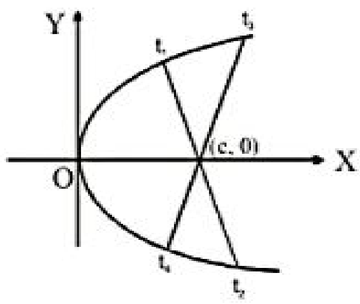
which is required parabola.

CHORD :

Line joining any two points on the parabola is called its chords.

Let the points $P(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_2)$, lie on the parabola then equation of chord is

$$(y - 2at_1) = \frac{2at_2 - 2at_1}{at_2^2 - at_1^2}\ (x - at_1^2)$$



$$\Rightarrow \quad y - 2at_1 = \frac{2}{t_1 + t_2}\ (x - at_1^2)$$

$$\Rightarrow \quad (t_1 + t_2)\ y = 2x + 2at_1 t_2$$

If this chord meet the x-axis at point (c, 0) then from above equation

$$c + at_1t_2 = 0 \text{ i.e. } t_1t_2 = -c/a.$$

Note:

(i) If the chord joining t_1, t_2 & t_3, t_4 pass through a point (c, 0) on the axis, then $t_1t_2 = t_3t_4 = -c/a$.

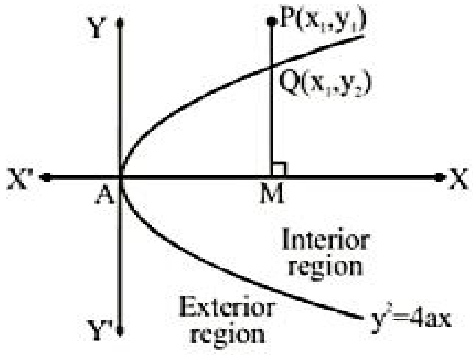
(ii) If PQ is a focal chord then $t_1t_2 = -1$ or $t_2 = -\frac{1}{t_1}$. which is required relation.

Hence if one extremity of a focal chord is $(at^2, 2at)$ then the other extremity will be $\left(\frac{a}{t^2}, -\frac{2a}{t}\right)$.

Position of a point with respect to a parabola $y^2 = 4ax$:

Let $P(x_1, y_1)$ be a point. From P draw PM $\perp$ AX (on the axis of parabola) meeting the parabola $y^2 = 4ax$ at Q (x_1, y_2) where Q (x_1, y_2) lie on the parabola therefore

$$y_2^2 = 4ax_1 \qquad ...(1)$$



Now, P will be outside, on or inside the parabola $y^2 = 4ax$ acoording as

PM >, = , or < QM

$\Rightarrow$ $(PM)^2 >, =, \text{ or } < (QM)^2$

$\Rightarrow$ $y_1^2 >, =, \text{ or } < y_2^2$

$\Rightarrow$ $y_1^2 >, =, \text{ or } < 4ax_1$ (from(1))

Hence $y_1^2 - 4ax_1 >, =, \text{ or } < 0$

Hence in short, equation of parabola $S(x, y) = y^2 - 4ax$.

(i) If $S(x_1, y_1) > 0$ then $P(x_1, y_1)$ lie outside the parabola.

(ii) If $S(x_1, y_1) < 0$ then $P(x_1, y_1)$ lie inside the parabola.

(iii) If $S(x_1, y_1) = 0$ then $P(x_1, y_1)$ lie on the parabola.

This result holds true for circle, parabola and ellipse.

Illustration :

Show that the point (2, 3) lies outside the parabola $y^2 = 2x$.

Sol. *Let* $S(x, y) \equiv y^2 - 2x$

$\therefore$ $S(2, 3) = 9 - 2.2 = 5$ = *positive*

$\Rightarrow$ *(2, 3) lie outside the parabola*

Illustration :

Find the position of the point (– 2, 2) with respect to the parabola $y^2 - 4y + 9x + 11 = 0$.

Sol. Let $S(x, y) = y^2 - 4y + 9x + 11$

$\therefore$ $S(-2, 2) = -11$ = negative

$\Rightarrow$ $(-2, 2)$ lie inside the parabola

Illustration :

If the point $(at^2, 2at)$ *be the extremity of a focal chord of parabola* $y^2 = 4ax$ *then show that the length of the focal chord is* $a\left(t + \frac{1}{t}\right)^2$.

Sol. Since one extremity of focal chord is $P(at^2, 2at)$ then the other extremity will be

$$Q\left(\frac{a}{t^2}, -\frac{2a}{t}\right) \qquad \text{(Replacing } t \text{ by } -1/t)$$

$\therefore$ Length of focal chord = PQ

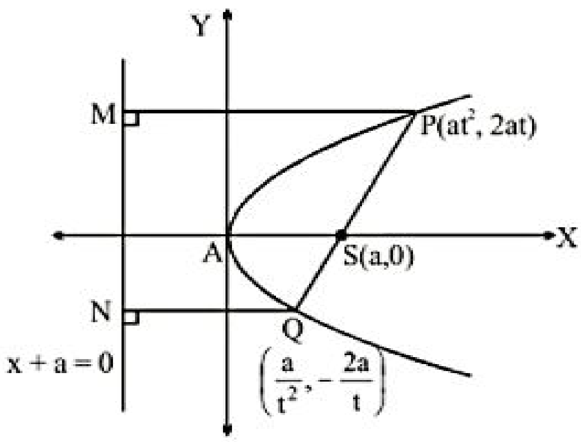
$= SP + SQ \qquad (\because SP = PM \text{ and } SQ = QN)$

$= PM + QN$

$$= at^2 + a + \frac{a}{t^2} + a$$

$$= \left(t^2 + \frac{1}{t^2} + 2\right)a$$

$$= a\left(t + \frac{1}{t}\right)^2.$$



Note :

(i) The length of focal chord having parameters t_1 and t_2 for its end points is $a(t_2 - t_1)^2$.

(ii) $\because$ $\left|t + \frac{1}{t}\right| \geq 2$ for all $t \neq 0 \qquad (\because AM \geq GM)$

$\therefore$ $a\left(t + \frac{1}{t}\right)^2 \geq 4a$

$\Rightarrow$ Length of focal chord $\geq$ latus rectum

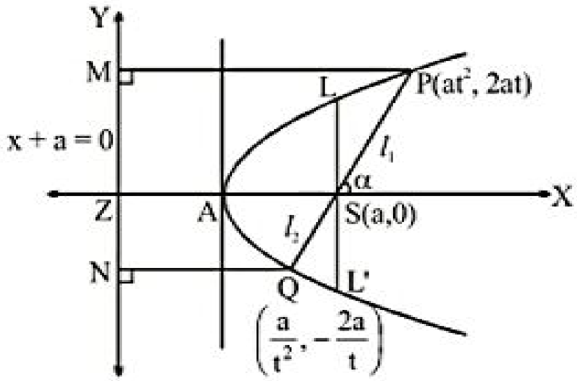
i.e., The length of smallest focal chord of the parabola is 4a, which is the latus rectum of a parabola.

Illustration :

Prove that the semi latus rectum of the parabola $y^2 = 4ax$ is the harmonic mean between the segments of any focal chord of the parabola.

Sol. *Let parabola be $y^2 = 4ax$*

If PQ be the focal chord then



$$P = (at^2, 2at) \text{ and } Q = \left(\frac{a}{t^2}, \frac{-2a}{t}\right)$$

If segment of focal chord are l_1 and l_2

then $l_1 = SP = PM = a + at^2 = a(1 + t^2)$

and $l_2 = SQ = QN = a + \dfrac{a}{t^2} = \dfrac{a(1+t^2)}{t^2}$

$\therefore$ *Harmonic mean of l_1 and l_2*

$$= \frac{2l_1 l_2}{l_1 + l_2} = \frac{1}{\frac{1}{l_2} + \frac{1}{l_1}} = \frac{2}{\frac{t^2}{a(1+t^2)} + \frac{1}{a(1+t^2)}} = \frac{2}{1/a} = 2a = \text{semi latus rectum.}$$

Note : *If l_1 and l_2 are the length of segments of a focal chord of a parabola, then its latus rectum is $\dfrac{4l_1 l_2}{l_1 + l_2}$.*

Illustration :

Show that the focal chord of parabola $y^2 = 4ax$ makes an angle α with the x-axis, then its length is equal to $4a\,\text{cosec}^2\alpha$.

Sol. *Let $P(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_2)$ be the end points of a focal chord PQ which makes an angle α with the axis of the parabola. Then*

$PQ = a(t_2 - t_1)^2$

$= a((t_2 + t_1)^2 - 4t_1t_2)$

$= a((t_2 + t_1)^2 + 4)$ $\quad (\because t_1t_2 = -1)$ ....(1)

$\therefore$ $\tan\alpha = \text{slope of } PQ = \dfrac{2at_2 - 2at_1}{at_2^2 - at_1^2} = \dfrac{2}{t_2 + t_1}$

$t_2 + t_1 = 2\cot\alpha$ ....(2)

Substituting the value of $t_2 + t_1$ from (2) in (1) then

$PQ = a(4\cot^2\alpha + 4) = 4a\,\text{cosec}^2\alpha.$

Practice Problem

Single correct question

Q.1 The vertex of parabola $y^2 + 6x - 2y + 13 = 0$ is

(A) $(-2, 1)$ (B) $(2, -1)$ (C) $(1, 1)$ (D) $(1, -1)$

Q.2 The value of p such that the vertex of $y = x^2 + 2px + 13$ is 4 units above the x-axis is

(A) ± 2 (B) 4 (C) ± 3 (D) 5

Q.3 The length of the latus rectum of the parabola whose focus is (3, 3) and directrix is $3x - 4y - 2 = 0$, is

(A) 1 (B) 2 (C) 4 (D) 8

Q.4 If the vertex and focus of a parabola are (3, 3) and (–3, 3) respectively, then its equation is

(A) $x^2 - 6x + 24y - 63 = 0$ (B) $x^2 - 6x + 24y + 81 = 0$

(C) $y^2 - 6x + 24x - 63 = 0$ (D) $y^2 - 6y - 24x + 81 = 0$

Q.5 The parabola having its focus at (3, 2) and directrix along the y-axis has its vertex at

(A) $\left(\frac{3}{2}, 1\right)$ (B) $\left(\frac{3}{2}, 2\right)$ (C) $\left(\frac{3}{2}, \frac{1}{2}\right)$ (D) $\left(\frac{3}{2}, \frac{-1}{2}\right)$

Q.6 The directrix of the parabola $x^2 - 4x - 8y + 12 = 0$ is

(A) $y = 0$ (B) $x = 1$ (C) $y = -1$ (D) $x = -1$

Q.7 Which one of the following equations represented parametrically, represents equation to a parabolic profile?

(A) $x = 3\cos t\,;\ y = 4\sin t$ (B) $x^2 - 2 = -2\cos t\,;\ y = 4\cos^2\frac{t}{2}$

(C) $\sqrt{x} = \tan t\,;\ \sqrt{y} = \sec t$ (D) $x = \sqrt{1 - \sin t}\ ;\ y = \sin\frac{t}{2} + \cos\frac{t}{2}$

Q.8 The locus of the point of trisection of all the double ordinates of the parabola $y^2 = lx$ is a parabola whose latus rectum is

(A) $\frac{l}{9}$ (B) $\frac{2l}{9}$ (C) $\frac{4l}{9}$ (D) $\frac{l}{36}$

Q.9 The straight line $y = m(x - a)$ will meet the parabola $y^2 = 4ax$ in two distinct real points if

(A) $m \in R$ (B) $m \in [-1, 1]$

(C) $m \in (-\infty, 1] \cup [1, \infty)R$ (D) $m \in R - \{0\}$

Q.10 The length of the intercept on $y-$axis cut off by the parabola, $y^2-5y=3x-6$ is
(A) 1 (B) 2 (C) 3 (D) 5

Q.11 If the line $x-1=0$ is the directrix of the parabola $y^2-kx+8=0$ then one of the values of 'k' is
(A) 1/8 (B) 8 (C) 4 (D) 1/4

Q.12 Length of the latus rectum of the parabola $25[(x-2)^2+(y-3)^2]=(3x-4y+7)^2$ is
(A) 4 (B) 2 (C) 1/5 (D) 2/5

Q.13 Maximum number of common chords of a parabola and a circle can be equal to
(A) 2 (B) 4 (C) 6 (D) 8

Multiple correct type question

Q.14 The locus of the mid point of the focal radii of a variable point moving on the parabola, $y^2=8x$ is a parabola whose
(A) Latus rectum is half the latus rectum of the original parabola
(B) Vertex is (1, 0)
(C) Directrix is y-axis
(D) Focus has the co-ordinates (2, 0)

Q.15 If from the vertex of a parabola $y^2=4x$ a pair of chords be drawn at right angles to one another and with these chords as adjacent sides a rectangle be made, then the locus of the further end of the rectangle is
(A) an equal parabola
(B) a parabola with focus at (9, 0)
(C) a parabola with directrix as $x-7=0$
(D) a parabola having tangent at its vertex $x=8$

Integer type question

Q.16 Find the vertex, focus, latus rectum, axis and the directrix of the parabola $x^2+8x+12y+4=0$.

Q.17 A parabola $y=ax^2+bx+c$ crosses the $x-$axis at $(\alpha, 0)$ $(\beta, 0)$ both to the right of the origin. A circle also passes through these two points. Then find length of a tangent from the origin to the circle.

Answer key

Q.1	A	Q.2	C	Q.3	B	Q.4	C	Q.5	B
Q.6	C	Q.7	B	Q.8	A	Q.9	D	Q.10	A
Q.11	C	Q.12	D	Q.13	C				
Q.14	A, B, C, D	Q.15	A, B, C, D						

Q.16 V(–4, 1) S(–4, –2), axis : $x+4=0$, directrix : $y-4=0$, LR = 12 Q.17 $\sqrt{\frac{c}{a}}$

INTERACTION BETWEEN THE LINE AND PARABOLA :

Let the parabola be $y^2 = 4ax$ ... (i)

and the given line be $y = mx + c$ ... (ii)

then line may cut, touch or does not meet parabola.

The points of intersection of the line (1) and the parabola (2) will be obtained by solving the two equations simultaneously. By solving equation (i) and (ii), we get

$$my^2 - 4ay + 4ac = 0$$

this equation has two roots and its nature will be decided by the discriminent $D = 16a(a - cm)$

Now, if $D > 0$ i.e., $c < \frac{a}{m}$, then line intersect the parabola at two distinct points.

If $D = 0$ i.e., $c = \frac{a}{m}$, then line touches the parabola. **(It is condition of tangency)**

If $D < 0$ i.e., $c > \frac{a}{m}$, then line neither touch nor intersect the parabola.

EQUATION OF TANGENT :

1. Point Form :

Equation of parabola is $y^2 = 4ax$ ....(1)

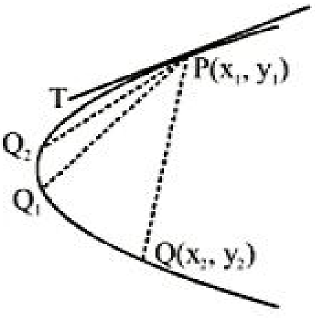
Let $P \equiv (x_1, y_1)$ and $Q = (x_2, y_2)$ be any two points on parabola (1), then

$$y_1^2 = 4ax_1 \quad ....(2)$$

and $$y_2^2 = 4ax_2 \quad ....(3)$$

Subtracting (2) from (3) then

$$y_2^2 - y_1^2 = 4a(x_2 - x_1)$$



or $$\frac{y_2 - y_1}{x_2 - x_1} = \frac{4a}{y_2 + y_1} \quad ....(4)$$

Equation of PQ is

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1) \quad ....(5)$$

From (4) and (5), then

$$y - y_1 = \frac{4a}{y_2 + y_1}(x - x_1) \quad ......(6)$$

Now for tangent at P, Q → P, i.e., $x_2 \to x_1$ and $y_2 \to y_1$ then equation (6) becomes

$$y - y_1 = \frac{4a}{2y_1}(x - x_1)$$

or $\quad yy_1 - y_1^2 = 2ax - 2ax_1$

or $\quad yy_1 = 2ax + y_1^2 - 2ax_1$

or $\quad yy_1 = 2ax + 4ax_1 - 2ax_1$ [From (2)]

or $\quad yy_1 = 2ax + 2ax_1$

which is the required equation of tangent at (x_1, y_1).

The equation of tangent at (x_1, y_1) can also be obtained by replacing x^2 by xx_1, y^2 by yy_1, x by $\frac{x + x_1}{2}$, y by $\frac{y+y_1}{2}$ and xy by $\frac{xy_1 + yx_1}{2}$ and without changing the constant (if any) in the equation of curve.

This method (standard substitution) is apply to all conic when point lie on the conic.

Equation of tangent of standard parabola :

Equation of Parabolas	Tangent at (x_1, y_1)
$y^2 = 4ax$	$yy_1 = 2a(x + x_1)$
$y^2 = -4ax$	$yy_1 = -2a(x + x_1)$
$x^2 = 4ay$	$xx_1 = 2a(y + y_1)$
$x^2 = -4ay$	$xx_1 = -2a(y + y_1)$

2. Slope Form :

The equation of tangent to the parabola $y^2 = 4ax$ at (x_1, y_1) is

$$yy_1 = 2a(x + x_1) \qquad \text{........(1)}$$

Since m is the slope of the tangent then

$$m = \frac{2a}{y_1} \quad \text{or} \quad y_1 = \frac{2a}{m}$$

Since (x_1, y_1) lies on $y^2 = 4ax$ therefore

$$y_1^2 = 4ax_1 \quad \text{or} \quad \frac{4a^2}{m^2} = 4ax_1 \quad \Rightarrow \quad x_1 = \frac{a}{m^2}.$$

Substituting the values of x_1 and y_1 in (1), we get

$$y = mx + \frac{a}{m} \qquad \text{........(2)}$$

Thus, $y = mx + \frac{a}{m}$ is a tangent to the parabola $y^2 = 4ax$ for all values of m, where m is the slope of the tangent and the co-ordinates of the point of contact is $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$.

Thus $y = mx + c$ is the tangent of $y^2 = 4ax$ for all values of m if only if $c = \frac{a}{m}$

and $(y - k) = m(x - h) + \frac{a}{m}$ is tangent to the parabola $(y - k)^2 = 4a(x - h)$.

The equation of tangent, condition of tangency and point of contact in terms of slope (m) of standard parabolas are shown below in the table.

Equation of parabolas	Point of contact in terms of slope (m)	Equation of tangent in terms of slope (m)	Condition of tangnecy
$y^2 = 4ax$	$\left(\frac{a}{m^2}, \frac{2a}{m}\right)$	$y = mx + \frac{a}{m}$	$c = \frac{a}{m}$
$x^2 = 4ay$	$(2am, am^2)$	$y = mx - am^2$	$c = -am^2$

Alternative method :

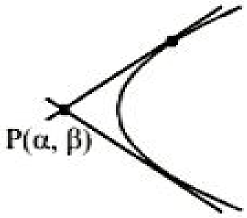
Let the parabola be $y^2 = 4ax$ ....(1)
and the given line by $y = mx + c$ ....(2)
Putting the value of y from (2) in (1), we get

$$m^2x^2 + 2x(mc - 2a) + c^2 = 0 \quad ....(3)$$

The line $y = mx + c$ is a tangent to parabola $y^2 = 4ax$ if the roots of equation (3) are equal. The condition for this is $4(mc - 2a)^2 - 4m^2c^2 = 0$ (Discriminant of the quadratic equation = 0)

or $-4mca + 4a^2 = 0$ or $c = \frac{a}{m}$, which is the required condition of tangency.

Two tangents can be drawn from a point P (α, β) to a parabola if P lies outside the parabola :



Let the parabola be $y^2 = 4ax$ ....(1)
Let P (α, β) be the given point

The equation of a tangent to parabola (1) is $y = mx + \frac{a}{m}$ ....(2)

If line (2) passes through P (α, β), then $\beta = m\alpha + \frac{a}{m}$ or $m^2\alpha - \beta m + a = 0$ ....(3)

There will be two tangents to parabola (1) from P (α, β) if roots of equation (3) are real and distinct i.e., $D > 0$ i.e. if $\beta^2 - 4\alpha a > 0 \Rightarrow$ P(α, β) lies outside parabola (1).

We can also find the angle between two tangents from point P(α, β) using the formula

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$$

3. Parametric Form :

We have to find the equation of tangent to the parabola $y^2 = 4ax$ at the point $(at^2, 2at)$ or 't'

Since the equation of tangent of the parabola $y^2 = 4ax$ at (x_1, y_1) is

$$yy_1 = 2a(x + x_1) \qquad \text{........(1)}$$

replacing x_1 by at^2 and y_1 by 2at, then (1) becomes

$$y(2at) = 2a(x + at^2)$$

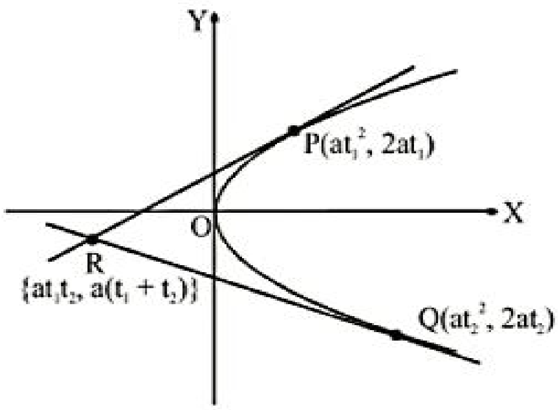
$$ty = x + at^2$$

Point of intersection of tangents at any two points on the parabola :

Let the given parabola be $y^2 = 4ax$ and

two points on the parbaola are

$$P = (at_1^2, 2at_1) \text{ and } Q = (at_2^2, 2\,at_2)$$



Equation of tangents at

$$P\,(at_1^2, 2at_1) \text{ and } Q(at_2^2, 2at_2)$$

are $\quad t_1y = x + at_1^2 \qquad \text{.........(1)}$

and $\quad t_2y = x + at_2^2 \qquad \text{.........(2)}$

Solving these equations we get

$$x = at_1t_2,\ y = a(t_1 + t_2)$$

Thus, the co-ordinates of the point of intersection of tangents at

$$P(at_1^2, 2at_1) \text{ and } Q(at_2^2, 2at_2) \text{ are } R(at_1t_2, a(t_1 + t_2)).$$

Note :

(i) The Arithmetic mean of the y-co-ordinates of P and Q $\left(\text{i.e., } \dfrac{2at_1 + 2at_2}{2} = a(t_1 + t_2)\right)$ is the y - co-ordinate of the point of intersection of tangents at P and Q on the parabola.

(ii) The Geometric mean of the x-co-ordinates of P and Q $\left(\text{i.e., } \sqrt{at_1^2 \times at_2^2} = at_1t_2\right)$ is the x co-ordinate of the point of intersection of tangents at P and Q on the parabola.

Illustration :

Show that the locus of the points of intersection of the tangents to a parabola at the extremity of focal chord are perpendicular and always meet at the directrix of the parabola.

Sol. Let the points $(P(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_2)$ on the parabola $y^2 = 4ax$ and tangents at P and Q are

$t_1y = x + at_1^2$ ....(1)

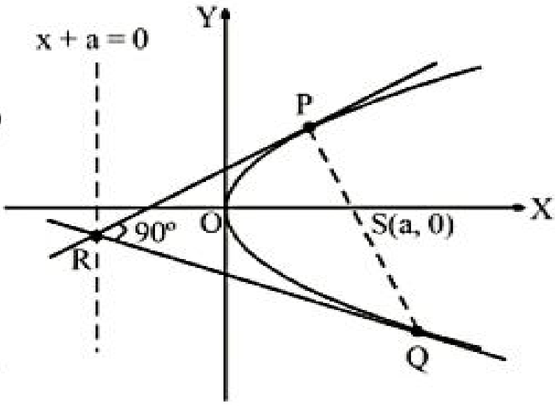
and $t_2y = x + at_2^2$ ....(2)

$\because$ point of intersection of these tangents is $(at_1t_2, a(t_1 + t_2))$

Let this point is (h, k)

then $h = at_1t_2$ ....(3)

and $k = a(t_1 + t_2)$ ....(4)



Slope of tangents (1) and (2) are $\frac{1}{t_1}$ and $\frac{1}{t_2}$ respectively.

$\therefore$ Product of slope $= \frac{1}{t_1} \times \frac{1}{t_2} = -1$ (Since PQ is focal chord)

$\Rightarrow h = -a$ from equation (3)

$\therefore$ Locus is $x + a = 0$ i.e. Directrix of parabola

Directrix is also called the Directer Circle of the parabola.

Illustration :

If the line $2x - 3y = k$ touches the parabola $y^2 = 6x$, then find the value of k.

Sol. Given $x = \frac{3y+k}{2}$ ....(1)

and $y^2 = 6x$ ....(2)

Solve (1) and (2), we get

$\Rightarrow y^2 = 6\left(\frac{3y+k}{2}\right) \Rightarrow y^2 = 3(3y + k)$

$\Rightarrow y^2 - 9y - 3k = 0$ ...(3)

If line (1) touches parabola (2) then roots of quadratic equation (3) is equal i.e. $D = 0$

$\therefore (-9)^2 = 4 \times 1 \times (-3k) \Rightarrow k = -\frac{27}{4}$

Alternative method :

Line $y = \frac{2}{3}x - \frac{k}{3}$ touch is $y^2 = 6x$ then use the condition of tangency

$c = \frac{a}{m} \Rightarrow -\frac{k}{3} = \frac{9}{4} \Rightarrow k = -\frac{27}{4}$

Illustration :

Find the equation to the tangents to the parabola $y^2 = 9x$ which goes through the point (4, 10).

Sol. *Equation of tangent to parabola $y^2 = 9x$ is*

$$y = mx + \frac{9}{4m}$$

Since it passes through (4, 10)

$$\therefore \quad 10 = 4m + \frac{9}{4m} \Rightarrow 16\,m^2 - 40\,m + 9 = 0$$

$$m = \frac{1}{4}, \frac{9}{4}$$

$\therefore$ *equation of tangent's are* $y = \frac{x}{4} + 9$ & $y = \frac{9}{4}x + 1.$ ***Ans.***

Illustration :

If the tangents at P and Q on a parabola (whose focus is S) meet in the point T, then prove that SP, ST and SQ are in geometric progression.

Sol. *Let P $(at_1^2, 2at_1)$ and Q $(at_2^2, 2at_2)$ be any two points on the parabola $y^2 = 4ax$, then point of intersection of tangents at P and Q will be*

$$T = [at_1t_2, a(t_1 + t_2)]$$

Now $SP = a(t_1^2 + 1)$

$SQ = a(t_2^2 + 1)$

$ST = a\sqrt{(t_1^2+1)(t_2^2+1)}$ *(use distance between two points formula)*

$\therefore \quad ST^2 = SP.SQ$

$\therefore$ *SP, ST and SQ are in G.P.* ***Ans.***

Illustration :

If two tangents are drawn from the pont (h, k) to the parabola $y^2 = 4x$ such that the slope of one tangent is double of the other, then prove that $9h = 2k^2$.

Sol. *Tangent to parabola $y^2 = 4x$ is*

$y = mx + \frac{1}{m}$ *and it passes through (h, k), so* $k = mh + \frac{1}{m}$

i.e. $hm^2 - km + 1 = 0$

Its roots are m_1 and $2m_1$

$$\therefore \quad m_1 + 2m_1 = \frac{k}{h} \quad \Rightarrow \quad 3m_1 = \frac{k}{h} \qquad ....(i)$$

$$m_1 \cdot 2m_1 = \frac{1}{h} \quad \Rightarrow \quad 2m_1^2 = \frac{1}{h} \qquad ....(ii)$$

from (i) and (ii) eliminate m, we get

$9h = 2k^2$ ***Ans.***

Illustration :

A tangent to the parabola $y^2 = 8x$ makes an angle of 45° with the straight line $y = 3x + 5$. Then find the equation of tangent and point of contact also.

Sol. Here $a = 2$, equation of tangent at $(2t^2, 4t)$ is $yt = x + 2t^2$

slope of tangent $m = \frac{1}{t}$

$$\therefore \quad \tan 45° = \pm \frac{\frac{1}{t} - 3}{1 + \frac{1}{t} \cdot 3} \Rightarrow \quad t = -\frac{1}{2} \text{ or } 2.$$

when $t = -\frac{1}{2}$, then equation of tangent is $\left(-\frac{1}{2}\right) y = x + 2 \cdot \frac{1}{4} \Rightarrow 2x + y + 1 = 0$ and point of contact is $\left(\frac{1}{2}, -2\right)$. When $t = 2$, the tangent is $2y = x + 8$ and point of contact is $\left(2 \cdot (2)^2, 4 \cdot 2\right)$ i.e. $(8, 8)$.

Illustration :

Find the common tangent of the parabola $y^2 = 8ax$ and the circle $x^2 + y^2 = 2a^2$

Sol. Any tangent to parabola is $y = mx + \frac{2a}{m}$

Solving with the circle $x^2 + (mx + \frac{2a}{m})^2 = 2a^2$

$$x^2 (1 + m^2) + 4ax + \left(\frac{4a^2}{m^2} - 2a^2\right) = 0$$

The condition of tangency is $B^2 - 4AC = 0$ gives $m^4 + m^2 - 2 = 0 \Rightarrow m = \pm 1$
Hence equation of tangent is $y = x + 2a$ and $y = -x - 2a$.

Illustration :

Find the equations to the common tangents of the parabolas $y^2 = 4ax$ and $x^2 = 4by$.

Sol. Equation of tangent to $y^2 = 4ax$ is $y = mx + \frac{a}{m}$ ....(i)
If this line also touches parabola $x^2 = 4by$ ......(ii)
then solve (i) and (ii), we get

$$x^2 - 4bmx - \frac{4ab}{m} = 0$$

Now condition of tangency is $D = 0$.

this gives $m^3 = -\frac{a}{b}$ or $m = -\left(\frac{a}{b}\right)^{1/3}$

$\therefore$ equation of common tangent is $y = \left(-\frac{a}{b}\right)^{1/3} x + a\left(-\frac{b}{a}\right)^{1/3}$. **Ans.**

EQUATION OF NORMALS :

1. Point form :

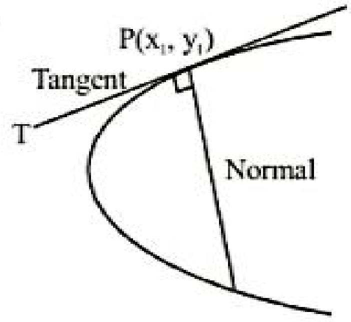
Since the equation of the tangent to the parabola $y^2 = 4ax$ at (x_1, y_1) is

$yy_1 = 2a(x + x_1)$ ....(1)

The slope of the tangent at $(x_1, y_1) = 2a/y_1$.

$\therefore$ Slope of the normal at $(x_1, y_1) = -y_1/2a$

Hence the equation of normal at (x_1, y_1) is



$$y - y_1 = -\frac{y_1}{2a}(x - x_1)$$

2. Slope form :

The equation of normal to the parabola $y^2 = 4ax$ at (x_1, y_1) is

$$y - y_1 = -\frac{y_1}{2a}(x - x_1) \quad ....(1)$$

Since m is the slope of the normal

then $m = -\frac{y_1}{2a}$ or $y_1 = -2am$

Since (x_1, y_1) lies on $y^2 = 4ax$ therefore

$y_1^2 = 4ax_1$ or $4a^2m^2 = 4ax_1$

$\therefore$ $x_1 = am^2$

Substituting the values of x_1 and y_1 in (1) we get

$y + 2am = m(x - am^2)$ ....(2)

Thus, $y = mx - 2am - am^3$ is a normal to the parabola $y^2 = 4ax$ where m is the slope of the normal. The co-ordinates of the point of contact are $(am^2, -2am)$.

Hence $y = mx + c$ will be normal to parabola. If and only if $c = -2am - am^3$

Note :

Equation of parabolas	Point of contact in terms of slope (m)	Equation of normals in terms of slope (m)	Condition of normality
$y^2 = 4ax$	$(am^2, -2am)$	$y = mx - 2am - am^3$	$c = -2am - am^3$
$x^2 = 4ay$	$\left(-\frac{2a}{m}, \frac{a}{m^2}\right)$	$y = mx + 2a + \frac{a}{m^2}$	$c = 2a + \frac{a}{m^2}$

3. Parametric form :

Equation of normal of the parabola $y^2 = 4ax$ at (x_1, y_1) is

$$y - y_1 = -\frac{y_1}{2a}(x - x_1) \quad ....(1)$$

Replacing x_1 by at^2 and y_1 by $2at$ then (1) becomes

$y - 2at = -t(x - at)^2$

or $y = -tx + 2at + at^3$

Three supplementry Results:

(a) Point of intersection of normals at any two points on the parabola :

Let the points P$(at_1^2, 2at_1)$ and Q $(at_2^2, 2at_2)$ lie on the parabola $y^2 = 4ax$

The equations of the normals at P$(at_1^2, 2at_1)$ and Q$(at_2^2, 2at_2)$ are

$$y = -t_1x + 2at_1 + at_1^3 \qquad ....(1)$$

and $$y = -t_2x + 2at_2 + at_2^3 \qquad ....(2)$$

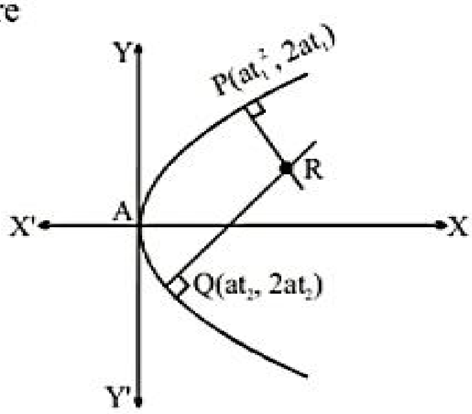
Hence point of intersection of above normals will be obtained by solving (1) and (2), we get

$$x = 2a + a(t_1^2 + t_2^2 + t_1t_2)$$

$$y = -at_1t_2(t_1 + t_2)$$

If R is the point of intersection then it is

$$R = [2a + a(t_1^2 + t_2^2 + t_1t_2), -at_1t_2(t_1 + t_2)]$$



(b) Relation between t_1 and t_2 if normal at t_1 meets the parabola again at t_2 :

Let the parabola be $y^2 = 4ax$, equation of normal at P$(at_1^2, 2at_1)$ is

$$y = -t_1x + 2at_1 + at_1^3 \qquad ....(1)$$

Since normal meet the parabola again at Q$(at_2^2, 2at_2)$

$\therefore \quad 2at_2 = -at_1t_2^2 + 2at_1 + at_2^3$

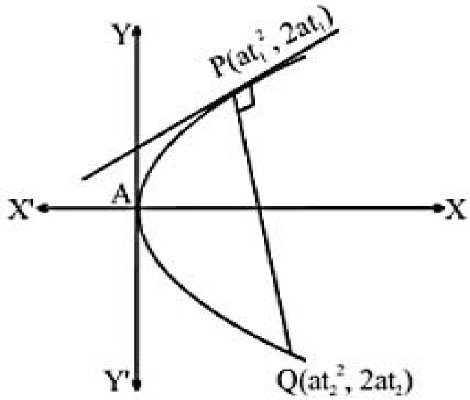
$\Rightarrow \quad 2a(t_2 - t_1) + at_1(t_2^2 - t_1^2) = 0$

$\Rightarrow \quad a(t_2 - t_1)[2 + t_1(t_2 + t_1)] = 0$

$\because \quad a(t_2 - t_1) \neq 0 \qquad (\because t_1$ and t_2 are different)

$\therefore \quad 2 + t_1(t_2 + t_1) = 0$

$\therefore \quad t_2 = -t_1 - \dfrac{2}{t_1}$



(c) If normal to the parabola $y^2 = 4ax$ drawn at any point $(at^2, 2at)$ meet the parabola at t_3 then

$$t_3 = -t - \frac{2}{t}$$

$\Rightarrow \quad t^2 + t\,t_3 + 2 = 0 \qquad ...(i)$

It has two roots t_1 & t_2. Hence there are two such point P(t_1) & Q(t_2) on the parabola from where normals are drawn and which meet parabola at R(t_3)

$\Rightarrow \quad t_1 + t_2 = -t_3 \quad \& \quad t_1t_2 = 2$

Thus the line joining P(t_1) & Q(t_2) meet x-axis at $(-2a, 0)$

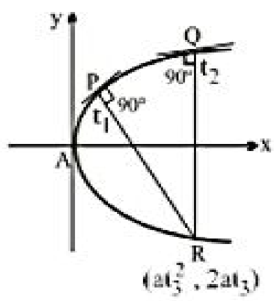


Illustration :

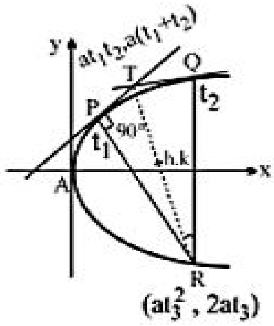
TP and TQ are tangents to the parabola $y^2 = 4ax$ and the normals at P and Q meet at a point R on the curve, prove that the centre of the circle circumscribing the triangle TPQ lies on the parabola $2y^2 = a(x - a)$.

Sol. Normal at two points $P(t_1)$ & $P(t_2)$ meet parabola at $R(t_3)$.

$\Rightarrow \quad t_1 + t_2 = -t_3 \quad \& \quad t_1t_2 = 2$

$\therefore \quad T(at_1t_2,\ a(t_1 + t_2)) \equiv T(2a, -at_3)$

Let centre of circle is (h, k) then it is mid point of T and R.



$\therefore \quad 2h = 2a + at_3^2 \qquad ...(i)$

$\quad 2k = -at_3 + 2at_3 \qquad ...(ii)$

i.e. $\quad 2k = at_3 \quad \Rightarrow \quad t_3 = \dfrac{2k}{a}$

Put value of t_3 in equation (i) we get the required locus. Now replace h by x and k by y

We get $2y^2 = a(x - a)$. **Ans.**

Illustration :

Find the equation of a normal at the parabola $y^2 = 4x$ which passes through (3,0).

Sol. Equation of Normal $y = mx - 2am - am^3$

Here $a = 1$ and it passes through (3,0)

$0 = 3m - 2m - m^3$

$\Rightarrow \quad m^3 - m = 0$

$\Rightarrow \quad m = 0, \pm 1$

for $\quad m = 0 \quad \Rightarrow \quad y = 0$

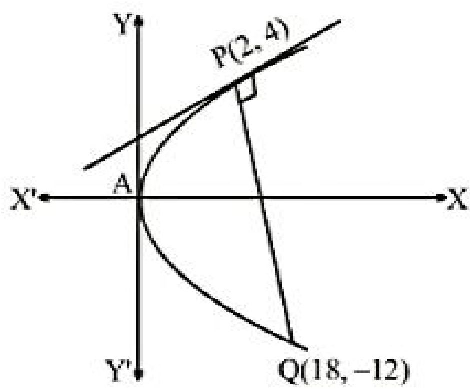
$\quad m = 1 \quad \Rightarrow \quad y = x - 3$

$\quad m = -1 \quad \Rightarrow \quad y = -x + 3$ **Ans.**

Illustration :

Show that normal to the parabola $y^2 = 8x$ at the point (2, 4) meets it again at (18, – 12). Find also the length of the normal chord.

Sol. Comparing the given parabola with $y^2 = 4ax$



$\therefore \quad 4a = 8$

$\therefore \quad a = 2$

$P(at_1^2, 2at_1) \equiv (2, 4)$

$\Rightarrow \quad t_1 = 1$

$\Rightarrow \quad$ parameter at $Q(t_2) = -t_1 - \dfrac{2}{t_1} = -3$

$\therefore \quad Q(2(-3)^2, 2 \times 2(-3))$

i.e. $\quad Q(18, -12)$

$\therefore \quad$ Length of normal chord PQ = Distance between points P and Q.

$$= PQ = \sqrt{(18-2)^2 + (-12-4)^2} = 16\sqrt{2}.$$

Illustration :

If the normal to a parabola $y^2 = 4ax$, makes an angle ϕ with the axis show that it will cut the curve again at an angle $\tan^{-1}\left(\frac{1}{2}\tan\phi\right)$.

Sol. Let the normal at $P(at_1^2, 2at_1)$ be $y = -t_1x + 2at_1 + at_1^3$

$\therefore \quad \tan\phi = -t_1$ = slope of the normal ....(1)

It meet the curve again Q say $(at_2^2, 2at_2)$

$$\therefore \quad t_2 = -t_1 - \frac{2}{t_1} \quad ....(2)$$

Now angle between the normal and parabola = Angle between the normal and tangent at Q (i.e., $t_2y = x + at_2^2$)

If θ be the angle, then

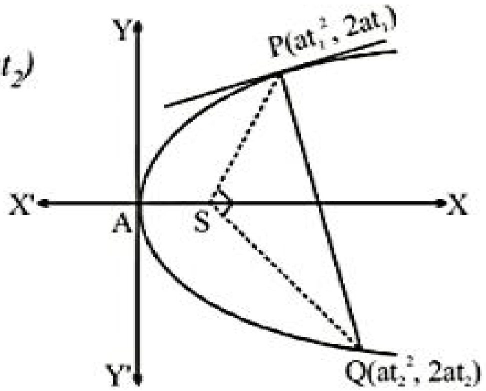
$$\tan\theta = \frac{m_1 - m_2}{1 + m_1m_2} = \frac{-t_1 - \frac{1}{t_2}}{1 + (-t_1)\left(\frac{1}{t_2}\right)} = -\frac{t_1t_2 + 1}{t_2 - t_1} = \frac{t_1\left(-t_1 - \frac{2}{t_2}\right) + 1}{-t_1 - \frac{2}{t_2} - t_1} \quad \text{\{from equation (2)\}}$$

$$= \frac{-t_1^2 - 1}{-2\left(\frac{1 + t_1^2}{t_1}\right)} = \frac{t_2}{2} = \frac{\tan\phi}{2} \qquad \text{\{from equation (1)\}}$$

$$\theta = \tan^{-1}\left(\frac{1}{2}\tan\phi\right).$$

Illustration :

Prove that the normal chord to a parabola $y^2 = 4ax$ at the point whose ordinate is equal to abscissa subtends a right angle at the focus.



Sol. Let the normal at $P(at_1^2, 2at_1)$ meet the curve at $Q(at_2^2, 2at_2)$

$\therefore \quad$ PQ is a normal chord.

$$\text{and} \quad t_2 = -t_1 - \frac{2}{t_1} \quad ....(1)$$

By given condition $\quad 2at_1 = at_1^2$

$\therefore \quad t_1 = 2$ from equation (1), $t_2 = -3$

then $\quad P(4a, 4a)$ and $Q(9a, -6a)$

but focus $S(a, 0)$

$$\therefore \quad \text{Slope of } SP = \frac{4a - 0}{4a - a} = \frac{4a}{3a} = \frac{4}{3} \text{ and slope of } SQ = \frac{-6a - 0}{9a - a} = \frac{6a}{8a} = -\frac{3}{4}$$

$$\because \quad \text{slope of } SP \times \text{slope of } SQ = \frac{4}{3} \times -\frac{3}{4} = -1$$

$$\therefore \quad \angle PSQ = \frac{\pi}{2} \quad \text{i.e., PQ subtends a right angle at the focus S.}$$

CO-NORMAL POINTS :

Maximum three normals can be drawn from a point to a parabola and their feet (points where the normal meet the parabola) are called co-normal points.

Let P(h, k) be any given point and $y^2 = 4ax$ be a parabola.

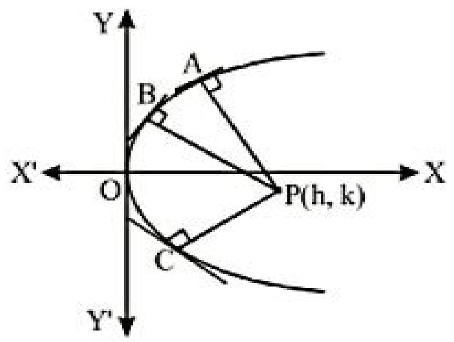
The equation of any normal to $y^2 = 4ax$ is

$$y = mx - 2am - am^3$$

If it passes through (h, k) then

$$k = mh - 2am - am^3$$

$\Rightarrow \quad am^3 + m(2a - h) + k = 0 \quad$ ......(i)



This is a cubic equation in m, so it has three roots, say m_1, m_2 and m_3.

$\therefore \quad m_1 + m_2 + m_3 = 0, \quad$ ...(ii)

$$m_1m_2 + m_2m_3 + m_3m_1 = \frac{(2a-h)}{a}, \quad \text{...(iii)}$$

$$m_1m_2m_3 = -\frac{k}{a} \quad \text{...(iv)}$$

Hence for any given point P(h, k), (i) has three real or imaginary roots. Correspoinding to each of these three roots, we have each normal passing through P(h, k). Hence we have three normals PA, PB and PC drawn through P to the parabola.

Points A, B, C in which the three normals from P(h, k) meet the parabola are called co-normal points.

Properties of co-normal points :

(1) The algebraic sum of the slopes of three concurrent normals is zero. This follows from equation (ii).

(2) The algebraic sum of ordinates of the feets of three normals drawn to a parabola from a given point is zero.

Let the ordiantes of A, B, C be y_1, y_2, y_3 respectively then

$$y_1 = -2\,am_1,\ y_2 = -2am_2 \text{ and } y_3 = -2am_3$$

$\therefore$ Algebraic sum of these ordinates is

$$\begin{aligned} y_1 + y_2 + y_3 &= -2am_1 - 2am_2 - 2am_3 \\ &= -2a(m_1 + m_2 + m_3) \\ &= -2a \times 0 \qquad \{\text{from equation (ii)}\} \\ &= 0 \end{aligned}$$

(3) If three normals drawn to any parabola $y^2 = 4ax$ from a given point (h, k) is real then $h > 2a$.

When normals are real, then all the three roots of equation (i) are real and in that case

$$m_1^2 + m_2^2 + m_3^2 > 0 \quad (\text{for any values of } m_1, m_2, m_3)$$

$\Rightarrow \quad (m_1 + m_2 + m_3)^2 - 2 \quad (m_1m_2 + m_2m_3 + m_3m_1) > 0$

$\Rightarrow \quad (0)^2 - \dfrac{2(2a-h)}{a} > 0$

$\Rightarrow \quad h - 2a > 0$

or $\quad h > 2a$

(4) The centroid of the triangle formed by the feet of the three normals lies on the axis of the parabola.
If $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ be vertices of ΔABC, then its centroid is

$$\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}\right) = \left(\frac{x_1+x_2+x_3}{3}, 0\right)$$

Since $y_1+y_2+y_3=0$ (from result-2). Hence the centroid lies on the x-axis, which is the axis of the parabola also.

Now $\frac{x_1+x_2+x_3}{3} = \frac{1}{3}(am_1^2+am_2^2+am_3^2) = \frac{a}{3}\left(m_1^2+m_2^2+m_3^2\right)$

$$= \frac{a}{3}\{(m_1+m_2+m_3)^2 - 2(m_1m_2+m_2m_3+m_3m_1)\}$$

$$= \frac{a}{3}\left\{(0)^2 - 2\left\{\frac{2a-h}{a}\right\}\right\} = \frac{2h-4a}{3}$$

$\therefore$ Centroid of ΔABC is $\left(\frac{2h-4a}{3}, 0\right)$

Illustration :

If from point P(h, k) three normals are drawn to the parabola $(y-1)^2 = 8(x-2)$ then find the condition

Sol. *Here $a = 2$ and abscissa of point from where three normals are drawn must be greater than 2a.*
i.e. $x - 2 > 2a$ *i.e.* $x > 6$ *Hence* $h > 6$ *Ans.*

Illustration :

The ordinate of points P and Q on the parabola $y^2 = 12x$ are in the ratio of 1 : 2. Find the locus of the point of intersection of the normal to the parabola at P and Q.

Sol. *Here $a = 3$,*
Let $P(3t_1^2, 6t_1)$ and $Q(3t_2^2, 6t_2)$ lie on the parabola

According to the question, $\frac{6t_1}{6t_2} = \frac{1}{2} \Rightarrow t_2 = 2t_1$ ......(i)

Let $P(\alpha, \beta)$ be the point of intersection of normal to parabola at P and Q then

$\alpha = 2a + a(t_1^2 + t_2^2 + t_1t_2) = 6 + 21t_1^2$ ....(ii)

and $\beta = -at_1t_2(t_1 + t_2) = -18t_1^3$ ....(iii)

from (ii) $t_1^6 = \left(\frac{\alpha-6}{21}\right)^3$ *and from (iii)*

$$t_1^6 = \frac{\beta^2}{324}$$

equate t_1^6, we get

$$343\beta^2 = 12(\alpha-6)^3$$

$\therefore$ *locus is* $343y^2 = 12(x-6)^3$. ***Ans.***

Illustration :

Find the locus of points through which three normals to parabola $y^2 = 4ax$ passes and two of them are perperdicular to each other.

Sol. *Let P(h, k) be the point of intersection of three normals to be parabola $y^2 = 4ax$ then it will be*

$y = mx - 2am - am^3$ and it passes through P(h, k)

$k = mh - 2am - am^3$

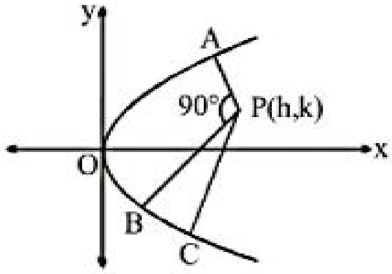
$\therefore \quad am^3 + m(2a - h) + k = 0 \qquad ....(i)$

It has three roots m_1, m_2 and m_3

$m_1 + m_2 + m_3 = 0 \qquad ....(ii)$

$m_1m_2m_3 = \dfrac{-k}{a} \qquad ....(iii)$

Given condition, $m_1m_2 = -1 \qquad ......(iv)$ *as given in figure.*



from (iii) and (iv)

$m_3 = \dfrac{k}{a}$

$\because \quad m_3$ *is also a root of equation (1) therefore it will satify equation (i)*

$$\therefore \quad a\left(\frac{k}{a}\right)^3 + (2a-h)\left(\frac{k}{a}\right) + k = 0$$

$\Rightarrow \quad k^3 + (2a - h)\, ka + ka^2 = 0$

$\Rightarrow \quad k^2 + 3a^2 - ah = 0$

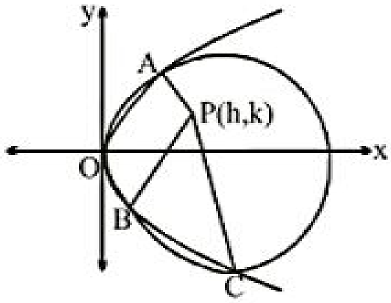
$\therefore$ *locus of P(h, k) is*

$y^2 + 3a^2 - ax = 0.$ ***Ans.***

Illustration :

Find the equation of circle passes through co-normal points.

Sol. *Let $A(am_1^2, -2am_1)$, $B(am_2^2, -2am_2)$ and $C(am_3^2, -2am_3)$ be the three points on the parabola and normal at these points intersect at P (h, k) then*



$am^3 + (2a - h)\, m + k = 0 \qquad ....(i)$

$m_1 + m_2 + m_3 = 0 \qquad ....(ii)$

$m_1m_2 + m_2m_3 + m_3m_1 = \dfrac{2a-h}{a} \qquad ......(iii)$

$m_1m_2m_3 = \dfrac{-k}{a} \qquad ......(iv)$

Let the equation of circle through A, B and C is $x^2 + y^2 + 2gx + 2fy + c = 0$. If the point $(am^2, -2am)$ lie on it then it will satisfy circle.

$\Rightarrow \quad a^2m^4 + (4a^2 + 2ag)m^2 - 4afm + c = 0 \qquad (v)$

It has four root m_1, m_2, m_3 and m_4

$\therefore \quad m_1 + m_2 + m_3 + m_4 = 0 \qquad$ ....(vi)

$\Rightarrow \quad m_4 = 0$ *using equation (ii)*

$\Rightarrow \quad (am_4^2, -2am_4) \equiv (0, 0)$. *This is conform that above circle passes through vertex of parabola*

$\Rightarrow$ *equation (v) becomes*

$\therefore \quad a^2m^4 + (4a^2 + 2ag)m^2 - 4afm = 0 \qquad$ *(No constant term)*

$\Rightarrow \quad am^3 + (4a + 2g)\, m - 4f = 0 \qquad$ .....(vi)

equation (i) and (vi) are identical

$$\Rightarrow \quad \frac{1}{1} = \frac{4a+2g}{2a-h} = -\frac{4f}{k} \Rightarrow 2g = -(2a + h) \Rightarrow 2f = -\frac{k}{2}$$

$\therefore$ *equation of circle is*

$$x^2 + y^2 - (2a + h)x - \frac{k}{2}y = 0.$$

Thus circle through co-normal points.

CHORD OF THE PARABOLA y^2=4ax WHOSE MIDDLE POINT IS GIVEN:

Equation of the parabola is $y^2 = 4ax$ ....(1)

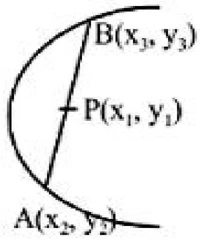
Let AB be a chord of the parabola whose middle point is P (x_1, y_1).

Equation of chord AB is $y - y_1 = m\,(x - x_1)$ ....(2)

where m = slope of AB

Let $A = (x_2, y_2)$ and $B = (x_3, y_3)$.

Since A and B lie on parabola (1)



$\therefore \quad y_2^2 = 4ax_2$ and $y_3^2 = 4ax_3$

$\therefore \quad y_2^2 - y_3^2 = 4a\,(x_2 - x_3)$ or $\dfrac{y_2 - y_3}{x_2 - x_3} = \dfrac{4a}{y_2 + y_3}$ ....(3)

But $P(x_1, y_1)$ is the middle point of AB $y_2 + y_3 = 2y_1$

$\therefore$ From (3), $\dfrac{y_2 - y_3}{x_2 - x_3} = \dfrac{4a}{2y_1} = \dfrac{2a}{y_1}$

$\therefore$ Slope of AB i.e., $m = \dfrac{2a}{y_1}$ ....(4)

From (2), equation of chord AB is $y - y_1 = \dfrac{2a}{y_1}\,(x - x_1)$

or $yy_1 - y_1^2 = 2ax - 2ax_1$ or $yy_1 - 2ax = y_1^2 - 2ax_1$

or $yy_1 - 2a\,(x - x_1) = y_1^2 - 4ax_1$ [Subtracting $2ax_1$ from both sides] ....(5)

(5) is the required equation. In usual notations, equation (5) can be written as $T = S_1$.

The same result holds true for cicle, ellipse and hyperbola also.

Illustration :

Find the locus of the mid points of the chords of the parabola $y^2 = 4ax$ which subtend a right angle at the vertex of the parabola.

Sol. *Let P(h, k) be the mid point of a chord QR of the parabola $y^2 = 4ax$ then equation of the chord QR is $T = S_1$*

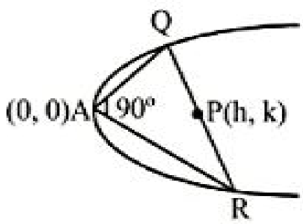
or $yk - 2a(x + h) = k^2 - 4ah$

$\Rightarrow$ $yk - 2ax = k^2 - 2ah$ ... (1)

If A is the vertex of the parabola. For combined equation of AQ and AR making homogeneous of $y^2 = 4ax$ with the help of (1)

$$y^2 = 4ax(1)$$

$$y^2 = 4ax\left(\frac{4k - 2ax}{k^2 - 2ah}\right)$$



$y^2(k^2 - 2ah) - 4akxy + 8a^2x^2 = 0$

Since $\angle QAF = 90^o$

$\therefore$ *Coefficient of* x^2 + *Co-efficient of* $y^2 = 0$

$k^2 - 2ah + 8a^2 = 0$

Hence the locus of P(h, k) is

$y^2 - 2ax + 8a^2 = 0$

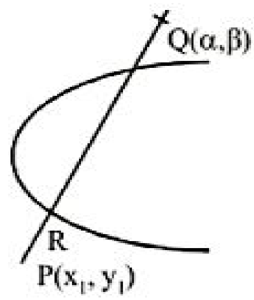
PAIR OF TANGENTS :

Let the parabola be $y^2 = 4ax$ ....(1)

Let $P(x_1, y_1)$ be a point outside the parabola.

Let a chord of the parabola through the point $P(x_1, y_1)$ cut the parabola at R and let $Q(\alpha, \beta)$ be an arbitrary point on line PR. Let R divide PQ in the ratio $\lambda : 1$,

then $R = \left(\frac{\lambda\alpha + x_1}{\lambda + 1}, \frac{\lambda\beta + y_1}{\lambda + 1}\right)$.



Since R lies on parabola (1), therefore,

$$\left(\frac{\lambda\beta + y_1}{\lambda + 1}\right)^2 - 4a\left(\frac{\lambda\alpha + x_1}{\lambda + 1}\right) = 0$$

or $(\lambda\beta + y_1)^2 - 4a(\lambda\alpha + x_1)(\lambda + 1) = 0$

or $(\beta^2 - 4a\alpha)\lambda^2 + 2[\beta y_1 - 2a(\alpha + x_1)]\lambda + (y_1^2 - 4ax_1) = 0$ ....(2)

Line PQ will become tangent to parabola (1) if roots of equation (2) are equal or if

$$4[\beta y_1 - 2a(\alpha + x_1)]^2 = 4(\beta^2 - 4a\alpha)(y_1^2 - 4ax_1)$$

Hence, locus of $Q(\alpha, \beta)$ i.e. equation of pair of tangents from $P(x_1, y_1)$ is

$$[yy_1 - 2a(x + x_1)]^2 = (y^2 - 4ax)(y_1^2 - 4ax_1)$$

$\Rightarrow$ $SS_1 = T^2$

where S, S_1 and T have usual meanings.

The same result holds true for cicle, ellipse and hyperbola also.

Illustration :

Tangents are drawn from the point $(-1, 2)$ on the parabola $y^2 = 4x$. The length, these tangents will intercept on the line $x = 2$ is :

(A) 6 *(B) $6\sqrt{2}$* *(C) $2\sqrt{6}$* *(D) none of these*

Sol. $SS_1 = T^2$

$(y^2 - 4x)(y_1^2 - 4x_1) = (y y_1 - 2(x + x_1))^2$

$(y^2 - 4x)(4 + 4) = [2y - 2(x - 1)]^2 = 4(y - x + 1)^2$

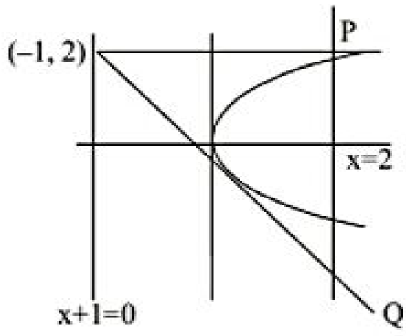
$2(y^2 - 4x) = (y - x + 1)^2$;

solving with the line $x = 2$ we get,

$2(y^2 - 8) = (y - 1)^2$ or $2(y^2 - 8) = y^2 - 2y + 1$

or $y^2 + 2y - 17 = 0$

where $y_1 + y_2 = -2$ and $y_1 y_2 = -17$



Now $|y_1 - y_2|^2 = (y_1 + y_2)^2 - 4y_1 y_2$

or $|y_1 - y_2|^2 = 4 - 4(-17) = 72$

$\therefore\ (y_1 - y_2) = \sqrt{72} = 6\sqrt{2}$

CHORD OF CONTACT OF POINT WITH RESPECT TO A PARABOLA :

Two tangents PA and PB are drawn to parabola, then line joining AB is called the chord of contact to the parabola with respect to point P.

Let the parabola be $y^2 = 4ax$ ....(1)

Let $P(\alpha, \beta)$ be a point outside the parabola.

Let PA and PB be the two tangents from $P(\alpha, \beta)$ to parabola (1).

Let $A = (x_1, y_1)$ and $B = (x_2, y_2)$

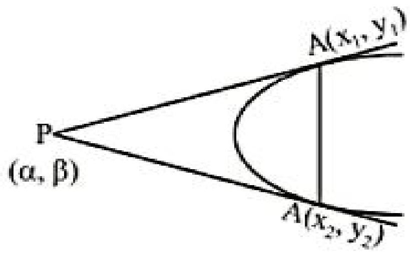
Equation of the tangent PA is $yy_1 = 2a(x + x_1)$ ....(2)

Equation of the tangent PB is $yy_2 = 2a(x + x_2)$ ....(3)

Since lines (2) and (3) pass through $P(\alpha, \beta)$, therefore

$\beta y_1 = 2a(a + x_1)$ ....(4)

and $\beta y_2 = 2a(a + x_2)$ ....(5)



Now we consider the equation $y\beta = 2a(x + \alpha)$ ......(6)

From (4) and (5), it follows that line (6) passes through $A(x_1, y_1)$ and $B(x_2, y_2)$.

Hence (6) is the equation of line AB which is the chord of contact of point $P(\alpha, \beta)$ with respect to parabola (1) i.e, chord of contact is $\mathbf{y\beta = 2a(x + \alpha)}$

The same result holds true for cicle, ellipse and hyperbola also.

Illustration :

Tangent are drawn to parabola $y^2 = 4ax$ at point where the line $lx + my + n = 0$ meets the parabola. Find the point of intersection of these tangents.

Sol. *Let the tangent intersect at $P(h, k)$, then $lx + my + n = 0$ will be the chord of contact of P. That means $lx + my + n = 0$ and $yk - 2ax - 2ah = 0$ will represent the same line. Thus,*

$$\frac{k}{m} = \frac{-2a}{l} = \frac{-2ah}{n} \quad \Rightarrow \quad h = \frac{n}{l}, k = -\frac{2am}{l}$$

POLAR & POLE :

Let P be any point inside or outside a parabola. Suppose in a straight line drawn through P intersect the parabola at Q and R. Then the locus of point of intersection of the tangents to the parabola at Q and R is called the polar of given point P with respect to the parabola and point P is called the pole of the polar.

Equation of the Polar of the point $P(x_1, y_1)$ w.r.t. the parabola $y^2 = 4ax$ is,

$$y\,y_1 = 2a(x + x_1)$$

The pole of the line $lx + my + n = 0$ w.r.t. the parabola $y^2 = 4ax$ is $\left(\frac{n}{l}, -\frac{2am}{l}\right)$.

Properties of polar :

(i) The polar of the focus of the parabola is the directrix.

(ii) When the point (x_1, y_1) lies without the parabola the equation to its polar is the same as the equation to the chord of contact of tangents drawn from (x_1, y_1) when (x_1, y_1) is on the parabola the polar is the same as the tangent at the point.

(iii) Two straight lines are said to be conjugated to each other w.r.t. a parabola when the pole of one lies on the other. Similarly two points P and Q are said to be conjugate points if polar of P passes through Q and vice versa.

(iv) Polar of a given point P w.r.t. any Conic is the locus of the harmonic conjugate of P w.r.t. the two points is which any line through P cuts the conic.

Illustration :

Prove that the area of the triangle formed by the tangets drawn from (x_1, y_1) to $y^2 = 4ax$ and their chord of contact is $(y_1^2 - 4ax_1)^{3/2} / 2a$

Sol. *Equation of QR (chord of contact) is*

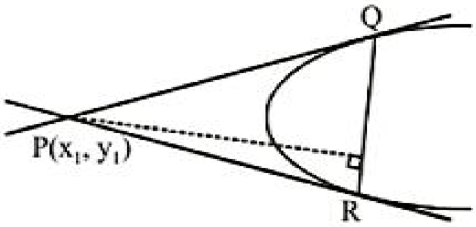
$$yy_1 = 2a(x + x_1)$$
$$yy_1 - 2a(x + x_1) = 0$$

$\because$ *PM = Length of perpendicular from $P(x_1, y_1)$ on QR*

$$= \frac{|y_1y_1 - 2a(x_1 + x_1)|}{\sqrt{(y_1^2 + 4a^2)}} = \frac{|(y_1^2 - 4ax_1)|}{\sqrt{(y_1^2 + 4a^2)}}$$

[Since $P(x_1, y_1)$ lies outside the parabola $\therefore y_1^2 - 4ax_1 > 0$]

Now area of $\Delta PQR = \frac{1}{2} QR \cdot PM$



$$= \frac{1}{2}\frac{1}{|a|}\sqrt{(y_1^2 - 4ax_1)(y_1^2 + 4a^2)}\,\frac{(y_1^2 - 4ax_1)}{\sqrt{y_1^2 - 4a^2}}$$

$= (y_1^2 - 4ax)^{3/2} / 2a$, *if* $a > 0$.

Note : *Length of the chord of contact QR* $= \dfrac{\sqrt{y_1^2 + 4a^2}\,\sqrt{y_1^2 - 4ax_1}}{a}$

Practice Problem

Single correct question

Q.1 From an external point P, pair of tangent lines are drawn to the parabola, $y^2 = 4x$. If θ_1 & θ_2 are the inclinations of these tangents with the axis of x such that, $\theta_1 + \theta_2 = \frac{\pi}{4}$, then the locus of P is :

(A) $x - y + 1 = 0$ (B) $x + y - 1 = 0$ (C) $x - y - 1 = 0$ (D) $x + y + 1 = 0$

Q.2 If m_1, m_2 are slopes of the two tangents that are drawn from (2, 3) to the parabola $y^2 = 4x$ then the value of $\frac{1}{m_1} + \frac{1}{m_2}$ is

(A) –3 (B) 3 (C) $\frac{2}{3}$ (D) $\frac{3}{2}$

Q.3 The points of contact Q and R of tangent from the point P (2, 3) on the parabola $y^2 = 4x$ are

(A) (9, 6) and (1, 2) (B) (1, 2) and (4, 4) (C) (4, 4) and (9, 6) (D) (9, 6) and $\left(\frac{1}{4}, 1\right)$

Q.4 If the lines $(y - b) = m_1(x + a)$ and $(y - b) = m_2(x + a)$ are the tangents to the parabola $y^2 = 4ax$, then

(A) $m_1 + m_2 = 0$ (B) $m_1m_2 = 1$ (C) $m_1m_2 = -1$ (D) $m_1 + m_2 = 1$

Q.5 If $2x + y + \lambda = 0$ is a normal to the parabola $y^2 = -8x$ then the value of λ is

(A) –24 (B) –16 (C) –8 (D) 24

Q.6 The set of points on the axis of the parabola $y^2 - 4x - 2y + 5 = 0$ from which all three normals to the parabola are real is

(A) $(\lambda, 0); \lambda > 1$ (B) $(\lambda, 1); \lambda > 3$ (C) $(\lambda, 2); \lambda > 6$ (D) $(\lambda, 3); \lambda > 8$

Q.7 The slope of a chord of the parabola $y^2 = 4ax$ which is normal at one end and which subtends a right angle at the origin is

(A) $\frac{1}{\sqrt{2}}$ (B) $\sqrt{2}$ (C) $-\frac{1}{\sqrt{2}}$ (D) $-\sqrt{2}$

Q.8 Equation of the other normal to the parabola $y^2 = 4x$ which passes through the intersection of those at $(4, -4)$ and $(9, -6)$ is

(A) $5x - y + 115 = 0$ (B) $5x + y - 135 = 0$

(C) $5x - y - 115 = 0$ (D) $5x + y + 115 = 0$

Q.9 Which one of the following lines cannot be the normals to $x^2 = 4y$?

(A) $x - y + 3 = 0$ (B) $x + y - 3 = 0$ (C) $x - 2y + 12 = 0$ (D) $x + 2y + 12 = 0$

Q.10 Normal to the parabola $y^2 = 8x$ at the point P (2, 4) meets the parabola again at the point Q. If C is the centre of the circle described on PQ as diameter then the coordinates of the image of the point C in the line $y = x$ are

(A) $(-4, 10)$ (B) $(-3, 8)$ (C) $(4, -10)$ (D) $(-3, 10)$

Q.11 Normals are concurrent drawn at points A, B, and C on the parabola $y^2 = 4x$ at P(h, k). The locus of the point P if the slope of the line joining the feet of two of them is 2, is

(A) $x + y = 1$ (B) $x - y = 3$ (C) $y^2 = 2(x - 1)$ (D) $y^2 = 2\left(x - \frac{1}{2}\right)$

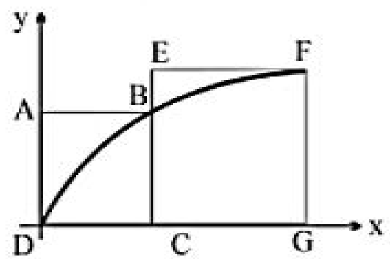
Q.12 If (a, b) is the mid-point of chord passing through the vertex of the parabola $y^2 = 4x$, then

(A) $a = 2b$ (B) $2a = b$ (C) $a^2 = 2b$ (D) $2a = b^2$

Q.13 Tangents are drawn from the points on the line $x - y + 3 = 0$ to parabola $y^2 = 8x$. Then the variable chords of contact pass through a fixed point whose coordinates are :

(A) (3, 2) (B) (2, 4) (C) (3, 4) (D) (4, 1)

Q.14 ABCD and EFGC are squares and the curve $y = k\sqrt{x}$ passes through the origin D and the points B and F. Then find the ratio of $\frac{FG}{BC}$.



(A) $\frac{\sqrt{5}+1}{2}$ (B) $\frac{\sqrt{3}+1}{2}$

(C) $\frac{\sqrt{5}+1}{4}$ (D) $\frac{\sqrt{3}+1}{4}$

Integer type question

Q.15 Find the y-intercept of the common tangent to the parabola $y^2 = 32x$ and $x^2 = 108y$.

Answer key

Q.1	C	Q.2	B	Q.3	B	Q.4	C	Q.5	A, D
Q.6	B	Q.7	D	Q.8	B	Q.9	D	Q.10	A
Q.11	B	Q.12	D	Q.13	C	Q.14	A	Q.15	– 12

DIAMETER OF A PARABOLA :

Diameter of a conic is the locus of middle points of a series of its parallel chords.

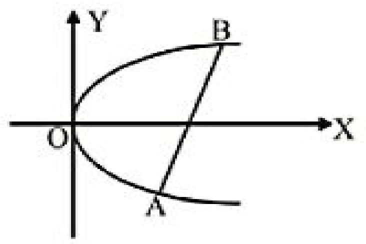
Equation of diameter of a parabola :

Let the parabola be $y^2 = 4ax$ ....(1)

Let AB be one of the chords of a series of parallel chords having slope m.

Let P(α, β) be the middle point of chord AB, then equation of AB will be $T = S_1$.

or $y\beta - 2a(x + \alpha) = \beta^2 - 4\alpha a$ ....(2)

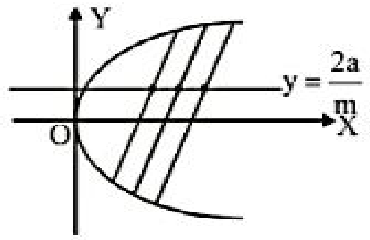


Slope of line (2) $= \dfrac{2a}{\beta}$

but slope of line (1) i.e. line AB is m.

$\therefore \quad \dfrac{2a}{\beta} = m \quad \text{or} \quad \beta = \dfrac{2a}{m}$

Hence locus of P(α, β) i.e. equation of diameter (which is the locus of a series of a parallel chords having slope m) is



$$y = \frac{2a}{m} \quad ....(3)$$

Clearly line (3) is parallel to the axis of the parabola. Thus a diameter of a parabola is parallel to its axis.

Length of tangent, subtangnet, normal and sub-normal :

Let the parabola is $y^2 = 4ax$. Let the tangent at any point P(x, y) meet the axis of parabola at T and G respectively and tangent makes an angle ψ with x-axis.

$\therefore \quad \tan\psi = \left(\dfrac{dy}{dx}\right)_{P(x,y)}$ and PN = y

$\therefore \quad$ PT = length of tangent = PN cosec ψ = y cosec ψ

PG = length of normal = y sec ψ

TN = length of sub-tangent = PN cot ψ = y cot ψ

NG = length of sub-normal = y tan ψ

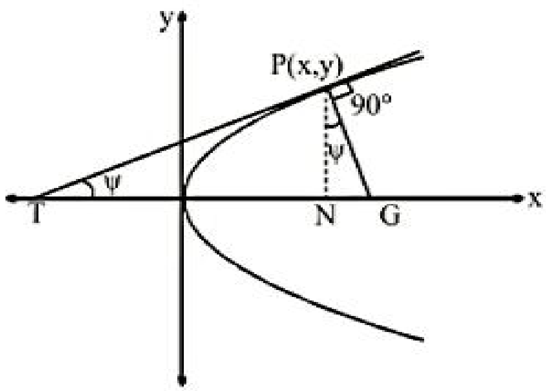
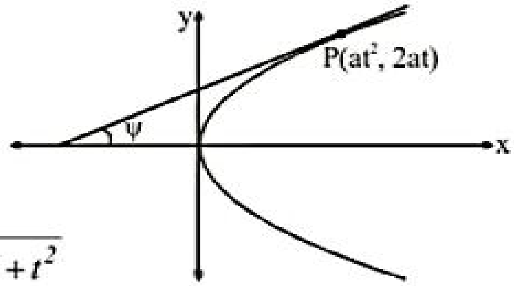


Illustration :

Find the length of tangent, sub-tangent, normal and sub-normal to $y^2 = 4ax$ at $(at^2, 2at)$.

Sol. $y^2 = 4ax$

$$2y\frac{dy}{dx} = 4a$$



$$\therefore \quad \left(\frac{dy}{dx}\right)_{(at^2, 2at)} = \frac{1}{t} = \tan\psi \Rightarrow \cot\psi = t$$

$$\therefore \quad l\,(\text{tangent}) = y\,\text{cosec}\,\psi = 2at\sqrt{1+\cot^2\psi} = 2at\sqrt{1+t^2}$$

$$l\,(\text{normal}) = y\sec\psi = 2at\sqrt{1+\tan^2\psi} = 2a\sqrt{1+t^2}$$

$$l\,(\text{sub-tangent}) = y\cot\psi = 2at \cdot t = 2at^2$$

$$l\,(\text{sub-normal}) = y\tan\psi = 2at \cdot \frac{1}{t} = 2a$$

Thus length of sub-normal of parabola is constant and equal to semi-latus rectum.

PROPERTIES OF PARABOLA :

(1) Circle described on the focal length (distance) as diameter touches the tangent at the vertex.

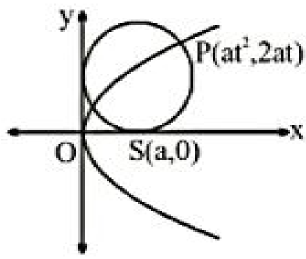
Equation of the circle described on SP as diameter is

$$(x - at^2)(x - a) + (y - 2at)(y - 0) = 0$$

Solve it with y-axis i.e. $x = 0$, we get

$$y^2 - 2aty + a^2t^2 = 0 \Rightarrow (y - at)^2 = 0$$

circle touches y-axis at $(0, at)$.



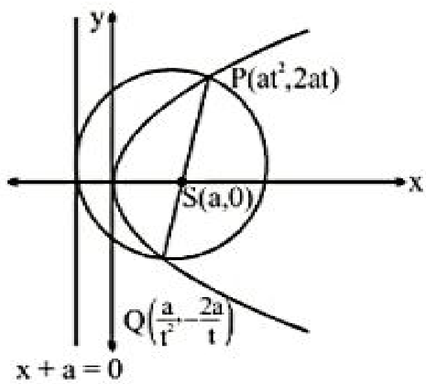
(2) Circle described on the focal chord as diameter touches directrix

Equation of the circle described on PQ as diameter is

$$(x - at^2)\left(x - \frac{a}{t^2}\right) + (y - 2at)\left(y + \frac{2a}{t}\right) = 0$$

Solving it with $x = -a$

$$(-a - at^2)\left(-a - \frac{a}{t^2}\right) + (y - 2at)\left(y + \frac{2a}{t}\right) = 0$$



$$\Rightarrow \quad y^2 - 2a\left(t - \frac{1}{t}\right)y + a^2\left(t - \frac{1}{t}\right)^2 = 0$$

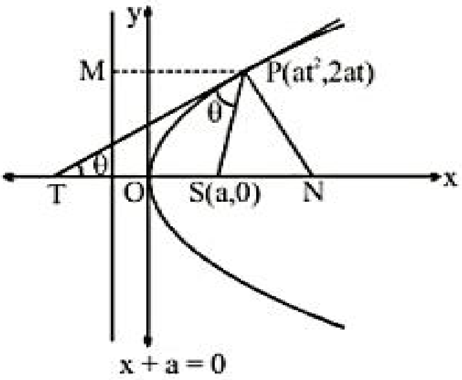
$$\Rightarrow \quad \left[y - a\left(t - \frac{1}{t}\right)\right]^2 = 0$$

$\Rightarrow$ circle tocuhes the directrix.

(3) Tangent at P is

$yt = x + at^2$, meet x-axis at T, then $T(-at^2, 0)$

Normal at P is $y + xt = 2at + at^3$, meet x-axis at N, then $N(2a + at^2, 0)$



$\Rightarrow$ $ST = SN = a + at^2 = PM = PS$

$\Rightarrow$ $\angle PTS = \angle TPS = \theta$

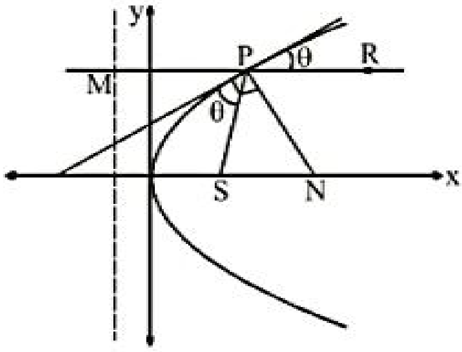
$\because$ $TS = PS = PM \Rightarrow \angle TPM = \theta$

Tangent and Normal at any point P bisect the angle between PS and PM internally and externally. This property leads to the *reflection property of parabola.*

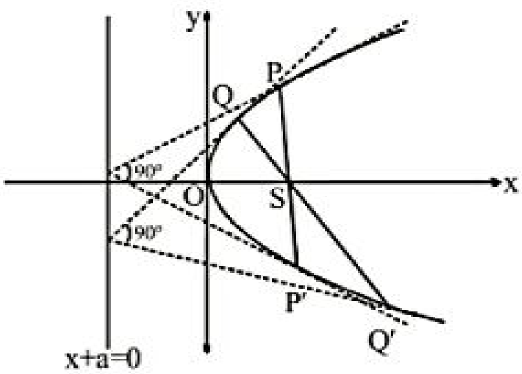
Circle circumscribing the triangle formed by any tangent normal and x-axis, has its centre at focus.

If we extend MP, then from figure $\angle RPN = \angle SPN = 90 - \theta$

Thus ray parallel axis meet parabola at P and after reflection from P it passes through the focus.



(4) The tangents at the extremities of a focal chord intersect at right angles on the directrix.



(5) The portion of tangent to the parabola intercepted between the directrix and the curve subtends a right angle at the focus.

tangent at $P(at^2, 2at)$ is $yt = x + at^2$ meet the directrix at $x = -a \Rightarrow Q\left(-a, \frac{at^2 - a}{t}\right)$ and $S(a, 0)$.

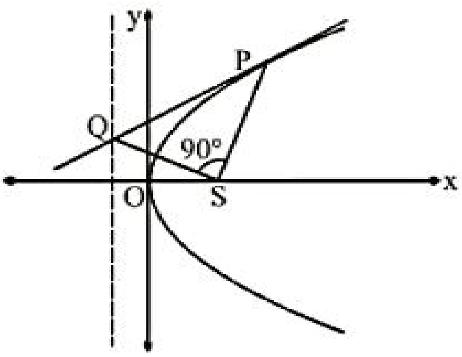
$$\text{Slope at SP} = \frac{2at - 0}{at^2 - a} = \frac{2t}{t^2 - 1} = m_1$$

$$\text{Slope at SQ} = \frac{\frac{at^2 - a}{t} - 0}{-a - 0} = \frac{t^2 - 1}{-2t} = m_2.$$

$\Rightarrow$ $m_1 m_2 = -1$

$\Rightarrow$ $SP \perp SQ$

$\Rightarrow$ $\angle PSQ = 90°$



(6) Tangent at P is $yt = x + at^2$ ......(i)

Line perpendicular to above line is $xt + y = \lambda$

and passes through (a, 0) gives $\lambda = at$

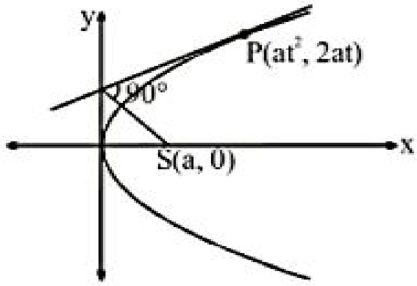
$\therefore$ perpendicular line will be $xt + y = at$ .....(ii)

Solve (i) and (ii), we get

$x = 0$

i.e., these two line intersect at y-axis i.e. tangnet at the vertex.

The foot of the perpendicular from the focus on any tangent to a parabola lies on the tangent at vertex.



(7) **Tangents and Normals** at the extremities of the latus rectum of a parabola $y^2 = 4ax$ **constitute a square**, their points of intersection being (– a, 0) & (3a, 0).

(8) The **circle circumscribing the triangle** formed by **any three tangents** to a parabola passes through the focus.

(9) The **orthocentre** of any triangle formed by three tangents to a parabola $y^2 = 4ax$ lies on the directrix & has the co–ordinates $(-a,\ a(t_1 + t_2 + t_3 + t_1t_2t_3))$.

(10) The area of the triangle formed by three points on a parabola is **twice the area** of the triangle formed by the tangents at these points.

Illustration :

If incident ray from point (– 3, 2) parallel to the axis of parabola $y^2 = 4x$ strike the parabola, then find the equation of reflected ray.

Sol. *Since incident ray strikes parabola at P(1, 2) i.e. extremity of latus rectum and it will pass through the focus of parabola therefore reflected ray will be parallel to y-axis and its equation will be x = 1.*

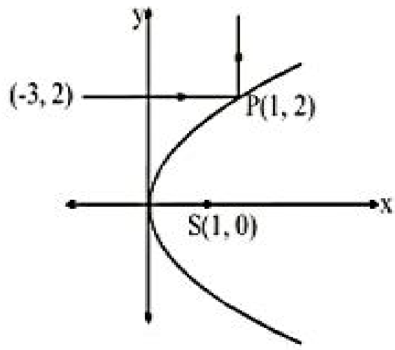


Illustration :

A ray of light moving parallel to the x-axis get reflected from a parabolic mirror $(y-2)^2 = 4(x + 1)$. Find the point on the axis of parabola through which the ray must pass after reflection.

Sol. *Axis of parabola is y = 2 i.e., parallel to x-axis. As we know if incident ray is parallel to x-axis then after reflextion it will pass through the focus of parabola and focus is (0, 2).* ***Ans.***

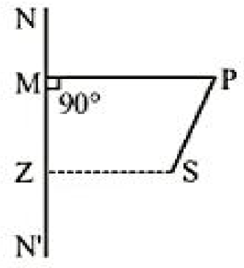
ELLIPSE

DEFINITION :

An ellipse is the locus of the point which moves in a plane such that the ratio of its distance from a fixed point (focus) to fixed straight line (directrix) is always constant (called eccentricity).

In the given figure, S is the focus and NN' is the directrix.
Let P be a point on the ellipse, then

$$\frac{PS}{PM} = e, \quad e < 1 \quad \text{(for ellipse)}$$



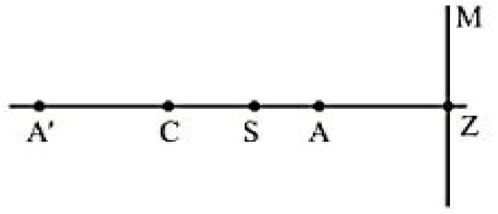
Thus, we can find the equation of an ellipse when the coordinates of its focus, equation of the directrix and eccentricity (e) are given.

STANDARD EQUATION OF AN ELLIPSE :

Let S be the focus & ZM is the directrix of an ellipse. Draw perpendicular from S to the directrix which meet it at Z. A moving point is on the ellipse such that

$$PS = ePM$$

then there is point lies on the line SZ and which divide SZ internally at A and externally at A' in the ratio of e : 1.



therefore $\quad SA = e\,AZ \quad$ ...(i)

$\quad SA' = eA'Z \quad$ ...(ii)

Let AA' = 2a & take C as mid point of AA'

$\therefore \quad CA = CA' = a$

Add (i) & (ii)

$$SA + SA' = e\,(AZ + A'Z)$$

$\Rightarrow \quad AA' = e[CZ - CA + CA' + CZ]$

$$2a = 2eCZ$$

$\Rightarrow \quad CZ = \dfrac{a}{e} \quad$ ...(iii)

Subtract (ii) & (i), we get

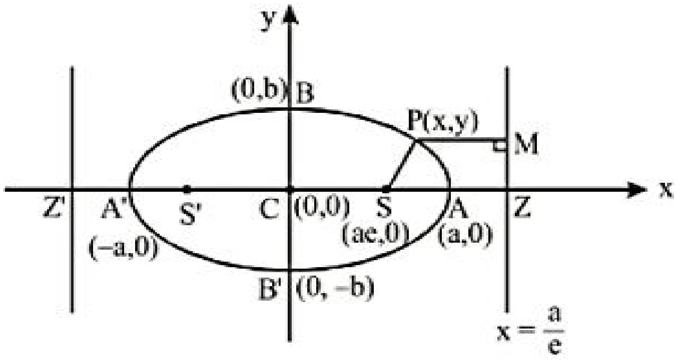
$$SA' - SA = e\,(A'Z - AZ)$$

$\Rightarrow \quad (CA' + CS) - (CA - CS)$

$\quad = e\,[(CA' + CZ) - (CZ - CA)]$

$\Rightarrow \quad 2CS = 2e\,CA$

$\therefore \quad CS = ae \quad$ ...(iv)



Result (iii) & (iv) are independent of axis.

Consider CZ line as x-axis, C as origin & perpendicular to this line & passes through C is considered as y-axis. Let P(x, y) is a moving point, then

By defintion of ellipse.

$PS = ePM \quad \Rightarrow \quad (PS)^2 = e^2 (PM)^2$

$$\Rightarrow \quad (x - ae)^2 + (y - 0)^2 = e^2\left(\frac{a}{e} - x\right)^2 \quad \Rightarrow \quad (x - ae)^2 + y^2 = (a - ex)^2$$

$$\Rightarrow \quad x^2 + a^2e^2 - 2xae + y^2 = a^2 + e^2x^2 - 2xae \quad \Rightarrow \quad x^2(1 - e^2) + y^2 = a^2(1 - e^2)$$

$$\Rightarrow \quad \frac{x^2}{a^2} + \frac{y^2}{a^2(1-e^2)} = 1 \quad \text{or} \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \text{where } b^2 = a^2(1 - e^2)$$

Tracing of an ellipse :

Given ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ ...(1)

(i) If we put $y = 0$ then we see that ellipse cuts x-axis at $(\pm a, 0)$.

(ii) If we put $x = 0$ then we see that ellipse cuts y-axis at $(0, \pm b)$.

(iii) Equation of ellipse does not change when y is replaced by $-y$. Hence, ellipse is symmetrical about x-axis. (Since equation contain even power of y therefore curve is symmetric about x-axis).

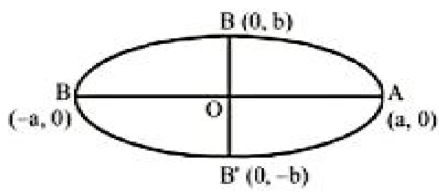
(iv) When x is replaced by $-x$, the equation of curve does not change therefore ellipse is symmetrical about y-axis. (Since equation contain even power of x therefore curve is symmetric about y-axis).

(v) From (1), $y = \pm \frac{b}{a}\sqrt{a^2 - x^2}$, Since y is real. $\therefore a^2 - x^2 \geq 0$ or $-a \leq x \leq a$

Also from (1), $x = \pm \frac{a}{b}\sqrt{b^2 - y^2}$, Since x is real. $\therefore b^2 - y^2 \geq 0$ or $-b \leq y \leq b$

Hence ellipse lies entirely between the lines $x = -a$ and $x = a$ and the lines $y = -b$ and $y = b$,

Thus an ellipse is a closed curve. Since curve is symmetrical about both axis, therefore first of all we draw its graph only in the first quadrant and then we will take its image in both axis.



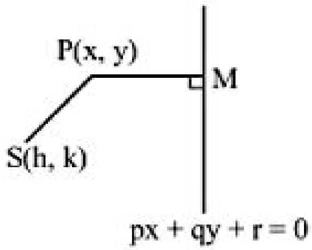
FACTS ABOUT AN ELLIPSE :

(1) By the symmetry of equation of ellipse, if we take second focus S'(–ae, 0) & second directrix $x = -\frac{a}{e}$ & perform same calculation, we get same equation of ellipse, therefore there are two focii & two directrix of an ellipse. The two focii of ellipse are (ae, 0) and (–ae, 0) and the two corresponding directrices are lines $x = \frac{a}{e}$ and $x = \frac{-a}{e}$. If focus of the ellipse is taken as (ae, 0), then corresponding directrix is $x = \frac{a}{e}$ and if focus is (–ae, 0), then corresponding directrix is $x = \frac{-a}{e}$.

(2) If equation of directrix is $px + qy + r = 0$ & focus is (h, k) then its equation will be

$$PS^2 = e^2\,PM^2$$

$$(x-h)^2 + (y-k)^2 = e^2 \cdot \left(\frac{px+qy+r}{\sqrt{p^2+q^2}}\right)^2$$



(3) Distance between foci $SS' = 2ae$ & distance between directrix $ZZ' = 2\frac{a}{e}$.

(4) Degree of flatness of an ellipse is also called on eccentricity & written as

$$e = \frac{CS}{CA} = \frac{\text{Distance from centre to focus}}{\text{Distance from centre to vertex}}$$

If $e \to 0 \Rightarrow b \to a \Rightarrow$ foci becomes closer & move towards centre and ellipse becomes circle.
If $e \to 1 \Rightarrow b \to 0 \Rightarrow$ ellipse get thinner & thinner

(5) Two ellipse are said to be similar if they have same eccentricity.

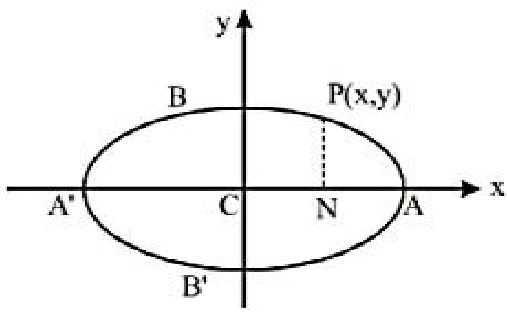
(6) Distance of focus from the extremity of minor axis is equal to 'a' because $a^2e^2 + b^2 = a^2$

(7) Let P(x, y) be any point on the ellipse.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad \therefore \qquad \frac{y^2}{b^2} = 1 - \frac{x^2}{a^2}$$

$$\Rightarrow \quad \frac{y^2}{b^2} = \frac{(a-x)(a+x)}{a^2} \quad \Rightarrow \quad \frac{PN^2}{b^2} = \frac{AN \cdot A'N}{a^2}$$

$$\Rightarrow \quad \frac{PN^2}{AN \cdot A'N} = \frac{b^2}{a^2}$$



(8) By definition of ellipse, the distance of any point P on the ellipse from focus = e (the distance of point P from the corresponding directrix).

BASIC TERMS RELATED TO AN ELLIPSE :

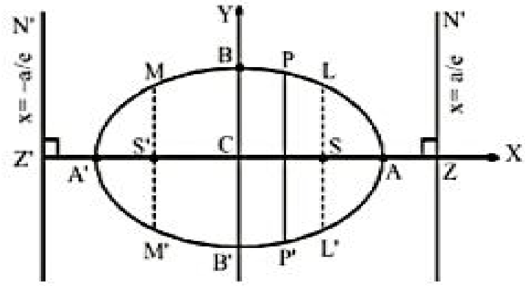
Let the equation of the ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$ ....(1)

1. Centre :

In the figure, C is the centre of the ellipse. All chords passing through C are called diameter and bisected at C.

2. Foci :

S and S' are the two foci of the ellipse and their coordinates are (ae, 0) and (–ae, 0) respecitvely. The line containing two foci are called the focal axis and the distance between S & S' the focal length



3. Directrices :

ZN and Z'N' are the two directices of the ellipse and their equations are $x=\frac{a}{e}$ and $x=\frac{-a}{e}$ respectively.

Here Z and Z' are called foot of directrix.

4. Axes :

The line segments A'A and B'B are called the major and minor axes respectively of the ellipse.
The point of intersection of major and minor axis is called centre of the ellipse. Major and minor axis together are called principal axis of ellipse.

Here Semi-major axis are $CA = CA' = a$
and Semi-minor axis are $CB = CB' = b$

5. Vertex :

The points where major axis meet the ellipse is called its vertices. In the given figure, A' and A are the vertices of the ellipse.

6. Ordinate and double ordinates :

Let P be a point on the ellipse. From P we draw PM perpendicular to major axis of the ellipse. Produce PM to meet the ellipse at P', then PM is called an ordinate and PMP' is called the double ordinate of the point P.
It is also defined as any chord perpendicular to major axis is called its double ordinate.

7. Latus rectum :

When double ordinate passes through focus then it is called the Latus rectum.
Let $L'L = 2k$, then $LS = k$ so $L = (ae, k)$.
Here LL' and MM' are called latus rectum.

Since L (ae, k) lies on the ellipse (1), therefore $\frac{a^2e^2}{a^2}+\frac{k^2}{b^2}=1$ or $\frac{k^2}{b^2}=1-e^2$

or $\quad k^2 = b^2(1-e^2) = b^2 \cdot \frac{b^2}{a^2} = \frac{b^4}{a^2} \qquad [\because b^2 = a^2(1-e^2)]$

$\therefore \quad k = \frac{b^2}{a}$

Hence length of semi latus rectum $LS = \frac{b^2}{a} = MS'$

i.e. length of the latus rectum LL' or $MM' = \frac{2b^2}{a} = \frac{(\text{minor axis})^2}{\text{major axis}}$

$= 2a(1-e^2)$

$= 2e$ (distance from focus to the corresponding directrix).

And the end points of latus rectum are $L\left(ae, \frac{b^2}{a}\right), L'\left(ae, \frac{-b^2}{a}\right), M\left(-ae, \frac{b^2}{a}\right) \& M'\left(-ae, -\frac{b^2}{a}\right)$

8. Focal chord :

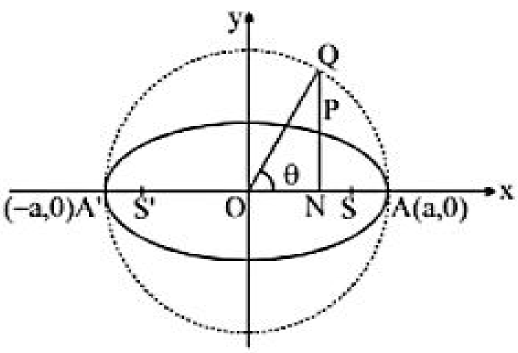
A chord of the ellipse passing through its focus is called a focal chord.

AUXILIARY CIRCLE AND ECCENTRIC ANGLE :

A circle described on major axis as diameter is called the **auxiliary circle** of the given ellipse & its equation is

$$x^2 + y^2 = a^2 \qquad \ldots (1)$$

and given ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad \ldots (2)$



Let Q be a point on the auxiliary circle $x^2 + y^2 = a^2$ then line through Q and perpendicular to x-axis meet the ellipse at P then P and Q are called the CORRESPONDING POINTS on the ellipse & the auxiliary circle respectively. Here $\angle QOA = \theta$ is called the ECCENTRIC ANGLE of the point P on the ellipse $(0 \le \theta < 2\pi)$.

Since Q lie on the circle therefore $Q(a\cos\theta, a\sin\theta)$

So coordinate of $P(a\cos\theta, y)$, which satisfy the equation of ellipse.

$\therefore \quad \frac{a^2\cos^2\theta}{a^2} + \frac{y^2}{b^2} = 1 \Rightarrow y = b\sin\theta$

$\therefore$ coordinate of P will be $(a\cos\theta, b\sin\theta)$ and this is called parametric equation of ellipse.

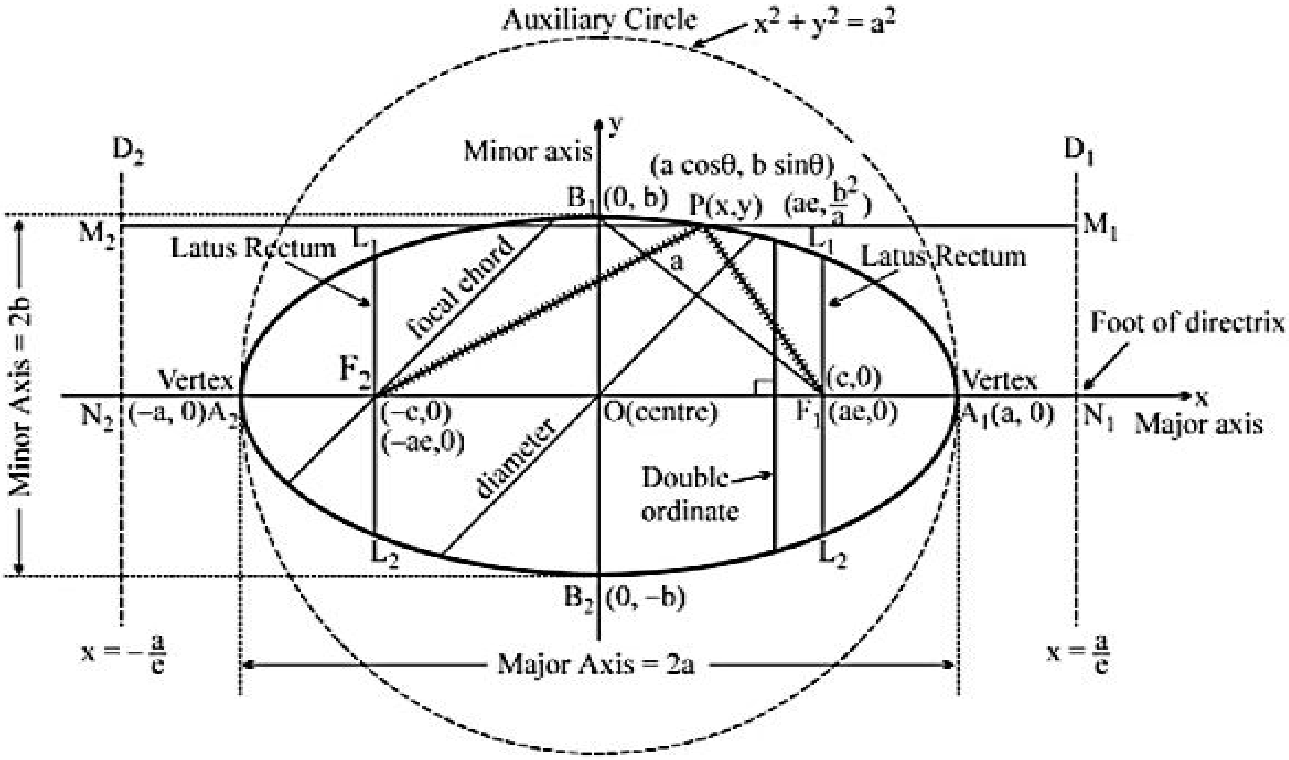
The equations $x = a\cos\theta$ & $y = b\sin\theta$ together represent the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Where θ is an eccentric angle of point P

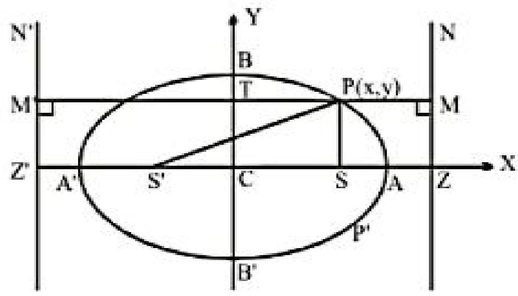
We observe that $\frac{\ell(PN)}{\ell(QN)} = \frac{b}{a} = \frac{\text{Semi minor axis}}{\text{Semi major axis}}$

Hence " If from each point of a circle perpendiculars are drawn upon a fixed diameter then the locus of the points dividing these perpendiculars in a given ratio is an ellipse of which the given circle is the auxiliary circle". This another defintion of ellipse.

ELLIPSE AT A GLANCE :



FOCAL DISTANCE OF A POINT :



Let P(x, y) be any point on the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \dots (1)$$

Then by definition of ellipse,

$$SP = ePM = e(MT - PT) = e\left(\frac{a}{e} - x\right) = a - ex$$

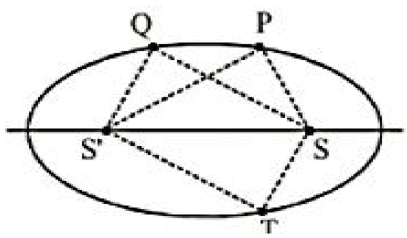
& $$S'P = ePM' = e(M'T + PT) = e\left(\frac{a}{e} + x\right) = a + ex$$

Hence $SP + S'P = 2a$

Because of the above property, **ellipse is also defined** as the locus of a point which moves in a plane such that the sum of its distance from two fixed points (called foci) is a contant (Length of major axis).

This defintion is called the physical definition of the ellipse.

Hence $PS + PS' = QS + QS' = TS + TS' =$ length of major axis



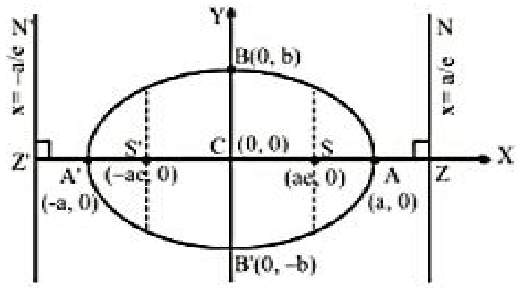
TWO STANDARD FORMS OF ELLIPSE :

There are two standard forms of ellipse with centre at the origin and axes along coordinate axes. The foci of the ellipse are either on the x-axis or on the y-axis.

1. Major axis along x-axis :

The equation of this type of ellipse is of the form $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, where $a>b>0$ and $b=a\sqrt{1-e^2}$.

For this ellipse :

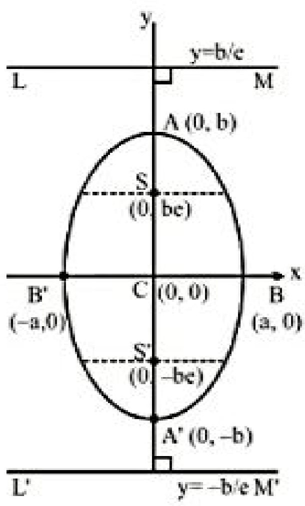


(i) Major axis is 2a
(ii) Minor axis is 2b.
(iii) Centre is (0, 0)
(iv) Vertices are $(\pm a, 0)$
(v) Foci are $(\pm ae, 0)$
(vi) Equation of directrices are $x=\pm\frac{a}{e}$
(vii) Equation of major axis is $y=0$
(viii) Equation of minor axis is $x=0$
(ix) Length of latus rectum $=\frac{2b^2}{a}$
(x) Extremity of latus rectum is $\left(\pm ae, \pm\frac{b^2}{a}\right)$

2. Major axis along y-axis :

The equation of this type of ellipse is of the form $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. where $0<a<b$ and $a=b\sqrt{1-e^2}$.

For this ellipse :



(i) Major axis is 2b
(ii) Minor axis is 2a.
(iii) Centre is (0, 0)
(iv) Vertices are $(0, \pm b)$
(v) Foci are $(0, \pm be)$
(vi) Equation of directrices are $y=\pm\frac{b}{e}$
(vii) Equation of major axis is $x=0$
(viii) Equation of minor axis is $y=0$
(ix) Length of latus rectum $=\frac{2a^2}{b}$
(x) Extremity of latus rectum is $\left(\pm\frac{a^2}{b}, \pm be\right)$

COMPARISON CHART BETWEEN STANDARD ELLIPSE :

Basic Elements	$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$	
	a > b	**a < b**
Centre	(0, 0)	(0, 0)
Vectrex	$(\pm a, 0)$	$(0, \pm b)$
Length of major axis	2a	2b
Length of minor axis	2b	2a
Foci	$(\pm ae, 0)$	$(0, \pm be)$
Equation of directrix	$x=\pm\frac{a}{e}$	$y=\pm\frac{b}{e}$
Relation among a, b, & e	$b^2=a^2(1-e^2)$	$a^2=b^2(1-e^2)$
Equation of major axis	$y=0$	$x=0$
Equation of minor axis	$x=0$	$y=0$
Length of latus rectum	$\frac{2b^2}{a}$	$\frac{2a^2}{b}$
End of latus rectum	$\left(\pm ae, \pm\frac{b^2}{a}\right)$	$\left(\pm\frac{a^2}{b}, \pm be\right)$
Focal distances of $P(x_1, y_1)$	$a \pm ex_1$	$b \pm ey_1$
SP + SP'	2a	2b
Distance between foci	2ae	2be
Distance between directrix	$\frac{2a}{e}$	$\frac{2b}{e}$
Parametric equation	$(a\cos\theta, b\sin\theta)$ $(0<\theta<2\pi)$	$(a\cos\theta, b\sin\theta)$

If the equation of the ellipse is given as $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ & nothing is mentioned then the rule is to assume that $a > b$.

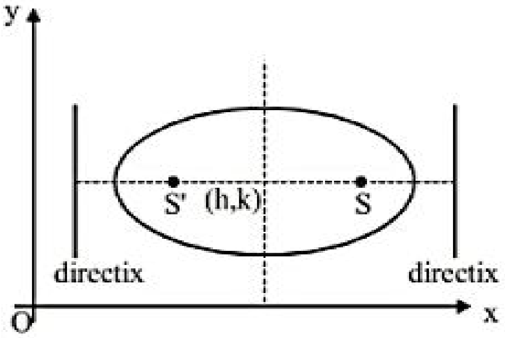
To find the Various Parameter of an ellipse :

Equation of an ellipse whose axis are parallel to coordiante axis & its centre is (h, k). The foci of the ellipse are either on x-axis or on the y-axis.

(I) Major axis parallel to x-axis :

Here the equation of ellipse is of the form $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$, where $a > b$ & $b^2=a^2(1-e^2)$

Here the equation of the ellipse is of the form $\frac{X^2}{a^2}+\frac{Y^2}{b^2}=1$, where $X=x-h$ and $Y=y-k$.



(1) Equation of major axis is $Y = 0$, i.e., $y - k = 0$
Equation of minor axis is $X = 0$, i.e., $x - h = 0$

(2) Coordinate of centre of the ellipse are given by
$X = 0$ and $Y = 0$ i.e, $x - h = 0$ and $y - k = 0$
$\therefore$ Centre of the ellipse is (h, k)

(3) Coordinate of foci of the ellipse are given by
$X = \pm ae, Y = 0$ i.e., $x - h = \pm ae$ and $y - k = 0$
$\therefore$ Hence foci of the ellipse are $(h \pm ae, k)$

(4) Equation of the directrices of the ellipse are $X = \pm\frac{a}{e}$, i.e., $x - h = \pm\frac{a}{e}$.

Thus directrices are $x = h \pm \frac{a}{e}$

(5) Coordinate of ends of latera recta are given by $X = \pm ae, Y = \pm\frac{b^2}{a}$ i.e. $x - h = \pm ae, y - k = \pm\frac{b^2}{a}$

Therefore ends of latera recta are given by $\left(h \pm ae, \ k \pm \frac{b^2}{a}\right)$

(6) Coordinate of vertices of the ellipse are given by $X = \pm a, Y = 0$ i.e., $x - h = \pm a, y - k = 0$.
Hence vertices are $(h \pm a, k)$

(II) Major axis parallel to y-axis :

Here the equation of ellipse is of the form $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$, where $a < b$ & $a^2 = b^2(1 - e^2)$

Equation (1) is of the form $\frac{X^2}{a^2} + \frac{Y^2}{b^2} = 1$, where $X = x - h$ and $Y = y - k$

(1) Equation of major axis is $X = 0$, i.e., $x - h = 0$
Equation of minor axis is $Y = 0$, i.e., $y - k = 0$

(2) Coordinate of centre of the ellipse are given by $X = 0$ and $Y = 0$
$\Rightarrow$ $x - h = 0$ and $y - k = 0$
$\therefore$ Centre of the ellipse is (h, k)

(3) Coordinate of foci of the ellipse are given by $X = 0, Y = \pm be$
$x - h = 0$ & $y - k = \pm be$
$x = h$ & $y = k \pm be$
$\therefore$ Foci are $(h, k \pm be)$

(4) Equation of the directrices of the ellipse are $Y = \pm\frac{b}{e}$, i.e,. $y - k = \pm\frac{b}{e}$

Thus directrices are $y = k \pm \frac{b}{e}$

(5) Coordinate of ends of latera recta are given by

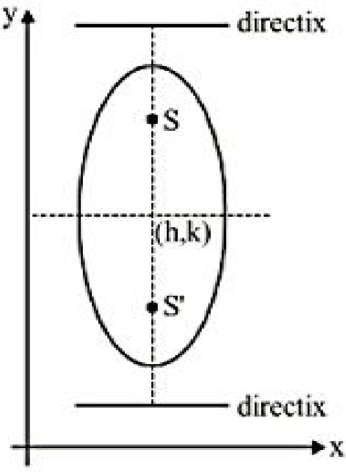
$$X = \pm\frac{a^2}{b} \quad \& \quad Y = \pm ae,$$

i.e. $x - h = \pm\frac{a^2}{b}$ & $y - k = \pm be,$

or $x = h \pm \frac{a^2}{b}$ & $y = k \pm be$

$\therefore$ Coordinates of ends of latera recta are given by

$$\left(h \pm \frac{a^2}{b},\ k \pm be\right)$$



(6) Coordinates of vertices of the ellipse is given by $X = 0$ & $Y = \pm b$
i.e., $x - h = 0$ and $y - k = \pm b$ therefore coordinates of vertex are $(h, k \pm b)$

Comparison chart between above two ellipse :

Basic Elements		$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$	
		a > b	**a < b**
1.	Length of major axis Length of minor axis	$2a$ $2b$	$2b$ $2a$
2.	Equation of major axis Equation of minor axis	$y - k = 0$ $x - h = 0$	$x - h = 0$ $y - k = 0$
3.	Centre of ellipse	(h, k)	(h, k)
4.	Eccentricity	$e = \sqrt{1 - \frac{b^2}{a^2}}$	$e = \sqrt{1 - \frac{a^2}{b^2}}$
5.	Foci	$(h \pm ae, k)$	$(h, k \pm be)$
6.	Equation of directrix	$x = h \pm \frac{a}{e}$	$y = k \pm \frac{b}{e}$
7.	Extremities of latus rectum	$\left(h \pm ae, k \pm \frac{b^2}{a}\right)$	$\left(h \pm \frac{a^2}{b}, k \pm be\right)$
8.	Vertices of an ellipse	$(h \pm a, k)$	$(h, k \pm b)$
9.	Length of latus rectum	$\frac{2b^2}{a}$	$\frac{2a^2}{b}$

EQUATION OF CHORD OF AN ELLIPSE :

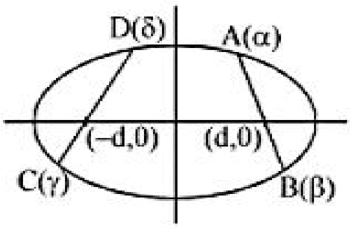
Equation of a chord of an ellipse joining two points P(α) and Q(β) on it is equal to

$$\frac{x}{a}\cos\left(\frac{\alpha+\beta}{2}\right)+\frac{y}{b}\sin\left(\frac{\alpha+\beta}{2}\right)=\cos\left(\frac{\alpha-\beta}{2}\right)$$

(use formula of line joining points P(a cosα, b sinα) and Q(a cosβ, b sinβ))

If this particular chord passes through (d, 0) then we have

$$\frac{d}{a}\cos\left(\frac{\alpha+\beta}{2}\right)=\cos\left(\frac{\alpha-\beta}{2}\right)\ ;\ \frac{\cos\left(\frac{\alpha+\beta}{2}\right)}{\cos\left(\frac{\alpha-\beta}{2}\right)}=\frac{a}{d}$$



Using componendo and dividendo rule

$$\frac{\cos\left(\frac{\alpha+\beta}{2}\right)-\cos\left(\frac{\alpha-\beta}{2}\right)}{\cos\left(\frac{\alpha-\beta}{2}\right)+\cos\left(\frac{\alpha-\beta}{2}\right)}=\frac{a-d}{a+d}$$

or $\quad -\dfrac{2\sin\alpha/2\,\sin\beta/2}{2\cos\alpha/2\,\cos\beta/2}=\dfrac{a-d}{a+d}$ i.e. $\quad \tan\dfrac{\alpha}{2}\tan\dfrac{\beta}{2}=\dfrac{d-a}{d+a}$

if $d=ae$ i.e. **PQ is a focal chord** then $\tan\dfrac{\alpha}{2}\tan\dfrac{\beta}{2}=\dfrac{e-1}{e+1}$

Illustration :

Find the equation of the ellipse (referred to its axis as the x-axis & y-axis) whose foci are (±2, 0) & eccentricity $=\dfrac{1}{2}$.

Sol. *Let* $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ *represent ellipse where* $e=\dfrac{1}{2}$

given $(\pm ae,\ 0)=(\pm 2,\ 0)$

$\Rightarrow\quad ae=2 \quad\Rightarrow\quad a=4 \quad\Rightarrow\quad b^2=a^2(1-e^2)$ *gives* $b^2=12$

$\therefore$ *Equation of ellipse is* $\dfrac{x^2}{16}+\dfrac{y^2}{12}=1$ ***Ans.***

Illustration :

Find the equation of an ellipse, referred to its axes as the axes of coordinates, with foci (± 2, 0) and latus rectum is 6 units.

Sol. $ae = 2$

Mid point of focus is centre (0, 0); $\frac{2b^2}{a} = 6 \Rightarrow b^2 = 3a$

$$e^2 = 1 - \frac{b^2}{a^2} \quad \Rightarrow \quad a^2e^2 = a^2 - b^2$$

$\Rightarrow \quad 4 = a^2 - 3a \quad \Rightarrow \quad a^2 - 3a - 4 = 0$

$\Rightarrow \quad (a-4)(a+1) = 0$

$a = 4, -1$

a cannot be negative, hence a = 4

$\therefore \quad b = 2\sqrt{3}$

So the equation of ellipse is $\frac{x^2}{16} + \frac{y^2}{12} = 1$

$\boxed{3x^2 + 4y^2 = 48}$ ***Ans.***

Illustration :

Find the eccentricity of an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ *whose latus rectum is half of its major axis.*

Sol. According to question $\frac{2b^2}{a} = a$

$\Rightarrow \; 2b^2 = a^2 \;\Rightarrow\; 2a^2(1 - e^2) = a^2 \;\Rightarrow\; e = \frac{1}{\sqrt{2}}$

Illustration :

If the focal distance of the end of minor axis of an ellipse is q & distance between its foci is 2p, then find its equation.

Sol. Let the equation of ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ $\quad (a^2 + b^2 = a^2e^2, b^2 = p^2 - q^2)$

According to question $2ae = 2p$ ...(i)

& $a = q$

$\Rightarrow \; b^2 = a^2e^2 - a^2 \Rightarrow \; b^2 = q^2 - p^2$

$\Rightarrow$ Equation is $\frac{x^2}{q^2} + \frac{y^2}{q^2 - p^2} = 1$ ***Ans.***

Illustration :

Find the equation of the ellipse having axes along the coordinate axes and passing through the points (4, 3) and (–1, 4).

Sol. Let the equation of the required ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ ... (1)

(4, –3) is on the ellipse $\therefore \frac{4^2}{a^2}+\frac{(-3)^2}{b^2}=1$ or $\frac{16}{a^2}+\frac{9}{b^2}=1$ ... (2)

(–1, 4) is also on the ellipse $\therefore \frac{(-1)^2}{a^2}+\frac{4^2}{b^2}=1$ or $\frac{1}{a^2}+\frac{16}{b^2}=1$ ... (3)

(3) × 16 $\Rightarrow \frac{9}{a^2}+\frac{256}{b^2}=16$ ... (4)

(4) – (2) $\Rightarrow \frac{247}{b^2}=15 \Rightarrow b^2=\frac{247}{15}$

From (3), $\frac{1}{a^2}+16\left(\frac{15}{247}\right)=1 \Rightarrow \frac{1}{a^2}=1-\frac{240}{247}=\frac{7}{247} \Rightarrow a^2=\frac{247}{7}$

$\therefore$ The equation of the required ellipse is $\frac{x^2}{\frac{247}{7}}+\frac{y^2}{\frac{247}{15}}=1$ or $7x^2+15y^2=247$

Illustration :

Find the equation of curve whose parametric equation are $x = 1 + 4\cos\theta, y = 2 + 3\sin\theta$. Also find its eccentricity.

Sol. $\cos\theta=\frac{x-1}{4}$ & $\sin\theta=\frac{y-2}{3}$

$\because \sin^2\theta+\cos^2\theta=1 \Rightarrow \frac{(x-1)^2}{16}+\frac{(y-2)^2}{9}=1$

$\therefore e=\sqrt{1-\frac{b^2}{a^2}}=\sqrt{1-\frac{9}{16}}=\frac{\sqrt{7}}{4}$ **Ans.**

Illustration :

Find the eccentric angle of the point on the ellipse $\frac{x^2}{6}+\frac{y^2}{2}=1$ whose distance from the centre is 2.

Sol. Any point on the ellipse is $\left(\sqrt{6}\cos\phi, \sqrt{2}\sin\phi\right)$, where ϕ is an eccentric angle of the point.
It's distance from the center (0, 0) is given and equal to 2, therefore

$6\cos^2\phi+2\sin^2\phi=4$ or $3\cos^2\phi+\sin^2\phi=2$

$2\cos^2\phi=1 \Rightarrow \cos\phi=\pm\frac{1}{\sqrt{2}}$; $\phi=\frac{\pi}{4},\frac{3\pi}{4},\frac{5\pi}{4},\frac{7\pi}{4}$. **Ans.**

Illustration :

Find the eccentric angles of the extremity of latus rectum lie in the first quadrant of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$

Sol. *The coordinate of any point on the ellipse* $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ *whose eccentric angle* θ *are* $(a\cos\theta, b\sin\theta)$ *.The coordinate of the end point of latus rectum are* $\left(ae, \pm\frac{b^2}{a}\right)$

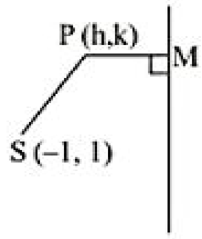
$\therefore$ *For Ist quadrant*

$a\cos\theta = ae$ and $b\sin\theta = \frac{b^2}{a}$; $\tan\theta = \frac{b}{ae} \Rightarrow \theta = \tan^{-1}\left(\frac{b}{ae}\right)$. ***Ans.***

Illustration :

Find the equation to the ellipse whose focus is the point (– 1, 1), whose directrix is the straight line $x - y + 3 = 0$ *and eccentricity is* $\frac{1}{2}$.

Sol. *Let* $P \equiv (h, k)$ *be moving point,*



$$e = \frac{PS}{PM} = \frac{1}{2}$$

$$\Rightarrow \quad (h+1)^2 + (k-1)^2 = \frac{1}{4}\left(\frac{h-k+3}{\sqrt{2}}\right)^2$$

$\Rightarrow$ *locus of P(h, k) is*

$$8\{x^2 + y^2 + 2x - 2y + 2\} = (x^2 + y^2 - 2xy + 6x - 6y + 9)$$

$$7x^2 + 7y^2 + 2xy + 10x - 10y + 7 = 0.$$ ***Ans.***

Illustration :

Find the equation to the ellipse whose centre is origin, axes are the axes of co-ordinate and passes through the points (2, 2) and (3, 1).

Sol. *Let the equation to the ellipse is* $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Since it passes through the points (2, 2) and (3, 1)

$$\therefore \quad \frac{4}{a^2} + \frac{4}{b^2} = 1 \qquad \text{..........(i)}$$

$$\text{and} \quad \frac{9}{a^2} + \frac{1}{b^2} = 1 \qquad \text{.........(ii)}$$

from (i) and (ii), we get

$$a^2 = \frac{32}{3} \quad \text{and} \quad b^2 = \frac{32}{5}$$

Ans.

Illustration :

Find the length and equation of major and minor axes, centre, eccentricity, foci, equation of directrices, vertices and length of the ellipse $16x^2 + y^2 = 16$.

Sol. *Given equaiton of the ellipse is* $16x^2 + y^2 = 16$ *or* $\frac{x^2}{1^2} + \frac{y^2}{4^2} = 1$ ... (1)

Here $a = 1$, $b = 4$ *and* $a < b$

Length of major axis $= 2b = 8$

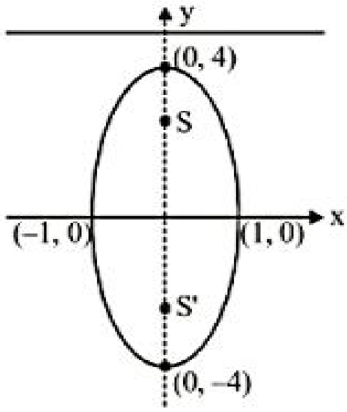
Length of minor axis $= 2a = 2$

Equation of major axis is $x = 0$

Equation of minor axis is $y = 0$

Coordiantes of centre are $(0, 0)$

Eccentricity of the ellipse,



$$e = \sqrt{1 - \frac{a^2}{b^2}} = \sqrt{1 - \frac{1}{16}} = \frac{\sqrt{15}}{4}$$

Coordinate of foci are given by

$$y = \pm be,\ x = 0 \qquad i.e., \qquad y = \pm\sqrt{15},\ x = 0$$

Hence foci are $(0, \pm\sqrt{15})$

Equation of directrices are $y = \pm\frac{b}{e}$ *or* $y = \pm\frac{16}{\sqrt{15}}$

Coordinates of vertices are given by $y = \pm b$ *and* $x = 0$ *i.e.,* $y = \pm 4$, $x = 0$

Hence vertices are $(0, \pm 4)$.

Illustration :

Find the equation of the ellipse whose focii are $(4, 0)$ *and* $(-4, 0)$ *and eccentricity is* $\frac{1}{3}$

Sol. *Since both focus lies on x-axis, therefore x-axis is major axis and mid point of focii is origin which is centre and a line perpendicular to major axis and passes throguh centre is minor axis which is y-axis, then equation of ellipse is*

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\because \quad 2ae = 8 \quad \text{and} \quad e = \frac{1}{3} \text{ (Given)}$$

$$\therefore \quad a = 12 \quad \text{and} \quad b^2 = a^2(1 - e^2)$$

$$\Rightarrow \quad b^2 = 144\left(1 - \frac{1}{9}\right) \quad \Rightarrow \quad b^2 = 16 \times 8 \quad \Rightarrow \quad b = 8\sqrt{2}$$

Equation of ellipse is $\frac{x^2}{144} + \frac{y^2}{128} = 1$ ***Ans.***

Illustration :

A rod of length 12cm moves with its ends always touching the coordiante axes. Determine the equation of the locus of a point on the rod, which is 3cm from the end in contact with the x-axis.

Sol. Let AB be the rod of length 12cm touching the coordiante axes at points A and B.

Let $A \equiv (a, 0)$, $B \equiv (0, b)$

Now $AB^2 = 12^2$

$\Rightarrow \quad a^2 + b^2 = 144 \qquad \dots (1)$

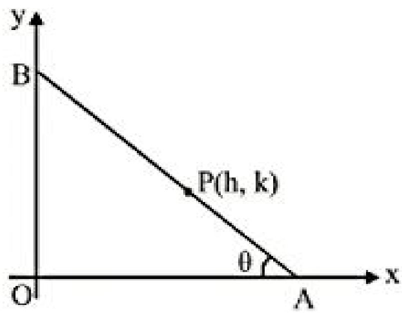
Let P be a point on AB such that

$AP = 3cm$

then $BP = 12cm - 3cm = 9cm$

$\therefore AP : PB = 1 : 3$

Hence P divides AB internally in the ratio 1 : 3.



$$\therefore \quad P \equiv \left(\frac{1.0+3.a}{1+3}, \frac{1.b+3.0}{1+3}\right) \quad or \quad P \equiv \left(\frac{3a}{4}, \frac{b}{4}\right)$$

$$\therefore \quad h = \frac{3a}{4} \text{ and } k = \frac{b}{4} \quad \Rightarrow \quad a = \frac{4h}{3} \text{ and } b = \frac{k}{3}$$

Put value of a and b in equaiton (1) we get $\dfrac{16h^2}{9} + 16k^2 = 144$

$\therefore \quad$ Locus of $P(h, k)$ is $\dfrac{16}{9}x^2 + 16y^2 = 144,$

i.e. $\quad x^2 + 9y^2 = 81$, which is the equation of the required locus.

Illustration :

If (5, 12) & (24, 7) are the foci of an ellipse passing through origin, then find the eccentricity of the ellipse.

Sol. Let the $S(5, 12)$ & $S(24, 7)$ are two foci & ellipse passes through origin O.

$\therefore \quad OS + OS' = 2a$

$\Rightarrow \quad \sqrt{25+144} + \sqrt{576+49} = 2a$

$\Rightarrow \quad 2a = 13 + 25 \quad \Rightarrow \quad a = 19 \qquad \& \qquad 2ae = SS' = \sqrt{386}$

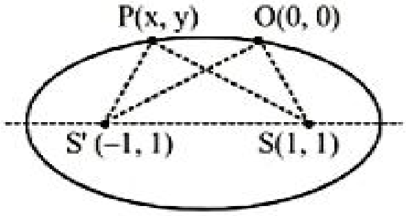
$\therefore \quad e = \dfrac{\sqrt{386}}{38}$ **Ans.**

Illustration :

Find the equaton of the ellipse that passes through the origin and has the foci at the points (–1, 1) and (1, 1).

Sol. *Let P(x, y) be any point on the ellipse and the foci be S(–1, 1) and S′(1, 1).*
and O lie on the ellipse
$OS + OS' = \text{constant} = 2a$

$$= \sqrt{(-1-0)^2+(1-0)^2} + \sqrt{(1-0)^2+(1-0)^2} = 2\sqrt{2}$$



Let P(x, y) be any point on the ellipse $\Rightarrow PS + PS' = 2\sqrt{2}$

$$\Rightarrow \quad \sqrt{(x+1)^2+(y-1)^2} + \sqrt{(x-1)^2+(y-1)^2} = 2\sqrt{2}$$

$$\Rightarrow \quad \sqrt{(x+1)^2+(y-1)^2} = 2\sqrt{2} - \sqrt{(x-1)^2+(y-1)^2}$$

$$\Rightarrow \quad (x+1)^2 + (y-1)^2 = 8 + [(x-1)^2 + (y-1)^2] - 4\sqrt{2}\sqrt{(x-1)^2+(y-1)^2}$$

$$\Rightarrow \quad x^2 + 2x + 1 + y^2 - 2y + 1 = 8 + x^2 - 2x + 1 + y^2 - 2y + 1 - 4\sqrt{2}\sqrt{(x-1)^2+(y-1)^2}$$

$$\Rightarrow \quad 4x - 8 = -4\sqrt{2}\sqrt{(x-1)^2+(y-1)^2}$$

$$\Rightarrow \quad x - 2 = -\sqrt{2}\sqrt{(x-1)^2+(y-1)^2}$$

$$\Rightarrow \quad x^2 - 4x + 4 = 2[x^2 - 2x + 1 + y^2 - 2y + 1]$$

$$\Rightarrow \quad 0 = x^2 + 2y^2 - 4y$$

$\Rightarrow \quad x^2 + 2y^2 - 4y = 0$ *is the required equation of the ellipse.*

Illustration :

Find the equation of the ellipse with foci at (±5, 0) and $x = \frac{36}{5}$ *as one directrix.*

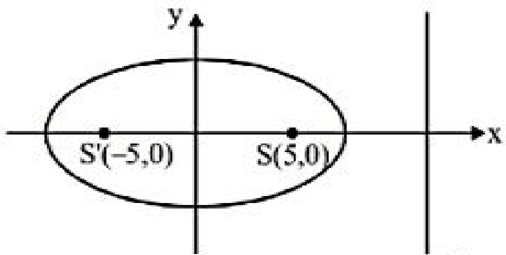
Sol. *Let S(5, 0) and S′ (–5, 0) be the two foci. Centre of the ellipse will be C(0, 0).*

Clearly, foci S and S′ lie on x-axis. Therefore, major axis of the ellipse will be the x-axis.
Let a and b be the length of semi-major and semi-minor axes respectively of the ellipse.
Then, $2ae = 10$ *or* $ae = 5$ ... (i)

Also equation of one directrix is given to be $x = \frac{36}{5}$

$$\therefore \quad \frac{a}{e} = \frac{36}{5} \qquad \ldots(ii)$$

$(i) \times (ii) \Rightarrow a^2 = 36 \ \therefore a = 6$ *From (i),* $e = \frac{5}{6}$



Now, $b^2 = a^2(1 - e^2) = 36\left(1 - \frac{25}{36}\right) = 11$

The required equation of the ellipse will be

$$\frac{(x-0)^2}{a^2} + \frac{(y-0)^2}{b^2} = 1 \text{ or } \frac{x^2}{36} + \frac{y^2}{11} = 1$$

Illustration :

Find the equation of the ellipse having major and minor axes along x and y axes respectively, the distance between whose foci is 8 units and the distance between the directrices is 18 units.

Sol. Given, 8 = distance between foci = $2ae$ ... (1)

and 18 = distance betwwen directrices $= \dfrac{2a}{e}$ ... (2)

$$(1) \times (2) \Rightarrow (8)\,(18) = (2ae)\left(\frac{2a}{e}\right) = 4a^2$$

$\Rightarrow \quad a^2 = 36 \Rightarrow a = 6$ ... (3)

Again, $8 = 2ae = 2(6)e = 12e$

$\Rightarrow \quad e = \dfrac{8}{12} = \dfrac{2}{3}$ ... (4)

Also, $b^2 = a^2(1 - e^2) = (6)^2\left[1 - \left(\dfrac{2}{3}\right)^2\right] = 36\left(\dfrac{5}{9}\right) = 20$ ... (5)

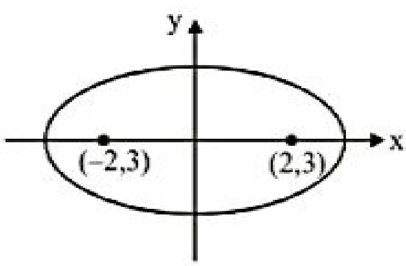
$\therefore$ Required equation of the ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \text{ or } \frac{x^2}{36} + \frac{y^2}{20} = 1 \qquad \text{[From (3) and (5)]}$$

Illustration :

Find the equation of the ellipse whose foci are (2, 3), (–2, 3) and whose semi-minor axis is of length $\sqrt{5}$.

Sol. Let $S_1(2, 3)$ and $S_2(-2, 3)$ be the two foci and let $2a$ and $2b$ denote the lengths of major and minor axes respectively, then, $b = \sqrt{5}$ and $2ae = S_1S_2 = 4$, where e is the eccentricity of the ellipse.



$\therefore \quad ae = 2$

$$b^2 = a^2(1 - e^2) \Rightarrow b^2 = a^2 - a^2c^2$$

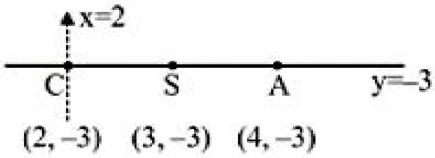
$\Rightarrow \quad S = a^2 - 4 \Rightarrow a = 3$

The major axes is $y = 3$ and centre is $(0, 3)$ – the mid point of the foci. Hence, equation of the ellipse is $\dfrac{x^2}{9} + \dfrac{(y-3)^2}{5} = 1$ **Ans.**

Illustration :

Find the equation of the ellipse having its centre at the point (2, –3), one focus at (3, –3) and one vertex at (4, –3).

Sol. Let $C = (2, -3)$, $S = (3, -3)$ and $A = (4, -3)$



Now $CA = \sqrt{(4-2)^2+(-3+3)^2} = 2 \therefore a = 2$

Again, $CS = \sqrt{(3-2)^2+(-3+3)^2} = 1$

$\therefore \quad ae = 1; \therefore e = \dfrac{1}{a} = \dfrac{1}{2} \quad \Rightarrow b^2 = a^2(1-e^2) = 4\left(1-\dfrac{1}{4}\right) = 3$

Major axis is $y = -3$ and parallel to x-axis

$\therefore$ Equation of ellipse is $\dfrac{(x-2)^2}{4}+\dfrac{(y+3)^2}{3} = 1$

Illustration :

Find the equation of axes, directrix, co-ordinate of focii, centre, vertices, length of latus - rectum and eccentricity of an ellipse $\dfrac{(x-3)^2}{25} + \dfrac{(y-2)^2}{16} = 1.$

Sol. Let $x - 3 = X$, $y - 2 = Y$, so equation of ellipse becomes as $\dfrac{X^2}{5^2} + \dfrac{Y^2}{4^2} = 1,$

$a = 5, b = 4$ and $a > b$

equation of major axis is $Y = 0 \quad \Rightarrow \quad y = 2.$

equation of minor axis is $X = 0 \quad \Rightarrow \quad x = 3.$

centre $(X = 0, Y = 0) \quad \Rightarrow \quad x = 3, y = 2$

$C \equiv (3, 2)$

Length of major axis $2a = 10$

Length of minor axis $= 2b = 8.$

Let 'e' be eccentricity, then

$\therefore \quad b^2 = a^2(1-e^2)$

$$e = \sqrt{\frac{a^2-b^2}{a^2}} = \sqrt{\frac{25-16}{25}} = \frac{3}{5}.$$

Length of latus rectum $= LL' = \dfrac{2b^2}{a} = \dfrac{2\times 16}{5} = \dfrac{32}{5}$

Co-ordinates focii are $X = \pm ae$, $Y = 0$

$x - 3 = \pm 5 \cdot \dfrac{3}{5}$ and $y - 2 = 0 \quad \Rightarrow \quad x = 3 \pm 3$ and $y = 2$ i.e. (6, 2) and (0, 2)

Illustration :

Find the centre, the length of the axes, and the eccentricity of the ellipse

$$2x^2+3y^2-4x+12y+13=0.$$

Sol. $2x^2+3y^2-4x+12y+13=0$

$2(x^2-2x)+3(y^2+4y)+13=0$

$2(x-1)^2+3(y+2)^2=1$

$$\frac{(x-1)^2}{\left(\frac{1}{2}\right)}+\frac{(y+2)^2}{\left(\frac{1}{3}\right)}=1$$

$\frac{X^2}{a^2}+\frac{Y^2}{b^2}=1$; where $X=x-1$, $Y=y+2$, $a=\frac{1}{\sqrt{2}}$, $b=\frac{1}{\sqrt{3}}$ & $a>b$

Centre is : $X=0$, $Y=0$; $x=1$, $y=-2$

Length of major axis $=2a=\sqrt{2}$ *and* *Length of minor axis* $=2b=\frac{2}{\sqrt{3}}$

If e denotes the eccentricity, then, $b^2=a^2(1-e^2)$

$\therefore \quad \frac{1}{3}=\frac{1}{2}(1-e^2)$; $e=\frac{1}{\sqrt{3}}$ ***Ans.***

Illustration :

Find the centre, the length of the axes, eccentricity and the foci of the ellipse.

$$12x^2+4y^2+24x-16y+25=0$$

Sol. *The given equation can be written in the form*

$$12(x+1)^2+4(y-2)^2=3$$

$$\Rightarrow \quad \frac{(x+1)^2}{1/4}+\frac{(y-2)^2}{3/4}=1 \qquad ...(1)$$

here $a^2=\frac{1}{4}$ *and* $b^2=\frac{3}{4}\Rightarrow a<b$

Co- ordinates of centre of the ellipse are given by $x+1=0$ *and* $y-2=0$

Hence centre of the ellipse is $(-1, 2)$

$\therefore$ *Length of major axis* $=2a=\sqrt{3}$ *and* *Length of minor axis* $=2b=1$

i.e. $a=\frac{\sqrt{3}}{2}$, $b=\frac{1}{2}$

Since $b^2=a^2(1-e^2)$ $\therefore 1/4=3/4(1-e^2)\Rightarrow e=\sqrt{2/3}$ $\quad\therefore ae=\frac{\sqrt{3}}{2}\times\frac{\sqrt{2}}{\sqrt{3}}=\frac{1}{\sqrt{2}}$

Co-ordinates of foci are given by $x+1=0$, $y-2=\pm ae$

Thus foci are $\left(-1, 2\pm\frac{1}{\sqrt{2}}\right)$ ***Ans.***

Illustration :

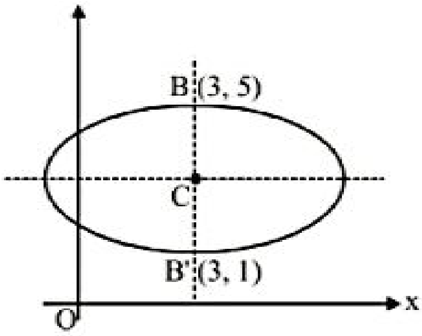
Find the equation of the ellipse the extremities of whose minor axis are (3, 1) and (3, 5) and whose eccentricity is 1/2.

Sol. Let C be the centre of the ellipse.

Let $B' \equiv (3, 1)$ and $B \equiv (3, 5)$, then $C \equiv (3, 3)$ [Since C is the mid-point of BB']

Also $BB' = 4$ $\therefore$ $2b = 4 \Rightarrow b = 2$

Also, slope of $BB' = \dfrac{5-1}{0}$ (not defined)



Hence minor axis is parallel to y-axis and therefore, major axis will be parallel to x-axis. Let a be the length of semi-major axis of the ellipse, then

$b^2 = a^2(1 - e^2)$

$$\therefore \quad 4 = a^2\left(1 - \frac{1}{4}\right) \quad \text{or} \quad a^2 = \frac{16}{3}$$

Since centre of the ellipse is (3, 3), therefore, its equation will be

$$\frac{(x-3)^2}{16/3} + \frac{(y-3)^2}{4} = 1$$

or $\quad 3x^2 + 4y^2 - 18x - 24y + 47 = 0$

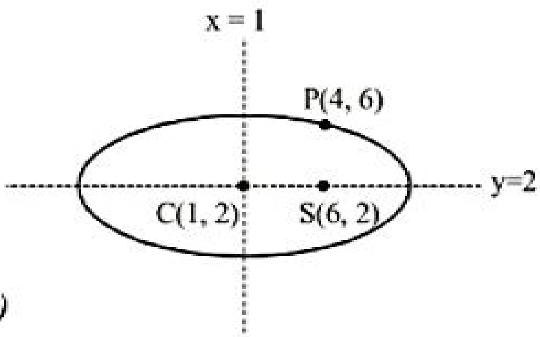
Illustration :

Find the equaiton of the ellipse with its centre at (1, 2), one focus at (6, 2) and passing through the point (4, 6).

Sol. Let $S \equiv (6, 2)$ and $C \equiv (1, 2)$. Sloep of $CS = 0$, therefore major axis of the ellipse is parallel to x-axis and minor axis is parallel to y-axis.

Since centre of the ellipse is (1, 2), therefore its equation will be of the form

$$\frac{(x-1)^2}{a^2} + \frac{(y-2)^2}{b^2} = 1 \qquad \ldots (1)$$



Since (4, 6) lies on (1), therefore $\dfrac{(4-1)^2}{a^2} + \dfrac{(6-2)^2}{b^2} = 1$

$$\text{or} \quad \frac{9}{a^2} + \frac{16}{b^2} = 1 \qquad \ldots (2)$$

Since ae = distnace between centre and focus = 5,

$\therefore \quad b^2 = a^2(1 - e^2) = a^2 - (ae)^2 = a^2 - 25 \qquad \ldots (3)$

Substituting this value in (2), we have

$$\frac{9}{a^2} + \frac{16}{a^2 - 25} = 1 \quad \text{or} \quad a^4 - 50a^2 + 225 = 0$$

or $\quad (a^2 - 45)(a^2 - 5) = 0$ or $a^2 = 45, 5$.

When $a^2 = 5$, from (3), $b^2 < 0$ (not possible)

$\therefore \quad a^2 = 45$, and from (3), $b^2 = 45 - 25 = 20$

Hence from (1), equation of required ellipse is $\dfrac{(x-1)^2}{45} + \dfrac{(y-2)^2}{20} = 1$

Practice Problem

Single correct question

Q.1 The eccentricity of an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ whose latus rectum is half of its major axis, is

(A) $\frac{1}{\sqrt{3}}$ (B) $\frac{1}{\sqrt{2}}$ (C) $\frac{\sqrt{3}}{2}$ (D) $\sqrt{\left(\frac{2}{3}\right)}$

Q.2 The length of the major axis of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is three times the length of minor axis, it eccentricity is

(A) $\frac{1}{\sqrt{2}}$ (B) $\frac{1}{\sqrt{3}}$ (C) $\frac{2\sqrt{2}}{3}$ (D) $\frac{2\sqrt{2}}{5}$

Q.3 The centre of the ellipse $\frac{(x+y-2)^2}{9}+\frac{(x-y)^2}{16}=1$, the locus of its pole is

(A) (0, 0) (B) (1, 0) (C) (0, 1) (D) (1, 1)

Q.4 The equation $\frac{x^2}{10-a}+\frac{y^2}{4-a}=1$ represents an ellipse, if

(A) $a<4$ (B) $a>4$ (C) $4<a<10$ (D) $a>10$

Q.5 The equation, $2x^2+3y^2-8x-18y+35=K$ represents

(A) no locus if $K>0$ (B) an ellipse if $K<0$

(C) a point if $K=0$ (D) a hyperbola if $K>0$

Q.6 If $\tan\alpha\tan\beta=-\frac{a^2}{b^2}$, then the chord joining two points α and β on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ will subtend a right angle at

(A) focus (B) centre (C) end of major axis (D) end of minor axis

Q.7 The eccentric angle of one end of a diameter of $x^2+3y^2=3$ is $\frac{\pi}{6}$, then the eccentric angle of the other end will be

(A) $\frac{5\pi}{6}$ (B) $-\frac{5\pi}{6}$ (C) $-\frac{2\pi}{3}$ (D) $\frac{2\pi}{3}$

Q.8 The eccentricity of the ellipse $(x-3)^2+(y-4)^2=\frac{y^2}{9}$ is

(A) $\frac{\sqrt{3}}{2}$ (B) $\frac{1}{3}$ (C) $\frac{1}{3\sqrt{2}}$ (D) $\frac{1}{\sqrt{3}}$

Q.9 F_1 and F_2 are the two foci of the ellipse $\frac{x^2}{9}+\frac{y^2}{4}=1$. Let P be a point on the ellipse such that $|PF_1| = 2|PF_2|$, where F_1 and F_2 are the two foci of the ellipses. The area of ΔPF_1F_2 is

(A) 3 (B) 4 (C) $\sqrt{5}$ (D) $\frac{\sqrt{13}}{2}$

Q.10 A circle has the same centre as an ellipse & passes through the foci F_1 & F_2 of the ellipse, such that the two curves intersect in 4 points. Let 'P' be any one of their point of intersection. If the major axis of the ellipse is 17 & the area of the triangle PF_1F_2 is 30, then the distance between the foci is :

(A) 11 (B) 12 (C) 13 (D) none

Q.11 Let S(5, 12) and S'(– 12, 5) are the foci of an ellipse passing through the origin. The eccentricity of ellipse equals

(A) $\frac{1}{2}$ (B) $\frac{1}{\sqrt{3}}$ (C) $\frac{1}{\sqrt{2}}$ (D) $\frac{2}{3}$

Q.12 An ellipse is inscribed in a circle and a point within the circle is chosen at random. If the probability that this point lies outside the ellipse is 2/3 then the eccentricity of the ellipse is :

(A) $\frac{2\sqrt{2}}{3}$ (B) $\frac{\sqrt{5}}{3}$ (C) $\frac{8}{9}$ (D) $\frac{2}{3}$

Multiple correct type question

Q.13 Consider the ellipse $\frac{x^2}{\tan^2\alpha}+\frac{y^2}{\sec^2\alpha}=1$ where $\alpha \in (0, \pi/2)$.

Which of the following quantities would vary as α varies?

(A) degree of flatness (B) ordinate of the vertex

(C) coordinates of the foci (D) length of the latus rectum

Integer type question

Q.14 Find the latus rectum, eccentricity, co-ordinates of the foci, co-ordinates of the vertices, the length of the axes and the centre of the ellipse

$$4x^2 + 9y^2 - 8x - 36y + 4 = 0$$

Q.15 The y-axis is the directrix of the ellipse with eccentricity $e = 1/2$ and the corresponding focus is at (3, 0), then find the equation to its auxilary circle.

Q.16 Find the eccentricity of the ellipse which meets the straight line $2x - 3y = 6$ on the X-axis and the straight line $4x + 5y = 20$ on the Y-axis and whose principal axes lie along the coordinate axes.

Q.17 A bar of length 20 units moves with its ends on two fixed straight lines at right angles. A point P marked on the bar at a distance of 8 units from one end describes a conic. Find its eccentricity.

Q.18 If one extremity of the minor axis of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and the foci form an equilateral triangle, then find its eccentricity.

Q.19 There are exactly two points on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ whose distance from the centre of the ellipse are greatest and equal to $\sqrt{\frac{a^2+2b^2}{2}}$. Then find the eccentricity of this ellipse.

Answer key

Q.1	B	Q.2	C	Q.3	D	Q.4	A	Q.5	C
Q.6	B	Q.7	B	Q.8	B	Q.9	B	Q.10	C
Q.11	C	Q.12	A	Q.13	A, B, D				

Q.14 $\frac{8}{3}, \frac{\sqrt{5}}{3}$ $(1 \pm \sqrt{5}, 2)$; $(-2, 2)$ and $(4, 2)$; 6 and 4; $(1, 2)$ Q.15 $x^2+y^2-8x+12=0$

Q.16 $e=\frac{\sqrt{7}}{4}$ Q.17 $e=\frac{\sqrt{5}}{3}$ Q.18 $e=\frac{1}{2}$ Q.19 $e=\frac{1}{\sqrt{2}}$

POSITION OF A POINT w.r.t. AN ELLIPSE :

Let $S(x, y)=\frac{x^2}{a^2}+\frac{y^2}{b^2}-1$ be the given ellipse

and $P(x_1, y_1)$ is the given point.

(i) If $S(x_1, y_1)>0$ then $P(x_1, y_1)$ lie outside the ellipse.

(ii) If $S(x_1, y_1)<0$ then $P(x_1, y_1)$ lie inside the ellipse.

(iii) If $S(x_1, y_1)=0$ then $P(x_1, y_1)$ lie on the ellipse.

This result holds true for circle and parabola also.

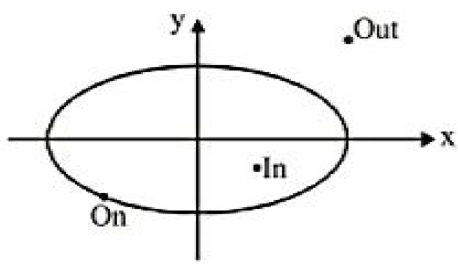


Illustration :

Check whether the point P(3, 2) lies inside, on or outside of the ellipse $\frac{x^2}{25}+\frac{y^2}{16}=1$.

Sol. $S(3, 2) \equiv \frac{9}{25}+\frac{4}{16}-1=\frac{9}{25}+\frac{1}{4}-1<0$

$\therefore$ *Point P(3, 2) lies inside the ellipse.* ***Ans.***

Illustration :

Find the set of values of 'α' for which the point P(α, – α) lies inside the ellipse $\frac{x^2}{16}+\frac{y^2}{9}=1$.

Sol. *If P(α, – α) lies inside the ellipse then*

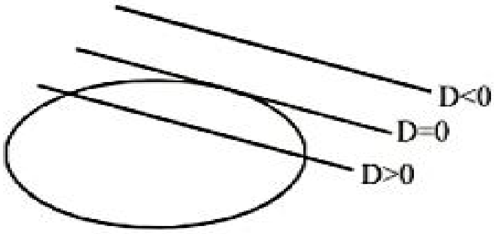
$$\therefore \quad S(\alpha, -\alpha) < 0 \Rightarrow \frac{\alpha^2}{16}+\frac{\alpha^2}{9}-1<0 \Rightarrow \frac{25}{144}.\alpha^2<1$$

$$\Rightarrow \quad \alpha^2 < \frac{144}{25}; \quad \therefore \alpha \in \left(-\frac{12}{5}, \frac{12}{5}\right). \quad \textit{Ans.}$$

INTERACTION OF A LINE AND AN ELLIPSE :

Let the equations of the line is $y = mx + c$ ......(1)

and equation of ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ ......(2)



The points of intersection of the line and the ellipse can be obtained by solving the two equations simultaneously. Hence by eliminating y from (1) & (2), we get

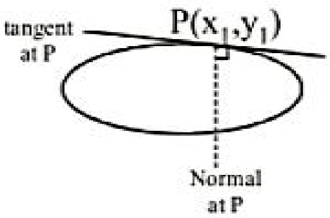
$$\frac{x^2}{a^2}+\frac{(mx+c)^2}{b^2}=1.$$

i.e. $(b^2 + a^2m^2)\,x^2 + 2a^2\,cmx + a^2\,(c^2 - b^2) = 0$ ......(3)

Let x_1, x_2 be the roots of the quadratic equation (3). The line meets the ellipse in real and distinct points if the roots x_1 and x_2 are real and different. The line is a tangent to the ellipse if $x_1 = x_2$ and the line does not meet the ellipse if the roots x_1 and x_2 are imaginary. All these will be decided by the discriminant of quadratic equation (3).

TANGENTS :

(i) Point form :



$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1$ is tangent to the ellipse at (x_1, y_1).

Since point (x_1, y_1) lie on the curve therefore we can use standard substitution to obtain the equation of tangent.

(ii) Slope form :

Let the given line is $y = mx + c$ and given ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

If line touch is the ellipse then by solving the two equations simultaneously (by eliminating y from (1) & (2)), we get $\frac{x^2}{a^2} + \frac{(mx+c)^2}{b^2} = 1$.

i.e. $(b^2 + a^2m^2)\, x^2 + 2a^2 cmx + a^2 (c^2 - b^2) = 0$ ......(3)

Since line is tangent to the ellipse therefore its $D = 0$

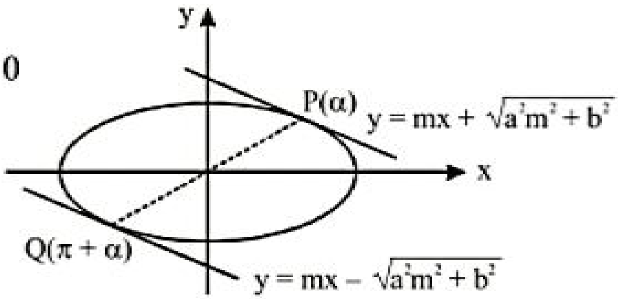
$4a^4c^2m^2 - 4\,(b^2 + a^2m^2) \cdot a^2(c^2 - b^2) = 0$

or $4a^2\,[a^2c^2m^2 - b^2c^2 - a^2c^2m^2 + b^4 + a^2b^2m^2] = 0$

or $b^2\,(-c^2 + b^2 + a^2m^2) = 0$

or $c^2 = b^2 + a^2m^2$ or $c = \pm\sqrt{a^2m^2 + b^2}$

which is the required **condition of tangency.**



Substituting this value of c in $y = mx + c$, we have

$y = mx + \sqrt{a^2m^2 + b^2}$ or $y = mx - \sqrt{a^2m^2 + b^2}$, which are tangents to the ellipse for all values of m.

Here ± sign represents two tangents to the ellipse having the same m, i.e. there are two tangents parallel to any given direction.

The equation of any tangent to the ellipse $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$.

$(y-k) = m\,(x-h) \pm \sqrt{a^2m^2 + b^2}$

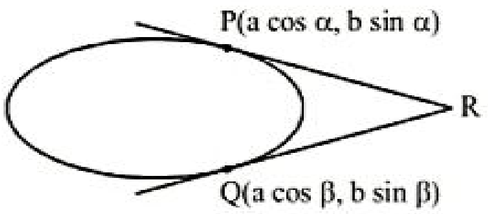
(iii) Parametric form :

$\frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1$ is tangent to the ellipse at the point $(a\cos\theta, b\sin\theta)$.

NOTE:

(i) Point of intersection of the tangents at the point α & β is $\left(a\frac{\cos\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}}, b\frac{\sin\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}}\right)$ can be deduced by comparing chord joining $P(\alpha)$ and $Q(\beta)$ with C.O.C. of the pair of tangents from $R(x_1, y_1)$ on the ellipse, where

$$x_1 = a\frac{\cos\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}}\ ;\ y_1 = b\frac{\sin\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}}$$



(ii) The eccentric angles of point of contact of two parallel tangents differ by π. Conversely if the difference between the eccentric angles of two points is π then the tangents at these points are parallel.

Illustration :

Find the equations of the tangents to the ellipse $3x^2 + 4y^2 = 12$ which are perpendicular to the line $y + 2x = 4$.

Sol. Slope of tangent to the given line = –2

Given ellipse is $\frac{x^2}{4} + \frac{y^2}{3} = 1$

Equation of tangent whose slope is 'm' is $y = mx \pm \sqrt{4m^2 + 3}$

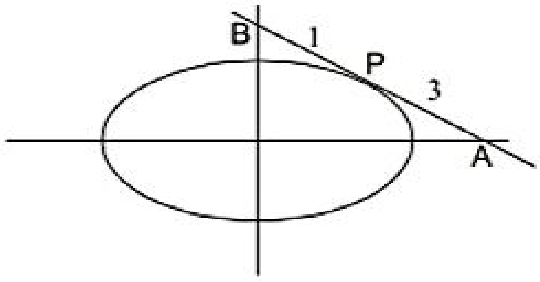
$\because \quad m = \frac{1}{2} \qquad \therefore \quad y = \frac{1}{2}x \pm \sqrt{1+3}$

$2y = x \pm 4$ **Ans.**

Illustration :

A tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ touches at the point P on it in the first quadrant and meets the co-ordinate axes in A and B respectively. If P divides AB in the ratio 3 : 1, find the equation of the tangent.

Sol. Let $P = (a\cos\theta, b\sin\theta)$



$\therefore$ equation of tangent is

$\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1$

$A = (a\sec\theta, 0)$

$B = (0, b\,\mathrm{cosec}\theta)$

$\because$ P divide AB internally in the ratio 3 : 1

$\therefore \quad a\cos\theta = \frac{a\sec\theta}{4} \Rightarrow \cos^2\theta = \frac{1}{4} \Rightarrow \cos\theta = \frac{1}{2}$

and $\quad b\sin\theta = \frac{3b\,\mathrm{cosec}\theta}{4} \Rightarrow \sin\theta = \frac{\sqrt{3}}{2}$

$\therefore$ tangent is $\frac{x}{2a} + \frac{\sqrt{3}y}{2b} = 1 \Rightarrow bx + \sqrt{3}\,ay = 2ab$ **Ans.**

Illustration :

How many real tangents can be drawn from the point (4, 3) to the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$. Find the equation these tangents & angle between them.

Sol. Given point $P = (4, 3)$

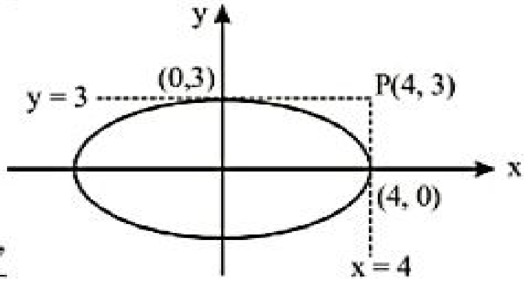
ellipse $\quad S = \frac{x^2}{16} + \frac{y^2}{9} - 1 = 0$

$\because \quad S_1 = \frac{16}{16} + \frac{9}{9} - 1 = 1 > 0$

$\Rightarrow$ Point $P = (4, 3)$ lies outside the ellipse.

$\therefore$ Two tangents can be drawn from the point $P(4, 3)$.

Equation of pair of tangents is



$$SS_1 = T^2 \Rightarrow \left(\frac{x^2}{16}+\frac{y^2}{9}-1\right) . 1 = \left(\frac{4x}{16}+\frac{3y}{9}-1\right)^2$$

$$\Rightarrow \frac{x^2}{16} + \frac{y^2}{9} - 1 = \frac{x^2}{16} + \frac{y^2}{9} + 1 + \frac{xy}{6} - \frac{x}{2} - \frac{2y}{3}$$

$\Rightarrow$ $-xy + 3x + 4y - 12 = 0$

$\Rightarrow$ $(4 - x)(y - 3) = 0 \Rightarrow x = 4$ & $y = 3$

and angle between them $= \dfrac{\pi}{2}$ **Ans.**

Illustration :

Find the equations of the tangents to the ellipse $x^2 + 16y^2 = 16$ each one of which makes an angle of 60^o with the x- axis.

Sol. We have, $x^2 + 16y^2 = 16 \Rightarrow \dfrac{x^2}{4^2}+\dfrac{y^2}{1^2}=1$

This is of the form $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, where

$a^2 = 16$ and $b^2 = 1$

So, the equations of the tangents are

$$y = mx \pm \sqrt{a^2m^2+b^2}$$

i.e. $y = \sqrt{3}\,x \pm \sqrt{16\times3+1} \Rightarrow y = \sqrt{3}\,x \pm 7$ **Ans.**

Illustration :

For what value of λ does the line $y = x + \lambda$ touches the ellipse $9x^2 + 16y^2 = 144$.

Sol. $\because$ Equation of ellipse is

$$9x^2 + 16y^2 = 144 \quad \text{or} \quad \frac{x^2}{16} + \frac{y^2}{9} = 1$$

Comparing this with $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ then we get $a^2 = 16$ and $b^2 = 9$

& comparing the line $y = x + \lambda$ with $y = mx + c$

$\therefore$ $m = 1$ and $c = \lambda$

therefore condition of tangency $c^2 = a^2m^2 + b^2 \Rightarrow \lambda^2 = 16 \times 1^2 + 9 \Rightarrow \lambda^2 = 25$

$\therefore$ $\lambda = \pm 5$ **Ans.**

Illustration :

A circle of radius r is concentric with the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$*. Prove that the common tangent is inclined to the major axis at an angle* $\tan^{-1}\left(\sqrt{\frac{r^2-b^2}{a^2-r^2}}\right)$.

Sol. *Let the equation of circle is* $x^2 + y^2 = r^2$ ......(i)

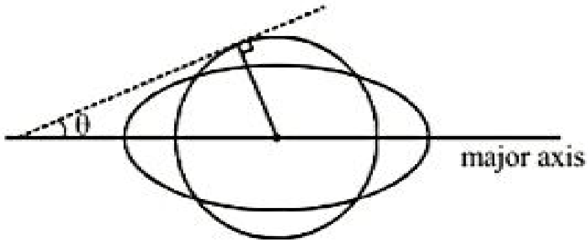
& $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ ......(ii)

Tangent to be ellipse is $y = mx + \sqrt{a^2m^2+b^2}$ *i.e.* $mx - y + \sqrt{a^2m^2+b^2} = 0$*. Since it is tangent to the circle also, therefore perpendicular distance from centre (0, 0) will be equal to radius r,*

$\therefore \quad \left|\frac{\sqrt{a^2m^2+b^2}}{\sqrt{m^2+1}}\right| = r$

$\Rightarrow \quad a^2m^2 + b^2 = r^2(1 + m^2)$

$\Rightarrow \quad m^2 = \frac{r^2-b^2}{a^2-r^2}$



$\therefore \quad m = \tan\theta = \sqrt{\frac{r^2-b^2}{a^2-r^2}}$

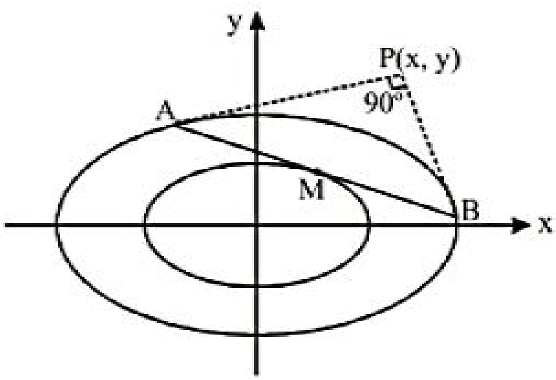
$\therefore \quad \theta = \tan^{-1}\sqrt{\frac{r^2-b^2}{a^2-r^2}}$ ***Ans.***

Illustration :

Show that the tangents drawn at those points of the ellipse $\frac{x^2}{a}+\frac{y^2}{b}=(a+b)$*, where it is cut by any tangent to* $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$*, intersect at right angles.*

Sol. *Given ellipse are* $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ ...... (i)

& $\frac{x^2}{a(a+b)}+\frac{y^2}{b(a+b)}=1$ ......(ii)



The chord of contact (x_1, y_1) *w.r.t. (ii) ellipse is*

$$\frac{xx_1}{a(a+b)}+\frac{yy_1}{b(a+b)}=1$$

i.e. $\quad y = \dfrac{-bx_1}{ay_1}x + \dfrac{b(a+b)}{y_1}$ ......(iii)

(iii) is tangent to ellipse (i) which is also given as

$$y = mx + \sqrt{a^2m^2 + b^2} \qquad ...(iv)$$

Hence, (iii) & (iv) is identical

$\therefore \quad m = \dfrac{bx_1}{ay_1}$ ...(v) $\quad$ & $\quad a^2m^2 + b^2 = \dfrac{b^2(a+b)^2}{y_1^2}$ ...(vi)

Now eliminate m from (v) & (vi), we get

$$x_1^2 + y_1^2 = (a+b)^2$$

$\therefore \quad$ Locus of $P(x_1, y_1)$ is $x^2 + y^2 = (a+b)^2$ **Ans.**

Illustration :

Prove that in the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ the locus of the middle points of the portions of tangents included between the axes is the curve $\dfrac{a^2}{x^2} + \dfrac{b^2}{y^2} = 4$.

Sol. Equation of any tangent to the given ellipse may be taken as

$$\frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1 \qquad ...(1)$$

Let this tangent meet the x-axis in P and y-axis in Q.

Putting $y = 0$ in (1), we get $x = a\sec\theta$.

$\therefore \quad$ Co-ordinates of P are $(a\sec\theta, 0)$

Similarly co-ordinates of Q are $(0, b\operatorname{cosec}\theta)$

Let (h, k) be the mid. point of PQ.

$$\therefore \quad h = \frac{a\sec\theta + 0}{2} = \frac{a\sec\theta}{2} \quad \text{and} \quad k = \frac{0 + b\operatorname{cosec}\theta}{2} = \frac{b\operatorname{cosec}\theta}{2}$$

Hence $\quad 2\cos\theta = \dfrac{a}{h}$ ... (2)

and $\quad 2\sin\theta = \dfrac{b}{k}$ ... (3)

Squaring and adding (2) and (3), we have $4 = \dfrac{a^2}{h^2} + \dfrac{b^2}{k^2}$.

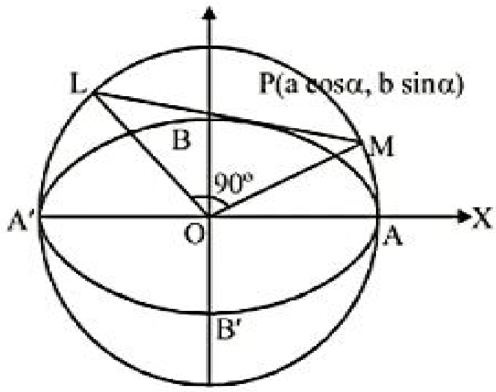
$\therefore$ Locus of (h, k) is $\dfrac{a^2}{x^2} + \dfrac{b^2}{y^2} = 4$.

Illustration :

The tangent at any point on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ *meets the auxiliary circle in two points which subtend a right angle at the centre. Show that the eccentricity of the ellipse is* $\frac{1}{\sqrt{1+\sin^2\alpha}}$.

Sol. *Given ellipse is*

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \qquad \ldots (1)$$



Its auxiliary circle is

$$x^2 + y^2 = a^2 \qquad \ldots (2)$$

Let $P \equiv (a \cos \alpha, b \sin \alpha)$

Equation of tangent to the ellipse at $P(a \cos \alpha, b \sin \alpha)$ *is*

$$\frac{x\cos\alpha}{a}+\frac{y\sin\alpha}{b}=1 \qquad \ldots(3)$$

Making equation (2) homogeneous with the help of (3), we get

$$x^2 + y^2 - a^2\left(\frac{x\cos\alpha}{a}+\frac{y\sin\alpha}{b}\right)^2=0$$

or $$(1-\cos^2\alpha)x^2+\left(1-\frac{a^2}{b^2}\sin^2\alpha\right)y^2-2\frac{a}{b}\cos a \sin a\, xy=0 \ldots (4)$$

(4) is the joint equaiton of OL and OM.

Since $\angle LOM = 90^o$

$\therefore$ Coefficient of x^2 + coeff. of $y^2 = 0$

$$\therefore \quad 1-\cos^2\alpha+1-\frac{a^2}{b^2}\sin^2\alpha=0$$

or $$\sin^2\alpha\left(\frac{a^2}{b^2}-1\right) \text{ or } \sin^2\alpha\left(\frac{1}{1-e^2}-1\right)=1 \qquad [\because b^2=a^2(1-e^2)]$$

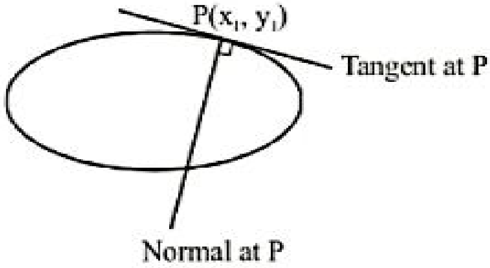
or $$e^2\sin^2\theta = 1-e^2 \text{ or } e^2(1+\sin^2\alpha)=1$$

or $$e=\frac{1}{\sqrt{1+\sin^2\alpha}}$$

NORMALS :

(i) Point form :

Equation of the tangent to the ellipse at (x_1, y_1) is $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$.



The slope of the tangent at $(x_1, y_1) = \frac{-x_1}{a^2} \times \frac{b^2}{y_1}$

$\therefore$ Slope of the normal at $(x_1, y_1) = \frac{a^2}{x_1} \times \frac{y_1}{b^2} = \frac{a^2 y_1}{b^2 x_1}$

Hence the equation of the normal at (x_1, y_1) is $y - y_1 = \frac{a^2 y_1}{b^2 x_1}(x - x_1)$

or $\quad \dfrac{x - x_1}{\frac{x_1}{a^2}} = \dfrac{y - y_1}{\frac{y_1}{b^2}}$

(ii) Parametric form :

In above equation if we put $x = a\cos\theta$ and $y = b\sin\theta$ then we will get normal equation in parametric form.

$\therefore \quad ax\sec\theta - by\,\text{cosec}\,\theta = a^2 - b^2 = a^2e^2$

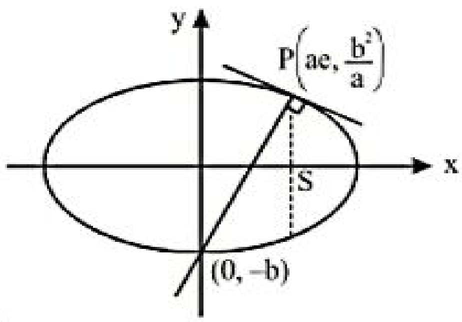
This is equation of normal in parametric form.

(iii) Equation of a normal in terms of its slope 'm' is $y = mx - \frac{(a^2 - b^2)m}{\sqrt{a^2 + b^2m^2}}$.

Illustration :

If the normal at one end of a latus-rectum of an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ passes through other extremity of the minor axis, then show that $e^2 = \frac{\sqrt{5} - 1}{2}$.

Sol. *Equation of normal at* $P\left(ae, \frac{b^2}{a}\right)$ *is* $\dfrac{x - ae}{\frac{ae}{a^2}} = \dfrac{y - \frac{b^2}{a}}{\frac{b^2/a}{b^2}}$



$\Rightarrow \quad \frac{ax}{e} - ya = a^2 - b^2$

Since it passes through (0, –b)

$\Rightarrow \quad 0 + ab = a^2 - b^2 \qquad \Rightarrow \quad a^2b^2 = (a^2 - b^2)^2$

$\Rightarrow \quad a^2 \cdot a^2(1 - e^2) = (a^2 \cdot e^2)^2 \qquad \Rightarrow \quad e^4 + e^2 - 1 = 0$

$\Rightarrow \quad e^2 = \frac{\sqrt{5} - 1}{2}$ ***Ans.***

Illustration :

Find the condition that the line lx + my = n may be a normal to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Sol. *Equation of normal to the given ellipse at (a cosθ, b sinθ) is*

$$\frac{ax}{\cos\theta}-\frac{by}{\sin\theta}=a^2-b^2 \qquad ...(1)$$

If the line lx + my = n is also normal to the ellipse then there must be a value of θ for which line (1) and line lx + my = n are identical. For that value of θ we have

$$\frac{l}{\left(\frac{a}{\cos\theta}\right)}=\frac{m}{\left(\frac{-b}{\sin\theta}\right)}=\frac{n}{(a^2-b^2)} \qquad \text{or} \qquad \frac{l}{a}\cos\theta=-\frac{m\sin\theta}{b}=\frac{n}{(a^2-b^2)}$$

$$\therefore \qquad \cos\theta=\frac{an}{l.(a^2-b^2)} \qquad ...(3)$$

$$\text{and} \qquad \sin\theta=\frac{-bn}{(a^2-b^2)m} \qquad ...(4)$$

Squaring and adding (3) and (4), we get

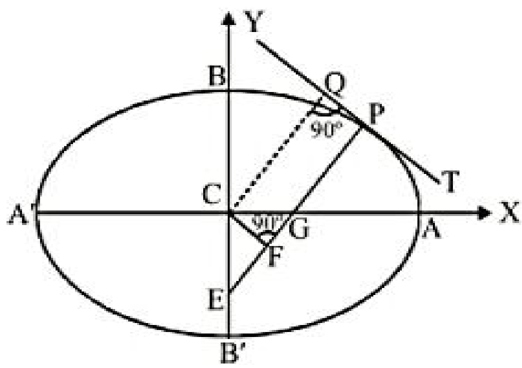
$$1=\frac{n^2}{(a^2-b^2)^2}\left(\frac{a^2}{l^2}+\frac{b^2}{m^2}\right)$$ *which is the required solution.*

Illustration :

If the nromal at any point P of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ *meets the major and minor axes at G and E. respectively, and if CF is perpendicular upon this normal from the centre C of the ellipse, show that : PF.PG =* b^2 *and PF.PE =* a^2

Sol. *Given ellipse is*

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \qquad ...(1)$$



Let P(a cos θ, b sin θ) be any point on ellipse (1)
Equation of normal to ellipse (1) at P is

$$ax\sec\theta-by\,\mathrm{cosec}\theta=a^2-b^2 \qquad ...(2)$$

since line (2) meets the major axis (x – axis) and minor axis (y – axis) at G and E respectively, therefore

$$G\equiv\left(\frac{a^2-b^2}{a}\cos\theta,0\right) \quad \text{and } E\equiv\left(0,\frac{-(a^2-b^2)}{b}\sin\theta\right)$$

PF = CQ ≡ length of perp. form C(0, 0) on the tangent at P,

i.e. on the line $\frac{x\cos\theta}{a}+\frac{y\sin\theta}{b}-1=0$

$$=\frac{|-1|}{\sqrt{\frac{\cos^2\theta}{a^2}+\frac{\sin^2\theta}{b^2}}}=\frac{ab}{\sqrt{b^2\cos^2\theta+a^2\sin^2\theta}} \quad \text{... (3)}$$

Also $PG=\sqrt{\left\{\frac{(a^2-b^2)\cos\theta}{a}-a\cos\theta\right\}^2+(0-b\sin\theta)^2}$

$$=\sqrt{\frac{b^4\cos^2\theta}{a^2}+b^2\sin^2\theta}=\frac{b}{a}\sqrt{b^2\cos^2\theta+a^2\sin^2\theta} \quad \text{... (4)}$$

$$PE=\sqrt{a^2\cos^2\theta+\left(b\sin\theta+\frac{a^2-b^2}{b}\sin\theta\right)^2}$$

$$=\sqrt{a^2\cos^2\theta+\frac{a^4\sin^2\theta}{b^2}}=\frac{a}{b}\sqrt{b^2\cos^2\theta+a^2\sin^2\theta} \quad \text{... (5)}$$

From (3) and (4), PF.PG = b^2

From (3) and (5), PF.PE = a^2

DIRECTOR CIRCLE :

Locus of the point of intersection of the tangents which meet at right angles is called the **Director Circle**.

Let equation of any tangent is $y=mx+\sqrt{a^2m^2+b^2}$

If it passes through (h, k) then

$$k=mh\pm\sqrt{a^2m^2+b^2}$$

$$(k-mh)^2=a^2m^2+b^2$$

$$(h^2-a^2)m^2-2khm+k^2-b^2=0 \quad \text{....(3)}$$

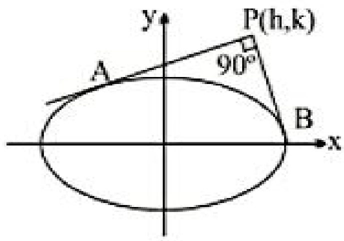
Equation (3) has two roots m_1 & m_2

$$m_1+m_2=\frac{2hk}{h^2-a^2} \quad \text{... (4)}$$

$$m_1m_2=\frac{k^2-b^2}{h^2-a^2} \quad \text{... (5)}$$

Hence passing through a given point there can be a maximum of two tangents.

If $PA \perp PB$ then $m_1 m_2 = -1$



i.e. $m_1 m_2 = \dfrac{k^2 - b^2}{h^2 - a^2} = -1$

i.e. $k^2 - b^2 = a^2 - h^2$; i.e. $x^2 + y^2 = a^2 + b^2$

which is the director circle of the ellipse. Hence **director circle of an ellipse is a circle** whose centre is the centre of ellipse and whose radius is the length of the line joining the ends of the major and minor axis.

Equation (3) can be used to determine the locus of the point of intersection of two tangents enclosing.

If from any point P (h, k) pair of tangents are drawn to the ellipse which include an angle α, then

$$\tan\alpha = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right| = \frac{\sqrt{(m_1 + m_2)^2 - 4m_1 m_2}}{1 + m_1 m_2}$$

By putting value of $m_1 + m_2$ and $m_1 m_2$ in above equation we will get the angle between pair of tangents.

Note :

If a right triangle, right angled at A circumscribes an ellipse then locus of the point A is the director circle of the ellipse.

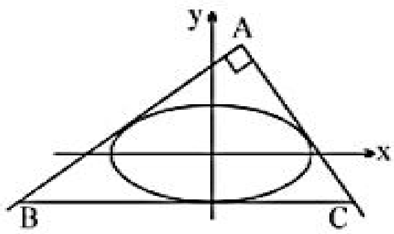
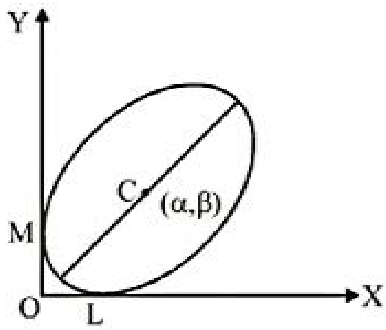


Illustration :

An ellipse slides between two lines at right angles to another. Show that the locus of its centre is a circle.

Sol. *Let the two given perpendicular lines be taken as the x and y axes respectivley.*



Let $C(\alpha, \beta)$ be the centre of the ellipse in any position. Here the position of centre C changes as the ellipse slides.

Let a and b be the semimajor and minor axes of the elipse.

Equation of the director circle of the ellipse is

$$(x - \alpha)^2 + (y - \beta)^2 = a^2 + b^2 \qquad \text{... (1)}$$

Since OX and OY are mutually perpendicular tangents to sliding ellipse for all its positions, therefore, O(0, 0) will lie on its director circle (1)

$\therefore \quad \alpha^2 + \beta^2 = a^2 + b^2$

Hence locus of C(a, b) is $x^2 + y^2 = a^2 + b^2$ .... (2)

CHORD OF CONTACT :

Pair of tangents drawn from outside point P (x_1, y_1) to the ellipse which meet it at A and B. Now line joining A and B is called the chord of contact of point P (x_1, y_1) w.r.t. the ellipse.
The equation of chord of contact is

$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$$

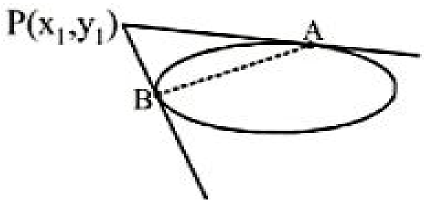


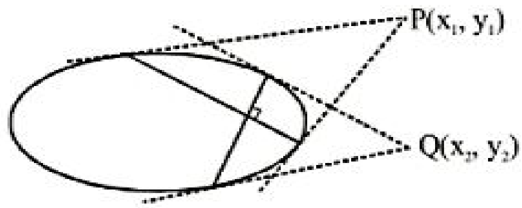
Illustration :

If the chords of contact of tangents from two points (x_1, y_1) and (x_2, y_2) to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ are at right angles, then Find the value of $\frac{x_1 x_2}{y_1 y_2}$.

Sol. *The equations of the chords of contact of tangents drawn from (x_1, y_1) and (x_2, y_2) to the ellipse*

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \text{ are } \frac{x x_1}{a^2} + \frac{y y_1}{b^2} = 1 \qquad \text{... (i)}$$

$$\frac{x x_2}{a^2} + \frac{y y_2}{b^2} = 1 \qquad \text{... (ii)}$$



It is given that (i) and (ii) are at right angles.

$$\therefore \frac{-b^2}{a^2}\frac{x_1}{y_1} x \frac{-b^2}{a^2}\frac{x_2}{y_2} = -1 \quad \Rightarrow \quad \frac{x_1 x_2}{y_1 y_2} = -\frac{a^4}{b^4} \qquad \textit{Ans.}$$

PAIR OF TANGENTS :

Pair of tangents PA and PB are drawn from outside point P (x_1, y_1), which is shown below. Hence joint equation of line PA and PB is given by $SS_1 = T^2$.

Here $S \equiv \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1$

$$S_1 \equiv \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1$$

$$T \equiv \frac{xx_1}{a^2} + \frac{yy_1}{b^2} - 1$$

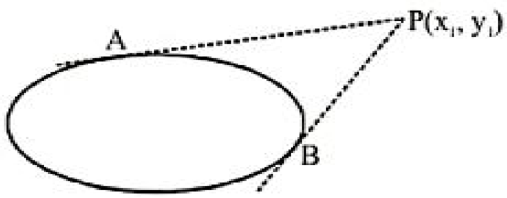


Illustration :

Find the locus of point of intersection of perpendicular tangents to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

Sol. *Let P(h, k) be the point of intersection of two perpendicular tangents*

$$\Rightarrow \quad \left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)\left(\frac{h^2}{a^2}+\frac{k^2}{b^2}-1\right)=\left(\frac{hx}{a^2}+\frac{ky}{b^2}-1\right)^2$$

$$\Rightarrow \quad \frac{x^2}{a^2}\left(\frac{k^2}{b^2}-1\right)+\frac{y^2}{a^2}\left(\frac{h^2}{a^2}-1\right)+\ldots\ldots=0 \qquad \ldots\ldots\ldots(i)$$

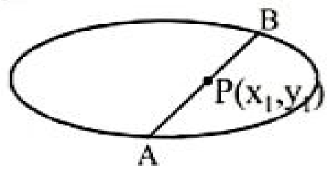
Since equation (i) represents two perpendicular lines

$$\therefore \quad \frac{1}{a^2}\left(\frac{k^2}{b^2}-1\right)+\frac{1}{b^2}\left(\frac{h^2}{a^2}-1\right)=0$$

$$\Rightarrow \quad k^2-b^2+h^2-a^2=0 \quad \Rightarrow \quad \text{locus is } x^2+y^2=a^2+b^2$$ ***Ans.***

CHORD WITH A GIVEN MIDDLE POINT :

Here chord AB is shown in the figure whose mid point is $P(x_1, y_1)$.
Then equation of this chord AB is $T = S_1$



Here $S_1 \equiv \frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1$

$T \equiv \frac{xx_1}{a^2}+\frac{yy_1}{b^2}-1$

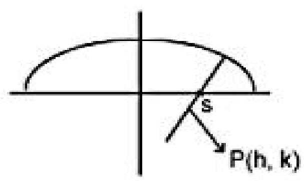
Illustration :

Find the locus of the mid - point of focal chords of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$

Sol. *Let P = (h, k) be the mid-point*

$\therefore$ *Equation of chord whose mid-point is given* $\frac{xh}{a^2}+\frac{yk}{b^2}-1=\frac{h^2}{a^2}+\frac{k^2}{b^2}-1$

Since it is a focal chord, therefore it passes through focus, either (ae, 0) or (–ae, 0)
If it passes through (ae, 0)



$\therefore$ *locus is* $\frac{ex}{a}=\frac{x^2}{a^2}+\frac{y^2}{b^2}$

If it passes through (–ae, 0)

$\therefore$ *locus is* $-\frac{ex}{a}=\frac{x^2}{a^2}+\frac{y^2}{b^2}$ ***Ans.***

Practice Problem

Single correct question

Q.1 If $\frac{x}{a}+\frac{y}{b}=\sqrt{2}$ touches the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ then its, eccentricity angle θ is equal to

(A) 0 (B) 45° (C) 60° (D) 90°

Q.2 The equations of the tangents to the ellipse $3x^2+y^2=3$ making equal intercepts on the axes are

(A) $y=\pm x\pm 2$ (B) $y=\pm x\pm 4$ (C) $y=\pm x\pm \sqrt{30}$ (D) $y=\pm x\pm \sqrt{35}$

Q.3 The common tangent of $x^2+y^2=4$ and $2x^2+y^2=2$ is

(A) $x+y+4=0$ (B) $x-y+7=0$ (C) $2x+3y+8=0$ (D) None of these

Q.4 If the normal at any point P on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ meets the axes in G and g respectively, then

PG : Pg =

(A) a : b (B) $a^2 : b^2$ (C) b : a (D) $b^2 : a^2$

Q.5 From the point $(\lambda, 3)$ tangents are drawn to $\frac{x^2}{9}+\frac{y^2}{4}=1$ and are perpendicular to each other than λ is

(A) ±1 (B) ±2 (C) ±3 (D) ±4

Q.6 The locus of the point of intersection of two perpendicular tangents of the ellipse $\frac{x^2}{9}+\frac{y^2}{4}=1$ is

(A) $x^2+y^2=4$ (B) $x^2+y^2=9$ (C) $x^2+y^2=13$ (D) $x^2+y^2=5$

Q.7 The angle between the pair of tangents drawn from the point (1, 2) to the ellipse $3x^2+2y^2=5$ is

(A) $\tan^{-1}\left(\frac{12}{5}\right)$ (B) $\tan^{-1}\left(\frac{6}{\sqrt{5}}\right)$ (C) $\tan^{-1}\left(\frac{12}{\sqrt{5}}\right)$ (D) $\tan^{-1}(12\sqrt{5})$

Q.8 If a quadrilateral is formed by four tangents to the ellipse $\frac{x^2}{9}+\frac{y^2}{4}=1$ is a square,

then the area of the square is equal to

(A) 26 (B) 24 (C) 22 (D) 20

Q.9 The Locus of the middle point of chords of an ellipse $\frac{x^2}{16}+\frac{y^2}{25}=1$ passing through P(0, 5) is another ellipse E. The coordinates of the foci of the ellipse E, is

(A) $\left(0, \frac{3}{5}\right)$ and $\left(0, \frac{-3}{5}\right)$ (B) (0, – 4) and (0, 1)

(C) (0, 4) and (0, 1) (D) $\left(0, \frac{11}{2}\right)$ and $\left(0, \frac{-1}{2}\right)$

Q.10 The area of the rectangle formed by the perpendiculars from the centre of the standard ellipse to the tangent and normal at its point whose eccentric angle is $\pi/4$ is :

(A) $\frac{\left(a^2-b^2\right)ab}{a^2+b^2}$ (B) $\frac{(a^2-b^2)}{(a^2+b^2)ab}$ (C) $\frac{\left(a^2-b^2\right)}{ab\left(a^2+b^2\right)}$ (D) $\frac{a^2+b^2}{\left(a^2-b^2\right)ab}$

Integer type question

Q.11 $x-2y+4=0$ is a common tangent to $y^2=4x$ & $\frac{x^2}{4}+\frac{y^2}{b^2}=1$. Then find the value of b and also find the other common tangent.

Q.12 If a tangent having slope 2 of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ $(a>0, b>0)$ is normal to the circle $x^2+y^2+4x+1=0$, then find the maximum value of ab.

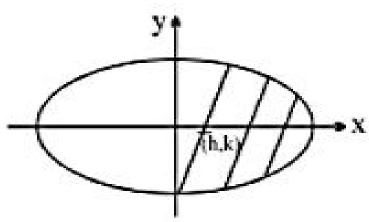
Answer key

Q.1	B	Q.2	A	Q.3	D	Q.4	D	Q.5	B
Q.6	B	Q.7	C	Q.8	A	Q.9	C	Q.10	A
Q.11	$b=\sqrt{3}$; $x+2y+4=0$			Q.12	4				

DIAMETER :

The locus of the middle points of a system of parallel chords with slope 'm' of an ellipse is a straight line passing through the centre of the ellipse, called its diameter and has the equation

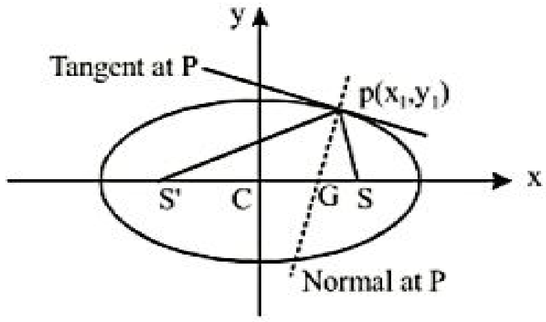
$$y=-\frac{b^2}{a^2m}x.$$



Chord of contact, pair of tangents, chord with a given middle point, pole & polar are to be interpreted as same as they are in parabola.

Properties of the ellipse :

(1) Let the equation of an ellipse is



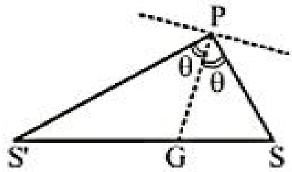
$$\frac{x^2}{a^2}+\frac{y^2}{b^2} \qquad \ldots(i)$$

& Normal at $P(x_1, y_1)$ is

$$\frac{x-x_1}{x_1/a^2}=\frac{y-y_1}{y_1/b^2} \qquad \ldots(ii)$$

The normal meet x-axis at G $\Rightarrow$ Put y = 0 in equation (ii) we get

$$CG = x = \left(\frac{a^2-b^2}{a^2}\right)x_1 = e^2x_1$$



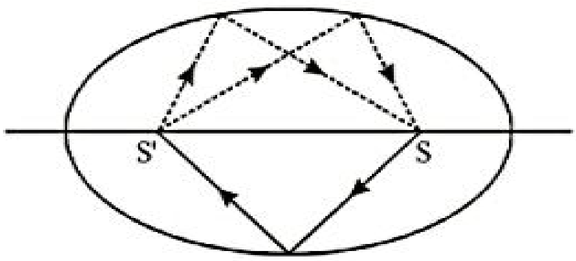
$\therefore \quad SG = CS - CG = ae - e^2x_1 = e(a - ex_1) = eSP$

Similarly $S'G = eS'P$

$$\therefore \quad \frac{SG}{S'G}=\frac{eSP}{eS'P}=\frac{SP}{S'P}$$

$\Rightarrow \quad$ PG is bisector of angle $\angle P$ in $\Delta S'PS$.

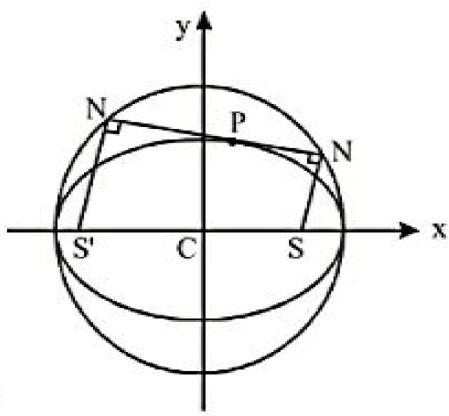
Tangent & normal at any point P bisect the external & internal angles between the focal distances of SP & S'P.



This lead to reflextion property of ellipse.

If incoming light ray passes through focus S'(or S), strike the concave side of ellipse the after reflextion it will pass through other focus.

(2) Let ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \qquad \ldots(i)$



its tangent is $y - mx = \sqrt{a^2m^2+b^2} \qquad \ldots(ii)$

Equation of perpendicular to above the passes through focus (ae, 0) is

$$my + x = ae \qquad \ldots(iii)$$

Eliminate m from (ii) & (iii) we will get focus of intersection point.

For that square & add (ii) & (iii) we will get an answer

$\therefore \quad y^2 + m^2x^2 - 2mxy + x^2 + my^2 + 2mxy = a^2m^2 + b^2 + a^2e^2$

$\Rightarrow \quad x^2(1 + m^2) + y^2(1 + m^2) = a^2m^2 + a^2$

$\Rightarrow \quad x^2 + y^2 = a^2$ which is the auxilairy circle

The locus of the point of interseciton of feet of perpendicular from foci on any tangent to an ellipse is the auxiliary circle.

(3) From previous equation of any tangent is $mx - y + \sqrt{a^2m^2 + b^2} = 0$

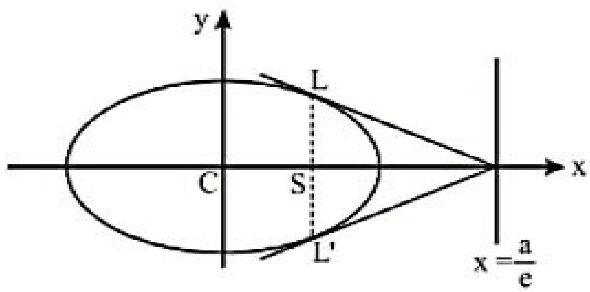
$$SN = \left|\frac{\sqrt{a^2m^2+b^2} + ame}{\sqrt{1+m^2}}\right| \quad \& \quad S'N' = \left|\frac{\sqrt{a^2m^2+b^2} - ame}{\sqrt{1+m^2}}\right|$$

$$SN \cdot S'N' = \frac{(a^2m^2+b^2) - a^2m^2e^2}{(1+m^2)} = \frac{a^2m^2+b^2-(a^2-b^2)m^2}{1+m^2} = b^2 \qquad \because a^2e^2 = a^2 - b^2$$

The product of perpendicular distance from the foci to any tangent of an ellipse is equal to square of the semi minor axis.

(4) Equation of tangent at $L\left(ae, \frac{b^2}{a}\right)$ is

$$\frac{x(ae)}{a^2} + \frac{y\left(\frac{b^2}{a}\right)}{b^2} = 1 \Rightarrow xe + y = a \quad \ldots(i)$$



& equation of tangent at $L'\left(ae, -\frac{b^2}{a}\right)$ is

$$ex - y = a \quad \ldots(ii)$$

Solve (i) & (ii) we get $x = \frac{a}{e}$ & $y = 0$ i.e. at the directrix of ellipse.

Tangents at the extremities of latus-rectum of an ellipse intersect on the corresponding directrix.

(5) If P be any point on the ellipse with S & S′ as its foci then $\ell(SP) + \ell(S'P) = 2a$.

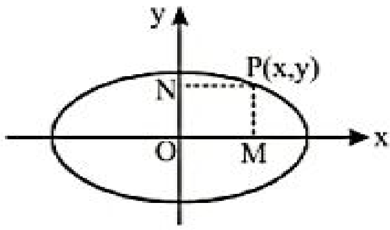
(6) If the normal at any point P on the ellipse with centre C meet the major & minor axes in G & g respectively & if CF be perpendicular upon this normal then

(i) $PF.\ PG = b^2$ (ii) $PF.\ Pg = a^2$

(7) The portion of the tangent to an ellipse between the point of contact & the directrix subtends a right angle at the corresponding focus.

Equation of an ellipse referred to two perpendicular lines :

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \text{ is given ellipse}$$



Let P(x, y) be any point on the ellipse, then PM = y & PN = x

$\therefore$ above equation can be written as $\dfrac{PN^2}{a^2}+\dfrac{PM^2}{b^2}=1$

From above we conclude that if perpendicular distances p_1 & p_2 of a moving point P(x, y) from two mutually perpendicular straight lines $L_1 \equiv lx+my+n_1=0$ & $L_2 \equiv mx-ly+n_2=0$ respectively then equation of ellipse in the plane of line will be

$$\frac{p_2^2}{a^2}+\frac{p_1^2}{b^2}=1$$

$$\Rightarrow \frac{\left(\frac{mx-ly+n_2}{\sqrt{l^2+m^2}}\right)^2}{a^2}+\frac{\left(\frac{lx+my+n_1}{\sqrt{l^2+m^2}}\right)^2}{b^2}=1$$

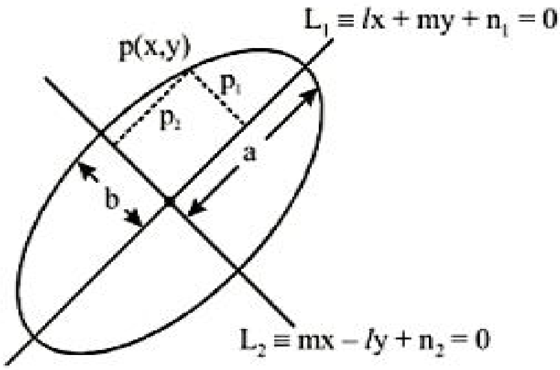


Illustration :

Find the equation the ellipse whose axis are of length 6 & $2\sqrt{6}$ & their equations are $x-3y+3=0$ & $3x+y-1=0$.

Sol. *Equation of ellipse will be*

$$\frac{\left(\frac{x-3x+3}{\sqrt{10}}\right)^2}{(\sqrt{6})^2}+\frac{\left(\frac{3x+y-1}{\sqrt{10}}\right)^2}{(3)^2}=1$$

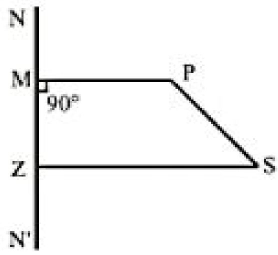
HYPERBOLA

DEFINITON OF HYPERBOLA :

A hyperbola is the locus of a point which moves in a plane such that the ratio of its distance from a fixed point and a given straight line is always constant.

The fixed point is called the focus, the fixed line is called the directrix and the constant ratio is called the eccentricity of the hyperbola and denoted by e.

In the given figure, S is the focus and N'N the directrix.



Let P be any point on the hyperbola, then $\frac{PS}{PM} = e, e > 1$.

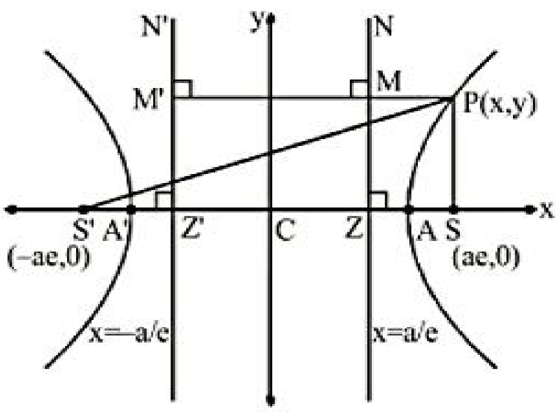
Equation of a hyperbola can be obtained if the coordinates of its focus, equation of its directrix and eccentricity are given.

STANDARD EQUATION OF A HYPERBOLA :

Let S be the focus & ZN is the directrix of an ellipse. Draw perpendicular from S to the directrix which meet it at Z. A moving point is on the hyperbola such that

$PS = ePM$

then there is point lies on the line SZ and which divide SZ internally at A and externally at A' in the ratio of e : 1.



therefore $SA = e\,AZ$ ...(i)

$SA' = eA'Z$ ...(ii)

Let $AA' = 2a$ & take C as mid point of AA'

$\therefore \quad CA = CA' = a$

Add (i) & (ii)

$SA + SA' = e\,(AZ + A'Z)$

$(CS - CA) + (CA' + CS) = e\,[CA - CZ + CA' + CZ]$

$2CS = 2e.\ CA$

$CS = ae$

Subtract (ii) & (i), we get

$SA' - SA = e\,(A'Z - AZ)$

$(CA' + CS) - (CS - CA) = e\,[(CA' + CZ) - (CA - CZ)]$

$2CA = 2e \cdot CZ \Rightarrow CZ = \frac{a}{e}$.

Consider CZ line as x-axis, C as origin & perpendicular to this line & passes through C is considered as y-axis. Now represent important parameters on coordinates plane. Let P(x, y) is a moving point, then By defintion of ellipse.

PS = ePM $\Rightarrow$ $(PS)^2 = e^2\,(PM)^2$

$$\Rightarrow \quad (x - ae)^2 + (y - 0)^2 = e^2\left(x - \frac{a}{e}\right)^2 \Rightarrow (x - ae)^2 + y^2 = (a - ex)^2$$

$$\Rightarrow \quad x^2 + a^2e^2 - 2xae + y^2 = a^2 + e^2x^2 - 2xae \Rightarrow x^2\,(1 - e^2) + y^2 = a^2\,(1 - e^2)$$

$$\Rightarrow \quad \frac{x^2}{a^2} + \frac{y^2}{a^2(1-e^2)} = 1 \quad \text{or} \quad \frac{x^2}{a^2} - \frac{y^2}{a^2(e^2-1)} = 1 \quad \text{where} \quad b^2 = a^2\,(e^2 - 1)$$

Hence equation of hyperbola is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ where $b^2 = a^2\,(e^2 - 1)$

TRACING OF HYPERBOLA :

Equation of given hyperbola is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ ....(1)

(i) If we put $y = 0$, then we see that hyperbola cuts x-axis at $(\pm a, 0)$

(ii) If we put $x = 0$, then $y^2 = -b^2$. Hence hyperbola does not cut y-axis.

(iii) When y is replaced by $-y$, the equation of hyperbola does not change, hence hyperbola is symmetric about x-axis. (Since equation contain even power of y therefore curve will be symmetric about x-axis.

(iv) When x is replaced by $-x$, then equation of hyprbola does not change, hence hyperbola is symmetric about y-axis. (Since equation contain even power of x therefore curve will be symmetric about y-axis.

(v) From (1), $y = \pm\frac{b}{a}\sqrt{x^2 - a^2}$, since y is real $\Rightarrow x^2 - a^2 \geq 0$ or $x \in (-\infty, -a] \cup [a, \infty)$

$\Rightarrow$ curve don't lie in $(-a, a)$

For each $x \geq a$ or $x \leq -a$ there are two values of y symmetrically situated on both side of x-axis. Hence, the curve denoted by (1) consist of two symmetrical branches, each extending to infinite in two direction.

FACTS ABOUT THE HYPERBOLA :

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$

(1) By symmetry of equation of hyperbola, if we take second focus (–ae, 0) and second directrix $x=-\frac{a}{e}$ and perform same calculation then we get same equation of hyperbola. This suggest that their are two foci are (ae, 0) and (– ae, 0) and the two corresponding directrices as $x=\frac{a}{e}$ and $x=-\frac{a}{e}$. If focus is (ae, 0), then corresponding directrix is $x=\frac{a}{e}$ and if focus is (– ae, 0), then corresponding directrix is $x=-\frac{a}{e}$.

(2) By definition of hyperbola, the distance of any point P on the hyperbola from focus
= e(the distance of P from the corresponding directrix)

(3) Distance between foci $SS'=2ae$ & distance between directrix $ZZ'=2\frac{a}{e}$.

(4) Two hyperbola are said to be similar if they have same eccentricity.

(5) The hyperbola has two branches neither of them cut the y-axis (conjugate axis).

(6) Since the fundamental equation to the hyperbola only differs from that to the ellipse is having $-b^2$ instead of b^2. It is observed that many propositions for the hyperbola are derived from those for the ellipse by simply changing the sign of b^2.

(7) Eccentricity $e=\sqrt{1+\frac{b^2}{a^2}}$. Also, $b^2=a^2(e^2-1)$.

The smaller the e, the smaller will be the value of b for a given a. Therefore, as e decreases for a given a, the branches of the hyperbola would be bending towards x-axis. As e increases, the branches open up.

(8) Equation of hyperbola when its transverse and conjugate axes are x and y-axes respectively is

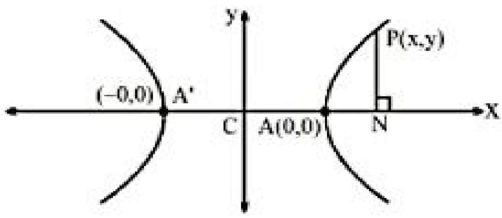
$$\frac{x^2}{a^2}-\frac{y^2}{a^2(e^2-1)}=1 \quad \text{or} \quad x^2-\frac{y^2}{(e^2-1)}=a^2$$

If e kept constant and $a\to 0$, then hyperbola will tend to pair of straight lines $x^2-\frac{y^2}{(e^2-1)}=0$, both passing through the origin.

Thus in the situated limiting case of a hyperbola is a pair of straight lines.

(9) Since $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

$$\therefore \quad \frac{y^2}{b^2}=\frac{x^2}{a^2}-1=\frac{(x-a)(x+a)}{a^2}$$



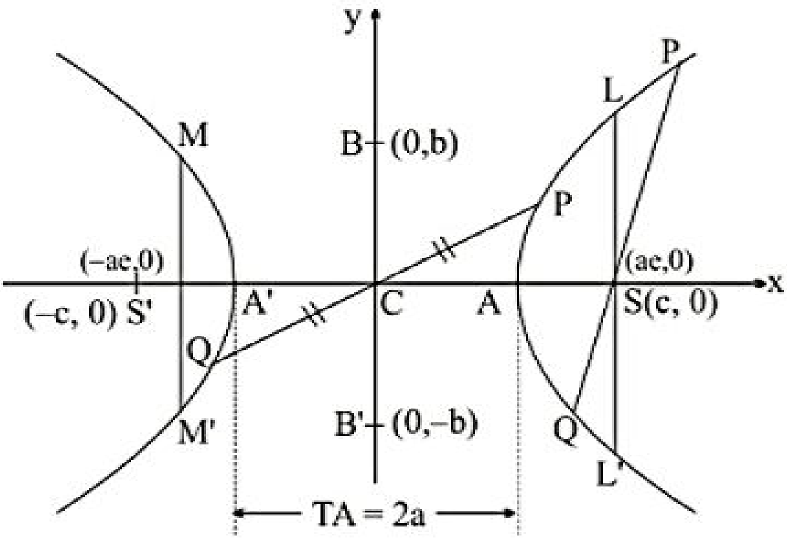
From figure, AN = CN – CA = x – a
A'N = CN + CA = x + a and PN = y

$$\therefore \quad \frac{(PN)^2}{}=\frac{b^2}{}$$

TERMS RELATED TO HYPERBOLA :

(1) Centre :

In the figure, C is the centre of the ellipse. All chords passing through C are called diameter and bisected at C.



(2) Foci :

S(ae, 0) and S'(–ae, 0) are two foci of hyperbola. Line containing the fixed points S and S' (called Foci) is called Transverse Axis (TA) or Focal Axis and the distance between S and S' is called Focal Length.

(3) Axes :

The line AA' is called transverse axis and the line BB' is perpendicular to it and passes through the centre (0, 0) of the hyperbola is called conjugate axis.

The length of transverse and conjugate axes are taken as 2a and 2b respectively.

The transverse and conjugate axes together are called principal axes of hyperbola and their intersection point is called the centre of hyperbola.

The points of intersection of the directrix with the transverse axis are known as Foot of the directrix (Z and Z').

(4) Vertex :

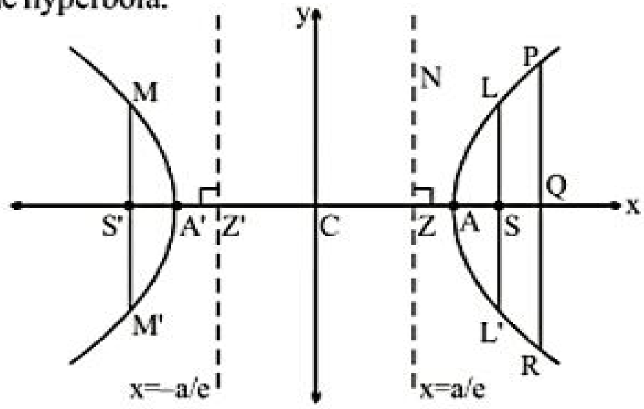
The points of intersection (A, A') of the curve with the transverse axis are called Vertices of the hyperbola.

(5) Double ordiante :

Any chord perpendicular to the Transverse axis is called a Double Ordinate.

(6) Latus-rectum :

When double ordinates passes through the focus of parabola then it is called the latus rectum. In the given figure LL' and MM' are the latus-rectums of the hyperbola.



let LL' = 2k then LS = L'S = k

Let L(ae, k) lie on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

$\therefore \quad \frac{a^2e^2}{a^2} - \frac{k^2}{b^2} = 1$

or $\quad k^2 = b^2(e^2 - 1) = b^2\left(\frac{b^2}{a^2}\right) \qquad [\because b^2 = a^2(e^2 - 1)]$

$\therefore \quad k = \frac{b^2}{a} \qquad (\because k > 0)$

$\therefore \quad 2k = \frac{2b^2}{a} = LL'$

Length of latus rectum $= LL' = \frac{2b^2}{a} = \frac{(CA)^2}{TA} = 2a(e^2 - 1) = 2e\left(ae - \frac{a}{e}\right)$

= (2e) (distance between the focus and the foot of the corresponding directrix)

End points of latus -rectums are

$L = \left(ae, \frac{b^2}{a}\right), L' = \left(ae, -\frac{b^2}{a}\right); M = \left(-ae, \frac{b^2}{a}\right); M' = \left(-ae, -\frac{b^2}{a}\right)$ respectively.

(7) Focal chord :

A chord of hyperbola passing through its focus is called a focal chord.

ECCENTRICITY :

For the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ we have

$b^2 = a^2(e^2 - 1)$

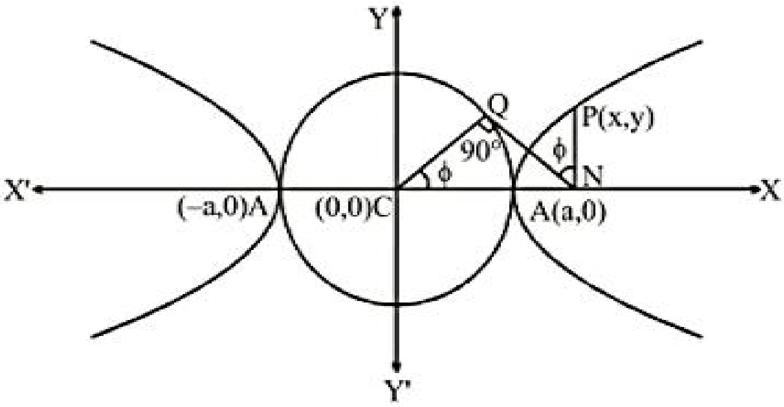
$\Rightarrow \quad e = \sqrt{1 + \left(\frac{b^2}{a^2}\right)} \Rightarrow e = \sqrt{\left\{1 + \frac{(\text{conjugate axis})^2}{(\text{transverse axis})^2}\right\}}$

Eccentricity defines the curvature of the hyperbola and is mathematically spelled as :

$$e = \frac{\text{distance from centre to focus}}{\text{distance from centre to vertex}}$$

PARAMETRIC EQUATIONS OF THE HYPERBOLA :

Let $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ be the hyperbola with centre C and transverse axis A'A. Therefore circle drawn with centre C and segment A'A as a diameter is called auxiliary circle of the hyperbola.



$\therefore$ Equation of the auxiliary circle is

$$x^2 + y^2 = a^2$$

Let P(x, y) be any point on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. Draw PN perpendicular to x-axis.

Let NQ be a tangent to the auxiliary circle $x^2 + y^2 = a^2$. Join CQ and let $\angle QCN = \phi$ then P and Q are the corresponding points of the hyperbola and the auxiliary circle. Here ϕ is the eccentric angle of P. $(0 \le \phi < 2\pi)$.

Since $\quad Q = (a \cos \phi, a \sin \phi)$

Now $\quad x = CN = CQ \sec \phi = \sec \phi \cdot a$

$\therefore \quad P(x, y) = (a \sec \phi, y)$

$\because \quad$ P lies on $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

$$\therefore \quad \frac{a^2 \sec^2 \phi}{a^2} - \frac{y^2}{b^2} = 1 \text{ or } \frac{y^2}{b^2} = \sec^2 \phi - 1 = \tan^2 \phi$$

$\therefore \quad y = \pm b \tan \phi$

$\therefore \quad y = b \tan \phi \qquad$ (P lies in I quadrant)

The equations of $x = a \sec \phi$ and $y = \tan \phi$ are known as the parametric equations of the hyperbola.

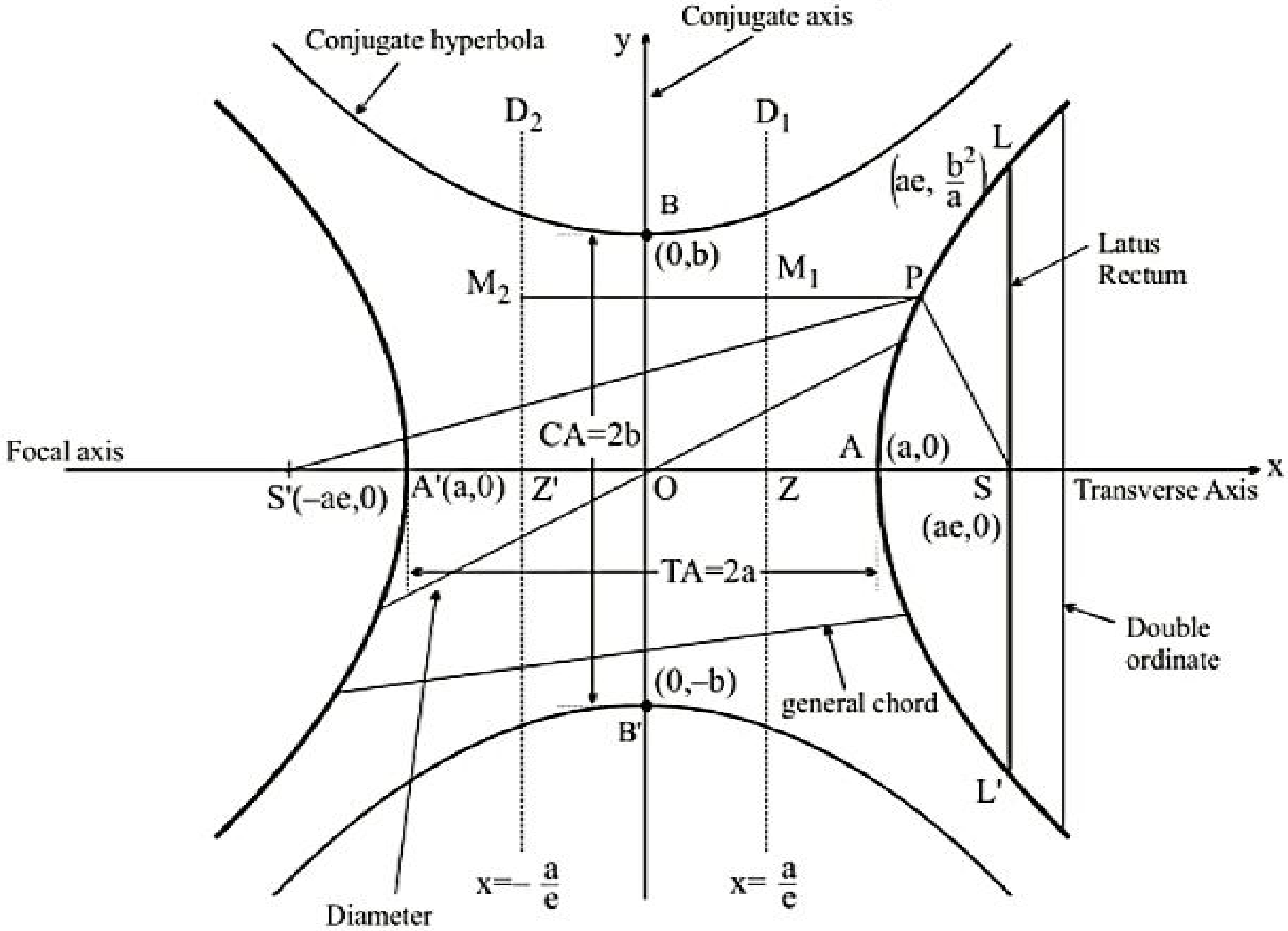
$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Position of points Q an auxiliary circle and corresponding point P which describes the hyperbola or shown below in the table. Here $0 \le \phi < 2\pi$.

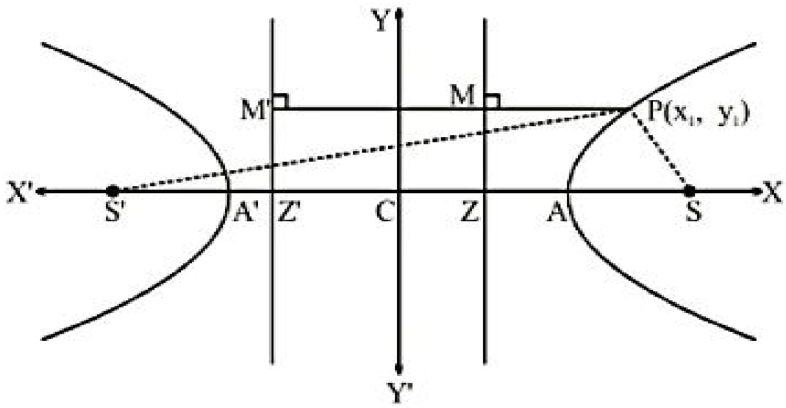
ϕ varies from	$Q(a\cos\phi, a\sin\phi)$	$P(a\sec\phi, b\tan\phi)$
0 to $\frac{\pi}{2}$	I	I
$\frac{\pi}{2}$ to π	II	III
π to $\frac{3\pi}{2}$	III	II
$\frac{3\pi}{2}$ ro 2π	IV	IV

HYPERBOLA AT A GLANCE :

Parametric coordinates x = a secθ and y = b tanθ



FOCAL DISTANCE OF A POINT ON HYPERBOLA :



The hyperbola is $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$ ....(1)

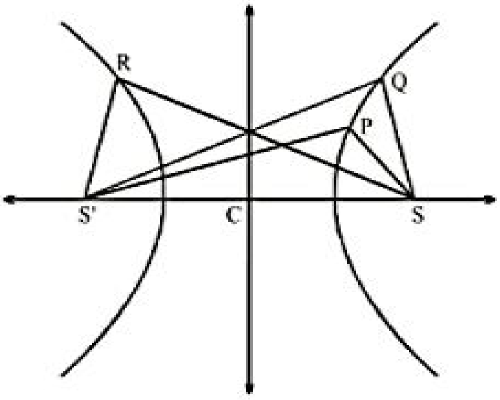
The foci S and S' are (ae, 0) and (– ae, 0) & corresponded directrix are $x=\dfrac{a}{e}$ and $x=-\dfrac{a}{e}$ respectively.

Let $P(x_1, y_1)$ be any point on (1).

Now $\quad SP = ePM = e\left(x_1-\dfrac{a}{e}\right)=ex_1-a$

and $\quad S'P = ePM' = e\left(x_1 + \frac{a}{e}\right) = ex_1 + a$

$\therefore \quad S'P - SP = (ex_1 + a) - (ex_1 - a) = 2a$
$= AA' = \text{Transverse axis}$



Thus hyperbola is the locus of a point which moves in a plane such that the difference of its distances from two fixed point (foci) is constant and always equal to transverse axis.

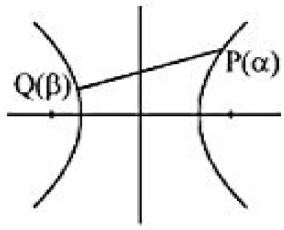
Hence, in the given figure

PS' – PS = QS' – QS = RS – RS' = length of transverse axis.

CHORD OF HYPERBOLA :

Chord joining two points with eccentric angles α and β is given by

$$\frac{x}{a}\cos\frac{\alpha-\beta}{2} - \frac{y}{b}\sin\frac{\alpha+\beta}{2} = \cos\frac{\alpha+\beta}{2} \qquad \text{....(1)}$$



if (1) passes through (d, 0) then

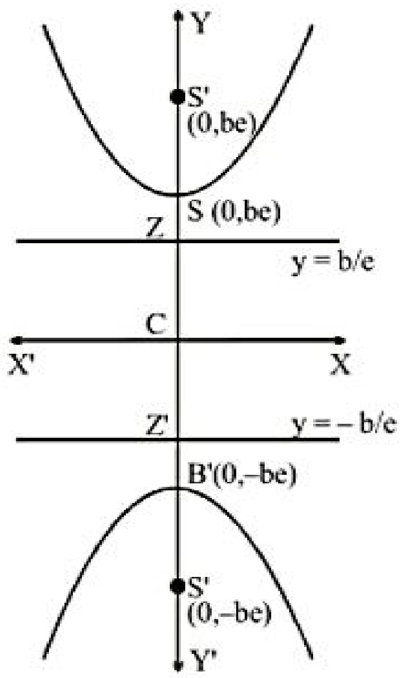
$$\frac{d}{a}\cos\frac{\alpha-\beta}{2} = \cos\frac{\alpha+\beta}{2}$$

$$\frac{d}{a} = \frac{\cos\frac{\alpha}{2}\cos\frac{\beta}{2} - \sin\frac{\alpha}{2}\sin\frac{\beta}{2}}{\cos\frac{\alpha}{2}\cos\frac{\beta}{2} + \sin\frac{\alpha}{2}\sin\frac{\beta}{2}}$$

$$\frac{d+a}{d-a} = -\frac{\cos\frac{\alpha}{2}\cos\frac{\beta}{2}}{\sin\frac{\alpha}{2}\sin\frac{\beta}{2}} \quad \Rightarrow \quad \tan\frac{\alpha}{2}\tan\frac{\beta}{2} = \frac{a-d}{a+d}$$

CONJUGATE HYPERBOLA :

Corresponding to every hyperbola there exist a hyperbola such that, the conjugate axis and transverse axis of one is equal to the transverse axis and conjugate axis of other, such hyperbolas are known as conjugate to each other.



Hence for the hyperbola, $\quad \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \qquad \text{.....(1)}$

the conjugate hyperbola is, $\quad \frac{x^2}{a^2} - \frac{y^2}{b^2} = -1 \qquad \text{....(2)}$

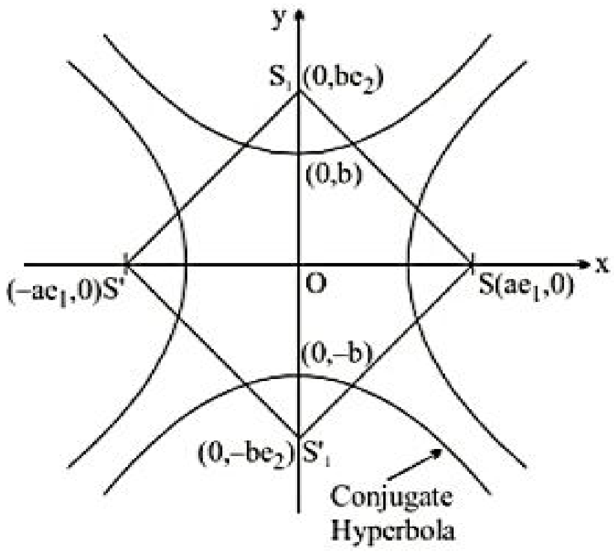
Comparison between hyperbola and its conjugate hyperbola

Basic Elements	Hyperbola	Conjugate Hyperbola
	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$
Centre	(0, 0)	(0, 0)
Length of transeverse axis	2a	2b
Length of conjugate axis	2b	2a
Eccentricity	$b^2 = a^2 (e^2 - 1)$	$a^2 = b^2 (e^2 - 1)$
Foci	$(\pm ae, 0)$	$(0, \pm be)$
Equation of directrix	$x = \pm \frac{a}{e}$	$y = \pm \frac{b}{e}$
Length of latus rectum	$\frac{2b^2}{a}$	$\frac{2a^2}{b}$
Parametric coordinate	$(a \sec \phi, b \tan \phi)$ $0 < \phi < 2\pi$	$(a \tan \phi, b \sec \phi)$ $0 \le \phi < 2\pi$
Focal distances	$ex_1 \pm a$	$ey_1 \pm b$
Diffference of focal distances	2a	2b
Equation of transverse axis	y = 0	x = 0
Equation of conjugates	x = 0	y = 0
Tangent at vertices	$x = \pm a$	$y = \pm b$

Note :

(1) The foci of a hyperbola and its conjugate are concylic and form the vertices of a square.

(2) If e_1 & e_2 are the eccentrcities of the hyperbola & its conjugate then $e_1^{-2} + e_2^{-2} = 1$.



RECTANGULAR HYPERBOLA :

If the lengths of transverse and conjugate axes of any hyperbola be equal then it is called rectangular or equilateral hyperbola.

Since length of transverse axis and conjugate axis are same i.e. $a = b$

then, $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ becomes $x^2 - y^2 = a^2$.

Eccentricity, $e = \sqrt{\left(1 + \dfrac{b^2}{a^2}\right)} = \sqrt{2}$.

All the results of $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ are applicable to the hyperbola $x^2 - y^2 = a^2$ after changing b by a.

FIND ALL THE PARAMETERS OF A HYPERBOLA $\dfrac{(x-h)^2}{a^2} - \dfrac{(y-k)^2}{b^2} = 1$:

When equation of the hyperbola is $\dfrac{(x-\alpha)^2}{a^2} - \dfrac{(y-\beta)^2}{b^2} = 1$

This equation is the form of $\dfrac{X^2}{a^2} - \dfrac{Y^2}{b^2} = 1$, where $X = x - h$ and $Y = y - k$

(1) Length of semi-transverse axis = a, length of semi-conjugate axis = b

(2) Equation of transverse axis is $Y = 0$, i.e, $y - k = 0$
Equation of conjugate axis is $X = 0$, i.e, $x - h = 0$

(3) Coordinates of centre is given by $X = 0$ and $Y = 0$, i.e., $x - h = 0$ and $y - k = 0$
Therefore, centre is (h, k)

(4) Eccentricity of the hyperbola $e = \sqrt{1 + \dfrac{b^2}{a^2}}$

(5) Coordinates of vertices of the hyperbola are given by $X = \pm a, Y = 0$ i.e., $x - h = \pm a, y - k = 0$. Hence vertices are $(h \pm a, k)$.

(6) Coordinate of foci are given by $X = \pm ae, Y = 0$
i.e., $x - h = \pm ae, y - k = 0$. Hence foci are $(h \pm ae, k)$

(7) Equation of directrices of the hyperbola are $X = \pm \frac{a}{e}$, i.e., $x - h = \pm \frac{a}{e}$.

Hence directrices are $x = h \pm \frac{a}{e}$

(8) Length of latus rectum $= \frac{2b^2}{a}$

(9) Coordinate of ends of latera recta are given by $X = ae, Y = \pm \frac{b^2}{a}$ i.e. $x - h = \pm ae, y - h = \pm \frac{b^2}{a}$

$\therefore$ end if LR is $\left(h \pm ae, k \pm \frac{b^2}{a}\right)$

Illustration :

Find the equation of the hyperbola whose directrix is $2x + y = 1$, focus $(1, 2)$ and eccentricity $\sqrt{3}$.

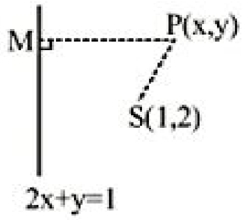
Sol. *Let $P(x, y)$ be any point on the hyperbola.*
Draw PM perpendicular from P on the directrix.
Then by definition $SP = e\,PM$

$$\Rightarrow \quad (SP)^2 = e^2 (PM)^2$$

$$\Rightarrow \quad (x-1)^2 + (y-2)^2 = 3\left\{\frac{2x+y-1}{\sqrt{4+1}}\right\}^2$$

$$\Rightarrow \quad 5(x^2 + y^2 - 2x - 4y + 5) = 3(4x^2 + y^2 + 1 + 4xy - 2y - 4x)$$

$$\Rightarrow \quad 7x^2 - 2y^2 + 12xy - 2x + 14y - 22 = 0$$



Which is the required hyperbola.

Illustration :

Find the eccentricity of the hyperbola whose latus-rectum is half of its transverse axis.

Sol. *Let the equation of hyperbola be* $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

Then transverse axis = 2a and latus-rectum $= \frac{2b^2}{a}$

According to question $\frac{2b^2}{a} = \frac{1}{2}(2a)$

$\Rightarrow 2b^2 = a^2$ $\quad (\because b^2 = a^2(e^2 - 1))$

$\Rightarrow 2a^2(e^2 - 1) = a^2 \quad \Rightarrow \quad 2e^2 - 2 = 1 \quad \Rightarrow \quad e^2 = \frac{3}{2}$

$\therefore e = \sqrt{\frac{3}{2}}$.

Hence the required eccentricity is $\sqrt{\frac{3}{2}}$.

Illustration :

Find the equaiton of the hyperbola, the length of whose latus rectum is 8, eccentricity is $\frac{3}{\sqrt{5}}$ and whose transverse and conjugate axes are along the x and y axes respectively.

Sol. Given, $8 = \frac{2b^2}{a} = \frac{2a^2(e^2 - 1)}{a}$

$$\Rightarrow \quad 8 = 2a(e^2 - 1) = 2a\left[\left(\frac{3}{\sqrt{5}}\right)^2 - 1\right] = 2a\left[\frac{9}{5} - 1\right] = \frac{8a}{5}$$

$\Rightarrow \quad a = 5$

Again, $8 = \frac{2b^2}{a} = 8 \times 5 \quad \therefore b^2 = 20$

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad$ or $\quad \frac{x^2}{25} - \frac{y^2}{20} = 1$

Illustration :

Find the equaiton of the hyperbola whose eccentricity is $\sqrt{2}$ and the distance between the foci is 16, taking transverse and conjugate axes of the hyperbola as x and y-axes respectively.

Sol. Let the equation of the hyperbola be

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \qquad ...(1)$$

The coordinates of the foci are (ae, 0) and (–ae, 0)

Given $2ae = 16$ or $2.a\sqrt{2} = 16 \quad \therefore a = 4\sqrt{2}$

Also $\quad b^2 = a^2(e^2 - 1) = 32(2 - 1) = 32 \quad [\because e = \sqrt{2}]$

$\therefore$ The required equation of the hyperbola is $\frac{x^2}{32} - \frac{y^2}{32} = 1$ or $x^2 - y^2 = 32$

Illustration :

Prove that the point $\left\{\frac{a}{2}\left(t+\frac{1}{t}\right), \frac{b}{2}\left(t-\frac{1}{t}\right)\right\}$ *lies on the hyperbola for all values of t* ($t \neq 0$).

Sol. Let $x = \frac{a}{2}\left(t+\frac{1}{t}\right)$ or $\frac{2x}{a} = t + \frac{1}{t}$ or $\left(\frac{2x}{a}\right)^2 = t^2 + \frac{1}{t^2} + 2$ ....(1)

and let $y = \frac{b}{2}\left(t-\frac{1}{t}\right)$ or $\frac{2y}{b} = t - \frac{1}{t}$ or $\left(\frac{2y}{b}\right)^2 = t^2 + \frac{1}{t^2} - 2$ ....(2)

Subtracting (2) from (1), $\frac{4x^2}{a^2} - \frac{4y^2}{b^2} = 4$ or $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

Which is hyperbola.

Illustration :

Find the lengths of transverse axis, conjugate axis, eccentricity, the co-ordinates of foci, vertices, lengths of the latus-rectum and equations of the directrices of the following hyperbolas.

(i) $9x^2 - y^2 = 1$ (ii) $16x^2 - 9y^2 = -144$.

Sol.

(i) The equation $9x^2 - y^2 = 1$ can be written as $\frac{x^2}{(1/9)} - \frac{y^2}{1} = 1 \Rightarrow a = \frac{1}{3}, b = 1$

The length of transverse, axis $= 2a = \frac{2}{3}$

The length of conjugate axis $= 2b = 2$.

Eccentricity $e = \sqrt{\left(1+\frac{b^2}{a^2}\right)} \sqrt{1+\frac{1}{(1/9)}} = \sqrt{10}$

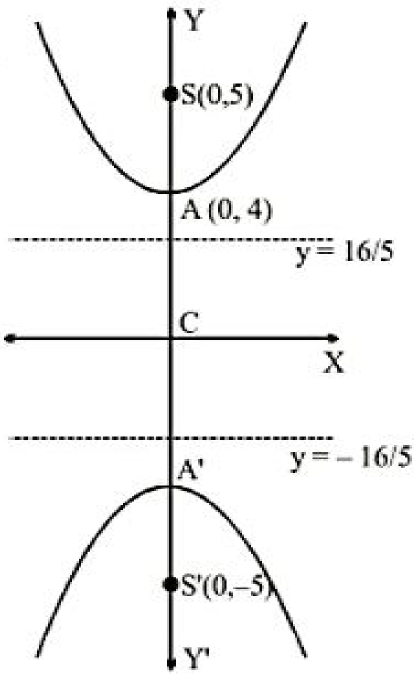
The co-ordinates of the foci are $(\pm ae, 0)$ i.e., $\left(\pm\frac{\sqrt{10}}{3}, 0\right)$.

The co-ordinates of the vertices are $(\pm a, 0)$ i.e., $\left(\pm\frac{1}{3}, 0\right)$

The length of latus - rectum $= \frac{2b^2}{a} = \frac{2(1)^2}{1/3} = 6$.

The equations of the directrices are

$$x = \pm\frac{a}{e} \text{ i.e., } x = \pm\frac{1/3}{\sqrt{10}} \text{ or } x = \pm\frac{1}{3\sqrt{10}}$$



(ii) *The equation $16x^2 - 9y^2 = -144$ can be written as* $\dfrac{x^2}{9} - \dfrac{y^2}{16} = -1$.

$a = 3$, $b = 4$, This conjugate hyperbola
The length of transverse axis = $2b = 8$.
The length of conjugate axis = $2a = 6$.

Eccentricity $e = \sqrt{\left(1 + \dfrac{a^2}{b^2}\right)} = \sqrt{\left(1 + \dfrac{9}{16}\right)} = \dfrac{5}{4}$

The co-ordinates of the foci are $(0, \pm be)$ i.e., $(0, \pm 5)$
The co-ordinates of the vertices are $(0, \pm b)$ i.e., $(0, \pm 4)$.

The length of latus-rectum $= \dfrac{2a^2}{b} = \dfrac{2(3)^2}{4} = \dfrac{9}{2}$

The equation of directrices are $y = \pm \dfrac{b}{e} = \pm \dfrac{4}{(5/4)} \Rightarrow y = \pm \dfrac{16}{5}$.

Illustration :

If e and e' are the eccentricities of the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ *and its conjugate hyperbola,*

prove that $\dfrac{1}{e^2} + \dfrac{1}{e'^2} = 1$.

Sol. *Given hyperbola is* $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ ....(1)

The eccentricity e of hyperbola (1) is given by

$$b^2 = a^2(e^2 - 1)$$

$$\therefore \quad \frac{b^2}{a^2} = e^2 - 1 \qquad ....(2)$$

The equation of the conjugate hyperbola is $\dfrac{y^2}{b^2} - \dfrac{x^2}{a^2} = 1$.

$\therefore$ *Its eccentricity e' is given by* $a^2 = b^2(e'^2 - 1)$

$$\therefore \quad (e'^2 - 1) = \frac{a^2}{b^2} \qquad ....(3)$$

Multiple (2) and (3), we get

$$(e^2 - 1) \times (e'^2 - 1) = \frac{a^2}{b^2} \times \frac{b^2}{a^2}$$

$$\therefore \quad e^2e'^2 - e^2 - e'^2 + 1 = 1$$

$$\therefore \quad e^2e'^2 = e^2 + e'^2$$

$$\therefore \quad 1 = \frac{1}{e^2} + \frac{1}{e'^2}.$$

Illustration :

If the foci of the ellipse $\frac{x^2}{16}+\frac{y^2}{b^2}=1$ *and hyperbola* $\frac{x^2}{144}-\frac{y^2}{81}=\frac{1}{25}$ *coincide, then the value of* b^2 *is-*

(A) 1 (B) 5 (C) 7 (D) 9

Sol. *For hyperbola* $\frac{x^2}{144}-\frac{y^2}{81}=\frac{1}{25} \Rightarrow \frac{x^2}{144/25}-\frac{y^2}{81/25}=1$

$e^2 = 1+\frac{b^2}{a^2} = 1+\frac{81}{144} = \frac{225}{144}$; $e = \frac{15}{12} = \frac{5}{4}$

Hence the foci are

$(\pm ae,\ 0) = \left(\pm\frac{12}{5}.\frac{5}{4},0\right) = (\pm 3,\ 0)$

Now the foci coincide therefore for ellipse

$ae = 3$ or $a^2e^2 = 9$ or $a^2\left(1-\frac{b^2}{a^2}\right) = 9$

$a^2 - b^2 = 9$ or $16 - b^2 = 9 \Rightarrow b^2 = 7.$ **Ans. [C]**

Illustration :

The foci of a hyperbola coincides with the foci of the ellipse $\frac{x^2}{25}-\frac{y^2}{9}=1$. *Find the equation of the hyperbola if its eccentricity is 2.*

Sol. *The given ellipse is* $\frac{x^2}{25}+\frac{y^2}{9}=1$ or $\frac{x^2}{5^2}+\frac{y^2}{3^2}=1$ ... (1)

Here $5 > 3$ $\therefore a = 5$ and $b = 3$

$\therefore$ *The foci of the ellipse are on the x-axis.*

$\therefore$ *Eccentricity of the ellipse,* $e = \sqrt{1-\frac{b^2}{a^2}} = \sqrt{1-\frac{9}{25}} = \frac{4}{5}$

$\therefore$ *Foci of the ellipse are* $(\pm\ ae,\ 0)$ or $(\pm 4,\ 0)$

Let the equation of the hyperbola be $\frac{x^2}{A^2}-\frac{y^2}{B^2}=1$.

$\therefore$ *Foci are* $(\pm Ae,\ 0)$ or $(\pm 2A,\ 0)$ $[\because e = 2]$

$\therefore$ $2A = 4$ i.e., $A = 2$ and $B = A\sqrt{e^2-1} = 2\sqrt{(2)^2-1} = 2\sqrt{3}$.

$\therefore$ *The equation of the hyperbola is* $\frac{x^2}{(2)^2}-\frac{y^2}{(2\sqrt{3})^2}=1$ or $\frac{x^2}{4}-\frac{y^2}{12}=1$

or $3x^2 - y^2 = 12.$

Illustration :

The equation $16x^2 - 3y^2 - 32x + 12y - 44 = 0$ represents a hyperbola

(A) the length of whose transverse axis is $2\sqrt{3}$ *(B) the length of whose conjugate axis is 8.*

(C) whose centre is (1, 2) *(D) whose eccentricity is $\dfrac{\sqrt{19}}{3}$*

Sol. *We have, $16(x^2 - 2x) - 3(y^2 - 4y) = 44 \Rightarrow 16(x-1)^2 - 3(y-2)^2 = 48$*

$$\Rightarrow \frac{(x-1)^2}{3} - \frac{(y-2)^2}{16} = 1,$$

centre is $x - 1 = 0$ & $y - 2 = 0 \Rightarrow (1, 2)$

$a^2 = 3 \Rightarrow a = \sqrt{3}$, $b^2 = 16 \Rightarrow b = 4$.

$\therefore$ $l(TA) = 2a = 2\sqrt{3}$ and $l(CA) = 2b = 2.4 = 8$.

This equation represents a hyperbola with eccentricity given

$$e = \sqrt{1 + \left(\frac{\text{Conjugate axis}}{\text{Transverse axis}}\right)^2} = \sqrt{1 + \left(\frac{4}{\sqrt{3}}\right)^2} = \sqrt{\frac{19}{3}}$$

Ans.[A, B, C]

Illustration :

The equation $9x^2 - 16y^2 - 18x + 32y - 151 = 0$ represent a hyperbola -

(A) The length of the transverse axes is 4 *(B) Length of latus rectum is 9*

(C) Equation of directrix is $x = \dfrac{21}{5}$ and $x = -\dfrac{11}{5}$ *(D) None of these*

Sol. *We have $9x^2 - 16y^2 - 18x + 32y - 151 = 0$*

$9(x^2 - 2x) - 16(y^2 - 2y) = 151$

$9(x^2 - 2x + 1) - 16(y^2 - 2y + 1) = 144$

$9(x - 1)^2 - 16(y - 1)^2 = 144$

$$\frac{(x-1)^2}{16} - \frac{(y-1)^2}{9} = 1$$

Shifting the origin at (1, 1) without rotating the axes

$$\frac{X^2}{16} - \frac{Y^2}{9} = 1$$

where $X = x - 1$ and $Y = y - 1$

This is of the form $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$

where $a^2 = 16$ and $b^2 = 9$ so

The length of the transverse axes = $2a = 8$; $l(CA) = 2b = 6$.

The length of the letus rectum $= \dfrac{2b^2}{a} = \dfrac{2 \cdot 9}{4} = \dfrac{9}{2}$ *and* $e = \sqrt{1 + \dfrac{b^2}{a^2}} = \sqrt{1 + \dfrac{9}{16}} = \dfrac{5}{4}$

The equaiton of the directrix $X = \pm \frac{a}{e}$

$$x - 1 = \pm \frac{16}{5} \Rightarrow x = \pm \frac{16}{5} + 1$$

$$x = \frac{21}{5} \text{ and } x = -\frac{11}{5}$$

Equation of directrix is $x = \frac{21}{5}$ and $x = -\frac{11}{5}$ **Ans. [C]**

Illustration :

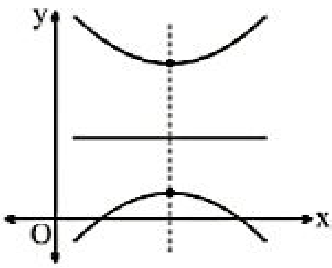
Show that the equation $7y^2 - 9x^2 + 54x - 28y - 116 = 0$ represent a hyperbola. Find the co-ordinates of the centre, lengths of transverse and conjugate axes, eccentricity, latus-rectum, co-ordinate of foci, vertices and equations of the directrices of the hyperbola.

Sol. We have $7y^2 - 9x^2 + 54x - 28y - 116 = 0$

or $7(y^2 - 4y) - 9(x^2 - 6x) - 116 = 0$

or $7(y^2 - 4y + 4) - 9(x^2 - 6x + 9) = 116 + 28 - 81$

or $7(y - 2)^2 - 9(x - 3)^2 = 63$



or $\frac{(y-2)^2}{9} - \frac{(x-3)^2}{7} = 1$

or $\frac{Y^2}{9} - \frac{X^2}{7} = 1.$

where $X = x - 3$ and $Y = y - 2$.

This equation represents conjugate hyperbola. Comparing it with $\frac{Y^2}{b^2} - \frac{X^2}{a^2} = 1.$

We get $b^2 = 9$ and $a^2 = 7$

$\therefore b = 3$ and $a = \sqrt{7}$.

Centre : $X = 0, Y = 0$ i.e., $x - 3 = 0, y - 2 = 0$

$\therefore$ Centre is $(3, 2)$

Length of transverse axis $= 2b = 6$

Length of conjugate axis $= 2a = 2\sqrt{7}$.

The eccentricity e is given by $e = \sqrt{1 + \frac{a^2}{b^2}} = \sqrt{1 + \frac{7}{9}} = \frac{4}{3}$.

The length of latus-rectum $= \frac{2a^2}{b} = \frac{2(7)}{3} = \frac{14}{3}$

The co-ordinates of foci $X = 0, Y = \pm be$

$\Rightarrow \quad x - 3 = 0, y - 2 = \pm 3 \times \frac{4}{3}$ or $(3, 2 \pm 4)$

i.e., $(3, -2)$ and $(3, 6)$

The co-ordinates of vertices are

or $X = 0, Y = \pm b$

or $(3, 2 \pm 3)$

or vertices are $(3, -1)$ and $(3, 5)$

The equation of directrices are $Y = \pm\frac{b}{e}$

$$\Rightarrow \quad y - 2 = \pm\frac{3}{4/3} \Rightarrow y = \left(2 \pm \frac{9}{4}\right)$$

i.e., $y = \frac{17}{4}$ and $y = \frac{-1}{4}$.

Illustration :

Find the equation of the hyperbola having $e = \frac{3}{2}$ and foci at $(\pm 3, 0)$

Sol. The foci are at $(\pm 3, 0)$. These are on the x-axis.
Since centre of the hyperbola is the mid-point of the line segement joining the foci, therefore centre is $(0, 0)$.

Let the equation of the hyperbola be $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ ... (1)

where $a, b > 0$ and $b^2 = a^2 (e^2 - 1)$.

The foci of this hyperbola are $(\pm ae, 0)$. $\therefore ae = 3$ & $e = \frac{3}{2} \Rightarrow a = 2$

$$\therefore \quad b^2 = a^2 (e^2 - 1) = 4\left(\frac{9}{4} - 1\right) = 5.$$

$\therefore$ From (1), the requird equation of the hyperbola is $\frac{x^2}{4} - \frac{y^2}{5} = 1$.

Illustration :

Find the equation of the hyperbola having eccentricity $e = \frac{4}{3}$ and vertices at $(0, \pm 7)$.

Sol. The vertices of the hyperbola are at $(0, \pm 7)$ and these are on the y-axis. Centre of the hyperbola will be the mid point of the vertex (focus also) $\Rightarrow C(0, 0)$

Let the equation of the hyperbola be $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$,

The vertices of this hyperbola are $(0, \pm b)$ $\therefore b = 7$

$$\text{Now, } a^2 = b^2 (e^2 - 1) = 7^2 \left[\left(\frac{4}{3}\right)^2 - 1\right] = 49\left(\frac{16 - 9}{9}\right) = \frac{343}{9}$$

$\therefore$ From (1), the equaiton of this hyperbola is $\frac{y^2}{(7)^2} - \frac{x^2}{\frac{343}{9}} = 1$ or $\frac{y^2}{49} - \frac{9x^2}{343} = 1$

Illustration :

Find the equation of the hyperbola having vertices at (±5, 0) and foci at (±7, 0).

Sol. *The foci of the hyperbola are at (±7, 0). They are on the x-axis.*
Also centre of the hyperbola will be (0, 0).

Let the equaiton of the hypebola be $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ ... (1)

where $a, b > 0$ *and* $b^2 = a^2(e^2 - 1)$
The vertices of this hyperbola are $(\pm a, 0)$
But it is given that vertices are $(\pm 5, 0)$ $\therefore a = 5$
The foci of this hyperbola are $(\pm ae, 0)$ $\therefore ae = 7$
$\Rightarrow 5e = 7 \Rightarrow e = 7/5$

Now, $b^2 = a^2(e^2 - 1) = 25\left(\frac{49}{25} - 1\right) = 24$

From (1), the required equation of the hyperbola is

$$\frac{x^2}{(5)^2} - \frac{y^2}{(\sqrt{24})^2} = 1 \text{ or } \frac{x^2}{25} - \frac{y^2}{24} = 1$$

Illustration :

Find the equation of the hyperbola whose foci are $(0, \pm\sqrt{10})$ *and which passes through the point (2, 3).*

Sol. *The foci of the hyperbola are at* $(0, \pm\sqrt{10})$*. These are on the y-axis*
Clearly, centre of the hyperbola will be (0, 0)

Let the equaiton of the hyperbola be $\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$ ... (1)

where $a, b > 0$ *and* $a^2 = b^2(e^2 - 1) \Rightarrow a^2 + b^2 = a^2e^2$ *i.e.* $a^2 + b^2 = 10$

$$\frac{9}{b^2} - \frac{4}{10 - b^2} = 1 \Rightarrow b^4 - 23b^2 + 90 = 0 \Rightarrow b^2 = 18 + 5$$

when $b^2 = 18$ *then* $a^2 = -8$ *not possible*
when $b^2 = 5 \Rightarrow a^2 = 5$ *possible.*

Hence the equaiton of hyperbola is $\frac{y^2}{5} - \frac{x^2}{5} = 1$ *or* $y^2 - x^2 = 5$

Illustration :

Find the equation of the hyperbola whose foci are (6, 4) and (–4, 4) and eccentricity is 2.

Sol. *The centre of the hyperbola is the mid-point of the line joining the two foci. So the co-ordinates of the centre are* $\left(\frac{6-4}{2}, \frac{4+4}{2}\right)$ *i.e., (1, 4)*

Let 2a and 2b be the lengths of transverse and conjugate axes and let e be the eccentricity.

Then equation of hyperbola is $\frac{(x-1)^2}{a^2}-\frac{(y-4)^2}{b^2}=1$

$\because$ *Distance between the foci = 2ae*

$$\sqrt{(6+4)^2+(4-4)^2}=2a\times 2 \quad \Rightarrow \quad 10=4a \Rightarrow a=\frac{5}{2}$$

$$\therefore\ b^2=a^2(e^2-1)=\frac{25}{4}(4-1)=\frac{75}{4}$$

Thus the equation of the hyperbola is $\frac{(x-1)^2}{\left(\frac{25}{4}\right)}-\frac{(y-4)^2}{\left(\frac{75}{4}\right)}=1.$

or $\quad 12(x-1)^2-4(y-4)^2=75$

or $\quad 12(x^2-2x+1)-4(y^2-8y+16)=75$

or $\quad 12x^2-4y^2-24x+32y-127=0.$

Illustration :

Obtain the equation of a hyperbola with co-ordinate axes as principal axes and given that the distances of one of its vertices from the foci are 9 and 1 units.

Sol. *Let equation of hyperbola is* $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1.$ ....(1)

If vertices are A (a, 0) and A′ (–a, 0) and foci are S (ae, 0) and S′ (–ae, 0)

Given $l(S'A)=9$ and $l(SA)=1$

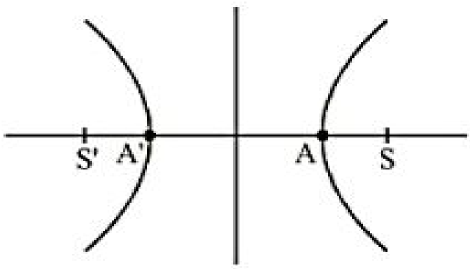
$\Rightarrow \quad a+ae=9$ and $ae-a=1$

or $\quad a(1+e)=9$ and $a(e-1)=1$

$$\therefore \quad \frac{a(1+e)}{a(e-1)}=\frac{9}{1} \Rightarrow 1+e=9e-9 \Rightarrow e=\frac{5}{4}$$

$\because \quad a(1+e)=9$

$$\therefore \quad a\left(1+\frac{5}{4}\right)=9 \Rightarrow a=4$$



$$b^2=a^2(e^2-1)=16\left(\frac{25}{16}-1\right) \Rightarrow b^2=9$$

From (1) equation of hyperbola is $\frac{x^2}{16}-\frac{y^2}{9}=1.$

Practice Problem

Single correct question

Q.1 The eccentricity of the conic represented by $x^2 - y^2 - 4x + 4y + 16 = 0$ is

(A) 1 (B) $\frac{1}{2}$ (C) -1 (D) $\sqrt{2}$

Q.2 If e_1 and e_2 are the eccentricities of the conic sections $16x^2 + 9y^2 = 144$ and $9x^2 - 16y^2 = 144$, then

(A) $e_1^2 - e_2^2 = 1$ (B) $e_1^2 + e_2^2 < 3$ (C) $e_1^2 + e_2^2 = 3$ (D) $e_1^2 + e_2^2 > 3$

Q.3 Eccentricity of the hyperbola conjugate to the hyperbola $\frac{x^2}{4} - \frac{y^2}{12} = 1$ is

(A) $\frac{2}{\sqrt{3}}$ (B) 2 (C) $\sqrt{3}$ (D) $\frac{4}{3}$

Q.4 Locus of the centre of the circle which touches the two circles $x^2 + y^2 + 8x - 9 = 0$ and $x^2 + y^2 - 8x + 7 = 0$ externally, is

(A) $x^2 + y^2 = 16$ (B) $\frac{x^2}{1} + \frac{y^2}{15} = 1$ (C) $\frac{x^2}{1} - \frac{y^2}{12} = 1$ (D) $\frac{x^2}{1} - \frac{y^2}{15} = 1$

Q.5 If the eccentricity of the hyperbola $x^2 - y^2 \sec^2 \alpha = 5$ is $\sqrt{3}$ times the eccentricity of the ellipse $x^2 \sec^2 \alpha + y^2 = 25$, then a value of α is :

(A) $\pi/6$ (B) $\pi/4$ (C) $\pi/3$ (D) $\pi/2$

Q.6 The foci of the ellipse $\frac{x^2}{16} + \frac{y^2}{b^2} = 1$ and the hyperbola $\frac{x^2}{144} - \frac{y^2}{81} = \frac{1}{25}$ coincide. Then the value of b^2 is

(A) 5 (B) 7 (C) 9 (D) 4

Q.7 The focal length of the hyperbola $x^2 - 3y^2 - 4x - 6y - 11 = 0$, is

(A) 4 (B) 6 (C) 8 (D) 10

Q.8 The equation $\frac{x^2}{29 - p} + \frac{y^2}{4 - p} = 1$ $(p \neq 4, 29)$ represents

(A) an ellipse if p is any constant greater than 4.

(B) a hyperbola if p is any constant between 4 and 29.

(C) a rectangular hyperbola if p is any constant greater than 29.

(D) no real curve if p is less than 29.

Q.9 If $\frac{x^2}{\cos^2\alpha} - \frac{y^2}{\sin^2\alpha} = 1$ represents family of hyperbolas where 'α' varies then
(A) distance between the foci is constant
(B) distance between the two directrices is constant
(C) distance between the vertices is constant
(D) distances between focus and the corresponding directrix is constant

Q.10 Let the major axis of a standard ellipse equals the transverse axis of a standard hyperbola and their director circles have radius equal to 2R and R respectively. If e_1 and e_2 are the eccentricities of the ellipse and hyperbola then the correct relation is
(A) $4e_1^2 - e_2^2 = 6$ (B) $e_1^2 - 4e_2^2 = 2$ (C) $4e_2^2 - e_1^2 = 6$ (D) $2e_1^2 - e_2^2 = 4$

Q.11 The equation to the chord joining two points (x_1, y_1) and (x_2, y_2) on the rectangular hyperbola $xy = c^2$ is

(A) $\frac{x}{x_1+x_2} + \frac{y}{y_1+y_2} = 1$ (B) $\frac{x}{x_1-x_2} + \frac{y}{y_1-y_2} = 1$

(C) $\frac{x}{y_1+y_2} + \frac{y}{x_1+x_2} = 1$ (D) $\frac{x}{y_1-y_2} + \frac{y}{x_1-x_2} = 1$

More than one

Q.12 Which of the following equations in parametric form can represent a hyperbolic profile, where 't' is a parameter.

(A) $x = \frac{a}{2}\left(t+\frac{1}{t}\right)$ & $y = \frac{b}{2}\left(t-\frac{1}{t}\right)$ (B) $\frac{tx}{a} - \frac{y}{b} + t = 0$ & $\frac{x}{a} + \frac{ty}{b} - 1 = 0$

(C) $x = e^t + e^{-t}$ & $y = e^t - e^{-t}$ (D) $x^2 - 6 = 2\cos t$ & $y^2 + 2 = 4\cos^2\frac{t}{2}$

Q.13 Let p and q be non-zero real numbers. Then the equation $(px^2 + qy^2 + r)(4x^2 + 4y^2 - 8x - 4) = 0$ represents
(A) two straight lines and a circle, when $r = 0$ and p, q are of the opposite sign.
(B) two circles, when $p = q$ and r is of sign opposite to that of p.
(C) a hyperbola and a circle, when p and q are of opposite sign and $r \neq 0$.
(D) a circle and an ellipse, when p and q are unequal but of same sign and r is of sign opposite to that of p.

Q.14 Let $C_1 : 9x^2 - 16y^2 - 18x + 32y - 23 = 0$ and $C_2 : 25x^2 + 9y^2 - 50x - 18y + 33 = 0$ are two conics then

(A) eccentricity of C_1 is $\frac{5}{4}$.

(B) eccentricity of C_2 is $\frac{5}{3}$.

(C) area of the quadrilateral with vertices at the foci of the conics is $\frac{8}{9}$.

(D) latus rectum of C_1 is greater than latus rectum of C_2.

Integer type question

Q.15 Find the centre, eccentricity and length of the axes of the hyperbola

$$3x^2 - 5y^2 - 6x + 20y - 32 = 0$$

Q.16 Consider the hyperbola $9x^2 - 16y^2 + 72x - 32y - 16 = 0$. Find the following:

(a) centre **(b)** eccentricity

(c) focii **(d)** equation of directrix

(e) length of the latus rectum **(f)** equation of auxilary circle

(g) equation of director circle

Answer key

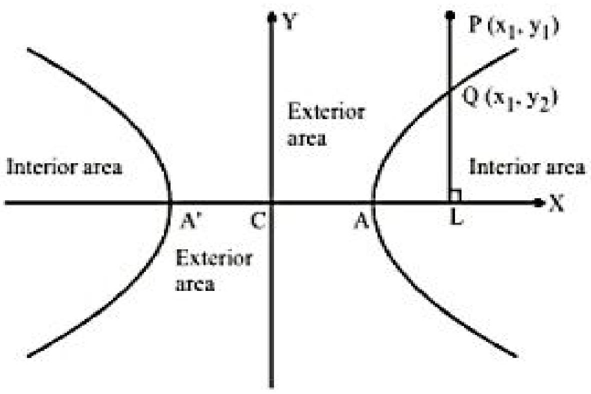
Q.1	D	Q.2	B	Q.3	A	Q.4	D	Q.5	B
Q.6	B	Q.7	C	Q.8	B	Q.9	A	Q.10	C
Q.11	A	Q.12	A, C, D	Q.13	A, B, C, D	Q.14	A, C, D		

Q.15 $(1, 2); 2\sqrt{\left(\frac{2}{5}\right)}; 2\sqrt{5}; 2\sqrt{3}$

Q.16 **(a)** $(-4, -1)$; **(b)** $\frac{5}{4}$; **(c)** $(1, -1), (-9, -1)$; **(d)** $5x + 4 = 0, 5x + 36 = 0,$

(e) $\frac{9}{2}$; **(f)** $(x + 4)^2 + (y + 1)^2 = 16$; **(g)** $(x + 4)^2 + (y + 1)^2 = 7$

POSITION OF A POINT WITH RESPECT TO HYPERBOLA :



Let $S(x, y) \equiv \frac{x^2}{a^2} - \frac{y^2}{b^2} - 1$ be the given hyperbola and $P(x_1, y_1)$ is the given point.

(i) If $S(x_1, y_1) > 0$ then $P(x_1, y_1)$ lie inside the ellipse.

(ii) If $S(x_1, y_1) < 0$ then $P(x_1, y_1)$ lie outside the ellipse.

(iii) If $S(x_1, y_1) = 0$ then $P(x_1, y_1)$ lie on the ellipse.

Illustration :

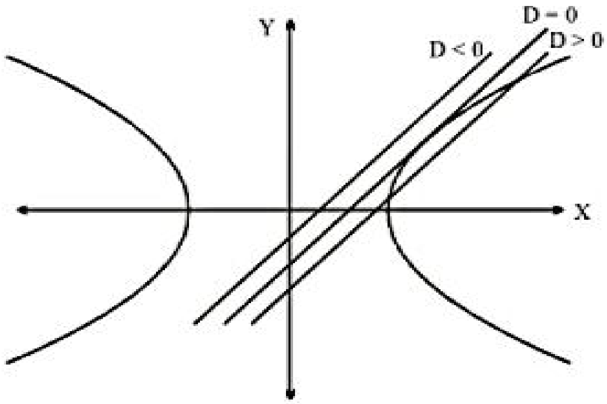
Find the position of the point (5, – 4) relative to the hyperbola $9x^2 - y^2 = 1$.

Sol. *Here* $S(x, y) \equiv 9x^2 - y^2 - 1$

and $S(5, -4) = 9(5)^2 - (-4)^2 - 1 = 255 - 16 - 1 = 208 > 0$

So the point (5, – 4) inside the hyperbola $9x^2 - y^2 = 1$.

INTERACTION OF A LINE AND A HYPERBOLA :



Let the hyperbola be $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ ....(1)

and the given line be $y = mx + c$ ....(2)

The point of intersection of line and hyperbola can be obtained by solving (1) and (2), therefore

Eliminating y from (1) and (2), we get $\frac{x^2}{a^2} - \frac{(mx+c)^2}{b^2} = 1$

$\Rightarrow b^2x^2 - a^2(mx + c)^2 = a^2b^2 \Rightarrow (a^2m^2 - b^2)x^2 + 2mca^2x + a^2(b^2 + c^2) = 0$ ....(3)

Above equation is a quadratic in x and gives two values of x. It shows that every straight line will cut the hyperbola in two points, may be real, coincident or imaginary according as discriminant of (3) >, =, < 0.

i.e., $4m^2c^2a^4 - 4(a^2m^2 - b^2)a^2(b^2 + c^2) >, =, < 0$

or $-a^2m^2 + b^2 + c^2 >, =, < 0$

or $c^2 >, =, < a^2m^2 - b^2$ ....(4)

EQUATIONS OF TANGENT :

(1) Point form :

The equation of the tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at (x_1, y_1) is $\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$.

Since point (x_1, y_1) lie on the hyperbola therefore we can use standard substitution to obtain the equation of tangent.

(2) Parametric form :

The equation of tangent to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ at $(a\sec\theta, b\tan\theta)$ is $\frac{x}{a}\sec\theta-\frac{y}{b}\tan\theta=1$.

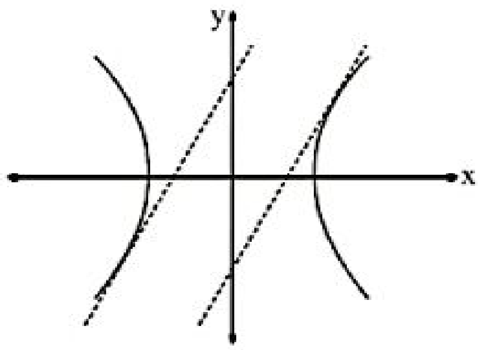
Note : Point of intersection of the tangents at θ_1 & θ_2 is $x = a\dfrac{\cos\frac{\theta_1-\theta_2}{2}}{\cos\frac{\theta_1+\theta_2}{2}}$, $y = b\dfrac{\sin\frac{\theta_1+\theta_2}{2}}{\cos\frac{\theta_1+\theta_2}{2}}$

(3) Slope form :

Given hyperbola is $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ ......(1)

and the given line $y = mx + c$ touches hyperbola then solve (1) and (2), we get

$\frac{x^2}{a^2}-\frac{(mx+c)^2}{b^2}=1$ or $b^2x^2 - a^2(mx + c)^2 = a^2b^2$



or $(b^2 - a^2m^2)x^2 - 2a^2mcx - a^2(c^2 + b^2) = 0$ ......(3)

Since line (2) will be tangent to hyperbola (1)

if roots of equation (3) are equal i.e. $D = 0$

$4a^4m^2c^2 + 4a^2(b^2 - a^2m^2)(c^2 + b^2) = 0$

or $a^2m^2c^2 + b^2c^2 - a^2c^2m^2 + b^4 - a^2b^2m^2 = 0$

or $b^2c^2 + b^4 - a^2b^2m^2 = 0$

or $c^2 + b^2 - a^2m^2 = 0$

or $c^2 = a^2m^2 - b^2$ or $c = \pm\sqrt{a^2m^2-b^2}$. This is the required **condition of tangency**

Note :

(i) Equation of any tangent to hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ may be taken as $y = mx + \sqrt{a^2m^2-b^2}$ or $y = mx - \sqrt{a^2m^2-b^2}$. The co-ordinates of the points of contact are $\left(\pm\frac{a^2m}{\sqrt{(a^2m^2-b^2)}}, \pm\frac{b^2}{\sqrt{(a^2m^2-b^2)}}\right)$.

(ii) The equation of any tangent to the hyperbola $\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$

may be taken as $(y - k) = m(x - h) \pm \sqrt{a^2m^2-b^2}$.

Tangents drawn from outside point :

$y = mx \pm \sqrt{a^2m^2 - b^2}$ is a tangent to the standard hyperbola. ....(1)

If above tangent passes through (h, k) then

$(k - mh)^2 = a^2m^2 - b^2$

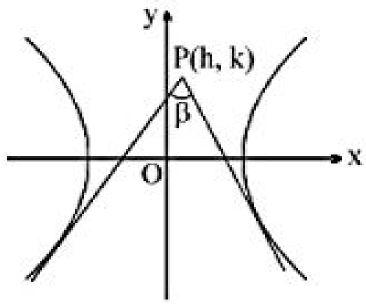
$(h^2 - a^2)m^2 - 2kmh + k^2 + b^2 = 0$ ....(2)

Above equation is quadratic in m therefore it has two roots m_1 and m_2.

Hence passing through a given point (h, k) there is a maximum of two tangents can be drawn to the hyperbola

therefore $m_1 + m_2 = \dfrac{2kh}{h^2 - a^2}$ ....(3)

$m_1m_2 = \dfrac{k^2 + b^2}{h^2 - a^2}$ ....(4)



Equations (3) and (4) are used to find the locus of the point of intersection of a pair of tangents which enclose an angle β.

Now $\tan^2\beta = \dfrac{(m_1 + m_2)^2 - 4m_1m_2}{(1 + m_1m_2)^2}$

(substituting the values of $m_1 + m_2$ and m_1m_2 to get the locus)

If $\beta = 90°$ then $m_1m_2 = -1$, hence from (4)

$k^2 + b^2 = a^2 - h^2$

$x^2 + y^2 = a^2 - b^2$ which is the equation of director circle, of the given hyperbola.

Illustration :

Find the equation of the tangent to the hyperbola $x^2 - 4y^2 = 36$ which is perpendicular to the line $x - y + 4 = 0$.

Sol. *Let m be the slope of the tangent. Since the tangent is perpendicular to the line $x - y + 4 = 0$.*

$m \times 1 = -1 \Rightarrow m = -1$

Given hyperbola $x^2 - 4y^2 = 36$ *or* $\dfrac{x^2}{36} - \dfrac{y^2}{9} = 1$, $a^2 = 36$, $b^2 = 9$

$\therefore$ *tangent to above hyperbola is* $y = mx \pm \sqrt{36m^2 - 9}$

here $m = -1$.

equation of tangent is $y = -x \pm \sqrt{27}$ *or* $x + y \pm 3\sqrt{3} = 0$.

Illustration :

For what value of λ does the line $y = 2x + \lambda$ touches the hyperbola $16x^2 - 9y^2 = 144$?

Sol. *Equation of hyperbola is $16x^2 - 9y^2 = 144$ or $\frac{x^2}{9} - \frac{y^2}{16} = 1$*

Comparing this with $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ we get $a^2 = 9$, $b^2 = 16$

and comparing this line $y = 2x + \lambda$ with $y = mx + c$.

$\therefore$ $m = 2$ *and* $c = \lambda$

If the line $y = 2x + \lambda$ touches the hyperbola $16x^2 - 9y^2 = 144$ then $c^2 = a^2m^2 - b^2$

$\Rightarrow \lambda^2 = 9(2)^2 - 16 = 36 - 16 = 20$

$\therefore \lambda = \pm\, 2\sqrt{5}$.

Illustration :

Find the locus of the mid points of the chords of the circle $x^2 + y^2 = 16$, which are tangent to the hyperbola $9x^2 - 16y^2 = 144$.

Sol. *Any tangent to hyperbola $\frac{x^2}{16} - \frac{y^2}{9} = 1$ is $y = mx + \sqrt{(16m^2 - 9)}$ ...(1)*

Let (x_1, y_1) be the mid- point of the chord of the circle $x^2 + y^2 = 16$, then equation of the chord is $(T = S_1)$

$xx_1 + yy_1 - (x_1^2 + y_1^2) = 0$ ...(2)

Since (1) and (2) are same, therefore

we get $\frac{m}{x_1} = -\frac{1}{y_1} = \frac{\sqrt{16m^2 - 9}}{-(x_1^2 + y_1^2)}$ $\Rightarrow m = -\frac{x_1}{y_1}$ *and* $(x_1^2 + y_1^2)^2 = y_1^2(16m^2 - 9)$

Eliminating m and replacing (x_1, y_1) by (x, y), we get the required locus is $(x^2 + y^2)^2 = 16x^2 - 9y^2$.

Ans.

Illustration :

Find the area of a triangle formed by the lines $x - y = 0$, $x + y = 0$ and any tangent to the hyperbola $x^2 - y^2 = a^2$.

Sol. *Any tangent to the hyperbola at $P(a \sec\theta, a \tan\theta)$ is*

$x \sec\theta - y \tan\theta = a$ ...(i)

Also $x - y = 0$ ...(ii)

$x + y = 0$ ...(iii)

Solving the above three lines in pairs, we get the point A, B, C as

$$\left(\frac{a}{\sec\theta - \tan\theta}, \frac{a}{\sec\theta - \tan\theta}\right), \left(\frac{a}{\sec\theta + \tan\theta}, \frac{-a}{\sec\theta + \tan\theta}\right) \text{ and } (0, 0)$$

Since the one vertex is the origin therefore the area of the triangle ABC is

$$= \frac{1}{2} |(x_1y_2 - x_2y_1)|$$

$$= \left| \frac{a^2}{2} \left(\frac{-1}{\sec^2\theta - \tan^2\theta} - \frac{1}{\sec^2\theta - \tan^2\theta} \right) \right|$$

$$= \left| \frac{a^2}{2}(-2) \right| = a^2.$$

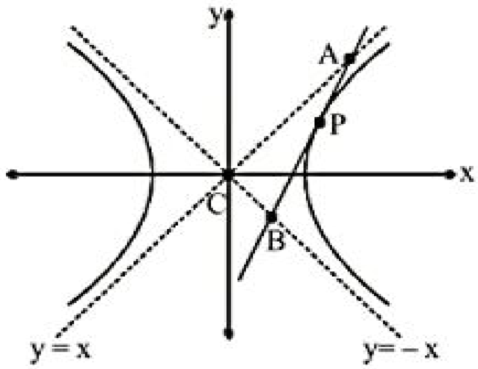


Illustration :

Find the equation and the length of the common tangents to hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \text{ and } \frac{y^2}{a^2} - \frac{x^2}{b^2} = 1.$$

Sol. *Tangent at $(a \sec\phi, b\tan\phi)$ on the 1st hyperbola is*

$$\frac{x}{a}\sec\phi - \frac{y}{b}\tan\phi = 1 \quad ....(1)$$

Similarly tangent at any point $(b\tan\theta, a\sec\theta)$ on 2nd hyperbolas is

$$\frac{y}{a}\sec\theta - \frac{x}{b}\tan\theta = 1 \quad ....(2)$$

If (1) and (2) are common tangents then they should be identical therefore

$$\frac{\frac{\sec\phi}{a}}{\frac{-\tan\theta}{b}} = \frac{\frac{-\tan\phi}{b}}{\frac{\sec\theta}{a}} = \frac{1}{1} \quad \Rightarrow \quad \frac{\sec\theta}{a} = -\frac{\tan\phi}{b}$$

or $\quad \sec\theta = -\frac{a}{b}\tan\phi \quad ....(3)$

and $\quad -\frac{\tan\theta}{b} = \frac{\sec\phi}{a} \quad$ *or* $\quad \tan\theta = -\frac{b}{a}\sec\phi \quad ....(4)$

$\because \quad \sec^2\theta - \tan^2\theta = 1$

$\Rightarrow \quad \frac{a^2}{b^2}\tan^2\phi - \frac{b^2}{a^2}\sec^2\phi = 1 \quad$ {*from (3) and (4)*}

or $\quad \frac{a^2}{b^2}\tan^2\phi - \frac{b^2}{a^2}\tan^2\phi = 1 + \frac{b^2}{a^2} \quad$ *or* $\quad \left(\frac{a^2}{b^2} - \frac{b^2}{a^2}\right)\tan^2\phi = 1 + \frac{b^2}{a^2}$

$\therefore \quad \tan^2\phi = \frac{b^2}{a^2 - b^2} \quad$ *and* $\quad \sec^2\phi = 1 + \tan^2\phi = \frac{a^2}{a^2 - b^2}$

Hence the points of contact are

$$\left\{\pm\frac{a^2}{\sqrt{(a^2-b^2)}},\pm\frac{b^2}{\sqrt{(a^2-b^2)}}\right\} \text{ and } \left\{\pm\frac{b^2}{\sqrt{(a^2-b^2)}},\pm\frac{a^2}{\sqrt{(a^2-b^2)}}\right\} \quad \{\text{from (3) and (4)}\}$$

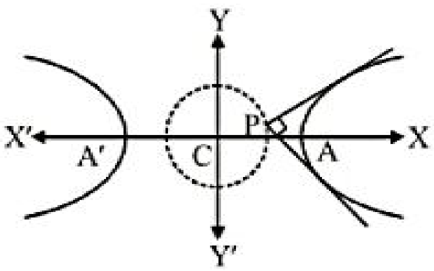
Length of common tangent i.e., the distance between the above points is $\sqrt{2}\,\dfrac{(a^2+b^2)}{\sqrt{(a^2+b^2)}}$ *and*

equation of common tangent on putting the values of $\sec\theta$ *and* $\tan\theta$ *in (1) is*

$$\pm\frac{x}{\sqrt{(a^2+b^2)}} \mp \frac{y}{\sqrt{(a^2-b^2)}} = 1 \qquad or \qquad x\mp y = \pm\sqrt{(a^2-b^2)}.\ \textit{Ans.}$$

DIRECTOR CIRCLE :

The locus of the point of intersection of the tangents to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, which are perpendicular to each other, is called the director circle.



Any tangent to hyperbola is $y = mx + \sqrt{a^2m^2-b^2}$ ....(1)

If it passes through P(h, k), then $k - mh = \sqrt{a^2m^2-b^2}$

$k^2 + m^2h^2 - 2mk = a^2m^2 - b^2$

$m^2(h^2-a^2) - 2mhk + (k^2+b^2) = 0$

It is quadratic in m therefore it has two roots m_1 and m_2. Hence two tangents (real or imaginary) can be drawn from P(h, k).

If pair of perpendicular tangents are drawn from P(h, k) then $m_1 m_2 = \frac{k^2+b^2}{h^2-a^2} = -1$.

$\Rightarrow \quad h^2 + k^2 = a^2 - b^2$

$\therefore \quad$ locus of P(h, k) is $x^2 + y^2 = a^2 - b^2$.

Note :

(i) If $b^2 < a^2$ this circle is real

(ii) If $b^2 = a^2$ the radius of the circle is zero & it reduces to a point circle at the origin. In this case the centre is the only point from which the tangents at right angles can be drawn to the curve.

(iii) If $b^2 > a^2$, the radius of the circle is imaginary, so that there is no such circle & so no tangents at right angle can be drawn to the curve.

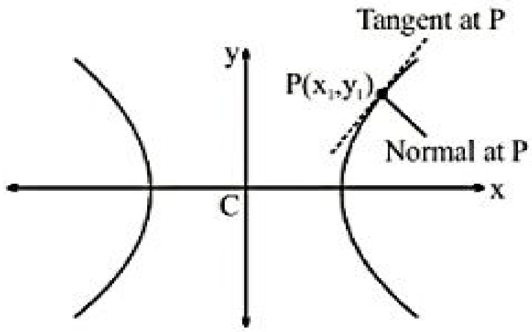
EQUATION OF NORMALS :

(1) Point form :

Equation of normal to a

hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ at a point (x_1, y_1) is

$$\frac{a^2x}{x_1}-\frac{b^2y}{y_1}=a^2+b^2$$



(2) Parametric form :

The equation of normal at $(a\sec\phi, b\tan\phi)$ to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is i

$$ax\cos\phi + by\cot\phi = a^2+b^2$$

Illustration :

Prove that the line $lx + my - n = 0$ will be a normal to the hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1 \text{ if } \frac{a^2}{l^2}-\frac{b^2}{m^2}=\frac{(a^2+b^2)^2}{n^2}.$$

Sol. *The equation of any normal to $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is*

$$ax\cos\phi + by\cot\phi = a^2+b^2 \text{ or } ax\cos\phi + by\cot\phi - (a^2+b^2) = 0 \qquad ....(1)$$

Since the straight line $lx+my-n=0$ is given normal to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, then (1) and $lx+my-n=0$ represent the same line

$$\frac{a\cos\phi}{l}=\frac{b\cot\phi}{m}=\frac{(a^2+b^2)}{n}$$

or $\sec\phi = \dfrac{na}{l(a^2+b^2)}$ *and* $\tan\phi = \dfrac{nb}{m(a^2+b^2)}$

$\because \quad \sec^2\phi - \tan^2\phi = 1$

$$\therefore \quad \frac{n^2a^2}{l^2(a^2+b^2)^2}-\frac{n^2b^2}{m^2(a^2+b^2)^2}=1$$

$$\Rightarrow \quad \frac{a^2}{l^2}-\frac{b^2}{m^2}=\frac{(a^2+b^2)^2}{n^2}.$$

Illustration :

Find the locus of the foot of perpendicular from the centre upon any normal to the hyperbola

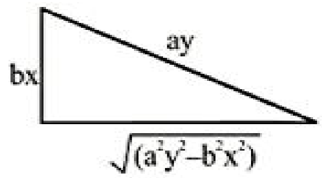
$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

Sol. *Normal at $P(a \sec \phi, b \tan \phi)$ is*

$$ax \cos \phi + by \cot \phi = a^2 + b^2 \quad ....(1)$$

and equation of line perpendicular to (1) and passing through origin is

$$bx - ay \sin \phi = 0 \qquad ....(2)$$

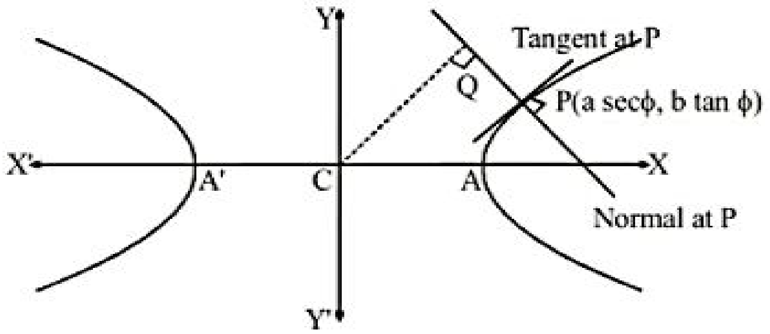


From (2)

$$\sin \phi = \frac{bx}{ay}$$

$$\therefore \quad \cos \phi = \frac{\sqrt{(a^2y^2 - b^2x^2)}}{ay}$$

$$\text{and} \quad \cot \phi = \frac{\sqrt{(a^2y^2 - b^2x^2)}}{bx}$$



Now by puttting value of $\cos \phi$ and $\cot \phi$ in equation (1), ϕ will be elminated and get the locus

$$\therefore \quad \text{from (1)}, \quad ax \times \frac{\sqrt{a^2y^2 - b^2x^2}}{ay} + by \times \frac{\sqrt{(a^2y^2 - b^2x^2)}}{bx} = a^2 + b^2$$

$$\Rightarrow \quad (x^2 + y^2)\sqrt{(a^2y^2 - b^2x^2)} = (a^2 + b^2)xy$$

or $\quad (x^2 + y^2)^2 (a^2y^2 - b^2x^2) = (a^2 + b^2)^2 x^2y^2$

which is required locus.

PROPERTIES :

(1) Normal and tangent at any point $P(x_1, y_1)$ meet the transverse axis at G and T respectively.

The equation of normal at P on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is

$$\frac{a^2x}{x_1} + \frac{b^2y}{y_1} = a^2 + b^2 \quad ....(1)$$

The normal (1) meets the x-axis i.e. then put $y = 0$ in (1) we will get co-ordinates of G

i.e. $\quad G\left(\frac{(a^2+b^2)}{a^2}x_1, 0\right)$ or $\quad (e^2x_1, 0)$

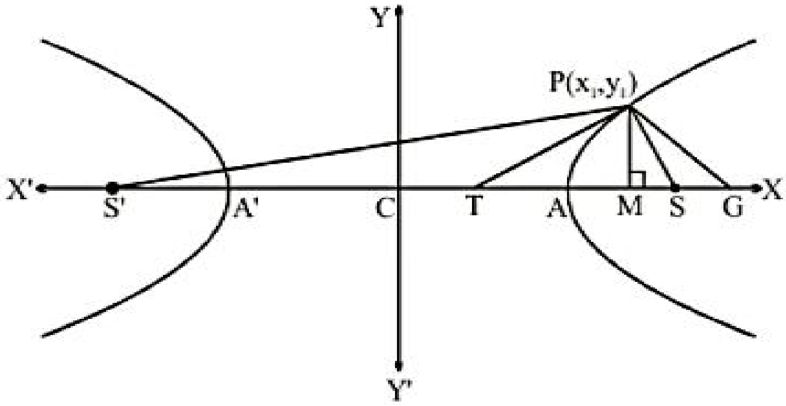
$\therefore \quad CG = e^2x_1$

Now,

$$SG = CG - CS = e^2x_1 - ae = e(ex_1 - a) = e \cdot SP$$

Similarly, $\quad S'G = e \cdot S'P$

$$\therefore \quad \frac{SG}{S'G} = \frac{SP}{S'P}$$

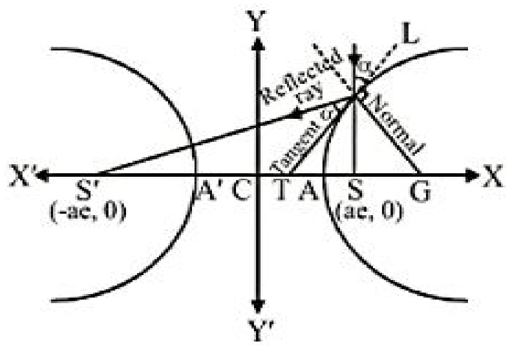


This relation shows that the normal PG is the external bisector of the angle SPS'. The tangent PT (perpendicular to PG) is therefore the internal bisector of the angle SPS'.

The tangent & normal at any point of a hyperbola bisect the angle between the focal radii.

Note : that the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and the hyperbola $\frac{x^2}{a^2 - k^2} - \frac{y^2}{k^2 - b^2} = 1 \, (a > k > b > 0)$ are confocal and therefore orthogonal.

(2) The tangent & normal at any point of a hyperbola bisect the angle between the focal radii. This spells the reflection property of the hyperbola as **"An incoming light ray "** aimed towards one focus is reflected from the outer surface of the hyperbola towards the other focus. It follows that if an ellipse and a hyperbola have the same foci, they cut at right angles at any of their common point.



(3) Locus of the feet of the perpendicular drawn from focus of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ upon any tangent is its auxiliary circle i.e. $x^2 + y^2 = a^2$ & the product of the feet of these perpendiculars is $b^2 \cdot$ i.e., square of semi-conjugate axis.

Similar property exists in ellipse also.

(4) The portion of the tangent between the point of contact & the directrix subtends a right angle at the corresponding focus.

(5) The foci of the hyperbola and the points P and Q in which any tangent meets the tangents at the vertices are concyclic with PQ as diameter of the circle.

PAIR OF TANGENTS :

The combined equation of the pair of tangents drawn from a point $P(x_1, y_1)$, lying outside the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is

$$\left(\frac{x^2}{a^2}-\frac{y^2}{b^2}-1\right)\left(\frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}-1\right)=\left(\frac{xx_1}{a^2}-\frac{yy_1}{b^2}-1\right)^2 \text{ or } SS_1 = T^2$$

where $S=\frac{x^2}{a^2}-\frac{y^2}{b^2}-1$; $S_1=\frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}-1$ and $T=\frac{xx_1}{a^2}-\frac{yy_1}{b^2}-1$.

CHORD OF CONTACT :

If the tangents from a point $P(x_1, y_1)$ to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ touch the hyperbola at Q and R, then the equation of the chord of contact QR is

$$\frac{xx_1}{a^2}-\frac{yy_1}{b^2}=1.$$

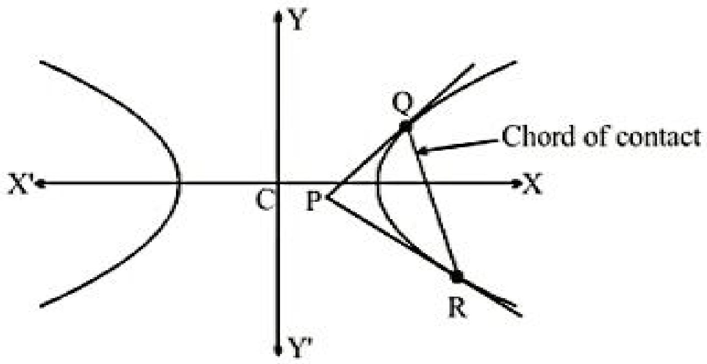


Illustration :

If tangent to the parabola $y^2 = 4ax$ intersect the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ at A and B, then find the locus of point of inersection of tangents at A and B.

Sol. *Let $P = (h, k)$ be the point of intersection of tangents at A and B. Therefore, the equation of chord of contact AB is of hyperbola is*

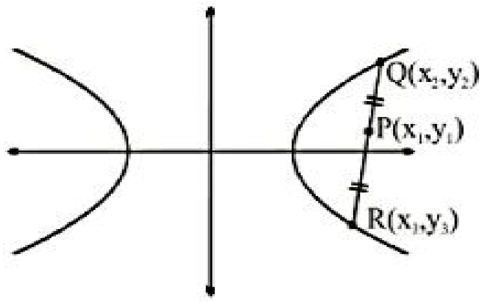
$$\frac{xh}{a^2}-\frac{yk}{b^2}=1$$

or $y=\frac{xb^2h}{ka^2}-\frac{b^2}{k}$ *touches the given parabola*

Therefore $$-\frac{b^2}{k}=\frac{a}{\left(\frac{b^2h}{a^2k}\right)}$$

$\Rightarrow$ *Locus of point P(h, k) is* $y^2=-\frac{b^4}{a^3}$

Equation of the chord bisected at a given point :



The equation of the chord of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, bisected at the point $P(x_1, y_1)$ is

$$\frac{xx_1}{a^2}-\frac{yy_1}{b^2}-1=\frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}-1$$

or $\quad T = S_1$, where $T=\frac{xx_1}{a^2}-\frac{yy_1}{b^2}-1$ ans $S_1=\frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}-1$.

Illustration :

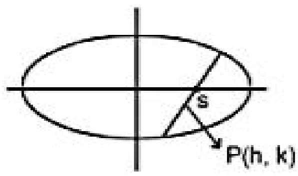
Find the locus of the mid - point of focal chords of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$.

Sol. *Let $P = (h, k)$ be the mid-point*

$\therefore$ *equation of chord whose mid-point is given is $T = S_1$ i.e.* $\frac{xh}{a^2}-\frac{yk}{b^2}-1=\frac{h^2}{a^2}-\frac{k^2}{b^2}-1$

since it is a focal chord,

$\therefore$ *it passes through focus, either $(ae, 0)$ or $(-ae, 0)$*

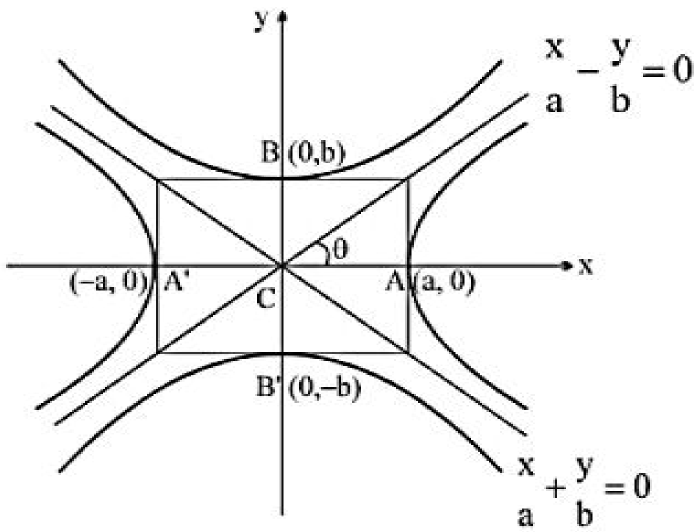


If it passes trhrough $(ae, 0)$

$\therefore$ *locus is* $\frac{ex}{a}=\frac{x^2}{a^2}-\frac{y^2}{b^2}$.

ASYMPTOTES OF HYERPBOLA :

If the length of the perpendicular let fall from a point on a hyperbola to a straight line tends to zero as the point on the hyperbola moves to infinity along the hyperbola, then the straight line is called the Asymptote of the Hyperbola.



In short asymptote is tangent to hyperbola at infinity.

Let $y = mx + c$ is the asymptote of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$.

Solving these two we get the quadratic as

$$(b^2 - a^2m^2)\,x^2 - 2a^2mcx - a^2(b^2 + c^2) = 0 \qquad ....(1)$$

In order that $y = mx + c$ be an asymptote, both roots of equation (1) must approach at infinity, the conditions for which are

coeff of $x^2 = 0$ & coeff of $x = 0$.

$\Rightarrow \quad b^2 - a^2m^2 = 0$ or $m = \pm\dfrac{b}{a}$ &

$a^2 mc = 0 \Rightarrow c = 0.$

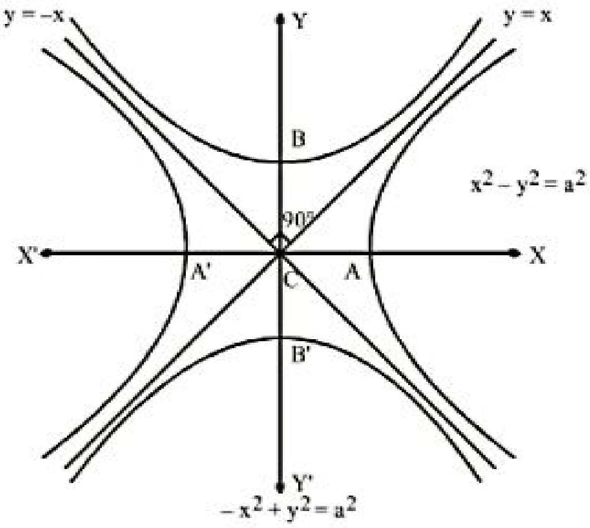
$\therefore$ equations of asymptote are $\dfrac{x}{a} + \dfrac{y}{b} = 0$ and $\dfrac{x}{a} - \dfrac{y}{b} = 0$.

combined equation to the asymptotes $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 0$.

As asymptotes of any hyperbola or a curve is a straight line which touches in it two points at infinity.

Note :

(i) If $b = a$, then $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ reduces $x^2 - y^2 = a^2$. The asymptotes of rectangular hyperbola $x^2 - y^2 = a^2$ are $y = \pm x$ which are at right angles.



(ii) A hyperbola and its conjugate have the same asymptote.

(iii) The asymptotes pass through the centre of the hyperbola & the bisectors of the angles between the asymptotes are the axes of the hyperbola.

(iv) Asymptotes are the tangent to the hyperbola from the centre.

(v) A simple method to find the coordinates of the centre of the hyperbola is expressed as a general equation of degree 2.

i.e. let $f(x, y) = 0$ represents a hyperbola.

Find $\dfrac{\partial f}{\partial x}$ & $\dfrac{\partial f}{\partial y}$. Then the point of intersection of $\dfrac{\partial f}{\partial x} = 0$ & $\dfrac{\partial f}{\partial y} = 0$.

gives the centre of the hyperbola.

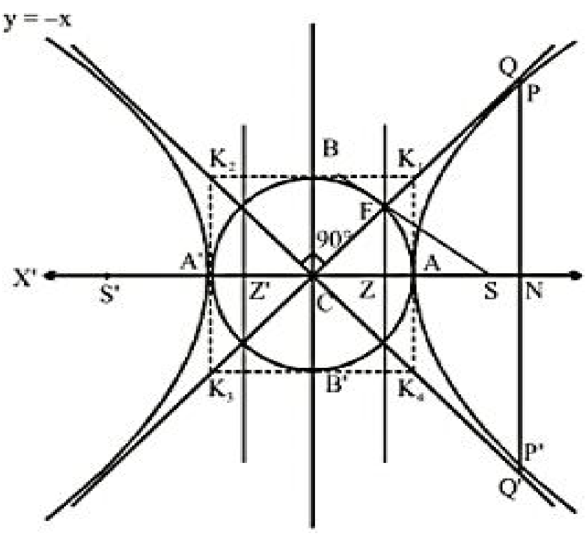
(vi) Let $H \equiv \frac{x^2}{a^2} - \frac{y^2}{b^2} - 1 = 0$; $A \equiv \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and $C = \frac{x^2}{a^2} - \frac{y^2}{b^2} + 1 = 0$

be the equation of the hyperbola, asymptotes and the conjugate hyperbola respectively, then clearly $\mathbf{C + H = 2A}$ i.e., all the equation different only by constant term and the constant term of H, A and C are in A.P.

(vii) The asymptotes of a hyperbola are the diagonals of the rectangle formed by the lines drawn through the extremities of each axis parallel to the other axis. Here area of rectangle is 4ab.

PROPERTIES OF ASYMPTOTES :

(i) Perpendicular from the foci on either asymptote meet it in the same points as the corresponding directrix & the common points of intersection lie on the auxiliary circle.



(ii) The angle between the asymptotes of $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is $2\tan^{-1}\left(\frac{b}{a}\right)$.

and if the angle between the asymptote of a hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is 2θ then $e = \sec\theta$.

(iii) If from any point on the asymptote a straight line be drawn perpendicular to the transverse axis, the product of the segments of this line, intercepted between the point & the curve is always equal to the square of the semi conjugate axis.

(iv) The tangent at any point P on a hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ with centre C, meets the asymptotes in Q and R and cuts off a Δ CQR of constant area equal to ab from the asymptotes & the portion of the tangent intercepted between the asymptote is bisected at the point of contact.

Illustration :

Find the asymptotes of the hyperbola $xy - 3y - 2x = 0$.

Sol. *Since equation of a hyperbola and its asymptotes differ in constant terms only.*

$\therefore$ *Pair of asymptotes is given by* $xy - 3y - 2x + \lambda = 0$ ......(1)

above equation represents two straight lines.

$\therefore \quad abc + 2fgh - af^2 - bg^2 - ch^2 = 0$

$$\Rightarrow \quad 0 + 2 \times \frac{-3}{2} \times -1 \times \frac{1}{2} - 0 - 0 - \lambda\left(\frac{1}{2}\right)^2 = 0 \Rightarrow \lambda = 6.$$

From (1), the asymptotes of given hyperbola are given by

$$xy - 3y - 2x + 6 = 0 \text{ or } (y - 2)(x - 3) = 0.$$

$\therefore$ *Asymptotes are* $x - 3 = 0$ *and* $y - 2 = 0$.

Illustration :

The asymptotes of a hyperbola having centre at the point (1, 2) are parallel to the lines $2x + 3y = 0$ and $3x + 2y = 0$. If the hyperbola passes through the point (5, 3), show that its equation is $(2x + 3y - 8)(3x + 2y + 7) = 154$.

Sol. *Let the asymptotes be $2x + 3y + \lambda = 0$ and $3x + 2y + \mu = 0$. Since asymptotes passes through (1, 2), then $\lambda = -8$ and $\mu = -7$.*

Thus the equation of asymptotes are $2x + 3y - 8 = 0$ and $3x + 2y - 7 = 0$.

Let the equation of hyperbola be $(2x + 3y - 8)(3x + 2y - 7) + \lambda = 0$ ......(1)

Hyperbola passes through (5, 3) then $(10 + 9 - 8)(15 + 6 - 7) + \lambda = 0$

$\therefore \lambda = -154$.

Putting the value of λ in (1), we get $(2x + 3y - 8)(3x + 2y - 7) - 154 = 0$ which is the equation of required hyperbola.

The rectangular hyperbola $xy = c^2$

When the centre of any rectangular hyperbola be at the origin and its asymptotes coincide with the co-ordinate axes then equation of hyperbola is $xy = c^2$.

Here the equation of asymptotes is $xy = 0$ and the equation conjugate hyperbola is $xy = -c^2$.

Note :

(i) The equation of a rectangular hyperbola having co-ordinate axes as its asymptotes is $xy = c^2$. If the asymptotes of a rectangular hyperbola are $x = \alpha$, $y = \beta$ then its equation is $(x - \alpha)(y - \beta) = c^2$ or $xy - \alpha y - \beta x + \lambda = 0$.

(ii). Parametric equation of $xy = c^2$ is $x = ct$ and $y = \dfrac{c}{t}$.

$\therefore \ (x, y) = \left(ct, \dfrac{c}{t}\right)$ $(t \neq 0)$ is called a 't' point on the rectangular hyperbola.

Properties of rectangular hyperbola $xy = c^2$.

(a) Equation of a chord joining the points $P(t_1)$ & $Q(t_2)$ is $x + t_1t_2y = c(t_1 + t_2)$ with slope $m = -\dfrac{1}{t_1t_2}$.

(b) Equation of the tangent at $P(x_1, y_1)$ is $\dfrac{x}{x_1} + \dfrac{y}{y_1} = 2$ & at $P(t)$ is $\dfrac{x}{t} + ty = 2c$.

(c) Point of intersection of tangents at 't_1' and 't_2' is $\left(\dfrac{2ct_1t_2}{t_1+t_2}, \dfrac{2c}{t_1+t_2}\right)$.

(d) Equation of normal is $y - \dfrac{c}{t} = t^2(x - ct)$ or $xt^3 - yt - ct^4 + c = 0$

(e) Equation of normal (x_1, y_1) is $xx_1 - yy_1 = x_1^2 - y_1^2$.

(f) Chord with a given middle point as (h, k) is $kx + hy = 2hk$.

Illustration :

If the normal at the point 't_1' to the rectangular hyperbola $xy = c^2$ meets it again at the point 't_2', prove that $t_1^3 t_2 = -1$.

Sol. *Since the equation of normal at* $\left(ct_1, \dfrac{c}{t_1}\right)$ *to the hyperbola* $xy = c^2$ *is* $xt_1^3 - yt_1 - ct_1^4 + c = 0$.

but this passes through $\left(ct_2, \dfrac{c}{t_2}\right)$ *then* $ct_2t_1^3 - \dfrac{c}{t_2}t_1 - ct_1^4 + c = 0$

$\Rightarrow t_2^2t_1^3 - t_1 - t_1^4t_2 + t_2 = 0 \Rightarrow t_2t_1^3(t_2 - t_1) + (t_2 - t_1) = 0$

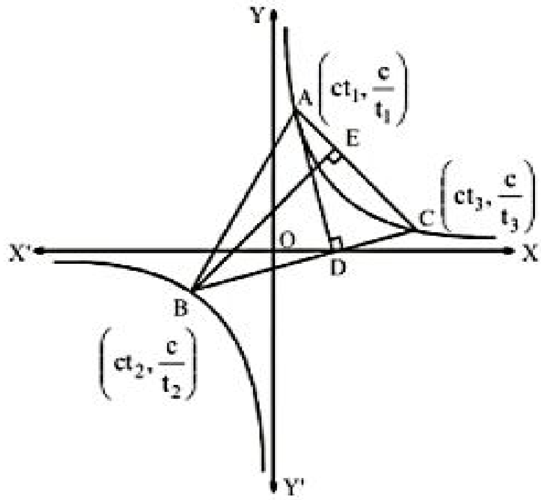
$\Rightarrow (t_1^3t_2 + 1)(t_2 - t_1) = 0 \Rightarrow t_1^3t_2 + 1 = 0 \qquad [\because t_2 \neq t_1]$

$\therefore\ t_1^3t_2 = -1.$

Illustration :

A triangle has its vertices on a rectangular hyperbola. Prove that the orthocentre of the triangle also lies on the same hyperbola.

Sol. $\therefore$ *Co-ordinates of A, B and C are* $\left(ct_1, \dfrac{c}{t_1}\right), \left(ct_2, \dfrac{c}{t_2}\right)$ *and* $\left(ct_3, \dfrac{c}{t_3}\right)$ *respectively are the vertices of a triangle lie on the rectangular hyperbola* $xy = c^2$.



Now slope of BC is $\dfrac{\frac{c}{t_3}-\frac{c}{t_2}}{ct_3-ct_2} = \dfrac{-1}{t_2t_3}$

$\therefore$ *Slope of AD is* t_2t_3

Equation of altitude AD is $y-\dfrac{c}{t_1} = t_2t_3\,(x-ct_1)$

or $t_1y - c = xt_1t_2t_3 - ct_1^2t_2t_3$ ......(1)

Similarly equation of altitude BE is

$t_2y - c = x\,t_1\,t_2\,t_3 - c\,t_1t_2^2\,t_3$ ......(2)

Solving (1) and (2), we get the orthocentre $\left(\dfrac{-c}{t_1t_2t_3}, -ct_1t_2t_3\right)$ *which also lies on* $xy = c^2$.

Illustration :

If the normals at (x_i, y_i), $i = 1, 2, 3, 4$ *on the rectangular hyperbola,* $xy = c^2$, *meet at the point* (α, β) *then show that*

(i) $\Sigma x_i = \alpha$ *(ii)* $\Sigma y_i = \beta$ *(iii)* $\Pi x_i = \Pi y_i = -c^4$ *(iv)* $\Sigma x_i^2 = \alpha^2$ *(v)* $\Sigma y_i^2 = \beta^2$

Sol. *Let* $(x_i, y_i) = \left(ct_i, \dfrac{c}{t_i}\right)$, $i = 1, 2, 3, 4$ *are the points on the rectangular hyperbola* $xy = c^2$.

Equation of normal to the hyperbola $xy = c^2$ *at* $\left(ct, \dfrac{c}{t}\right)$ *is* $ct^4 - t^3x + ty - c = 0$

It passes through (α, β), *then* $ct^4 - t^3\alpha + t\beta - c = 0$

it is biquadratic equation in t. Let the roots of this equation are t_1, t_2, t_3, t_4 *then* $\Sigma t_1 = \dfrac{\alpha}{c}$

$$\Sigma t_1t_2 = 0,\ \Sigma t_1t_2t_3 = \frac{-\beta}{c},\ t_1t_2t_3t_4 = -1$$

Now (i) $\Sigma x_i = c\,\Sigma t_1 = \alpha$

(ii) $\Sigma y_i = c\left(\Sigma\dfrac{1}{t_i}\right) = \left(\dfrac{\Sigma t_1t_2t_3}{t_1t_2t_3t_4}\right) = \beta$

(iii) $\Pi x_i = c^4\,\Pi t_i = -c^4$

and $\Pi y_i = c^4\left(\dfrac{1}{\Pi t_i}\right) = -c^4$

(iv) $\Sigma x_i^2 = c^2\left(\Sigma t_i^2\right) = c^2\left((\Sigma t_1)^2 - 2\Sigma t_1t_2\right) = \alpha^2$

(v) $\Sigma y_i^2 = \left((\Sigma y_1)^2 - \Sigma y_1y_2\right) = b^2 - 2c^2\,\Sigma\left(\dfrac{1}{t_1t_2}\right) = b^2 - 2c\cdot\dfrac{\Sigma t_1t_2}{t_1t_2t_3t_4} = \beta^2.$

COMPLEX NUMBER

1. INTRODUCTION :

Indian mathematician Mahavira (850 A.D.) was first to mention in this work 'Ganitasara Sangraha' As in nature of things a negative (quantity) is not a square (quantity), it has, therefore, no square root'. Hence there is no real number x which satisfies the polynomial equation $x^2 + 1 = 0$.

A symbol $\sqrt{-1}$, denoted by letter i was introduced by Swiss Mathematician, Leonhard Euler (1707-1783) in 1748 to provide solutions of equation $x^2 + 1 = 0$. i was regarded as a fictitious or imaginary number which could be manipulated algebrically like an ordinary real number, except that its square was -1. The letter i was used to denote $\sqrt{-1}$, possibly because i is the first letter of the Latin word 'imaginarius'.

To permit solutions of such polynomial equations, the set of complex numbers is introduced. We can consider a complex number as having the form $a + ib$ where a and b are real number.
It is denoted by z i.e. $z = a + ib$. 'a' is called as real part of z which is denoted by Re(z) and 'b' is called as imaginary part of z which is denoted by Im (z).

1.1 Classification of complex number :

In fact every complex can be classified as

$z = a + ib$

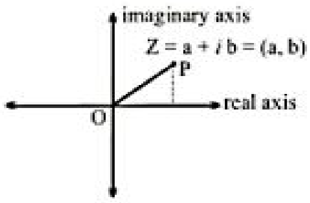
purely real	purely imaginary	imaginary
if $b = 0$	if $a = 0$	if $b \neq 0$

Hence, $0 + 0i$ is both a purely real as well as a purely imaginary but not imaginary.

1.2 Geometrical representation of a complex number :

Master Argand had done a systematic studies on complex numbers and represented every complex number as a set of ordered pair (a, b) on a plane called complex plane / argand plane.

All complex numbers lying on the real axis were called as purely real and those lying on imaginary axis as purely imaginary.



1.3 Integral Powers of i :

We have $i = \sqrt{-1}$ so $i^2 = -1$, $i^3 = -i$, $i^4 = 1$
or $i^{4n+1} = i$, $i^{4n+2} = -1$ for any $n \in I$,
$i^{4n+3} = -i$, $i^{4n} = 1$
Thus any integral power of i can be expressed as ± 1 or $\pm i$.

Illustration :

Find the value of $\dfrac{i^{592}+i^{590}+i^{588}+i^{586}+i^{584}}{i^{582}+i^{580}+i^{578}+i^{576}+i^{574}}+2.$

Sol. $\dfrac{i^{592}+i^{590}+i^{588}+i^{586}+i^{584}}{i^{582}+i^{580}+i^{578}+i^{576}+i^{574}}+2=\dfrac{i^{584}}{i^{574}}+2$

$= i^{10} + 2 = -1 + 2 = 1$

Note :

(a) The set R of real number is a proper subset of the Complex Numbers. Hence the complete number system is $N \subset W \subset I \subset Q \subset R \subset C$.

(b) Zero is purely real as well as purely imaginary but not imaginary.

(c) $\sqrt{a}\sqrt{b}=\sqrt{a\,b}$ only if atleast one of a or b is non-negative.

(d) $z_1^2+z_2^2=0 \not\Rightarrow z_1=0=z_2$ i.e., $z_1 = 1+i$ and $z_2 = 1-i$

2. ALGEBRA OF COMPLEX NUMBER :

2.1 Equality of complex number :

Let there be two complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$.

If $z_1 = z_2$ then $\text{Re}(z_1) = \text{Re}(z_2)$ and $\text{Im}(z_1) = \text{Im}(z_2)$.

i.e., if $x_1 + iy_1 = x_2 + iy_2$

$\Rightarrow$ $x_1 = x_2$ and $y_1 = y_2$ simultaneously.

Illustration :

Find the real values of x and y for which the equation $(x^4 + 2xi) - (3x^2 + yi) = (3 - 5i) + (1 + 2yi)$ *is satisfied.*

Sol. *Given equation* $(x^4 + 2xi) - (3x^2 + yi) = (3 - 5i) + (1 + 2yi)$

$\Rightarrow$ $(x^4 - 3x^2) + i\,(2x - 3y) = 4 - 5i$

Equating real and imaginary parts, we get

$x^4 - 3x^2 = 4$ ... (i)

and $2x - 3y = -5$ ...(ii)

From (i) and (ii), we get $x = \pm 2$ *and* $y = 3, \dfrac{1}{3}$

Note : Inequality in complex numbers are never talked. If $x_1 + iy_1 > x_2 + iy_2$ has to be meaningful $\Rightarrow y_1 = y_2 = 0$. Equalities however in complex numbers are meaningful. Two complex numbers z_1 and z_2 are said to be equal if

$\text{Re } z_1 = \text{Re } z_2$ and $\text{Im}(z_1) = \text{Im}(z_2)$ (i.e. they occupy the same position on complex plane)

2.2 Addition :

$z_1 + z_2 = (x_1 + i\,y_1) + (x_2 + i\,y_2) = (x_1 + x_2) + i\,(y_1 + y_2) \in C.$

It is easy to oberve that the sum of two complex numbers is a complex number whose real (imaginary) part is the sum of the real (imaginary) parts of the given numbers :

$$\mathrm{Re}(z_1 + z_2) = \mathrm{Re}(z_1) + \mathrm{Re}(z_2);$$
$$\mathrm{Im}(z_1 + z_2) = \mathrm{Im}(z_1) + \mathrm{Im}(z_2)$$

2.3 Subtraction :

$z_1 - z_2 = (x_1 + i\,y_1) - (x_2 + i\,y_2) = (x_1 - x_2) + i\,(y_1 - y_2) \in C.$

That is

$$\mathrm{Re}(z_1 - z_2) = \mathrm{Re}(z_1) - \mathrm{Re}(z_2);$$
$$\mathrm{Im}(z_1 - z_2) = \mathrm{Im}(z_1) - \mathrm{Im}(z_2).$$

2.4 Multiplication :

$z_1 \cdot z_2 = (x_1 + i\,y_1)\,(x_2 + i\,y_2) = (x_1x_2 - y_1y_2) + i\,(x_1y_2 + x_2y_1) \in C.$

In other words

$$\mathrm{Re}(z_1z_2) = \mathrm{Re}(z_1) \cdot \mathrm{Re}(z_2) - \mathrm{Im}(z_1) \cdot \mathrm{Im}(z_2)$$

and $\quad \mathrm{Im}(z_1z_2) = \mathrm{Im}(z_1) \cdot \mathrm{Re}(z_2) + \mathrm{Im}(z_2) \cdot \mathrm{Re}(z_1)$

For a real number λ and a complex number $z = x + i\,y$.

$$\lambda \cdot z = \lambda\,(x + iy) = \lambda x + i\,\lambda y \in C$$

is the product of a real number with a complex number. The following properties are obvious :

(a) $\lambda\,(z_1 + z_2) = \lambda z_1 + \lambda z_2$

(b) $\lambda_1\,(\lambda_2 z) = (\lambda_1\lambda_2)z;$

(c) $(\lambda_1 + \lambda_2)z = \lambda_1 z + \lambda_2 z$ for all $z, z_1, z_2 \in C$ and $\lambda, \lambda_1, \lambda_2 \in R$.

Actually, relations (a) and (c) are special cases of the distributive law and relation (b) comes from the associative law of multiplication for complex numbers.

2.5 Division of Complex Number :

Let $\quad z_1 = x_1 + iy_1 \quad \& \quad z_2 = x_2 + iy_2$

Then $\quad \dfrac{z_1}{z_2} = \dfrac{x_1 + iy_1}{x_2 + iy_2} \Rightarrow \dfrac{(x_1 + iy_1)(x_2 - iy_2)}{(x_2 + iy_2)(x_2 - iy_2)}$

$$\Rightarrow \frac{(x_1x_2 + y_1y_2) + i(y_1x_2 - x_1y_2)}{(x_2^2 + y_2^2)} \Rightarrow \left\{\frac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2}\right\} + i\left\{\frac{y_1x_2 - x_1y_2}{x_2^2 + y_2^2}\right\}$$

$$\frac{z_1}{z_2} \Rightarrow \mathrm{Re}\left(\frac{z_1}{z_2}\right) + i\,\mathrm{Im}\left(\frac{z_1}{z_2}\right)$$

2.6 Square Root of Complex Number :

Let $z = x + iy$ be the given complex number and we have to obtain its square root.

Let $a + ib = (x + iy)^{1/2} \Rightarrow a^2 - b^2 + 2iab = x + iy$

$\Rightarrow x = a^2 - b^2$ and $y = 2ab$

$\Rightarrow x^2 = (a^2 + b^2)^2 - 4a^2b^2 \Rightarrow x^2 + y^2 = (a^2 + b^2)^2$

$\Rightarrow a^2 + b^2 = |z|$ ....(1) $\quad \Rightarrow a^2 - b^2 = x$ ....(2)

$$\Rightarrow a^2 = \frac{|z|+x}{2} \Rightarrow a = \pm\sqrt{\frac{|z|+x}{2}} \qquad \Rightarrow b^2 = \frac{|z|-x}{2} \Rightarrow b = \pm\sqrt{\frac{|z|-x}{2}}$$

$$\therefore \sqrt{x+iy} = a + ib = \pm\left(\sqrt{\frac{|z|+\text{Re}(z)}{2}} + i\sqrt{\frac{|z|-\text{Re}(z)}{2}}\right)$$

Replacing i by $-i$, we get

$$\sqrt{x-iy} = \pm\left(\sqrt{\frac{|z|+\text{Re}(z)}{2}} - i\sqrt{\frac{|z|-\text{Re}(z)}{2}}\right)$$

Illustration :

Find the square root of $3 + 4i$

Sol. *Let* $\sqrt{3+4i} = a + ib \quad \Rightarrow \quad 3 + 4i = a^2 - b^2 + 2ab \quad \Rightarrow \quad a^2 - b^2 = 3,\ 2ab = 4$

$\therefore \quad a^2 + b^2 = \sqrt{(a^2-b^2)^2 + 4a^2b^2} = \sqrt{9+16} = 5$

$\therefore \quad a + i\,b = \pm\,(2 + i)$

Alternative method :

Using formula $\sqrt{3+4i} = \pm\left(\sqrt{\frac{5+3}{2}} + i\sqrt{\frac{5-3}{2}}\right) = \pm\,(2 + i)$

Illustration :

If $z = x + iy$, $z^{1/3} = a - ib$ and $\frac{x}{a} - \frac{y}{b} = k(a^2 - b^2)$, then find the value of k.

Sol. $(x + iy)^{1/3} = a - ib$

$\Rightarrow \quad x + iy = (a - ib)^3 = (a^3 - 3ab^2) + i\,(b^3 - 3a^2b)$

$\Rightarrow \quad x = a^3 - 3ab^2,\ y = b^3 - 3a^2b \Rightarrow \quad \frac{x}{a} = a^2 - 3b^2$ *and* $\frac{y}{b} = b^2 - 3a^2$

$\Rightarrow \quad \frac{x}{a} - \frac{y}{b} = a^2 - 3b^2 - b^2 + 3a^2 = 4(a^2 - b^2)$

$\therefore \quad k = 4.$

Illustration :

Find the values of θ *if* $\dfrac{(3+2i\sin\theta)}{(1-2i\sin\theta)}$ *is purely real or purely imaginary.*

Sol. $z = \dfrac{3+2i\sin\theta}{1-2i\sin\theta}$

Multiplying numerator and denominator by conjugate,

$$z = \frac{(3+2i\sin\theta)(1+2i\sin\theta)}{1+4\sin^2\theta} = \frac{3-4\sin^2\theta+8i\sin\theta}{1+4\sin^2\theta}$$

Now z *is purely real if* $\sin\theta = 0$ *or* $\theta = n\pi,\ n \in I$. z *is purely imaginary if* $3 - 4\sin^2\theta = 0$

$$\Rightarrow \quad \sin\theta = \pm\frac{\sqrt{3}}{2} = \pm\sin\frac{\pi}{3} \quad \Rightarrow \quad \theta = n\pi \pm \frac{\pi}{3},\ n \in I.$$

Illustration :

Show that the polynomial $x^{4p} + x^{4q+1} + x^{4r+2} + x^{4s+3}$ *is divisible by* $x^3 + x^2 + x + 1$ *where* $p, q, r, s \in N$.

Sol. *Let* $f(x) = x^{4p} + x^{4q+1} + x^{4r+2} + x^{4s+3}$

$x^3 + x^2 + x + 1 = (x^2 + 1)(x + 1) = (x + i)(x - i)(x + 1)$

$f(i) = i^{4p} + i^{4q+1} + i^{4r+2} + i^{4s+3} = 1 + i^1 + i^2 + i^3 = 1 + i - 1 - i = 0$

$f(-i) = (-i)^{4p} + (-i)^{4q+1} + (-i)^{4r+2} + (-i)^{4s+3}$

$\quad = 1 + (-i)^1 + (-i)^2 + (-i)^3 = 1 - i - 1 + i = 0$

$f(-1) = (-1)^{4p} + (-1)^{4q+1} + (-1)^{4r+2} + (-1)^{4s+3} = 0$

Thus by division theorem $f(x)$ *is divisible by* $x^3 + x^2 + x + 1$.

Illustration :

If the expression $(1 + ir)^3$ *is of the form of* $s(1 + i)$ *for some real 's' where 'r' is also real and* $i = \sqrt{-1}$, *then the value of 'r' can be*

(A) $\cot\dfrac{\pi}{8}$ (B) $\sec\pi$ (C) $\tan\dfrac{\pi}{12}$ (D) $\tan\dfrac{5\pi}{12}$

Sol. *We have* $(1 + ri)^3 = s(1 + i)$

$1 + 3ri + 3r^2i^2 + r^3i^3 = s(1 + i)$

$1 - 3r^2 + i(3r - r^3) = s + si \quad \Rightarrow \quad 1 - 3r^2 = s = 3r - r^3$

Hence $1 - 3r^2 = 3r - r^3$

$\Rightarrow \quad r^3 - 3r^2 - 3r + 1 = 0 \;\Rightarrow\; (r^3 + 1) - 3r(r + 1) = 0 \;\Rightarrow\; (r + 1)(r^2 + 1 - r - 3r) = 0$

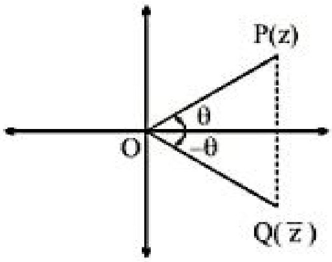
$\therefore \quad r = -1 \quad or \quad r^2 - 4r + 1 = 0$

$$\Rightarrow \quad r = \frac{4 \pm \sqrt{16-4}}{2} = \frac{4 \pm 2\sqrt{3}}{2} \quad \Rightarrow \quad r = 2+\sqrt{3} \ \text{ or } \ 2-\sqrt{3} \quad \Rightarrow \ \mathbf{B, C, D}$$

3. THREE IMPORTANT TERMS WITH RESPECT TO COMPLEX NUMBER :

3.1 Conjugate of Complex Number :

Conjugate of a complex number $z = a + ib$ is denoted and defined by $\bar{z} = a - ib$.
In a complex number if we replace i by $-i$, we get conjugate of the complex number. $\bar{z}$ is the mirror image of z about real axis on Argand's Plane.



Geometrical representation of conjugate of complex number

Note :

(a) $z + \bar{z} = 2\text{Re } z$

(b) $z - \bar{z} = 2i \text{ Im } z$

(c) $z\bar{z} = a^2 + b^2$, where $z = a + ib$

(d) If z lies in 1st quadrant then $\bar{z}$ lies in 4th quadrant and $-\bar{z}$ in the 2nd Quad.

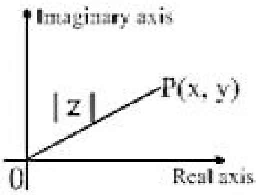
(e) If $x + iy = f(a + ib)$ then $x - iy = f(a - ib)$
Further , $g(x + iy) = f(a + ib) \Rightarrow g(x - iy) = f(a - ib)$
e.g. $\sin(\alpha + i\beta) = x + iy \Rightarrow \sin(\alpha - i\beta) = x - iy$

3.2 Modulus of Complex Number :

Modulus of complex number is a distance of the point on the argand plane representing the complex number z from the origin.

If P denotes a complex number $z = x + iy$

then $OP = |z| = \sqrt{x^2 + y^2}$



Note :

(i) $|z| > 0$.

(ii) All complex numbers having the same modulus lie on a circle with centre as origin and radius $r = |z|$.

Illustration :

Find the modulus of the following complex numbers

(i) $\frac{1}{2} + i\frac{\sqrt{3}}{2}$ *(ii)* $\frac{\sqrt{3}+1}{2\sqrt{2}} - i\,\frac{\sqrt{3}-1}{2\sqrt{2}}$ *(iii)* $1 + \cos\alpha + i\sin\alpha,\ \alpha \in \left(\pi, \frac{3\pi}{2}\right)$

Sol. *(i)* $z = \frac{1}{2} + i\frac{\sqrt{3}}{2}$

$$\therefore\ |z| = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = 1$$

(ii) $|z| \Rightarrow \sqrt{\left(\frac{\sqrt{3}+1}{2\sqrt{2}}\right)^2+\left(\frac{\sqrt{3}-1}{2\sqrt{2}}\right)^2} \Rightarrow \sqrt{\frac{3+1+2\sqrt{3}+3+1-2\sqrt{3}}{8}} = \sqrt{1} = 1.$ ***Ans.***

(iii) $z = 1 + \cos\alpha + i\sin\alpha \qquad = 2\cos^2\frac{\alpha}{2} + 2i\sin\frac{\alpha}{2}\cdot\cos\frac{\alpha}{2} = 2\cos\frac{\alpha}{2}\left[\cos\frac{\alpha}{2}+i\sin\frac{\alpha}{2}\right]$

$|z| = -\,2\cos\frac{\alpha}{2}$ ***Ans.*** $\left\{\frac{\alpha}{2} \in \left(\frac{\pi}{2},\frac{3\pi}{4}\right) \Rightarrow \cos\frac{\alpha}{2} < 0\right\}$

Illustration :

If the complex number z satisfying $z + |z| = 1 + 7i$ *then find the value of* $|z|^2$.

Sol. $z = x + iy$

$\therefore \quad x + iy + \sqrt{x^2+y^2} = 1 + 7i$

$x + \sqrt{x^2+y^2} = 1 \qquad ...(1)$

and $\quad y = 7 \qquad ...(2)$

$\therefore \quad x + \sqrt{x^2+49} = 1$

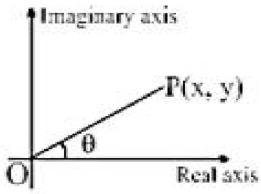
$x^2 + 49 = 1 + x^2 - 2x$

$2x = -\,48$

$x = -\,24$

$\therefore \quad |z|^2 = x^2 + y^2 = 625$ ***Ans.***

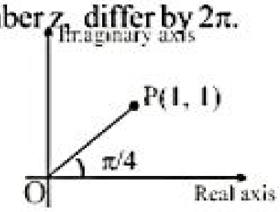
3.3 Argument of Complex Number :



Angle (θ) made by the line segment joining the point on the complex plane representing the complex number z to the origin from the positive real axis is called argument of complex number z which is denoted as arg(z) = θ.

(i) General Argument :

If OP makes an angle θ with real axis then θ is called one of the argument of z.
General values of argument of z are given by
$2n\pi + \theta,\ n \in I$. Note that any two argument of the same complex number z, differ by 2π.



e.g. If $z = 1 + i$ then $\arg(z) = \frac{\pi}{4}$

$\therefore$ General value of argument of $z = 2n\pi + \frac{\pi}{4},\ n \in I$

Note that by specifying the modulus and argument, a complex number is completely defined. However for the complex number $0 + 0\,i$ the argument is not defined and this is the only complex number which is completely defined by talking in terms of its modulus. i.e., $|z| = 0$.

(ii) Amplitude (Principal value of argument) :

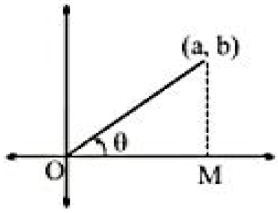
The unique value of θ such that $-\pi < \theta \leq \pi$ is called principal value of argument. Unless otherwise stated, amp z refers to the principal value of argument.

Working rule for finding principal argument of Complex number Z

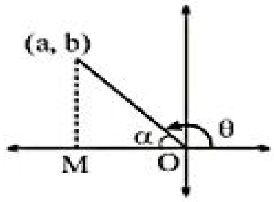
Let $z = a + ib$

First compute $\alpha = \tan^{-1}\left(\dfrac{|b|}{|a|}\right)$

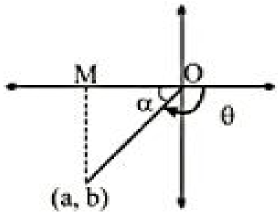
Case I: If z lies in I quadrant i.e. a, b > 0 then amp (z) = $\theta = \alpha$.



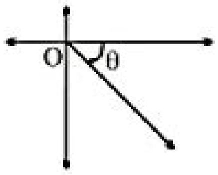
Case II : If z lies in II quadrant i.e. $a < 0, b > 0$ then amp(z) = $\theta = (\pi - \alpha)$



Case III: If z lies in III quadrant i.e. $a < 0, b < 0$ then amp(z) = $\theta = -(\pi - \alpha)$



Case IV: If z lies in IV quadrant i.e $a > 0, b < 0$ then amp(z) = $\theta = -\alpha$.



Note :

(i) If z is purely real positive complex number then amp(z) = 0.

(ii) If z is purely imaginary positive complex number then amp(z) = $\dfrac{\pi}{2}$.

(iii) If z is purely real negative complex number then amp(z) = π.

(iv) If z is purely imaginary neagative complex number then amp(z) = $\dfrac{-\pi}{2}$.

Illustration :

Find the amplitude of

(a) $-1-i\sqrt{3}$ (b) $\dfrac{1+\sqrt{3}i}{\sqrt{3}+i}$ (c) $\sin\alpha + i\,(1-\cos\alpha),\ 0<\alpha<\pi$

Sol.

(a) Let, $z = -1 - i\sqrt{3}$. The $\alpha = \tan^{-1}\left|\dfrac{b}{a}\right| = \tan^{-1}\left|\dfrac{\sqrt{3}}{1}\right| = \dfrac{\pi}{3}$

Clearly, z is in third quadrant.

Therefore argument is $\theta = -(\pi-\alpha) = -\left(\pi - \dfrac{\pi}{3}\right) = -\dfrac{2\pi}{3}$.

(b) $$amp\left(\frac{1+\sqrt{3}i}{\sqrt{3}+i}\right) = amp\left(\frac{1+\sqrt{3}i}{\sqrt{3}+i}\times\frac{\sqrt{3}-i}{\sqrt{3}-i}\right) = amp\left(\frac{2\sqrt{3}+2i}{4}\right) = amp\left(\frac{\sqrt{3}+i}{2}\right)$$

Complex number $\dfrac{\sqrt{3}+i}{2}$ lies in 1^{st} quadrant

$$\therefore\ amp(z) = \theta = \alpha = \frac{\pi}{6}$$

(c) $z = \sin\alpha + i(1-\cos\alpha),\ 0<\alpha<\pi$;
$z = \sin\alpha + i(1-\cos\alpha)$

$$\Rightarrow\quad amp\,(z) = \tan^{-1}\left(\frac{1-\cos\alpha}{\sin\alpha}\right) = \tan^{-1}\left(\frac{2\sin^2\frac{\alpha}{2}}{2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}}\right)$$

$$= \tan^{-1}\tan\left(\frac{\alpha}{2}\right) = \frac{\alpha}{2} \qquad \left\{0<\frac{\alpha}{2}<\frac{\pi}{2}\right\}$$

(iii) Least positive argument :

The value of θ such that $0<\theta\le 2\pi$ is called the least positive argument.

Illustration :

Find general argument, principal argument and least positive argument of the following complex numbers

(1) $z_1 = \sqrt{3} - 2i$ (2) $z_2 = -1 + i$ (3) $z_3 = -2 - 3i$

(4) $z_4 = \left(\sqrt{7} - 2\right)i$ (5) $z_5 = 2 - \sqrt{7}$ (6) $z_6 = \pi - e$

Sol.

S.No	Complex No.	General Argument	Principal Argument	Least Positive Argument
1.	$\sqrt{3} - 2i$	$2n\pi - \tan^{-1}\left(\frac{2}{\sqrt{3}}\right)$	$-\tan^{-1}\left(\frac{2}{\sqrt{3}}\right)$	$2\pi - \tan^{-1}\left(\frac{2}{\sqrt{3}}\right)$
2.	$-1 + i$	$2n\pi + \frac{3\pi}{4}$	$\frac{3\pi}{4}$	$\frac{3\pi}{4}$
3.	$-2 - 3i$	$2n\pi - \pi + \tan^{-1}\left(\frac{3}{2}\right)$	$-\left(\pi - \tan^{-1}\left(\frac{3}{2}\right)\right)$	$\pi + \tan^{-1}\left(\frac{3}{2}\right)$
4.	$\left(\sqrt{7} - 2\right)i$	$2n\pi + \frac{\pi}{2}$	$\frac{\pi}{2}$	$\frac{\pi}{2}$
5.	$\left(2 - \sqrt{7}\right)$	$2n\pi + \pi$	π	π
6.	$(\pi - e)$	$2n\pi$	0	2π

Practice Problem

Q.1 Find the value of $i^n + i^{n+1} + i^{n+2} + i^{n+3}$, $\forall$ n $\in$ I.

Q.2 If $\frac{(x-2)+(y-3)i}{1+i} = 1 - 3i$ find (x, y).

Q.3 If $f(x) = x^4 - 4x^3 + 4x^2 + 8x + 44$, find $f(3 + 2i)$.

Q.4 If $z = \frac{\sqrt{9+40i} + \sqrt{9-40i}}{\sqrt{9+40i} - \sqrt{9-40i}}$, find $|z|$ and z.

Q.5 A square $P_1P_2P_3P_4$ is drawn in the complex plane with P_1 at (1, 0) and P_3 at (3, 0). Let P_n denotes the point (x_n, y_n) n = 1, 2, 3, 4. Find the numerical value of the product of complex numbers $(x_1 + i\,y_1)(x_2 + i\,y_2)(x_3 + i\,y_3)(x_4 + i\,y_4)$.

Q.6 Solve the following equations over C and express the result in the form a + ib, a, b $\in$ R.
(a) $ix^2 - 3x - 2i = 0$ (b) $2(1 + i)x^2 - 4(2 - i)x - 5 - 3i = 0$

Q.7 Find the sum of the following series

$i + 2i^2 + 3i^3 + \ldots\ldots$ to 100 terms.

Q.8 If $z = (3 + 7i)(p + iq)$ where $p, q \in I - \{0\}$, is purely imaginary then minimum value of $|z|^2$ is

(A) 0 (B) 58 (C) $\frac{3364}{3}$ (D) 3364

Q.9 Match the equation in z, in **Column-I** with the corresponding values of arg(z) in **Column-II**.

	Column-I (equations in z)		**Column-II** (principal value of arg (z))
(A)	$z^2 - z + 1 = 0$	(P)	$-2\pi/3$
(B)	$z^2 + z + 1 = 0$	(Q)	$-\pi/3$
(C)	$2z^2 + 1 + i\sqrt{3} = 0$	(R)	$\pi/3$
(D)	$2z^2 + 1 - i\sqrt{3} = 0$	(S)	$2\pi/3$

Answer key

Q.1 0 Q.2 (6, 1) Q.3 5 Q.4 $\frac{\pi}{2}, \frac{4}{5}$ or $-\frac{\pi}{2}, \frac{5}{4}$

Q.5 15 Q.6 (a) $-i, -2i$; (b) $\frac{3-5i}{2}$ or $-\frac{1+i}{2}$ Q.7 $50(1-i)$

Q.8 D Q.9 (A) Q, R; (B) P, S; (C) Q, S; (D) P, R

4. REPRESENTATION OF A COMPLEX NUMBER IN DIFFERENT FORMS:

4.1 Cartesian Form / Algebraic Form :

Every complex number expressed in the form of $z = x + iy$ where $x, y \in R$ and $i = \sqrt{-1}$ is called cartesian form or algebraic form of complex number

for $z = x + iy$, $\text{Re}(z) = x$ and $\text{Im}(z) = y$

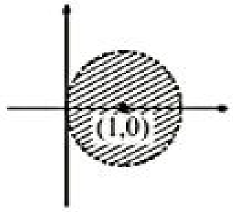
$$|z| = \sqrt{x^2 + y^2}, \quad \bar{z} = x - iy, \quad \arg(z) = \tan^{-1}\left(\frac{y}{x}\right)$$

Illustration :

If $Re\left(\frac{1}{z}\right) > \frac{1}{2}$ *then find the locus of z.*

Sol. Let $z = x + iy$

$$\frac{1}{z} = \frac{1}{x+iy} = \frac{x-iy}{x^2+y^2} = \frac{x}{x^2+y^2} - \frac{iy}{x^2+y^2}.$$



$\because \quad Re\left(\frac{1}{z}\right) > \frac{1}{2} \quad \Rightarrow \quad \frac{x}{x^2+y^2} > \frac{1}{2}$

$\Rightarrow \quad 2x > x^2 + y^2 \quad \Rightarrow \quad x^2 + y^2 - 2x < 0$

$\Rightarrow \quad (x-1)^2 + y^2 < 1.$

Hence, locus of z represents interior region of the circle whose centre is (1, 0) and radius is 1 unit.

Illustration :

Solve the equation $z^2 + |z| = 0$ *where* $z \in C$.

Sol. Let $z = x + iy$

$z^2 + |z| = 0$

$x^2 - y^2 + 2xy\,i + \sqrt{x^2+y^2} = 0$

Comparing real and imaginary parts

$\Rightarrow \quad x^2 - y^2 + \sqrt{x^2+y^2} = 0$ and $2xy = 0 \Rightarrow x = 0$ or $y = 0$

Case-I : *When* $x = 0$

$\Rightarrow \quad -y^2 + \sqrt{y^2} = 0 \quad \Rightarrow \quad y^2 = \sqrt{y^2} \quad \Rightarrow \quad y^2 = \pm y$

$\Rightarrow \quad y^2 \pm y = 0 \quad \Rightarrow \quad y(y \pm 1) = 0$

$y = 0, y = \pm 1$

$\therefore$ *Solutions are (0, 0), (0, 1) and (0, – 1)*

Case-II : *When* $y = 0$

$x^2 + \sqrt{x^2} = 0 \Rightarrow x^2 + |x| = 0 \Rightarrow x = 0$

$\therefore$ *Solution is (0, 0) already determined*

Hence, solutions of the equation are (0, 0), (0, 1), (0, – 1).

Illustration :

If z is a complex number satisfying the equation $|z - (1+i)|^2 = 2$ *and* $\omega = \frac{2}{z}$,

then the locus traced by 'ω' in the complex plane is

(A) $x - y - 1 = 0$ (B) $x + y - 1 = 0$ (C) $x - y + 1 = 0$ (D) $x + y + 1 = 0$

Sol. We have $|z - (1+i)|^2 = 2$

$\Rightarrow \quad (x-1)^2 + (y-1)^2 = 2 \qquad$ (Put $z = x + iy$)

$\Rightarrow \quad x^2 + y^2 = 2(x + y) \qquad$ ......(1)

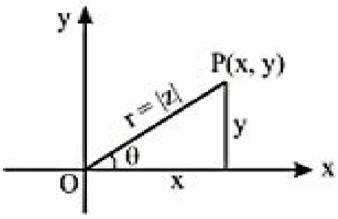
$$\text{Let}\quad \omega = h + ik = \frac{2}{z} = \frac{2}{x+iy} = \frac{2(x-iy)}{x^2+y^2}, \text{ so}$$

$$h = \frac{2x}{x^2+y^2}, \quad k = \frac{-2y}{x^2+y^2}$$

$$\Rightarrow \quad h - k = \frac{2(x+y)}{x^2+y^2} = 1 \text{ (from equation (1))}$$

$\therefore$ *Locus of the point* $\omega(h, k)$ *will be* $x - y = 1$ *Ans. (A)*

4.2 Trigonometrical Form / Polar Form :



Let the given complex number be $z = x + iy$
r and θ be the modulus and amp (z) respectively.
From the figure $x = r\cos\theta$, $y = r\sin\theta$

$\therefore \quad z = x + iy = r(\cos\theta + i\sin\theta) = r\,\text{cis}\,\theta$

Hence, $z = r(\cos\theta + i\sin\theta)$ is called polar / triangometrical form of the complex number.

Illustration :

Express the following complex number in polar form.

(i) $z_1 = -1 - i\sqrt{3}$ *(ii)* $z_2 = -2 + 3i$

Sol.

(i) $z_1 = -1 - i\sqrt{3} = r(\cos\theta + i\sin\theta)$

$$amp\,(z_1) = \theta = -\left(\pi - \frac{\pi}{3}\right) = \frac{-2\pi}{3}$$

$$r = \sqrt{1+3} = 2$$

$$\therefore \quad z_1 = -1 - i\sqrt{3} = 2\left(\cos\left(\frac{-2\pi}{3}\right) + i\sin\left(\frac{-2\pi}{3}\right)\right) = 2\,cis\left(\frac{-2\pi}{3}\right).$$

(ii) $z_2 = -2 + 3i$

$$amp\,(z_2) = \theta = \pi - \tan^{-1}\frac{3}{2}$$

$$r = \sqrt{4+9} = \sqrt{13}$$

$$\therefore \quad z_2 = -2 + 3i = \sqrt{13}\left(\cos\left(\pi - \tan^{-1}\frac{3}{2}\right) + i\sin\left(\pi - \tan^{-1}\frac{3}{2}\right)\right) = \sqrt{13}\;cis\left(\pi - \tan^{-1}\frac{3}{2}\right)$$

Illustration :

If $z = 1+\cos\frac{6\pi}{5}+i\sin\frac{6\pi}{5}$ *then find modulus and amplitude of z.*

Sol. $z = 1+\cos\frac{6\pi}{5}+i\sin\frac{6\pi}{5}$

$$2\cos^2\frac{3\pi}{5}+i\,2\sin\frac{3\pi}{5}\cdot\cos\frac{3\pi}{5} \quad\Rightarrow\quad 2\cos\frac{3\pi}{5}\left(\cos\frac{3\pi}{5}+i\sin\frac{3\pi}{5}\right)$$

$$\Rightarrow\quad 2\cos\left(\pi-\frac{2\pi}{5}\right)\left(\cos\frac{3\pi}{5}+i\sin\frac{3\pi}{5}\right) \quad\Rightarrow\quad -2\cos\frac{2\pi}{5}\left(\cos\frac{3\pi}{5}+i\sin\frac{3\pi}{5}\right)$$

$$\Rightarrow\quad 2\cos\frac{2\pi}{5}\left(\cos\left(\pi-\frac{3\pi}{5}\right)-i\sin\left(\pi-\frac{3\pi}{5}\right)\right)$$

$$\Rightarrow\quad 2\cos\frac{2\pi}{5}\left(\cos\frac{2\pi}{5}-i\sin\frac{2\pi}{5}\right) \quad\Rightarrow\quad 2\cos\frac{2\pi}{5}\left[\cos\left(\frac{-2\pi}{5}\right)+i\sin\left(\frac{-2\pi}{5}\right)\right]$$

$$\therefore\quad |z| = 2\cos\frac{2\pi}{5} \text{ and } amp\,(z) = \frac{-2\pi}{5}.$$

4.3 Exponential Form :

$z = x + iy = r(\cos\theta + i\sin\theta) = r\,e^{i\theta}$

$z = re^{i\theta}$ is called exponential form of the complex number.

where r is modulus of z and θ is amplitude of z.

Here, $\cos\theta + i\sin\theta = e^{i\theta}\ \forall\ \theta$ ....(1)

Replacing i by $-$i, we get

$\cos\theta - i\sin\theta = e^{-i\theta}\ \forall\ \theta$ ....(2)

Adding (1) and (2)

$$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} \text{ which is purely real}$$

subtracting (2) from (1)

$$i\sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2} \text{ which is purely imaginary.}$$

Illustration :

If $z = -2e^{i\left(\frac{-\pi}{3}\right)}$ *then find modulus and amplitude of z.*

Sol. $z = -2\left(\cos\left(\frac{-\pi}{3}\right)+i\sin\left(\frac{-\pi}{3}\right)\right)$

$$= -2\left(\cos\frac{\pi}{3}-i\sin\frac{\pi}{3}\right) = -2\left(\frac{1}{2}-i\,\frac{\sqrt{3}}{2}\right) = -1+i\sqrt{3} \text{ which lies in } 2^{nd} \text{ quadrant}$$

modulus $r = \sqrt{1+3} = 2$

$$amp\,(z) = \theta = \pi - \tan^{-1}\left(\sqrt{3}\right) = \pi-\frac{\pi}{3} = \frac{2\pi}{3}.$$

Illustration :

Find the real part of $z = e^{e^{i\theta}}$, $\theta \in R$

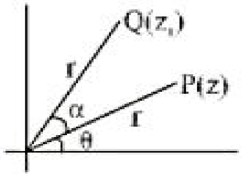
Sol. $z = e^{e^{i\theta}} = e^{(\cos\theta + i\sin\theta)} = e^{\cos\theta} \cdot e^{i\sin\theta} = e^{\cos\theta} \cdot (\cos(\sin\theta) + i\sin(\sin\theta))$

$\therefore \quad Re(z) = e^{\cos\theta} \cdot \cos(\sin\theta)$

4.4 Vectorial Representation of a Complex Number :

Every complex number can be considered as if it is the position vector of that point. If the point P represents the complex number z then,

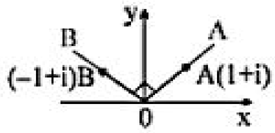
$\overrightarrow{OP} = z \;\; \& \;\; |\overrightarrow{OP}| = |z|$.



Geometrical meaning of $e^{i\alpha}$.

(i) If $\overrightarrow{OP} = z = r e^{i\theta}$ then $\overrightarrow{OQ} = z_1 = r e^{i(\theta + \alpha)} = z \, . \, e^{i\alpha}$.

If $\overrightarrow{OP}$ and $\overrightarrow{OQ}$ are of unequal magnitude then $\hat{OQ} = \hat{OP} e^{i\alpha}$



(ii) If $z = \overline{OA} = 1 + i$ and $\alpha = \dfrac{\pi}{2}$ then $z_1 = \overline{OB} = i(1 + i) = -1 + i$

(iii) Using the vetorial concept and section formula complex numbers corresponding to centroid, incentre, orthocentre and circumcentre for a triangle whose vertices are z_1, z_2, z_3 can be deduced.

Practice Problem

Q.1 Find the locus of z if it satisfies the equation $(z - 1)^2 + |z + 1|^2 = 2$.

Q.2 Find all the possible solutions of the equation $z^2 = \bar{z}$ where $z \in C$.

Q.3 Find the set of all points on the complex plane for which $z^2 + z + 1$ is real and positive.

Q.4 If $\dfrac{\pi}{2} < \theta < \dfrac{3\pi}{2}$ then find modulus and principal argument of $z = (1 + \cos 2\theta) + i\sin 2\theta$.

Answer key

Q.1 Point (0, 0)

Q.2 $(0, 0), (1, 0), \left(\dfrac{-1}{2}, \dfrac{\sqrt{3}}{2}\right), \left(\dfrac{-1}{2}, \dfrac{-\sqrt{3}}{2}\right)$

Q.3 $(\lambda, 0), \lambda \in R$ and $\left(\dfrac{-1}{2}, y\right)$ where $y \in \left(\dfrac{-\sqrt{3}}{2}, \dfrac{\sqrt{3}}{2}\right)$

Q.4 $-2\cos\theta, \theta - \pi$

5. PROPERTIES OF CONJUGATE, MODULUS AND ARGUMENT :

5.1 Properties of conjugate of complex Numbers :

(i) $(\bar{\bar{z}}) = z$ (ii) $|z| = |\bar{z}|$ (iii) $z + \bar{z} = 2\text{Re}(z)$

(iv) $z - \bar{z} = 2i\,\text{Im}(z)$ (v) If z is purely real $z = \bar{z}$ (vi) If z is purely imaginary $z = -\bar{z}$

(vii) $z\bar{z} = |z|^2 = |\bar{z}|^2$

(viii) $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$ In general, $\overline{z_1 + z_2 + \ldots\ldots + z_n} = \overline{z_1} + \overline{z_2} + \ldots\ldots + \overline{z_n}$

(ix) $\overline{z_1 - z_2} = \overline{z_1} - \overline{z_2}$ (x) $\overline{z_1 z_2} = \overline{z_1} \cdot \overline{z_2}$ (xi) $\overline{z^n} = (\bar{z})^n$ (xii) $\overline{\left(\dfrac{z_1}{z_2}\right)} = \left(\dfrac{\bar{z}_1}{\bar{z}_2}\right)$

(xiii) If $\alpha = f(z)$, then $\bar{\alpha} = \overline{f(z)} = f(\bar{z})$ where $\alpha = f(z)$ is a function in complex variable with real coefficients.

In other words if $f(x + iy) = a + ib$ then $f(x - iy) = a - ib$.

Explanation :

Let $f(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^2 + \ldots\ldots + a_n z$, where $a_0, a_1, a_2, \ldots\ldots, a_n$ are real numbers and z is a complex number. Then

$$f(\bar{z}) = a_0 + a_1 \bar{z} + a_2(\bar{z})^2 + a_3(\bar{z})^3 + \ldots\ldots + a_n(\bar{z})^n = \overline{a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \ldots\ldots + a_n z^n}.$$

5.2 Properties of modulus of complex numbers :

(i) $|z| = 0 \Rightarrow z = 0 + i0$

(ii) $|z| = |-z| = |\bar{z}| = |iz|$

(iii) $-|z| \le \text{Re}(z) \le |z|$ and $-|z| \le \text{Im}(z) \le |z|$

(iv) $z\bar{z} = |z|^2$ If z is unimodular i.e. $|z| = 1$, then $\bar{z} = \dfrac{1}{z}$

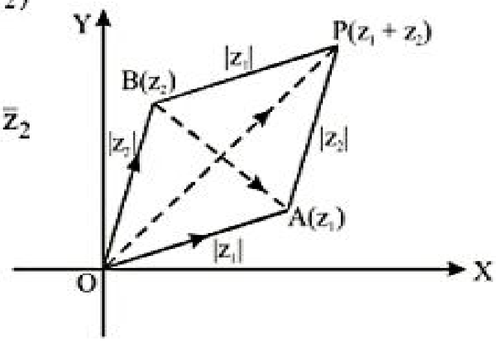
(v) $|z_1 z_2| = |z_1||z_2|$ In general $|z_1 \cdot z_2 \cdot z_3 \ldots\ldots z_n| = |z_1||z_2||z_3| \ldots\ldots |z_n|$

(vi) $|z^n| = |z|^n$

(vii) $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}$ $(z_2 \ne 0)$

(viii) $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2(|z_1|^2 + |z_2|^2)$

Proof :



$|z_1+z_2|^2+|z_1-z_2|^2 \Rightarrow (z_1+z_2)\left(\overline{z_1+z_2}\right)+(z_1-z_2)\left(\overline{z_1-z_2}\right)$

$\Rightarrow \quad (z_1+z_2)(\bar{z}_1+\bar{z}_2)+(z_1-z_2)(\bar{z}_1-\bar{z}_2)$

$\Rightarrow \quad z_1\bar{z}_1+z_1\bar{z}_2+z_2\bar{z}_1+z_2\bar{z}_2+z_1\bar{z}_1-z_1\bar{z}_2-z_2\bar{z}_1+z_2\bar{z}_2$

$\Rightarrow \quad 2(|z_1|^2+|z_2|^2)$

OAPB is a parallelogram.

By the vector law of addition

$$\overrightarrow{OP}=z_1+z_2 \Rightarrow \left|\overrightarrow{OP}\right|=|z_1+z_2|$$

$$\overrightarrow{BA}=z_1-z_2 \Rightarrow \left|\overrightarrow{BA}\right|=|z_1-z_2|$$

$\therefore \quad OP^2+BA^2=OA^2+AP^2+PB^2+OB^2$

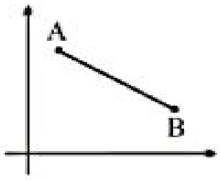
$|z_1+z_2|^2+|z_1-z_2|^2=2(|z_1|^2+|z_2|^2)$

The above identity indicates the sum of squars of diagonals of a parallelogram is equal to sum of square of its all four sides.

(ix) Let $z_1=x_1+iy_1$ and $z_2=x_2+iy_2$

$|z_1-z_2|$ denotes the distance between two points on the complex plane representing z_1 and z_2.

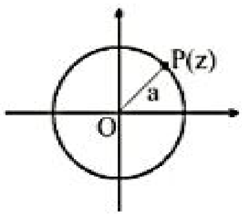
$AB=|z_1-z_2|$



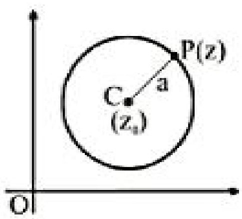
$$=\left|x_1+iy_1-(x_2+iy_2)\right|$$

$$=\left|x_1-x_2+i(y_1-y_2)\right|=\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}\,.$$

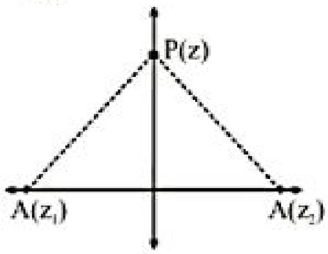
(x) $|z|=a,\ a\in R^+ \Rightarrow$ locus of z represents a circle whose centre is origin and radius is 'a'.



(xi) $|z-z_0|=a$ where z_0 is a fixed complex number and $a\in R^+ \Rightarrow$ locus of z represents a circle whose centre is z_0 and radius is a.



(xii) If $\left|\dfrac{z-z_1}{z-z_2}\right|=1$ where z_1 and z_2 are two fixed complex numbers then locus of z is the perpendicular bisector of joining the points representing z_1 and z_2.



(xiii) Triangle inequalities $\left|\,|z_1| - |z_2|\,\right| \le |z_1 \pm z_2| \le |z_1| + |z_2|$

Proof :

Method-I Algebraic Method :

Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

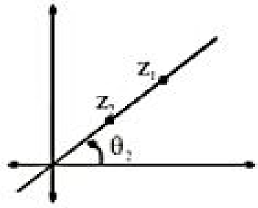
$z_1 + z_2 = r_1\cos\theta_1 + r_2\cos\theta_2 + i(r_1\sin\theta_1 + r_2\sin\theta_2)$

$$z_1 + z_2 = \sqrt{(r_1\cos\theta_1 + r_2\cos\theta)^2 + (r_1\sin\theta_1 + r_2\sin\theta_2)^2} = \sqrt{r_1^2 + r_2^2 + 2r_1r_2\cos(\theta_1 - \theta_2)}.$$

$|z_1 + z_2|$ will be maximum when $\cos(\theta_1 - \theta_2) = 1$

$\Rightarrow \quad \theta_1 - \theta_2 = 2n\pi$

$\Rightarrow \quad \theta_1 = \theta_2 + 2n\pi,\ n \in I$

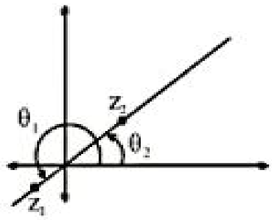


Hence, for the maximum value of $|z_1 + z_2|$ points representing complex numbers z_1, z_2 and the origin are collinear and z_1, z_2 must lie on the same side of the origin.

$|z_1 + z_2|$ will be minimum when $\cos(\theta_1 - \theta_2) = -1$

$\Rightarrow \quad \theta_1 - \theta_2 = 2n\pi + \pi,\ n \in I$

$\Rightarrow \quad \theta_1 = \theta_2 + 2n\pi + \pi,\ n \in I$



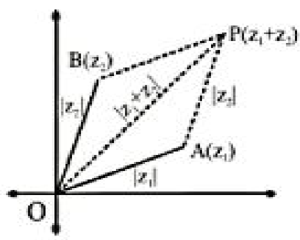
Hence, for the minimum value of $|z_1 + z_2|$ points representing the complex number z_1, z_2 and the origin are collinear and z_1, z_2 must lie on the opposite side of the origin.

In the similar manner minimum and maximum values of $|z_1 - z_2|$ can also be determined.

Method-II Geometrical Method :

Let A and B represent complex numbers z_1 and z_2 respectively.

A parallelogram OAPB is completed $\overline{OP} = z_1 + z_2$



$\therefore \quad |\overrightarrow{OP}| = |z_1 + z_2|$

In the ΔOAP, from the fact

(i) Sum of two sides is always greater than third side.

(ii) Absolute value of the difference of two sides is always less than third side.

$$\therefore \quad \left|\,|z_1| - |z_2|\,\right| \le |z_1 + z_2| \le |z_1| + |z_2|$$

Note : Sign of equality holds when z_1, z_2 and the origin are collinear.

5.3 Properties of Argument of complex number :

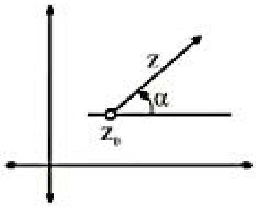
(i) amp $(z_1 z_2)$ = amp (z_1) + amp (z_2) + $2k\pi$, $k \in I$
In general amp $(z_1 \cdot z_2 \ldots\ldots z_n)$ = amp (z_1) + amp (z_2) + ... amp (z_n) + $2k\pi$, $k \in I$

(ii) amp $\left(\frac{z_1}{z_2}\right)$ = amp (z_1) − amp (z_2) + $2k\pi$, $k \in I$

(iii) amp (z^n) = n amp (z) + $2k\pi$, $k \in I$

Note :
In the above properties $2k\pi$, $k \in I$ is added in RHS and k is choosen in such a way so that value of the expresion in RHS belongs to $(-\pi, \pi]$

(iv) If amp $(z - z_0) = \alpha$ where z_0 is a fixed complex number then locus of z denotes a ray emanating from z_0 (z_0 is not included) and making an angle α from positive real axis.



Note : For any complex number z

$$\text{amp}(z) + \text{amp}(-\bar{z}) = \pi \quad \text{or} \quad \text{amp}(z) + \text{amp}(-1) + \text{amp}(\bar{z}) = \pi$$

Illustration :

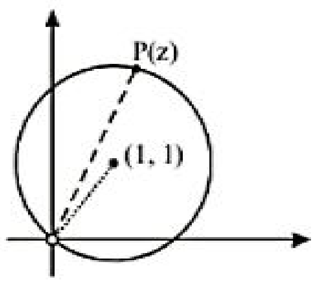
Represent the locus of z satisfying given equation or inequation on the argand's plane.

(i) $|z - 1 - i| = \sqrt{2}$ *and* $amp(z) = \frac{\pi}{3}$ *(ii)* $|amp\,(z - 2 - i)| < \frac{\pi}{3}$

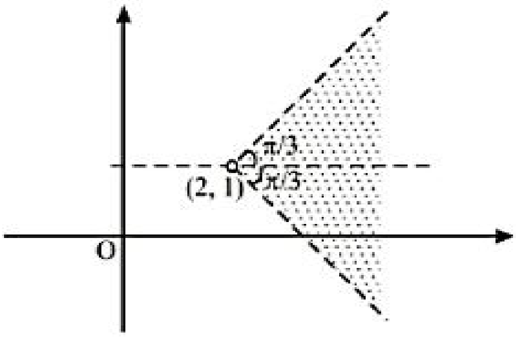
(iii) $3 \le |z - 4| \le 4$ *and* $|amp\,(z - 4)| \le \frac{\pi}{4}$

Sol.

(i) The complex number z repreting the point P is only the solution satisfying both the equations simultaneously



(ii) *Complex numbers corresponding to the points lie in the shaded region satisfy the given inequality.*



(iii) *All the complex number* z *satisfying both the inequalities simultaneously lies in the shaded reagion*

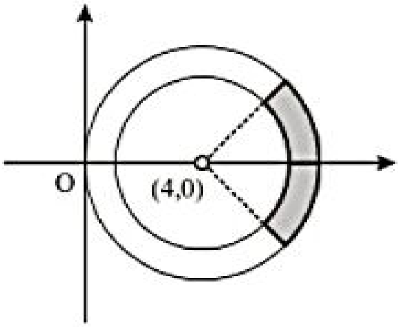


Illustration :

If $|z_1 - 1| \le 1,\ |z_2 - 2| \le 2,\ |z_3 - 3| \le 3$, *then find the greatest value of* $|z_1 + z_2 + z_3|$.

Sol. $|z_1 + z_2 + z_3| = |(z_1 - 1) + (z_2 - 2) + (z_3 - 3) + 6|$

$$\le |z_1 - 1| + |z_2 - 2| + |z_3 - 3| + 6 \le 1 + 2 + 3 + 6 = 12.$$

Illustration :

If $|z_1| = |z_2| = |z_3| = 1$, *prove that* $|z_1 + z_2 + z_3| = \left|\frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3}\right|$.

Sol. *We know that* $|z| = |\bar{z}|$

$\Rightarrow \quad |z_1 + z_2 + z_3| = |\bar{z}_1 + \bar{z}_2 + \bar{z}_3|$

$$= \left|\frac{\bar{z}_1 . z_1}{z_1} + \frac{\bar{z}_2 . z_2}{z_2} + \frac{\bar{z}_3 . z_3}{z_3}\right| = \left|\frac{|z_1|^2}{z_1} + \frac{|z_2|^2}{z_2} + \frac{|z_3|^2}{z_3}\right|$$

$$= \left|\frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3}\right| \qquad \left(\because |z_1|^2 = |z_2|^2 = |z_3|^2 = 1\right).$$

Illustration :

Consider two pairs of non-zero conjugate complex numbers (z_1, z_2) *and* (z_3, z_4).

Find the value of $arg\left(\frac{z_1}{z_3}\right) + arg\left(\frac{z_2}{z_4}\right)$.

Sol. $arg\left(\frac{z_1}{z_3}\right) + arg\left(\frac{z_2}{z_4}\right) = arg\left(\frac{z_1 z_2}{z_3 z_4}\right)$

$$= arg\left(\frac{|z_1|^2}{|z_3|^2}\right) \qquad (as\ z_2 = \bar{z}_1\ and\ \ z_4 = \bar{z}_3)$$

$= 0$ *(as argument of a positive real number is zero).*

Illustration :

If z and w are complex numbers satisfying $\bar{z} + i\,\overline{w} = 0$ *and* $Amp(zw) = \pi$, *then* $Amp(z)$ *is equal to*

(A) $\frac{\pi}{4}$ (B) $\frac{-\pi}{2}$ (C) $\frac{\pi}{2}$ (D) $\frac{3\pi}{4}$

Sol. Given $\bar{z} + i\,\overline{w} = 0 \quad \Rightarrow \quad \bar{z} = -i\,\overline{w} \quad or \quad z = i\,w$

$amp(z) - amp(w) = amp\ i = \frac{\pi}{2}$ ....(1)

also $amp(zw) = \pi$

$amp(z) + amp(w) = \pi$ ....(2)

(1) + (2), gives $2\ amp(z) = \frac{3\pi}{2} \quad \Rightarrow \quad amp(z) = \frac{3\pi}{4}$; Also $amp(w) = \frac{\pi}{4}$

Illustration :

If $Arg\ (z + a) = \frac{\pi}{6}$ *and* $Arg\ (z - a) = \frac{2\pi}{3}$; $a \in R^+$, *then*

(A) z is independent of a (B) $|\,a\,| = |\,z + a\,|$

(C) $z = a\ Cis\ \frac{\pi}{6}$ (D) $z = a\ Cis\ \frac{\pi}{3}$

Sol. *Refer the figure z lies on the point of intersection of the rays from A and B. ΔACB is a right angle and OBC is an equilateral triangle*

$\Rightarrow OC = a \ \Rightarrow z = a\ Cis\ \frac{\pi}{3} \ \Rightarrow (D)$

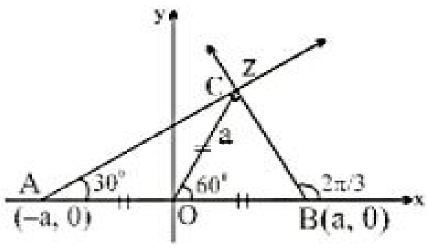


Illustration :

If $|z_1| = 1$, $|z_2| = 2$, $|z_3| = 3$ *and* $|9z_1z_2 + 4z_1z_3 + z_2z_3| = 12$,

then find the value of $|z_1 + z_2 + z_3|$.

Sol. $|z_1| = 1 \Rightarrow z_1\bar{z}_1, |z_2| = 2 \Rightarrow z_2\bar{z}_2 = 4, |z_3| = 3 \Rightarrow z_3\bar{z}_3 = 9$.

Also, $|9z_1z_2 + 4z_1z_3 + z_2z_3| = 12$

$\Rightarrow \quad |z_1z_2z_3\bar{z}_3 + z_1z_2z_3\bar{z}_2 + \bar{z}_1z_2z_3| = 12 \quad \Rightarrow \quad |z_1z_2z_3||\bar{z}_1 + \bar{z}_2 + \bar{z}_3| = 12$

$\Rightarrow \quad |z_1||z_2||z_3||\bar{z}_1 + \bar{z}_2 + \bar{z}_3| = 12 \quad \Rightarrow \quad 6|\bar{z}_1 + \bar{z}_2 + \bar{z}_3| = 12$

$\Rightarrow \quad |\bar{z}_1 + \bar{z}_2 + \bar{z}_3| = 2 \quad \Rightarrow \quad |z_1 + z_2 + z_3| = 2$.

Illustration :

Find modulus and principal argument of the complex number

$$z = (1 + i)\left(1 - i\sqrt{3}\right)(-2 - 2i)\,(i)\,(3).$$

Sol. $amp\ of\ (1 + i) = tan^{-1}\ 1 = \dfrac{\pi}{4}$

$amp\ of\ \left(1 - i\sqrt{3}\right) = -tan^{-1}\left|\sqrt{3}\right| = -\dfrac{\pi}{3}$

$amp\ of\ (-2 - 2i) = -(\pi - tan^{-1}\ 1) = -\dfrac{3\pi}{4}$

$amp\ \ of\ i = \dfrac{\pi}{2}$

$amp\ \ of\ 3 = 0$

$amp\ (z) = \dfrac{\pi}{4} - \dfrac{\pi}{3} + \left(\dfrac{-3\pi}{4}\right) + \dfrac{\pi}{2} + 0 \Rightarrow \quad \dfrac{-\pi}{3}$

$|z| = |1 + i|\left|1 - i\sqrt{3}\right|\,|-2-2i|\,|i||3|$

$= \sqrt{2}\cdot 2\cdot\sqrt{2}\cdot 1\cdot 3 = 24.$ ***Ans.***

Illustration :

If $|z - (5 + 7i)| = 9$ *then find the greatest and least value of* $|z - 2 - 3i|$.

Sol. $|z - 2 - 3i| = |z - (5 + 7i) + (3 + 4i)|$

$||z - (5 + 7i)| - |3 + 4i|| \le |z - (5 + 7i) + 3 + 4i| \le |z - (5 + 7i)| + |3 + 4i|$

$|9 - 5| \le |z - 2 - 3i| \le 9 + 5$

$4 \le |z - 2 - 3i| \le 14.$ ***Ans.***

Alternative method :

Z lies on the circle with centre C(5, 7) and radius equal to 9.
Now, we have to find maximum and minimum distance of the point A (2, 3) from the circle.
Maximum distance, $AP = AC + CP = 5 + 9 = 14$
Minimum distance, $AQ = CQ - AQ = 9 - 5 = 4$

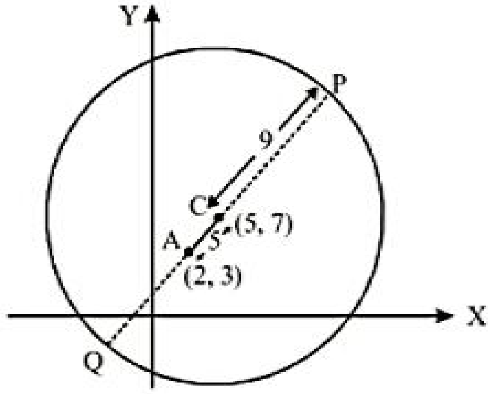


Illustration :

Find the greatest and least value of $|z|$ *is* z *satisfies* $\left|z-\frac{4}{z}\right| = 2.$

Sol. $\left||z|-\frac{4}{|z|}\right| \le \left|z-\frac{4}{z}\right| \le |z|+\frac{4}{|z|}$

$\left||z|-\frac{4}{|z|}\right| \le 2 \le |z|+\frac{4}{|z|}$

Put $|z| = r,\ r > 0$

$\left|r-\frac{4}{r}\right| \le 2 \le r+\frac{4}{r}$

$2 \le r+\frac{4}{r} \quad \Rightarrow \quad r^2 - 2r + 4 \ge 0$ *always true* $\quad \Rightarrow \quad r > 0$

$\left|r-\frac{4}{r}\right| \le 2$

$-2 \le r-\frac{4}{r} \le 2$

$r^2 + 2r - 4 \ge 0 \quad \Rightarrow \quad (r+1)^2 - 5 \ge 0 \quad \Rightarrow \quad \left(r+1-\sqrt{5}\right)\underbrace{\left(r+1+\sqrt{5}\right)}_{Positive} \ge 0$

$r \ge \sqrt{5} - 1$

$r^2 - 2r - 4 \le 0 \quad \Rightarrow \quad (r-1)^2 - 5 \le 0 \quad \Rightarrow \quad \left(r-1-\sqrt{5}\right)\left(r-1+\sqrt{5}\right) \le 0$

$\Rightarrow \quad r \in \left[1-\sqrt{5}, \sqrt{5}+1\right]$

$\therefore \quad r \in \left[\sqrt{5}-1, \sqrt{5}+1\right]$ ***Ans.***

Illustration :

Let z_1 *&* z_2 *be complex numbers such that* $z_1 \neq z_2$ *and* $|z_1| = |z_2|$, *then show that* $\dfrac{z_1+z_2}{z_1-z_2}$ *is purely imaginary.*

Sol. $|z_1| = |z_2|$, $Re\left(\dfrac{z_1+z_2}{z_1-z_2}\right) = \dfrac{1}{2}\left(\dfrac{z_1+z_2}{z_1-z_2} + \dfrac{\bar{z}_1+\bar{z}_2}{\bar{z}_1-\bar{z}_2}\right) = \dfrac{1}{2}\left(\dfrac{2|z_1|^2-2|z_2|^2}{|z_1-z_2|^2}\right) = 0 =$ *purely imaginary.*

Illustration :

It $\dfrac{z_1-2z_2}{2-z_1\bar{z}_2}$ *is unimodulus and* z_2 *is not unimodulus, then find* $|z_1|$.

Sol. $\left|\dfrac{z_1-2z_2}{2-z_1z_2}\right| = 1 \quad \Rightarrow \quad |z_1 - 2z_2|^2 = |2 - z\bar{z}_2|^2$

$\Rightarrow \quad (z_1 - 2z_2)(\bar{z}_1 - 2\bar{z}_2) = (2 - \bar{z}_1 z_2)(2 - z_1\bar{z}_2)$

$\Rightarrow \quad z_1\bar{z}_1 - 2\bar{z}z_2 - 2z_1\bar{z}_2 + 4z_2\bar{z}_2 = 4 - 2\bar{z}_1 z_2 - 2z_1\bar{z}_2 + z_1\bar{z}_1\, z_2\bar{z}_2$

$\Rightarrow \quad |z_1|^2 + 4|z_2|^2 = 4 + |z_1|^2\,|z_2|^2 \quad \Rightarrow \quad (|z_2|^2 - 1)(|z_1|^2 - 4) = 0$

$\Rightarrow \quad |z_1| = 2$ *or* $|z_2| \neq 1$. ***Ans.***

Illustration :

It is given that complex numbers z_1 *and* z_2 *satisfy* $|z_1| = 2$ *and* $|z_2| = 3$. *If the included angle of their corresponding vectors is 60° then* $\left|\dfrac{z_1+z_2}{z_1-z_2}\right|$ *can be expressed as* $\dfrac{\sqrt{N}}{7}$ *where N is natural number then N equals*

(A) 126 *(B) 119* *(C) 133* *(D) 19*

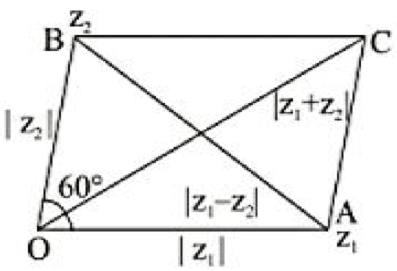
Sol. *Using cosine rule*

$$|z_1 + z_2| = \sqrt{|z_1|^2 + |z_2|^2 - 2|z_1||z_2|\cos 120^\circ}$$

$$= \sqrt{4 + 9 + 2\cdot 3} = \sqrt{19}$$

and $$|z_1 - z_2| = \sqrt{|z_1|^2 + |z_2|^2 - 2|z_1||z_1|\cos 60^\circ}$$

$$= \sqrt{4+9-6} = \sqrt{7}$$



$$\therefore \quad \left|\frac{z_1+z_2}{z_1-z_2}\right| = \sqrt{\frac{19}{7}} = \frac{\sqrt{133}}{7} \Rightarrow N = 133 \ \textbf{Ans.}$$

Illustration :

Let $z_1, z_2 \in C$ be complex numbers such that $|z_1 + z_2| = \sqrt{3}$ and $|z_1| = |z_2| = 1$. Compute $|z_1 - z_2|$.

Sol. $|z_1 + z_2| = \sqrt{3}$ and $|z_1| = |z_2| = 1$

By squaring

$(|z_1 + z_2|)^2 = (z_1 + z_2)(\bar{z}_1 + \bar{z}_2) = 3$

$= |z_1|^2 + |z_2|^2 + 2Re(z_1\bar{z}_2) = 3$

or $\quad 2Re(z_1\bar{z}_2) = 1$

$\therefore \quad |z_1 - z_2|^2 = (z_1 - z_2)(\bar{z}_1 - \bar{z}_2)$

$= |z_1|^2 + |z_2|^2 - 2Re(z_1\bar{z}_2) = 1 + 1 - 1$

$|z_1 - z_2|^2 = 1$

$\therefore |z_1 - z_2| = 1$

Illustration :

If $z = \dfrac{1 + i\sqrt{3}}{2i\left(\cos\dfrac{\pi}{3} + i\sin\dfrac{\pi}{3}\right)}$ *then find* $|z|$ *& amp (z)*

Sol. $z = \dfrac{1 + i\sqrt{3}}{(-\sqrt{3} + i)} \quad \Rightarrow \quad |z| = \dfrac{|1 + i\sqrt{3}|}{|-\sqrt{3} + i|} = 1$

$amp\ z = amp\left(1 + i\sqrt{3}\right) - amp\left(-\sqrt{3} + i\right) \quad \Rightarrow \quad \dfrac{\pi}{3} - \dfrac{5\pi}{6} \Rightarrow \dfrac{-\pi}{2}$. ***Ans.***

Practice Problem

Q.1 If $\dfrac{z-1}{z+1}$ is purely imaginary, then prove that $|z| = 1$.

Q.2 Prove that $z = x + iy$ which satisfy the equation $\left|\dfrac{z-5i}{z+5i}\right| = 1$ lie on the axis of x.

Q.3 If $\arg z = \dfrac{\pi}{4}$ and $|z + 3 - i| = 4$ then find z.

Q.4 If z is any complex number such that $|z + 4| \le 3$, then the least value and greatest value of $|z + 1|$ are.

Q.5 If $z\left(2 - 2\sqrt{3}i\right)^2 = i\left(\sqrt{3} + i\right)^4$, then find the principal argument of z.

Q.6 If $\arg(z) = 50°$ then find principal argument of (z^{100}).

Q.7 Show that all the roots of the equation $z^n \cos\theta_0 + z^{n-1}\cos\theta_1 + \ldots\ldots + \cos\theta_n = 2$ where $\theta_0, \theta_1, \theta_2 \ldots\ldots\theta_n \in R$ lies out side the circle $|z| = \frac{1}{2}$

Q.8 Find z if it satisfies $\text{amp}(z+2) = \frac{\pi}{4}$ and $\text{amp}(z-3+2i) = \frac{3\pi}{4}$.

Q.9 If $\text{amp}\left(\frac{(8+i)(7+i)}{5-i}\right)^6 = 6\tan^{-1}\left(\frac{1}{\lambda}\right)$ then find λ.

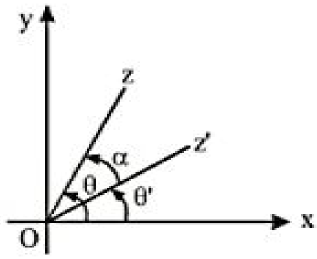
Q.10 If $|z_1| = |z_2| = 1$ and $z_1z_2 \neq -1$ then prove that $\frac{z_1+z_2}{1+z_1z_2}$ is a real number.

Answer key

Q.3	$z = 1 + i$	Q.4	0, 5	Q.5	$-\frac{\pi}{6}$	Q.6	$-40°$
Q.8	$-\frac{1}{2} + i\frac{3}{2}$	Q.9	2				

6. CONCEPT OF ROTATION :

If z and z' are two complex numbers then argument of $\frac{z}{z'}$ is the angle through which Oz' must be turned in order that it may lie along Oz.

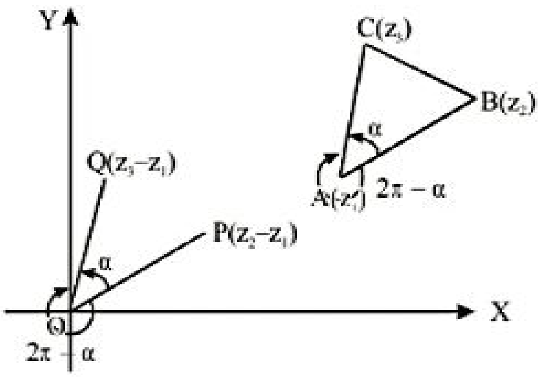


$$\frac{z}{z'} = \frac{|z|\,e^{i\theta}}{|z'|\,e^{i\theta'}} = \frac{|z|}{|z'|}e^{i(\theta-\theta')} = \frac{|z|}{|z'|}e^{i\alpha}$$

In general, let z_1, z_2, z_3 be the three vertices of a triangle ABC described in the counter-clock wise sense. Draw OP and OQ parallel and equal to AB and AC respectively. Then the point P is $z_2 - z_1$ and Q is $z_3 - z_1$ and

$$\frac{z_3 - z_1}{z_2 - z_1} = \frac{OQ}{OP}(\cos\alpha + i\sin\alpha)$$

$$= \frac{CA}{BA}\cdot e^{i\alpha} = \frac{|z_3 - z_1|}{|z_2 - z_1|}\cdot e^{i\alpha}$$

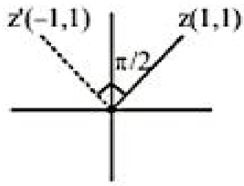


Note that arg. $(z_3 - z_1) - \arg(z_2 - z_1) = \alpha$ is the angle through which OP must be rotated in the anti-clockwise direction so that it concides with OQ.

Here we can write $\frac{z_3 - z_1}{z_2 - z_1} = \frac{|z_3 - z_1|}{|z_2 - z_1|} \cdot e^{-i(2\pi - \alpha)}$ also. In case we are rotating OP in clockwise direction by an angle $(2\pi - \alpha)$. Since the rotation is in clockwise direction, we are taking negative sign with angle $(2\pi - \alpha)$.

Note :

If a complex number (z) is mutliplied by i, it means z has been rotated through an angle $\frac{\pi}{2}$ in anticlockwise sense.



e.g $\quad z = 1 + i$

$\quad z' = (1 + i)\, e^{i\pi/2} = (1 + i)\, i = -1 + i.$

Illustration :

If area of triangle whose sides are reprsented by z, iz and z + iz is 200 square units then | z | is

Sol. $z = x + iy\ (x, y)$

$iz = ix + i^2y\ (-y, x)$

$\frac{1}{2}|z||iz| = 200$

$|z|^2 = 400$

$|z| = 20.$ ***Ans.***

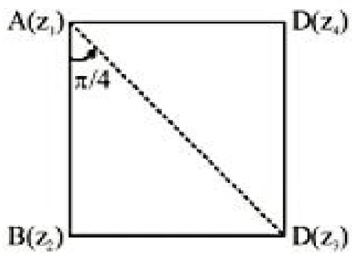
Illustration :

Consider a square ABCD such that z_1, z_2, z_3 and z_4 represent its vertices A, B, C and D respectively. Express 'z_3' and 'z_4' in terms of z_1 and z_2.

Sol. *Consider the rotation of AB about A through an angle $\frac{\pi}{4}$. We get*

$$\frac{z_3 - z_1}{z_2 - z_1} = \left|\frac{z_3 - z_1}{z_2 - z_1}\right| e^{i\pi/4}$$

$$= \sqrt{2}\left(\cos\frac{\pi}{4} + i \sin\frac{\pi}{4}\right) \Rightarrow z_3 = z_1 + (z_2 - z_1)(1 + i)$$



Similarly, $\quad \frac{z_4 - z_1}{z_2 - z_1} = \left|\frac{z_4 - z_1}{z_2 - z_1}\right| \left|\frac{z_4 - z_1}{z_2 - z_1}\right| e^{i\pi/2} = i$

$\Rightarrow \quad z_4 = z_1 + i\,(z_2 - z_1).$

Illustration :

If z_1, z_2, z_3 are the vertices of an equilateral triangle then prove that

$$z_1^2+z_2^2+z_3^2 = z_1z_2 + z_2z_3 + z_3z_1$$

and if z_0 is the circumcentre of the triangle then also prove that $3z_0^2 = z_1^2+z_2^2+z_3^2$.

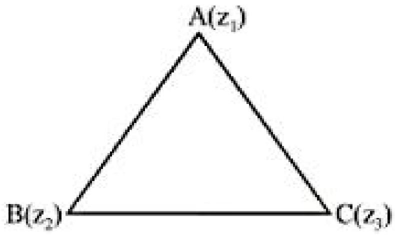
Sol. *Since ΔABC is equilateral*

$\therefore AB = BC = CA$

$|z_1-z_2| = |z_2-z_3| = |z_3-z_1|$ ... (1)

also $\angle CBA = \angle ACB = \alpha$

Rotate BC about B (z_2) in anticlockwise sense then



$$\Rightarrow \quad \frac{z_1-z_2}{z_3-z_2} = \frac{|z_1-z_2|}{|z_3-z_2|}\cdot e^{i\alpha} \quad ... (2)$$

again rotate side AB about A(z_1) in anticlockwise sense then

$$\Rightarrow \quad \frac{z_3-z_1}{|z_3-z_1|} = \frac{z_2-z_1}{|z_2-z_1|}\cdot e^{i\alpha} \quad ... (3)$$

By equation (2) and (3)

$$\frac{z_1-z_2}{z_3-z_2} = \frac{z_3-z_1}{z_2-z_1}$$

$\Rightarrow \quad z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_2z_3 + z_3z_1$ ...(4)

and if z_0 is the circumcentre for equilateral triangle

$$z_0 = \frac{z_1+z_2+z_3}{3} \quad ... (5)$$

or by equation (5)

$$9z_0^2 = z_1^2 + z_2^2 + z_3^2 + 2(z_1z_2 + z_2z_3 + z_3z_1)$$

or by equaiton (4)

$$z_1^2 + z_2^2 + z_3^2 = 3z_0^2$$

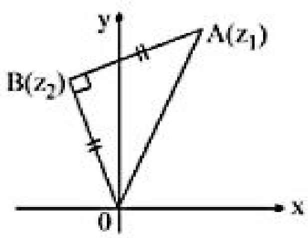
Illustration :

If z_1 & z_2 be non zero complex numbers satisfying the equation, $z_1^2 - 2z_1z_2 + 2z_2^2 = 0$ then find the geometrical nature of the triangle whose vertices are the origin and the points representing z_1 & z_2

(A) an isosceles right angled triangle *(B) a right angled triangle which is not isosceles*

(C) an equilateral triangle *(D) an isosceles triangle which is not right angled.*

Sol. $\frac{z_1}{z_2} = z \Rightarrow z^2 - 2z + 2 = 0 \Rightarrow z = 1 \pm i$



$\Rightarrow \quad \frac{z_1}{z_2} = 1 \pm i \Rightarrow z_1 = z_2 \pm z_2 i \Rightarrow z_1 - z_2 = \pm z_2 i$

$\Rightarrow \quad z_1 - z_2$ *is perpendicular to z_2 and* $|z_1 - z_2| = |z_2|$

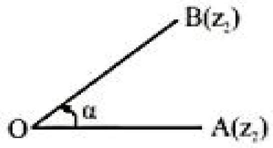
Illustration :

Let z_1 and z_2 be the roots of the equation $z^2 + pz + q = 0$, where the coefficients p and q may be complex number. Let A and B represent z_1 and z_2 in the complex plane. If $\angle AOB = \alpha \neq 0$ and $OA = OB$, where O is the origin, prove that $p^2 = 4q\cos^2\left(\frac{\alpha}{2}\right)$.

Sol. $z_1 + z_2 = -p$ ....(1)

and $z_1 z_2 = q$ ....(2)

Also $z_2 = z_1 e^{i\alpha}$ ....(3)



Now $p^2 = (z_1 + z_2)^2 = (z_1 + z_1 e^{i\alpha})^2$

$$= z_1^2 (1 + 2e^{i\alpha} + e^{i2\alpha})$$

$$= qe^{-i\alpha}(1 + 2e^{i\alpha} + e^{i2\alpha}) = q(2 + e^{i\alpha} + e^{i\alpha})$$

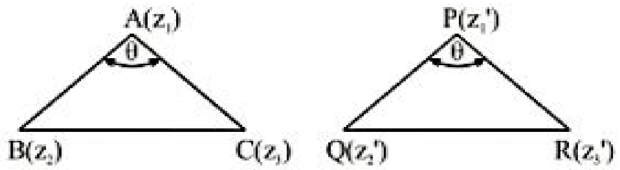
$$= q(2 + 2\cos\alpha) = 4q\cos^2\frac{\alpha}{2}.$$

Illustration :

If z_1, z_2, z_3 and z_1', z_2', z_3' represent the vertices of two similar triangles ABC and PQR, respectively then prove that

$$\left|\frac{\bar{z}_1'}{\bar{z}_2 - \bar{z}_1}\right|\left|\frac{z_2 - z_3}{z_3'}\right| + \left|\frac{z_2'}{z_2 - z_1}\right|\left|\frac{\bar{z}_3 - \bar{z}_1}{\bar{z}_3'}\right| \geq 1$$

Sol.



Since ΔABC and ΔPQR are similar, $\frac{AC}{AB} = \frac{PR}{PQ}$ and $\angle BAC = \angle QPR = \theta$.

In ΔABC, $\frac{z_3 - z_1}{z_2 - z_1} = \frac{AC}{AB}e^{i\theta}$ ......(1)

In ΔPQR, $\frac{z_3' - z_1'}{z_2' - z_1'} = \frac{PR}{PQ}e^{i\theta}$ .....(2)

From (1) and (2), $\dfrac{z_3 - z_1}{z_2 - z_1} = \dfrac{z_3' - z_1'}{z_2' - z_1'}$

$\Rightarrow \quad z_1'(z_2 - z_3) + z_2'(z_3 - z_1) = z_3'(z_2 - z_1)$

$\Rightarrow \quad |z_1'(z_2 - z_3) + z_2'(z_3 - z_1)| = |z_3'(z_2 - z_1)|$

$\Rightarrow \quad |z_1'(z_2 - z_3)| + |z_2'(z_3 - z_1)| \geq |z_3'(z_2 - z_1)|$

$$\Rightarrow \quad \left|\frac{z_1'}{z_2 - z_1} \cdot \frac{z_2 - z_3}{z_3'}\right| + \left|\frac{z_2'}{z_2 - z_1}\right|\left|\frac{z_3 - z_1}{z_3'}\right| \geq 1$$

$$\Rightarrow \quad \left|\frac{z_1'}{z_2 - z_1}\right|\left|\frac{z_2 - z_3}{z_3'}\right| + \left|\frac{z_2'}{z_2 - z_1}\right|\left|\frac{z_3 - z_1}{z_3'}\right| \geq 1$$

$$\Rightarrow \quad \left|\frac{\bar{z}_1'}{\bar{z}_2 - \bar{z}_1}\right|\left|\frac{z_2 - z_3}{z_3'}\right| + \left|\frac{z_2'}{z_2 - z_1}\right|\left|\frac{\bar{z}_3 - \bar{z}_1}{\bar{z}_3'}\right| \geq 1.$$

Practice Problem

Q.1 Complex number z_1, z_2, z_3 are the vertices A_1, B_1 C respectively of an isosceles right angle triangle with right angle at C. Show that $(z_1 - z_2)^2 = 2(z_1 - z_3)(z_3 - z_2)$

Q.2 Find the centre of the arc represented by $\arg\left[\dfrac{(z-3i)}{(z-2i+4)}\right] = \dfrac{\pi}{4}$.

Q.3 If z_r (r = 1, 2, ……, 6) are vertices of a regular hexagon and $\sum_{r=1}^{6} z_r^2 = kz_0^2$ then find the value of k.

Q.4 If z_1, z_2, z_3 are collinear then prove that $z_1|z_2 - z_3| - z_2|z_3 - z_1| + z_3|z_1 - z_2| = 0$.

Q.5 P is a point on the arranged diagram. On the circle with OP an diameter, two points Q & R are taken such that $\angle OOQ = \angle QOR = \theta$, if O is the origin and P, Q and R are respresented by the complex number z_1, z_2 and z_3 respectively show that $z_2^2 \cos 2\theta = z_1 z_3 \cos^2 \theta$.

Answer key

Q.2 $\left(\dfrac{9i-5}{2}\right)$ Q.3 $k = 6$

7. DEMOIVRE'S THEOREM (DMT) :

Case-I :

Statement :
If n is any integer then
(i) $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$
(ii) $(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)(\cos\theta_3 + i\sin\theta_2)(\cos\theta_3 + i\sin\theta_3)\ldots\ldots(\cos\theta_n + i\sin\theta_n)$
$= \cos(\theta_1 + \theta_2 + \theta_3 + \ldots\ldots \theta_n) + i\sin(\theta_1 + \theta_2 + \theta_3 + \ldots\ldots \theta_n)$

Case-II :

Statement : If $p, q \in Z$ and $q \neq 0$ then

$$(\cos\theta + i\sin\theta)^{p/q} = \cos\left(\frac{2k\pi + p\theta}{q}\right) + i\sin\left(\frac{2k\pi + p\theta}{q}\right)$$

where $k = 0, 1, 2, 3, \ldots\ldots, q-1$

Note : When index 'n' is integer then $(\cos\theta + i\sin\theta)^n$ has exactly one value which is $\cos n\theta + i\sin n\theta$ but when n is rational number (say $p/q, q \neq 0$) other than integer then $(\cos\theta + i\sin\theta)^{p/q}$ has exactly q different values.

Illustration :

Prove that $\left(\dfrac{1+\sin\theta + i\cos\theta}{1+\sin\theta - i\cos\theta}\right)^n = \cos n\left(\dfrac{\pi}{2} - \theta\right) + i\sin n\left(\dfrac{\pi}{2} - \theta\right)$.

Sol. $\dfrac{(1+\sin\theta) + i\cos\theta}{(1+\sin\theta) - i\cos\theta}$

$$\Rightarrow \left[\frac{1+\cos\left(\frac{\pi}{2}-\theta\right) + i\sin\left(\frac{\pi}{2}-\theta\right)}{1+\cos\left(\frac{\pi}{2}-\theta\right) - i\sin\left(\frac{\pi}{2}-\theta\right)}\right]^n \Rightarrow \left(\frac{1+\cos\alpha + i\sin\alpha}{1+\cos\alpha - i\sin\alpha}\right)^n \text{ where } \alpha = \frac{\pi}{2} - \theta$$

$$\Rightarrow \left(\frac{2\cos^2\frac{\alpha}{2} + i2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}}{2\cos^2\frac{\alpha}{2} - i\,2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}}\right)^n = \left(\frac{\cos\frac{\alpha}{2} + i\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2} - i\sin\frac{\alpha}{2}}\right)^n \Rightarrow \left(\frac{e^{i\alpha/2}}{e^{-i\alpha/2}}\right)^n \Rightarrow (e^{i\alpha})^n = e^{in\alpha}$$

$$\Rightarrow \cos n\left(\frac{\pi}{2} - \theta\right) + i\sin n\left(\frac{\pi}{2} - \theta\right).$$

Aliter : Let $\sin\theta + i\cos\theta = z$

$$\left(\frac{1+z}{1+\bar{z}}\right)^n \Rightarrow \left(\frac{1+z}{1+\frac{1}{z}}\right)^n, \ \bar{z} = \frac{1}{z} \Rightarrow \quad z^n \Rightarrow (\sin\theta + i\cos\theta)^n$$

$$\Rightarrow \quad \left(\cos\left(\frac{\pi}{2}-\theta\right)+i\sin\left(\frac{\pi}{2}-\theta\right)\right)^n \Rightarrow \quad \cos n\left(\frac{\pi}{2}-\theta\right)+i\sin n\left(\frac{\pi}{2}-\theta\right).$$

Illustration :

If $z = \left(1+i\sqrt{3}\right)^6 + \left(1-i\sqrt{3}\right)^6$ the find $|z|$.

Sol. $1 + i\sqrt{3} = 2\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)$

$$1 - i\sqrt{3} = 2\left(\cos\frac{\pi}{3}-i\sin\frac{\pi}{3}\right)$$

$$\therefore\ z = \left(1+i\sqrt{3}\right)^6 + \left(1-i\sqrt{3}\right)^6 = 2^6\left[\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)^6 + \left(\cos\frac{\pi}{3}-i\sin\frac{\pi}{3}\right)^6\right]$$

$$= 2^6\,[\cos 2\pi + i\sin 2\pi + \cos 2\pi - i\sin 2\pi] = 2^7$$

$$\therefore\ |z| = 2^7$$

Application of Demoivre's Theorem :

To find roots of a complex quantity is the main application of DMT.
Working rule for finding roots of a complex quantity.

Illustration :

Find all the roots of the equation $z^4 - (1 + i) = 0$.

Sol. $z^4 = 1 + i \quad \Rightarrow \quad z = (1 + i)^{1/4}$

Step-1 : *Express* $1 + i$ *in the polar form*

$$z = \left(\sqrt{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)\right)^{1/4} = 2^{1/8}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)^{1/4}$$

Step-2 : *Add* $2m\pi$ *in the principal argument.*

$$z = 2^{1/8}\left(\cos\left(2m\pi+\frac{\pi}{4}\right)+i\sin\left(2m\pi+\frac{\pi}{4}\right)\right)^{1/4}$$

Step-3 : *Apply DMT*

$$z = z^{1/8}\left(\cos\left(2m\pi + \frac{\pi}{4}\right)\cdot\frac{1}{4} + i\sin\left(2m\pi + \frac{\pi}{4}\right)\cdot\frac{1}{4}\right)$$

Step-4 : *Put m = 0, 1, 2, 3 to get all four roots of the equation*

$$m = 0,\ z_1 = 2^{1/8}\left(\cos\frac{\pi}{16} + i\sin\frac{\pi}{16}\right), \qquad m = 1,\ z_2 = 2^{1/8}\left(\cos\frac{9\pi}{16} + i\sin\frac{9\pi}{16}\right)$$

$$m = 2,\ z_3 = 2^{1/8}\left(\cos\frac{17\pi}{16} + i\sin\frac{17\pi}{16}\right), \qquad m = 3,\ z_4 = 2^{1/8}\left(\cos\frac{25\pi}{16} + i\sin\frac{25\pi}{16}\right).$$

7.1 Cube Roots of unity :

$$z^3 - 1 = 0 \Rightarrow z = (1)^{1/3} = (\cos 0 + i\sin 0)^{1/3} = (\cos 2m\pi + i\sin 2m\pi)^{1/3}$$

$$= \cos\frac{2m\pi}{3} + i\sin\frac{2m\pi}{3},\ m = 0, 1, 2$$

$$m = 0,\ z_1 = \cos 0 + i\sin 0 = 1$$

$$m = 1,\ z_2 = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} = \frac{-1}{2} + i\frac{\sqrt{3}}{2} = \omega$$

$$m = 2,\ z_3 = \cos\frac{4\pi}{3}, + i\sin\frac{4\pi}{3} = \frac{-1}{2} - i\frac{\sqrt{3}}{2} = \omega^2$$

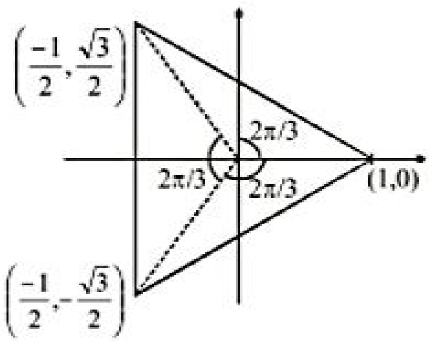
Note :

(i) $\omega^3 = 1$

(ii) $\omega^{3n} = 1, n \in I$

(iii) $1 + \omega + \omega^2 = 0$

(iv) $1 + \omega^r + \omega^{2r} = \begin{cases} 0 \text{ if r is not a multiple of 3.} \\ 3 \text{ if r is a multiple of 3.} \end{cases}$



(v) Representation of cube roots of unity on argand plane. Cube roots of unity form an equilateral Δ whose side is $\sqrt{3}$ units.

(vi) Some important facts

- $a^3 + b^3 = (a + b)(a^2 - ab + b^2) = (a + b)(a^2 + (\omega + \omega^2)ab) + b^2)$
  $= (a + b)[a^2 + ab\omega + ab\omega^2 + b^2\omega^3] = (a + b)[a(a + b\omega) + bw^2(a + b\omega)]$
  $\Rightarrow (a + b)(a + b\omega^2)(a + b\omega)$
- $a^3 - b^3 = (a - b)(a - b\omega^2)(a - b\omega)$
- $a^3 + b^3 + c^3 - 3abc = (a + b + c)(a + b\omega + c\omega^2)(a + b\omega^2 + c\omega)$

Illustration :

Find the sum of series $1\cdot(2-\omega)(2-\omega^2)+2\cdot(3-\omega)(3-\omega^2)+\ldots\ldots+(n-1)(n-\omega)(n-\omega^2)$ *where* ω *is one of the imaginary cube root of unity.*

Sol. $T_n = (n-1)(n-\omega)(n-\omega^2)$
$= (n-1)(n^2-n(\omega+\omega^2)+\omega^3)$
$= (n-1)(n^2+n+1)$
$T_n = n^3-1$

$$\therefore S_n = \sum_1^n T_n = \sum n^3 - \sum 1 = \left(\frac{n(n+1)}{2}\right)^2 - n \quad \textit{Ans.}$$

Illustration :

If $x = 1-i\sqrt{3}$, $y = 1+i\sqrt{3}$ *and* $z = 2$ *then prove that* $x^p + y^p = z^p$ *where P is a prime number > 3.*

Sol. $x = \left(1-i\sqrt{3}\right) = -2\left(\dfrac{-1+i\sqrt{3}}{2}\right) = -2\omega$

$\therefore x^p = (-2\omega)^p = -(2)^p\,\omega^p$

$y = 1+i\sqrt{3} = -2\left(\dfrac{-1-i\sqrt{3}}{2}\right) = -2\,\omega^2$

$\therefore y^p = (-2\omega^2)^p = -(2)^p\,\omega^{2p}$
$x^p + y^p = -2^p(\omega^p+\omega^{2p}) = -(2)^p(-1) = 2^p = z^p$

Illustration :

If $(a+\omega)^{-1}+(b+\omega)^{-1}+(c+\omega)^{-1} = 2\omega^{-1}$ *and* $(a+\omega^2)^{-1}+(b+\omega^2)^{-1}+(c+\omega^2)^{-1} = 2\omega^{-2}$ *where* ω *is the complex cube root of unity then show that* $(a+1)^{-1}+(b+1)^{-1}+(c+1)^{-1} = 2$. $a, b, c \in R$.

Sol. *Let* $\dfrac{1}{a+x}+\dfrac{1}{b+x}+\dfrac{1}{c+x} = \dfrac{2}{x}$

then $\dfrac{1}{a+x}+\dfrac{1}{b+x} = \dfrac{2}{x}-\dfrac{1}{c+x}$

$\Rightarrow$ $x^3 - x(ac+bc+ab) - 2abc = 0$ *(Let the roots of the euqation are* ω_1, ω_2 *and* α*)*
If equation has three roots then
sum of roots = 0
$\therefore$ $\omega+\omega^2+\alpha = 0$ *or* $\alpha = 1$

$\therefore$ $\dfrac{1}{a+1}+\dfrac{1}{b+1}+\dfrac{1}{c+1} = \dfrac{2}{1}$

Illustration :

Let z_1 and z_2 be the roots of the equation $x^2 - x + 1 = 0$. Compute
(i) $x_1^{2000} + x_2^{2000}$ (ii) $x_1^{1999} + x_2^{1999}$ (iii) $x_1^n + x_2^n$ for all $n \in N$.

Sol. *For equation $x^2 - x + 1 = 0$*
solutions are $-\omega$ and $-\omega^2$
$x_1 = -\omega$ *and* $x_2 = -\omega^2$
(i) $x_1^{2000} + x_2^{2000} = (-\omega)^{2000} + (-\omega^2)^{2000}$
$\Rightarrow \omega^2 + \omega = -1$
(ii) $x_1^{1999} + x_2^{1999} = (-\omega)^{1999} + (-\omega^2)^{1999}$
$\Rightarrow -(\omega + \omega^2) = +1$
(iii) $x_1^n + x_2^n = (-\omega)^n + (-\omega^2)^n,\ n \in N$
$n = 3\lambda \Rightarrow -1 - 1 = -2$
$n = 3\lambda + 1 \Rightarrow \omega + \omega^2 = -1$
$n = 3\lambda + 2 \Rightarrow \omega^2 + \omega = -1$ *or* 1

7.2 n, nth root of unity :

$z^n - 1 = 0 \quad \Rightarrow \quad z = (1)^{1/n} = (\cos 0 + i\sin 0)^{1/n} = (\cos 2m\pi + i\sin 2m\pi)^{1/n}$

$$\Rightarrow \quad \cos\left(\frac{2m\pi}{n}\right) + i\sin\left(\frac{2m\pi}{n}\right), \quad m = 0, 1, 2, 3, \ldots\ldots, (n-1)$$

$m = 0,\ z_1 = 1$

$$m = 1,\ z_2 = \cos\left(\frac{2\pi}{n}\right) + i\sin\left(\frac{2\pi}{n}\right) = e^{i\frac{2\pi}{n}} = \alpha \quad \text{(Let)}$$

$$m = 2,\ z_3 = \cos\left(\frac{4\pi}{n}\right) + i\sin\left(\frac{4\pi}{n}\right) = e^{i\frac{4\pi}{n}} = \alpha^2$$

$\vdots$

$\vdots$

$$m = n-1,\ z_n = \cos\ \cos\left(\frac{2(n-1)\pi}{n}\right) + i\sin\left(\frac{2(n-1)\pi}{n}\right) = e^{\frac{i2(n-1)\pi}{n}} = \alpha^{n-1}.$$

(1) $1, \alpha, \alpha^2, \alpha^3 + \ldots\ldots, \alpha^{n-1}$ are n, nth roots of unity which are in G.P. with common ratio α where $\alpha = e^{\frac{i2\pi}{n}}$.

(2) Sum of n, nth roots of unity is awlays zero
$1 + \alpha + \alpha^2 + \alpha^3 + \ldots\ldots\ \alpha^{n-1} = 0$

Proof : $S = \frac{1-\alpha^n}{1-\alpha}$

$$\alpha = e^{\frac{i2\pi}{n}} \Rightarrow \alpha^n = \left(e^{\frac{i2\pi}{n}}\right)^n = e^{i\,2\pi} = 1$$

$$\therefore S = \frac{1-1}{1-\alpha} = 0$$

(3) Sum of p^{th} powers of n, n^{th} roots of unity $= \begin{cases} n, \text{ if } p \text{ is an integral multiple of } n \\ 0, \text{ if } p \text{ is not an integral multiple of } n \end{cases}$

Proof : $1^p + (\alpha)^p + (\alpha^2)^p + (\alpha^3)^p + \ldots\ldots + (\alpha^{n-1})^p$

Case-I: When 'p' is an integral multiple of n, $p = n\lambda, \lambda \in I$

$$\alpha^p = \left(e^{\frac{i2\pi}{n}}\right)^p = e^{\frac{i2\pi}{n}p} = e^{i2\lambda\pi} = 1$$

$\alpha^{2p} = 1, \alpha^{3p} = 1, \ldots\ldots\ldots, \alpha^{(n-1)p} = 1$

$\therefore \quad 1 + \alpha^p + \alpha^{2p} + \alpha^{3p} + \ldots\ldots + \alpha^{(n-1)p}$

$\Rightarrow \quad 1 + 1 + 1 + 1 + \ldots\ldots + 1 = n$

Case-2 : When p is not an integral multiple of n :

$\Rightarrow \quad 1 + \alpha^p + \alpha^{2p} + \alpha^{3p} + \ldots\ldots\ldots + \alpha^{(n-1)p}$

$$= \frac{1-(\alpha^p)^n}{1-\alpha^p} \Rightarrow \frac{1-1}{1-\alpha^p} = 0 \qquad \{\because \alpha^{np} = 1\}$$

Illustration :

If $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ *and* α_5 *are 5, 5^{th} roots of unity then find the value of*

(i) $\alpha_1^4 + \alpha_2^4 + \alpha_3^4 + \alpha_4^4 + \alpha_5^4 = 0$

(ii) $\alpha_1^{20} + \alpha_2^{20} + \alpha_3^{20} + \alpha_4^{20} + \alpha_5^{20} = 5$

Sol.

(i) $\because$ *4 is not a multiple of 5.*

$$\therefore \quad \sum_{i=1}^{5} \alpha_i^4 = 0$$

(ii) *20 is a multiple of 5.*

$$\therefore \quad \sum_{i=1}^{5} \alpha_i^{20} = 5$$

Illustration :

Evaluate : $\sum_{\lambda=1}^{12}\left(\sin\frac{2\pi\lambda}{13}-i\cos\frac{2\pi\lambda}{13}\right)$

Sol. $\sum_{\lambda=1}^{12}\left(\sin\frac{2\pi\lambda}{13}-i\cos\frac{2\pi\lambda}{13}\right) \Rightarrow (-i)\sum_{\lambda=1}^{12}\left(\cos\frac{2\pi\lambda}{13}+i\sin\frac{2\pi\lambda}{13}\right)$

$$\Rightarrow \quad (-i)\sum_{\lambda=1}^{12} e^{\frac{i2\pi\lambda}{13}} \Rightarrow (-i)\left[1+e^{\frac{i2\pi}{3}}+e^{\frac{i4\pi}{13}}+e^{\frac{i6\pi}{13}}+.....+e^{\frac{i24\pi}{13}}-1\right]$$

$$\Rightarrow \quad (-i)(-1) = i. \ \textbf{Ans.}$$

Illustration :

If $\alpha_1, \alpha_2, \alpha_3,.........,\alpha_{n-1}$ are non- real n^{th} roots of unity then prove that

(i) $(1-\alpha_1)(1-\alpha_2)(1-\alpha_3).........(1-\alpha_{n-1}) = n$

(ii) $(1+\alpha_1)(1+\alpha_2)(1+\alpha_3)+......+(1+\alpha_{n-1}) = \begin{cases} 0, \text{if } n \text{ is even} \\ 1, \text{if } n \text{ is odd}\end{cases}$

Sol.

(i) $z^{n-1} = (z-1)(z-\alpha_1)(z-\alpha_2)(z-\alpha_3)......(z-\alpha_{n-1})$

$$\frac{z^n-1}{z-1} = (z-\alpha_1)(z-\alpha_2)(z-\alpha_3)........(z-\alpha_{n-1})$$

$1+z+z^2+........z^{n-1} = (z-\alpha_1)(z-\alpha_2).......(z-\alpha_{n-1})$

Put $z = 1$

$$\Rightarrow \quad \underbrace{1+1+1+...........1}_{n \text{ times}} = (1-\alpha_1)(1-\alpha_2).........(1-\alpha_{n-1})$$

Aliternative method :

$$\Rightarrow \quad (1-\alpha_1)(1-\alpha_2).........(1-\alpha_{n-1}) = n$$

$$\lim_{z\to 1}\frac{z^n-1}{z-1} = (1-\alpha_1)(1-\alpha_2).......(1-\alpha_{n-1})$$

$$\Rightarrow \quad \lim_{z\to 1}\frac{nz^n-1}{1} = (1-\alpha_1)(1-\alpha_2)........(1-\alpha_{n-1})$$

$$\Rightarrow \quad (1-\alpha_1)(1-\alpha_2).........(1-\alpha_{n-1}) = n$$

(ii) $\dfrac{z^n - 1}{z - 1} = (z - \alpha_1)(z - \alpha_2)(z - \alpha_3) \ldots\ldots (z - \alpha_{n-1})$

Put $z = -1$

$$(-1 - \alpha_1)(-1 - \alpha_2)(-1 - \alpha_3) \ldots\ldots (-1 - \alpha_{n-1}) = \frac{(-1)^n - 1}{-1 - 1}$$

$$\Rightarrow \quad (-1)^{n-1} [(1 + \alpha_n)(1 + \alpha_2) + \ldots\ldots (1 + \alpha_{n-1})] = \frac{(-1)^n - 1}{-2}$$

$$\Rightarrow \quad (1 + \alpha_1)(1 + \alpha_2) \ldots\ldots (1 + \alpha_{n-1}) = \frac{(-1)^n - 1}{-2} \cdot \frac{1}{(-1)^{n-1}} \Rightarrow \begin{bmatrix} 0, \text{even} \\ 1, \text{odd} \end{bmatrix}$$

Illustration :

If z_1, z_2, z_3, z_4 & z_5 are roots of the equation $z^5 + z^4 + z^3 + z^2 + z + 1 = 0$ then find them also evaluate

(i) $\sum_{i=1}^{5} z_i$ *(ii)* $\sum_{i=1}^{5} z_i^4$ *(iii)* $\sum_{i=1}^{5} z_i^{12}$ *(iv)* $\prod_{i=1}^{5} (2 - z_i)$

Sol. $(z^5 + z^4 + z^3 + z^3 + z^2 + z + 1)(z - 1) = 0$

$$\Rightarrow z^6 - 1 = 0 \Rightarrow z = (1)^{1/6} \Rightarrow \quad z = e^{\frac{i\,2m\pi}{6}}, \quad m = 0, 1, 2, \ldots\ldots, 5$$

Let roots of the equation are $1, z_1, z_2, z_3 \ldots\ldots z_5$.

$$z_1 = e^{\frac{i2\pi}{6}} = e^{\frac{i\pi}{3}}, \quad z_2 = e^{\frac{i4\pi}{6}} = e^{\frac{i2\pi}{3}}, \quad z_3 = e^{\frac{i6\pi}{6}} = e^{i\pi}, \quad z_4 = e^{\frac{i8\pi}{6}} = e^{\frac{i4\pi}{3}}, \quad z_5 = e^{\frac{i\,10\pi}{6}} = e^{\frac{i\,5\pi}{3}}$$

(i) $1 + z_1 + z_2 + z_3 + z_4 + z_5 = 0 \Rightarrow \sum_{i=1}^{5} z_i = -1$

(ii) $1^4 + z_1^4 + \ldots\ldots + z_5^4 = 0 \Rightarrow \sum_{i=1}^{5} z_i^4 = -1$ $\{\because$ 4 is not a mutliple of 6$\}$

(iii) $\sum_{i=1}^{5} z_i^{12}$

$\Rightarrow 1 + z_1^{12} + z_2^{12} + \ldots\ldots + z_5^{12} = 6$ $\{\because$ 12 is multiple of 6$\}$

$\Rightarrow z_1^{12} + z_2^{12} + \ldots\ldots + z_5^{12} = 5$

(iv) $\prod_{i=1}^{5}(2-z_i)$

$z^6 - 1 = (z-1)(z-z_1)(z-z_2)(z-z_3)(z-z_4)(z-z_5)$

Put $z = 2$

$$\frac{2^6-1}{2-1} = (2-z_1)(2-z_2) \ldots\ldots (2-z_5) \Rightarrow \prod_{i=1}^{5}(2-z_i) = 63 \quad \textbf{Ans.}$$

Representation of all the roots of the equation on argand plane :

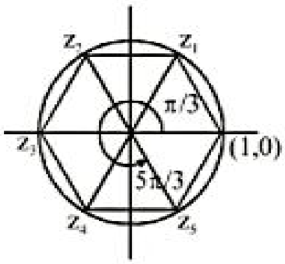


Illustration :

Factorize $z^7 + 1$ *into linear and quadratic factors with real coefficients.*

Sol. $z^7 + 1 = 0$

$\Rightarrow z = (-1)^{1/7} = (\cos\pi + i\sin\pi)^{1/7} = (\cos(2m\pi+\pi) + i\sin(2m\pi+\pi))^{1/7}$

$$= \cos\left(\frac{2m\pi+\pi}{7}\right) + i\sin\left(\frac{2m\pi+\pi}{7}\right) = e^{i\left(\frac{2m\pi+\pi}{7}\right)},\ m = 0, 1, 2, \ldots\ldots, 6$$

$z_1 = e^{i\pi/7}, z_2 = e^{i3\pi/7}, z_3 = e^{i5\pi/7}, z_4 = e^{i7\pi/7}, z_5 = e^{i9\pi/7}, z_6 = e^{i11\pi/7}, z_7 = e^{i13\pi/7}$

$z_7 = e^{i13\pi/7},\ = e^{-i\pi/7} = \bar{z}_1$

$z_6 = e^{i11\pi/7},\ = e^{-3i\pi/7} = \bar{z}_2$

$z_5 = e^{i9\pi/7},\ = e^{-i5\pi/7} = \bar{z}_3$

Now $z^7 + 1 = (z-z_1)(z-z_2) \ldots\ldots (z-z_7)$

$= (z-z_1)(z-z_2)(z-z_3)(z-z_4)(z-\bar{z}_3)(z-\bar{z}_2)(z-\bar{z}_1)$

$= \left(z^2 - z(z_1+\bar{z}_1) + |z_1|^2\right)\left(z^2 - z(z_2+\bar{z}_2) + |z_2|^2\right)\left(z^2 - z(z_3+\bar{z}_3) + |z_3|^2\right)(z-z_4)$

$$= \left(z^2 - z\cdot 2\cos\frac{\pi}{7} + 1\right)\left(z^2 - z\ 2\cos\frac{3\pi}{7} + 1\right)\left(z^2 - z\cdot 2\cos\frac{5\pi}{7} + 1\right)(z+1)$$

$$= (z+1)\left(z^2 - 2\ z\cos\frac{\pi}{7} + 1\right)\left(z^2 - 2\ z\cos\frac{3\pi}{7} + 1\right)\left(z^2 - 2\ z\cos\frac{5\pi}{7} + 1\right)$$

Practice Problem

Q.1 Find the value of the following expression $\left[\dfrac{1-\cos\frac{\pi}{10}+i\sin\frac{\pi}{10}}{1-\cos\frac{\pi}{10}-i\sin\frac{\pi}{10}}\right]^5$.

Q.2 If $x^2-x+1=0$ then the value of $\sum_{n=1}^{5}\left(x^n+\dfrac{1}{x^n}\right)^2$.

Q.3 If α is a non real fifth root of unity, then find the value of $3^{|1+\alpha+\alpha^2+\alpha^{-2}-\alpha^{-1}|}$ is.

Q.4 $z_1, z_2, z_3, \ldots\ldots, z_{n-1}$ are the non real n^{th} roots of unity, then the value of

$$\frac{1}{(3-z_1)}+\frac{1}{(3-z_2)}+\ldots\ldots+\frac{1}{(3-z_{n-1})}$$

Q.5 Find the number of roots of the equation $z^{10}-z^5-992=0$ whose real part is negative.

Q.6 Least positive argument of the 4^{th} root of the complex number $2-i\sqrt{12}$ is

(A) $\pi/6$ (B) $5\pi/12$ (C) $7\pi/12$ (D) $11\pi/12$

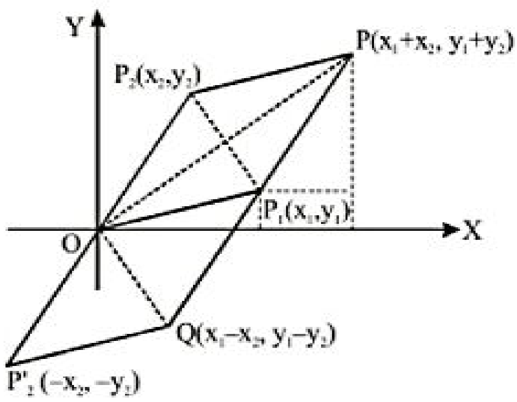
Answer key

Q.1	-1	Q.2	8	Q.3	9	Q.4	$\dfrac{n\cdot 3^{n-1}}{3^n-1}-\dfrac{1}{2}$
Q.5	5	Q.6	B				

8. GEOMETRY OF COMPLEX NUMBER :

8.1 Geometrical meaning of addition and subtraction :

Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ be two complex numbers represented by the point $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ respectively. By definition $z_1 + z_2$ should be represented by the point $(x_1 + x_2, y_1 + y_2)$. This point is the vertex which completes the parallelogram with the line segments joining the origin with OP_1 and OP_2 as the adjacent sides.



$\Rightarrow \quad |z_1 + z_2| = OP$

Also by definition $z_1 - z_2$ should be represented by the point $(x_1 - x_2, y_1 - y_2)$. This point is the vertex which completes the parallelogram with the line segments joining the origin with OP_1 and OP'_2 (where the point P'_2 represents $-z_2$; the point $-z_2$ can be obtained by producing the directed line P_2O by length $(|z_2|)$ asthe adjacent sides.

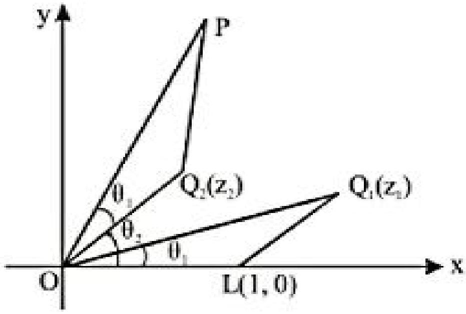
$\Rightarrow \quad |z_1 - z_2| = OQ = P_2P_1$

8.2 Geometrical Meaning of product and Division :

Let $z_1 = r_1 (\cos\theta_1 + i\sin\theta_1)$, $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$ be complex numbers represented by Q_1 and Q_2.

(i) Construction for the point representing the product z_1z_2 :

Let L be the point on OX which represents unity, so that OL = 1. Draw the triangle OQ_2P directly similar to the triangle OLQ_1. Then point P represents the product z_1z_2.



Explanation :

Due to similar triangles

$$\frac{OP}{OQ_1} = \frac{OQ_2}{OL}, \text{ that is } \frac{OP}{r_1} = \frac{r_2}{1} \Rightarrow OP = r_1r_2$$

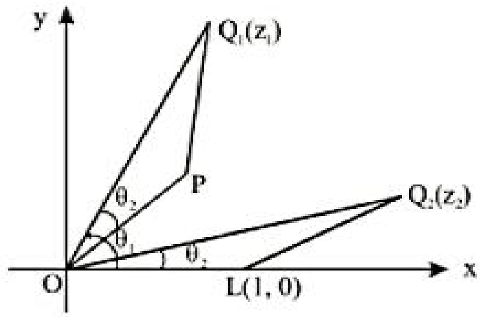
Also $\angle Q_2OP = \angle LOQ_1 = \theta_1 \Rightarrow \angle LOP = \theta_1 + \theta_2$

Since $z_1z_2 = r_1r_2\{\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\}$, P represents z_1z_2.

(ii) Construction for the point representing the quotient z_1/z_2 :

Draw the triangle OQ_1P directly similar to the triangle OQ_2L

Then P represents the quotient $\frac{z_1}{z_2}$



Explanation :

From the last construction,

$$\frac{OQ_1}{OQ_2} = \frac{OP}{OL} \Rightarrow \frac{r_1}{r_2} = \frac{OP}{1}$$

number represented by $P \cdot z_2 = z_1$ $\Rightarrow$ number represented by $P = \frac{z_1}{z_2}$

Remark :

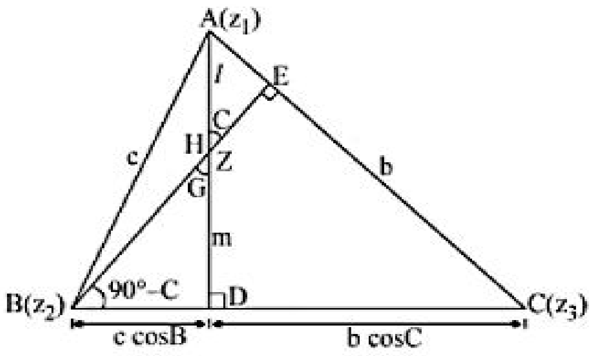
(i) If $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$, and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$,

then $z_1z_2 = r_1r_2 = e^{i(\theta_1+\theta_2)}$ and $\frac{z_1}{z_2} = \frac{r_1}{r_2} \cdot e^{i(\theta_1-\theta_2)}$

(ii) $\arg(z_1z_2) = \theta_1 + \theta_2 = \arg(z_1) + \arg(z_2)$

(iii) $\arg\left(\frac{z_1}{z_2}\right) = \theta_1 - \theta_2 = \arg(z_1) + \arg(z_2)$

8.3 Centroid, Incentre, Orthocentre & Circumcentre of a triangle on a complex plane :



(i) Centroid 'G' = $\dfrac{z_1 + z_2 + z_3}{3}$

(ii) Incentre 'I' = $\dfrac{a z_1 + b z_2 + c z_3}{a + b + c}$

(iii) Orthocentre :

$$Z_D = \frac{b \cos C \; z_2 + c \cos B \; z_3}{a}$$

Now AE = c cos A ;

$$l = AE \ \text{cosec}\, C = c \cos A \ \text{cosec}\, C$$

$$l = 2R \cos A \qquad \left(\frac{c}{\sin C} = 2R\right)$$

and $m = c \cos B \cot C$ or $m = 2R \cos B \cos C$

Hence $Z_H = \dfrac{m z_1 + \ell Z_D}{\ell + m}$

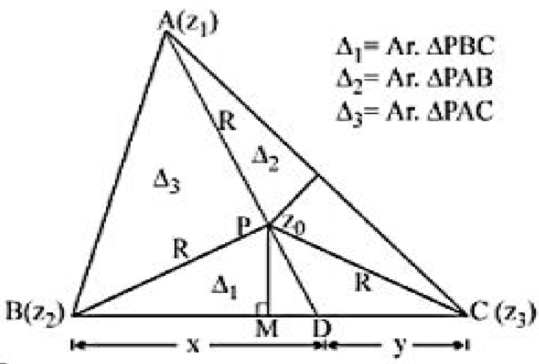
$$= \frac{2R\cos B\cos C z_1 + 2R\cos A\left(\dfrac{b\cos C z_2 + c\cos B z_1}{a}\right)}{2R(\cos A + \cos B\cos C)}$$

$$= \frac{a \cos B \cos C z_1 + b \cos A \cos C z_2 + c \cos A \cos B z_3}{a\left(-\cos(B+C) + \cos B \cos C\right)}$$

$$= \frac{z_1(\sin A \cos B \cos C) + (\sin B \cos C \cos A)\, z_2 + (\sin C \cos A \cos B)\, z_3}{\sin A\,(\sin B \sin C)}$$

$$Z_H = \frac{z_1 \tan A + z_2 \tan B + z_3 \tan C}{\Sigma \tan A} \quad \text{or} \quad \frac{z_1 \tan A + z_2 \tan B + z_3 \tan C}{\Pi \tan A}$$

(iv). Circumcentre :



We have z_0 being equidistant from the vertices gives,

$$|z_1 - z_0| = |z_2 - z_0| = |z_3 - z_0|$$

Consider, $|z_1 - z_0|^2 = |z_2 - z_0|^2$

$$(z_1 - z_0)(\bar{z}_1 - \bar{z}_0) = (z_2 - z_0)(\bar{z}_2 - \bar{z}_0)$$

$$\bar{z}_1(z_1 - z_0) - \bar{z}_2(z_2 - z_0) = \bar{z}_0\left[(z_1 - z_0) - (z_2 - z_0)\right]$$

$$\bar{z}_1(z_1 - z_0) - \bar{z}_2(z_2 - z_0) = \bar{z}_0(z_1 - z_2) \quad \text{....(1)}$$

Similarly 1st & 3rd gives

$$\bar{z}_1(z_1 - z_0) - \bar{z}_3(z_3 - z_0) = \bar{z}_0(z_1 - z_3) \quad \text{....(2)}$$

dividing (1) by (2) eliminate $\bar{z}_0$ and get z_0.

8.4 Different forms of equation of a straight line :

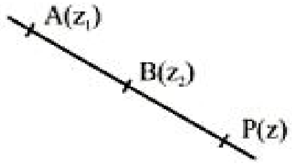
(i) Equation of straight line with line the help of coordinate geometry :

Writing $x = \dfrac{z+\bar{z}}{2}$, $y = \dfrac{z-\bar{z}}{2i}$ etc. in $\dfrac{y-y_1}{y_2-y_1} = \dfrac{x-x_1}{x_2-x_1}$ and re-arranging terms, we find that the equation of the line through z_1 and z_2 is given by.

$$\frac{z-z_1}{z_2-z_1} = \frac{\bar{z}-\bar{z}_1}{\bar{z}_2-\bar{z}_1} \qquad \text{or} \qquad \begin{vmatrix} z & \bar{z} & 1 \\ z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \end{vmatrix} = 0$$

(ii) Equation of a straight line with the help of rotation formula :

Let $A(z_1)$ and $B(z_2)$ be any two points lying on any line and we have to obtain the equation of this line. For this purpose let us take any point



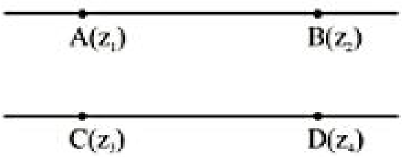
P(z) lying on this line. Since $\arg\left(\dfrac{z-z_1}{z_2-z_1}\right) = 0$ or π,

$\dfrac{z-z_1}{z_2-z_1}$ is purely real.

$$\Rightarrow \quad \frac{z-z_1}{z_2-z_1} = \frac{\bar{z}-\bar{z}_1}{\bar{z}_2-\bar{z}_1}$$

Conditon for which two lines which are parallel or perpendicular :

(a) For parallel :



Since AB and CD lines are parallel

$$\therefore \quad \arg\left(\frac{z_1-z_2}{z_3-z_4}\right) = 0 \text{ or } \pi \quad \Rightarrow \quad \frac{z_1-z_2}{z_3-z_4} \text{ is purely real} \quad \Rightarrow \quad \boxed{\frac{z_1-z_2}{z_3-z_4} = \frac{\bar{z}_1-\bar{z}_2}{\bar{z}_3-\bar{z}_4}}$$

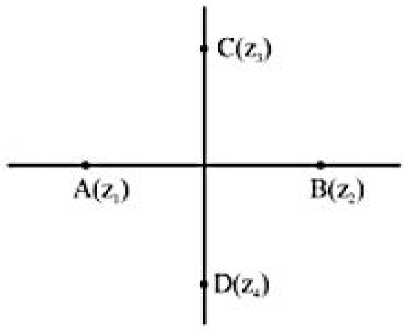
(b) For perpendicular :

Since lines AB and CD are perpendicular

$$\arg\left(\frac{z_1-z_2}{z_3-z_4}\right) = \pm\frac{\pi}{2}$$

$$\Rightarrow \quad \frac{z_1-z_2}{z_3-z_4} \text{ is purely imaginary}$$

$$\Rightarrow \quad \boxed{\frac{z_1-z_2}{z_3-z_4} = -\left(\frac{\bar{z}_1-\bar{z}_2}{\bar{z}_3-\bar{z}_4}\right)}$$



(c) General equation of the line :

From equation (i) we get, $z(\bar{z}_2 - \bar{z}_1) - z_1\bar{z}_2 + z_1\bar{z}_1 = \bar{z}(z_2 - z_1) - \bar{z}_1 z_2 + z_1\bar{z}_1$

$\Rightarrow \quad z(\bar{z}_2 - \bar{z}_1) + \bar{z}(z_1 - z_2) + \bar{z}_1 z_2 - z_1\bar{z}_2 = 0$

Here $\bar{z}_1 z_2 - z_1\bar{z}_2$ is a purely imaginary number as $\bar{z}_1 z_2 - z_1\bar{z}_2 = 2i\,\mathrm{Im}(\bar{z}_1 z_2)$.

Let $\quad \bar{z}_1 z_2 - z_1\bar{z}_2 = ib, \; b \in R$

$\Rightarrow \quad z(\bar{z}_2 - \bar{z}_1) + \bar{z}(z_1 - z_2) + ib = 0 \quad \Rightarrow \quad z\,i(\bar{z}_1 - \bar{z}_2) + \bar{z}\,i(z_2 - z_1) + b = 0$

Let $\quad a = i(z_2 - z_1) \quad \Rightarrow \quad \bar{a} = i(\bar{z}_1 - \bar{z}_2)$

$\Rightarrow \quad \bar{a}z + a\bar{z} + b = 0$

This is the general equation of a line in the complex plane.

8.5 Complex slope of a line :

(a) Complex slope of a line $\bar{a}z + a\bar{z} + b = 0$ is defined as

$$\omega = -\frac{a}{\bar{a}}$$

(b) Complex slope of a line passing through $A(z_1)$ & $B(z_2)$ is given by

$$\omega = \frac{z_1 - z_2}{\bar{z}_1 - \bar{z}_2}$$

(c) Complex slope of a line making an angle θ from positive real axis is given by

$$\omega = e^{i2\theta}$$

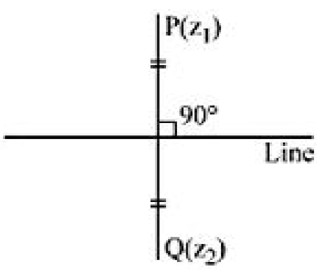
Note : Let ω_1 and ω_2 be the complex slopes of two lines

(i) If $\omega_1 = \omega_2$ then lines are parallel

(ii) If $\omega_1 + \omega_2 = 0$ then lines are perpendicular.

8.6 Reflection Points For A Line (Image of a point in a line) :

Two given points P & Q denoted by complex numbers z_1 and z_2 respectively are the reflection points in a given straight $\bar{\alpha}z + \alpha\bar{z} + r = 0$ if the given line is the right bisector of the line segment PQ. Two points $P(z_1)$ & $Q(z_2)$ will be the reflection points of each other in the given straight line, iff $\bar{\alpha}z_1 + \alpha\bar{z}_2 + r = 0$ where 'r' is real and α is non zero complex constant.

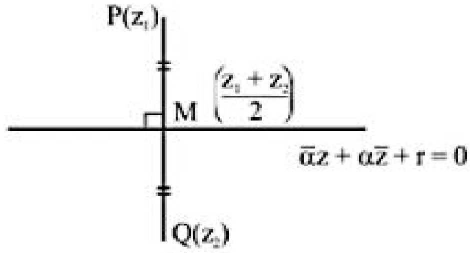


Proof : M is the mid point of PQ which lies on the line mirror

$$\therefore \quad \bar{\alpha}\left(\frac{z_1+z_2}{2}\right)+\alpha\left(\frac{\bar{z}_1+\bar{z}_2}{2}\right)+r=0$$

$$\bar{\alpha}\, z_1 + \bar{\alpha}\, z_2 + \alpha\, \bar{z}_1 + \alpha\, \bar{z}_2 + 2r = 0 \qquad \ldots(i)$$

Line PQ and the given line both are perpendicular



$$\therefore \quad \frac{z_1-z_2}{\bar{z}_1-\bar{z}_2}+\left(-\frac{\alpha}{\bar{\alpha}}\right)=0$$

$$z_1\bar{\alpha}-z_2\bar{\alpha}=\alpha\bar{z}_1-\alpha\bar{z}_2$$

$$\alpha\bar{z}_1+\bar{\alpha}z_2=\alpha\bar{z}_2+\bar{\alpha}z_1 \qquad \ldots(ii)$$

From (i) & (ii)

$$2(\bar{\alpha}z_2+\alpha\bar{z}_1)+2r=0$$

$$\Rightarrow \quad \alpha\bar{z}_1+\bar{\alpha}z_2+r=0$$

Illustration :

Find the image of the point $P(1-i)$ in the line mirror $(1+i)z-(1-i)\bar{z}+2i=0$

Sol. *The given line mirror is*

$$(1+i)z-(1-i)\bar{z}+2i=0$$

$$(-1+i)z-(1+i)\bar{z}-2=0$$

$$(1-i)z+(1+i)\bar{z}+2=0$$

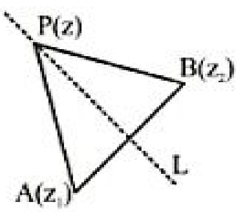
Let z_2 be the image of the given point $P(1-i)$

$$\Rightarrow \quad (1-i)z_2+(1+i)(1+i)+2=0 \qquad \Rightarrow \quad (1-i)z_2+1-1+2i+2=0$$

$$\Rightarrow \quad z_2=\frac{-2(1+i)}{1-i}=\frac{-2(1-1+2i)}{2}=-2i$$

8.7 Equation of perpendicular bisector :

Consider a line segment joining $A(z_1)$ and $B(z_2)$. Let the line 'L' be it's perpendicular bisector. If $P(z)$ be any point on the 'L'. we have



$$PA=PB \Rightarrow |z-z_1|=|z-z_2|$$

$$\Rightarrow \quad |z-z_1|^2=|z-z_2|^2$$

$$\Rightarrow \quad (z-z_1)(\bar{z}-\bar{z}_1)=(z-z_2)(\bar{z}-\bar{z}_2)$$

$$\Rightarrow \quad z\bar{z}-z\bar{z}_1-\bar{z}_1\bar{z}+z\bar{z}-z\bar{z}_2-z_2\bar{z}+z_2\bar{z}_2$$

$$\Rightarrow \quad z(\bar{z}_2-\bar{z}_1)+\bar{z}(\bar{z}_2-\bar{z}_1)+z_1\bar{z}_1-z_2\bar{z}_2=0$$

8.8 Distance of a given point from a given line :

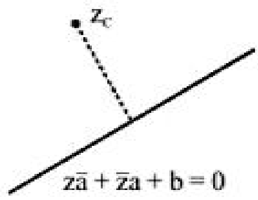
Let the given line be $z\bar{a}+\bar{z}a+b=0$, and the given point be z_c

Say $z_c = x_c + iy_c$.

Replacing z by x + iy, in the given equation,

we get, $x(a+\bar{a})+iy(\bar{a}-a)+b=0$

Distance of (x_c, y_c) from this line is,



$$\frac{|x_c(a+\bar{a})+iy_c(\bar{a}-a)+b|}{\sqrt{(a+\bar{a})^2-(a-\bar{a})^2}} = \frac{|z_c\bar{a}+\bar{z}_c a+b|}{\sqrt{4(\mathrm{Re}(a))^2+4(\mathrm{im}(a))^2}} = \frac{|z_c\bar{a}+\bar{z}_c a+b|}{2|a|}$$

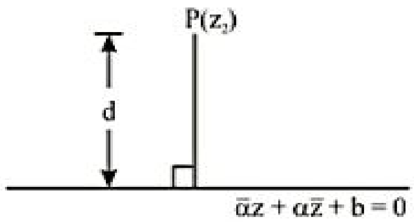
Illustration :

Consider a line $(1+i)z-(1-i)\bar{z}+2i=0$. Find the distance of a point $2i$ from the above line.

Sol. *The given line can be written as*

$$(1-i)z+(1+i)\bar{z}+2=0$$

Applying distance formula

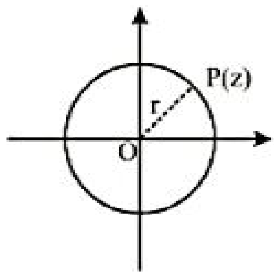


$$d = \frac{|z_c\bar{\alpha}+\bar{z}_c\alpha+b|}{2|\alpha|}$$

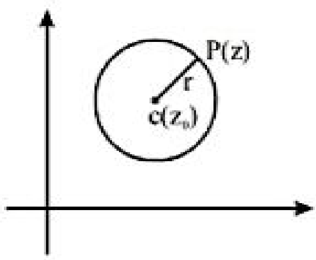
$$= \left|\frac{(1-i)2i+(1+i)(-2i)+2}{2\sqrt{2}}\right| = \left|\frac{2i+2-2i+2+2}{2\sqrt{2}}\right| = \frac{3}{\sqrt{2}} \quad Ans.$$

8.9 Different forms of equation of a circle :

(i) $|z| = r,\ r \in R^+$ then locus of z represent a circle whose centre is the origin and radius is equal to r.



(ii) $|z - z_0| = r,\ r \in R^+$ then locus of z represents a circle whose centre is z_0 and radius is equal to r.



(iii) Equation $z\bar{z} + a\bar{z} + \bar{a}z + b = 0$ represnt a circle whose centre is $-a$ and radius is equal $\sqrt{|a|^2 - b}$.

Equation of the circle with centre z_0 and radius 'r'.

$|z - z_0| = r \Rightarrow |z - z_0|^2 = r^2 \Rightarrow (z - z_0)(\bar{z} - \bar{z}_0) = r^2 \Rightarrow z\bar{z} - z\bar{z}_0 - \bar{z}z_0 + |z_0|^2 - r^2 = 0$

Putting $-z_0 = a$ and $|z_0|^2 - r^2 = b$ equation becomes

$$z\bar{z} + \bar{a}z + a\bar{z} + b = 0$$

$\therefore$ Centre $z_0 = -a$ and radius $r = \sqrt{|z_0|^2 - b} = \sqrt{|a|^2 - b}$

Illustration

Find the centre and radius of the circle

$$z\bar{z} - (3 - 4i)z - (3 + 4i)\bar{z} + 9 = 0$$

$$z\bar{z} + (-3 + 4i)z + (-3 - 4i)\bar{z} + 9 = 0$$

Sol. *Here,* $a = -3 - 4i,\ b = 9$

$\therefore$ *centre,* $-a = 3 + 4i$ *or* $(3, 4)$

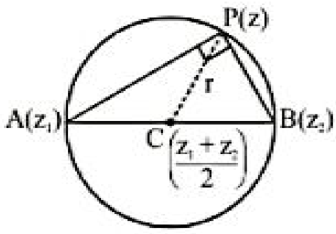
radius $r = \sqrt{|a|^2 - b} = \sqrt{|3 + 4i|^2 - 9} = \sqrt{25 - 9} = 4$

(iv) Diametric form of equation of circle :

Let $A(z_1)$ and $B(z_2)$ are the extremities of diameter of a circle and $P(z)$ be a variable point then

(a) $CP = r$ (r = radius of the circle)

$$\left|z - \frac{z_1 + z_2}{2}\right| = \left|\frac{z_1 - z_2}{2}\right|$$



(b) AP & BP are perpendicular

$\therefore \omega_1 + \omega_2 = 0 \Rightarrow \dfrac{z - z_1}{\bar{z} - \bar{z}_1} + \dfrac{z - z_2}{\bar{z} - \bar{z}_2} = 0$

(c) $AP^2 + BP^2 = AB^2$

$|z - z_1|^2 + |z - z_2|^2 = |z_1 - z_2|^2$

(v) Equation of circle passing through three non-collinear points :

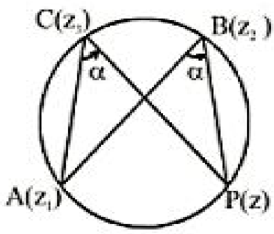
Let $A(z_1)$, $B(z_2)$ and $C(z_3)$ be three given non-collinear points and $P(z)$ be a variable point.

Angle substended by the arc AP at B and C are equal (say 'α')

Using rotation theorem :

$$\frac{z - z_2}{z_1 - z_2} = \lambda_1 e^{i\alpha} \qquad \ldots(i)$$

$$\frac{z - z_3}{z_1 - z_3} = \lambda_2 e^{i\alpha} \qquad \ldots(ii)$$



(i) ÷ (ii)

$$\frac{z-z_2}{z_1-z_2}\cdot\frac{z_1-z_3}{z-z_3}=\frac{\lambda_1}{\lambda_2}=\text{Purely real}$$

$$\therefore\quad \left(\frac{z-z_2}{z-z_3}\right)\left(\frac{z_1-z_3}{z_1-z_2}\right)=\left(\frac{\bar{z}-\bar{z}_2}{\bar{z}-\bar{z}_3}\right)\left(\frac{\bar{z}_1-\bar{z}_3}{\bar{z}_1-\bar{z}_2}\right)$$

The above equaiton represents a circle passing through $A(z_1)$, $B(z_2)$ and $C(z_3)$.

Imp. Note :

Let z_1 and z_2 be two given complex numbers and z be any complex number such that, $\arg\left(\dfrac{z-z_1}{z-z_2}\right)=\alpha$, where $\alpha\in(0,\pi)$

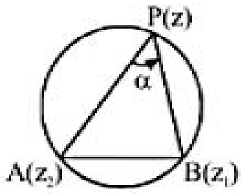


Figure-I

Then 'z' would be lie on an arc of segment of a circle on z_1z_2, containing angle α. Clearly if $\alpha\in\left(0,\dfrac{\pi}{2}\right)$, z would lie on the major arc (excluding the points z_1 and z_2) and if $\alpha\in\left(\dfrac{\pi}{2},\pi\right)$. 'z' would lie on the minor arc (excluding the points z_1 and z_2).

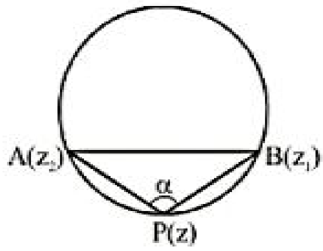


Illustration :

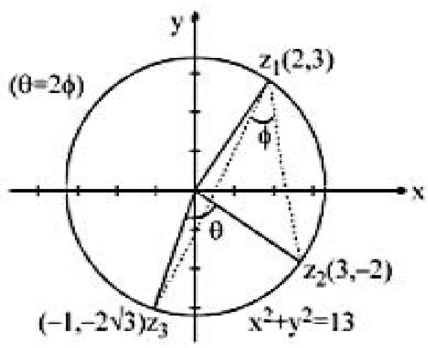
If $z_1=2+3i$, $z_2=3-2i$ and $z_3=-1-2\sqrt{3}i$ then which of the following is true?

(A) $\arg\left(\dfrac{z_3}{z_2}\right)=\arg\left(\dfrac{z_3-z_1}{z_2-z_1}\right)$ (B) $\arg\left(\dfrac{z_3}{z_2}\right)=\arg\left(\dfrac{z_2}{z_1}\right)$

(C) $\arg\left(\dfrac{z_3}{z_2}\right)=2\arg\left(\dfrac{z_3-z_1}{z_2-z_1}\right)$ (D) $\arg\left(\dfrac{z_3}{z_2}\right)=\dfrac{1}{2}\arg\left(\dfrac{z_3-z_1}{z_2-z_1}\right)$

Sol. *Note that $|z_1|=|z_2|=|z_3|=\sqrt{13}$*

Hence z_1, z_2, z_3 lies on a circle with centre (0, 0) and $r=\sqrt{13}$ as shown



now $\quad Arg\dfrac{z_2}{z_3}=2Arg\dfrac{z_2-z_1}{z_3-z_1}$

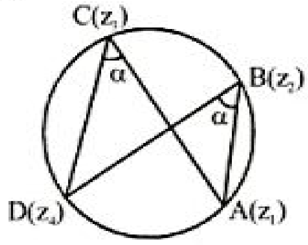
$\therefore\quad Arg\dfrac{z_3}{z_2}=2Arg\dfrac{z_3-z_1}{z_2-z_1}\quad\Rightarrow\quad$ *(C)*

8.10 Condition of concyclic of four points :

Four points $A(z_1), B(z_2), C(z_3)$ and $D(z_4)$ taken in order are cycyclic then

$$\frac{z_1 - z_2}{z_4 - z_2} = \lambda_1 e^{i\alpha} \qquad \ldots(i)$$

$$\frac{z_1 - z_3}{z_4 - z_3} = \lambda_2 e^{i\alpha} \qquad \ldots(ii)$$



From (i) & (ii)

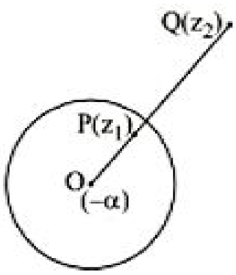
$$\frac{z_1 - z_2}{z_4 - z_2} \cdot \frac{z_4 - z_3}{z_1 - z_3} = \text{Purely real (Which is the required conditon)}$$

8.11 Inverse Point w.r.t. a circle :

Two points $P(z_1)$ & $Q(z_2)$ are said to be the inverse point of each other w.r.t. the circle if

(i) O, P, Q are collinear and lie on the same side of the centre 'O' and

(ii) $(OP)\cdot(OQ) = \rho^2$ when ρ is the radius of the circle.

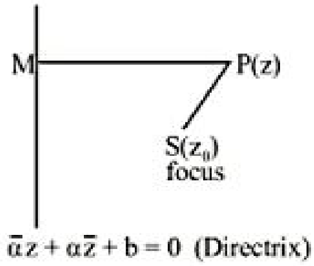


The points z_1 & z_2 will be the inverse point of each other w.r.t. the circle $z\bar{z} + \bar{\alpha}z + \alpha\bar{z} + r = 0$ if

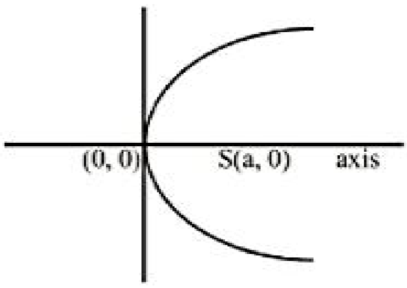
$$z_1\bar{z}_2 + \bar{\alpha}z_1 + \alpha\bar{z}_2 + r = 0$$

8.12 General locii on complex plane :

(a) (i) If $|z - z_0| = \left|\dfrac{\bar{\alpha}z + \alpha\bar{z} + b}{2|\alpha|}\right|$ then locus of z is a parabola whose focus is z_0 and directrix is the line $\bar{\alpha}z + \alpha\bar{z} + b = 0$ provided $\bar{\alpha}z_0 + \alpha\bar{z}_0 + b \neq 0$



(ii) If $(z - \bar{z})^2 + 8a(z + \bar{z}) = 0$ then locus of z represents a parabola whose focus is (a, 0), vertex is (0, 0) and real axis is the axis of parabola.



Put $\quad z = x + iy$

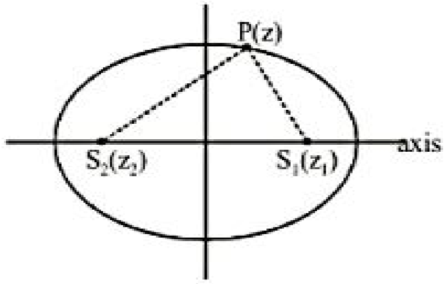
$\Rightarrow \quad (i\,2y)^2 + 8a.2x = 0$

$\Rightarrow \quad -4y^2 + 16ax = 0$

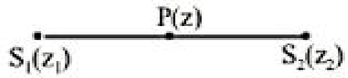
$\Rightarrow \quad y^2 = 4ax$

(b) $|z - z_1| + |z - z_2| = k$

(i) $k > |z_1 - z_2|$ then locus of z represents an ellipse whose foci are z_1 and z_2.



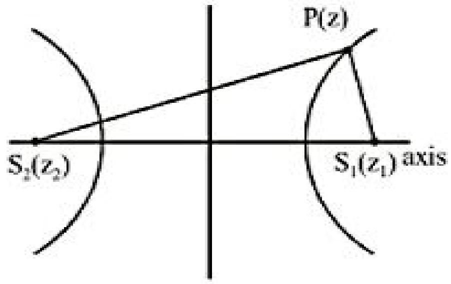
(ii) $k = |z_1 - z_2|$ then locus of z is line segment joining z_1 and z_2.



$PS_1 + PS_2 = S_1S_2 = |z_1 - z_2| \Rightarrow$ P lies on the line segement joining $S_1(z_1)$ and $S_2(z_2)$.

(c) $||z - z_1| - |z - z_2|| = k$

(i) $k < |z_1 - z_2|$ then locus of z represents a hyperbola whose foci are z_1 and z_2.



$|PS_1 - PS_2| < S_1S_2$

$||z - z_1| - |z - z_2| < |z_1 - z_2|$

(ii) $k = |z_1 - z_2|$ then locus of z is union of two rays emanting from z_1 and z_2.

$|PS_1 - PS_2| = S_1S_2$

$||z - z_1| - |z - z_2| = |z_1 - z_2|$

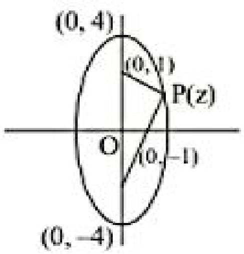
$\therefore$ P(z) lies on the rays emanting either z_1 or z_2.

Illustration :

If z is a complex number satisfying the equation $|z + i| + |z - i| = 8$, on the complex plane then maximum value of $|z|$ is

(A) 2 *(B) 4* *(C) 6* *(D) 8*

Sol. *If $|z + i| + |z - i| = 8$,*



$PF_1 + PF_2 = 8$

$\therefore \quad |z|_{max} = 4 \quad \Rightarrow \quad$ ***(B)***

Illustration :

Number of complex numbers satisfying the relation $|z+\bar{z}|+|z-\bar{z}|=2$ *and* $|z+i|+|z-i|=2$*, is*

(A) 1 *(B) 2* *(C) 3* *(D) 4*

Sol. *We have* $\left|\frac{z+\bar{z}}{2}\right|+\left|\frac{z-\bar{z}}{2}\right|=1 \Rightarrow |x|+|y|=1$

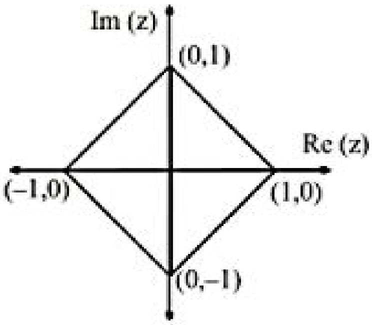
Also, $|z-i|+|z+i| = 2$

$\Rightarrow$ *A line segment between*

(0, 1) and (0, –1).

So, number of solution is 2

i.e., $z = i$ *and* $-i$



9. LOGARITHM OF COMPLEX QUANTITY :

Let u and z be two complex number such that

$u = e^z$ where $u = \alpha + i\beta$ and $z = x + iy$ or $\alpha + i\beta = e^{x+iy}$

then z is called the logarithm of u to the base e

$z = \log_e u$

or $x + iy = \log_e(\alpha + i\beta)$

Now $\alpha + i\beta = e^{x+iy} = e^x.e^{iy} = e^x(\cos y + i \sin y)$

Hence corresponding to some vlaues of x and y there is one and only one value of $\alpha + i\beta$.

Now we will prove that for a given value of u, there can be infinite values of z.

Here $z = \log u$

or $x + iy = \log(\alpha + i\beta)$

$= \log r(\cos\theta + i\sin\theta)$

[where $\alpha = r\cos\theta$, $\beta = r\sin\theta$ and $r = \sqrt{(\alpha^2+\beta^2)}$, $\theta = \tan^{-1}\beta/\alpha$]

$= \log r\{\cos(\theta + 2n\pi) + i\sin(\theta + 2n\pi)\}$

$= \log re^{i(\theta+2n\pi)}$, where $n \in I$

$= \log r + i(\theta + 2n\pi)$

$= \log\sqrt{(\alpha^2+\beta^2)} + i(2n\pi + \tan^{-1}\beta/\alpha) \ldots (1)$

From (1) it is clear that for different values of n, we have different values of x + iy.

Hence logrithim of a complex quantity is a multi-valued function and it is expressed as Log u i.e.,

$$\boxed{\text{Log } u = \text{Log}(\alpha + i\beta) = \frac{1}{2}\log(\alpha^2+\beta^2) + i\tan^{-1}\beta/\alpha + 2n\pi i}$$

If we put n = 0, we obtain the principal value of log u.

Hence the principal value of $\boxed{\log u = \frac{1}{2}\log(\alpha^2+\beta^2) + i\theta}$ and $\theta\,(=\tan^{-1}\beta/\alpha)$ should be such that $-\pi < \theta \le \pi$.

Illustration :

Separate into real and imaginary parts the following :
(i) $\log(1+i)$ *(ii)* $\log(-5)$

Sol.

(i) $\log(1+i) = \log(1+i) + 2n\pi i.$

Let $1 = r\cos\theta$ and $1 = r\sin\theta$

then $r = \sqrt{1^2+1^2} = \sqrt{2}$

and $\tan\theta = 1$ or $\theta = \pi/4$

$$\begin{aligned}\therefore \quad \log(1+i) &= \log(r\cos\theta + ir\sin\theta)\\ &= \log r(\cos\theta + i\sin\theta)\\ &= \log\sqrt{2}(\cos\pi/4 + ir\sin\pi/4)\\ &= \log(\sqrt{2}\,e^{i\pi/4})\\ &= \log\sqrt{2} + \log e^{i\pi/4}\\ &= \frac{1}{2}\log 2 + \frac{1}{4}i\pi\end{aligned}$$

Here
$$\begin{aligned}\text{Log}(1+i) &= \frac{1}{2}\log 2 + \frac{1}{4}i\pi + 2n\pi i\\ &= \frac{1}{2}\log 2 + i(2n\pi + \pi/4)\end{aligned}$$

(ii) $\text{Log}(-5) = \log(-5) + 2n\pi i$

Now
$$\begin{aligned}\log(-5) &= \log(-5 + i.0)\\ &= \log(r\cos\theta + ir\sin\theta)\end{aligned}$$

where $r\cos\theta = -5,\ r\sin\theta = 0,$

$\therefore \quad r = \sqrt{[(-5)^2 + 0^2]} = 5,\ r\sin\theta = 0,$

$$\begin{aligned}\therefore \quad \log(-5) &= \log 5(\cos\pi + i\sin\pi)\\ &= \log(5e^{i\pi})\\ &= \log 5 + i\pi\end{aligned}$$

Hence
$$\begin{aligned}\text{Log}(-5) &= \log 5 + i\pi + 2n\pi i\\ &= \log 5 + i(2n+1)\pi\end{aligned}$$

Illustration :

If $\tan\log(x+iy) = a + ib$ *and* $a^2 + b^2 \neq 1$, *then prove that*

$$\tan\log(x^2+y^2) = \frac{2a}{1-a^2-b^2}.$$

Sol. Here $\tan\log(x+iy) = a+ib$

$\therefore \quad \log(x+iy) = \tan^{-1}(a+ib) \qquad \dots(1)$

and $\quad \log(x-iy) = \tan^{-1}(a-ib) \qquad \dots(2)$

$\therefore \quad \log(x+iy) + \log(x+iy) = \tan^{-1}(a+ib) + \tan^{-1}(a-ib)$

or $\log\left\{(x+iy)(x-iy)=\tan^{-1}\left\{\dfrac{(a+ib)+(a-ib)}{1-(a+ib)(a-ib)}\right\}\right\}$

or $\log(x^2+y^2)=\tan^{-1}\left\{\dfrac{2a}{1-a^2-b^2}\right\}$

or $\tan\log(x^2+y^2)=\dfrac{2a}{1-a^2-b^2}$

Illustration :

Prove that : $i^i=\exp\{-(4n+1)\pi/2\}$

Sol. $i^i=\exp(i\log i)$
$=\exp\{i(\log i+2n\pi i)\}$
$=\exp[i\{2n\pi i+\log(0+1.i)\}]$
$=\exp[i\{2n\pi i+\log(\cos\pi/2+i\sin\pi/2)\}]$
$=\exp[i(2n\pi i+\log e^{(i\pi)/2}]$
$=\exp\{i(2n\pi i+i\pi/2)\}=\exp(-2n\pi-\pi/2)$
$=\exp\{-(4n+1)\pi/2\}$
where $n=0,1,2,3$ ....

Illustration :

If $i^{i^{i\ldots\infty}}=A+iB$ *(principal values only being considered then prove that)*

$$\tan\left(\frac{\pi A}{2}\right)=\frac{B}{A}\ \text{ and }\ A^2+B^2=e^{-B\pi}$$

Sol. *Here* $i^{i^{i\ldots\infty}}=A+iB$

$\therefore$ $i^{A+iB}=A+iB$

$\Rightarrow$ $(A+iB)\log\{\cos(\pi/2)+i\sin(\pi/2)\}=\log(A+iB)$

$\Rightarrow$ $(A+iB)\log e^{(i\pi/2)}=\dfrac{1}{2}\log(A^2+B^2)+i\tan^{-1}(B/A)$

$\Rightarrow$ $(A+iB)(i\pi/2)=\dfrac{1}{2}\log(A^2+B^2)+i\tan^{-1}(B/A)$

Equating real and imaginary parts of both sides

$$-\left(\frac{B\pi}{2}\right)=\frac{1}{2}\log(A^2+B^2)$$

$\Rightarrow$ $A^2+B^2=e^{-B\pi}$

and $A\pi/2=\tan^{-1}(B/A)\Rightarrow B/A=\tan(A\pi/2)$

$\Rightarrow$ $\tan\dfrac{A\pi}{2}=\dfrac{B}{A}$. *Hence proved.*

Practice Problem

Q.1 Find the equation of the line passing through $1+i$ and $2+i$ on argand plane and also find the complex numbers corresponding to the points on the line which are at a distance of $\sqrt{3}$ units from $1+i\cdot 0$.

Q.2 Find the equation of a line passing through $A(z_1)$ and perpendicular to OA where O is the origin.

Q.3 Locate the complex number $z=x+iy$ which satisfies the inequality $\log_{\sqrt{3}}\frac{|z^2|-|z|+1}{2+|z|}<2$.

Q.4 If A & B represent the complex numbers z_1 and z_2 such that $|z_1+z_2|=|z_1-z_2|$ then find the circumcentre of ΔOAB, where O is the origin.

Q.5 Locate the points representing complex number z for which

(i) $\frac{\pi}{3}<\arg z\leq\frac{3\pi}{2}$ (ii) $\arg\left(\frac{z-1-i}{z-2}\right)=\frac{\pi}{3}$

Q.6 Find the equation of a circle which touches the line $iz+\bar{z}+1+i=0$ and the lines $(z-i)z=(2+i)\bar{z}$ and $(2+i)z+(i-2)\bar{z}-4i=0$ are the normals to the circle.

Q.7 If $|z+2|+|z|\leq 8$, then the range of values of $|z-4|$.

Q.8 Prove the following

(a) $\text{Log}(-i)=2n\pi i-\frac{1}{2}i\pi$ (b) $\log(-i)=-\frac{1}{2}\pi i$ (c) $\log_i i=\frac{4m+1}{4n+1}$, m, n ∈ i

Q.9 $i\log\frac{x-i}{x+i}=\pi-2\tan^{-1}x$

Answer key

Q.1 $1+\sqrt{2}+i,\ 1-\sqrt{2}+i$ Q.2 $z\bar{z}_1+z_1\bar{z}-2|z_1|^2=0$

Q.3 z lies in the interior region of the circle whose centre is the origin and radius is equal to 5.

Q.4 $\frac{z_1+z_2}{2}$

Q.6 $|z-(1-2i)|=2\sqrt{2}$ Q.7 $[1,9]$

SOLVED EXAMPLES

Q.1 The value of $169\, e^{i\left(\pi + \sin^{-1}\frac{12}{13} + \cos^{-1}\frac{5}{13}\right)}$ is

(A) $119 - 120\,i$ (B) $120 + 119\,i$ (C) $119 + 120\,i$ (D) None of these

Sol. $169\, e^{i\left(\pi+\sin^{-1}\frac{12}{13}+\cos^{-1}\frac{5}{13}\right)}$

$$= -169\left[\cos\left(\cos^{-1}\frac{5}{13}\right) + i\sin\left(\cos^{-1}\frac{5}{13}\right)\right]\left[\cos\left(\sin^{-1}\frac{12}{13}\right) + i\sin\left(\sin^{-1}\frac{12}{13}\right)\right]$$

$$= -169\left[\frac{5}{13} + i\frac{12}{13}\right]\left[\frac{5}{13} + i\frac{12}{13}\right]$$

$= [119 - 120\,i] = -i\,[120 + 119\,i]$

Hence (A) is the correct answer.

Q.2 If $A(z_1), B(z_2)$ and $C(z_3)$ be the vertices of a triangle ABC in which $\angle ABC = \frac{\pi}{4}$ and $\frac{AB}{BC} = \sqrt{2}$, then the value of z_2 is equal are

(A) $z_3 + i(z_1 + z_3)$ (B) $z_3 - i(z_1 - z_3)$ (C) $z_3 + i(z_1 - z_3)$ (D) None of these

Sol. $\frac{AB}{BC} = \sqrt{2}$

Considering the rotation about B, we get

$$\frac{z_1 - z_2}{z_3 - z_2} = \left|\frac{z_1 - z_2}{z_3 - z_2}\right| e^{i\pi/4} = \frac{AB}{BC} e^{i\pi/4}$$

$\Rightarrow \quad \sqrt{2}\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right) = 1 + i \quad \Rightarrow \quad z_1 - z_2 = \{1 + i\}\,(z_3 - z_2)$

$\Rightarrow \quad z_1 - (1 + i)\,z_3 = z_2(1 - 1 - i) = -iz_3$

$\Rightarrow \quad z_2 = -z_3\,(z_3 + z_1)$

Q.3 If $f(x) = g(x^3) + xh(x^3)$ is divisible by $x^2 + x + 1$, then

(A) $g(x)$ is divisible by $(x - 1)$ but not by $h(x)$ (B) $h(x)$ is divisible by $(x - 1)$ but not by $g(x)$

(C) both $g(x)$ and $h(x)$ are divisible by $(x - 1)$ (D) None of these

Sol. $f(x) = g(x^3) + xh\,(x^3)$

Let $f_1(x) = 1 + x + x^2$

Clearly, the roots of $f_1(x) = 0$ and ω and ω^2 (where is a non-real cube root of unity). As $f_1(x)$ divides $f(x)$.

$\Rightarrow \quad f(\omega) = 0,\ f(\omega^2) = 0 \quad \Rightarrow \quad g(\omega^3) + \omega h(\omega^3) = 0$ and $g(\omega^6) + \omega^2 n(\omega^6) = 0$

$\Rightarrow \quad g(1) + \omega h(1) = 0,\ g(1) + \omega^2 h(1) = 0 \quad \Rightarrow \quad 2g(1) + h(1)\ (\omega + \omega^2) = 0$

$\Rightarrow \quad 2g(1) - h(1) = 0 \quad \Rightarrow \quad h(1) = 2g(1)$

$\Rightarrow \quad g(1) + \omega.\ 2g(1) = 0 \quad \Rightarrow \quad g(1)\ (1 + 2\omega) = 0 \ \Rightarrow \ g(1) = 0 \ \Rightarrow \ g(1) = 0$

$\Rightarrow$ $x = 1$ is the root of $g(x) = 0$ and $h(x) = 0$. Thus $g(x)$ and $h(x)$ both are divisible by $x - 1$.

Hence (C) is the correct answer.

Q.4 Find z such that $|z - 2 + 2i| \le 1$ and z has

(i) least absolute value (ii) numerically least amplitude.

Sol.

(i) Given equation represents a circle with centre $(2, -2)$ and radius 1.

Distance of $|z - 2 + 2i| = 1$ will be minimum from origin along the line which is normal to the circle passing through the origin, so it will passing through the centre of the circle also. $OP = \left(2\sqrt{2} - 1\right)$

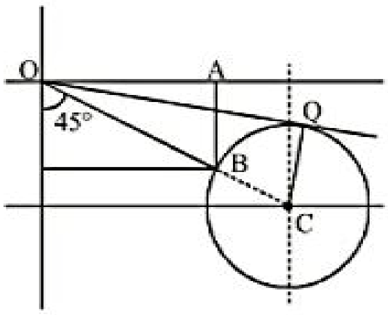
$OA = OP \cos 45°$ and $OB = OP \sin 45°$ so that

$$z = \frac{\left(2\sqrt{2}-1\right)}{\sqrt{2}} - i\left(\frac{2\sqrt{2}-1}{\sqrt{2}}\right)$$

(ii) We have to find the complex number z represent by Q where OQ is a tangent to the circle

Now $OQ = \sqrt{OC^2 - CQ^2} = |2 - 2i|^2 = 2\sqrt{2^2 + (-2)^2 - 1} = \sqrt{7}$

$\therefore \quad |z| = \sqrt{7}$



$\angle QOX = |\angle COX - \angle COQ|$

$$= -\left|\frac{\pi}{4} - \sin^{-1}\frac{1}{|2-2i|}\right| = -\left(\frac{\pi}{4} - \sin^{-1}\frac{1}{2\sqrt{2}}\right)$$

$$\Rightarrow \quad \text{amp } z = -\left(\frac{\pi}{4} - \sin^{-1}\frac{1}{2\sqrt{2}}\right)$$

amp $z = |z|\ \{\cos(\text{amp } z) + i \sin(\text{amp } z)\}$

$$= \text{amp } z = \left|\cos\left\{-\left(\frac{\pi}{4} - \sin^{-1}\frac{1}{2\sqrt{2}}\right)\right\} + i\sin\left\{-\left(\frac{\pi}{4} - \sin^{-1}\frac{1}{2\sqrt{2}}\right)\right\}\right|$$

$$= \text{amp } z = \left|\cos\left(\frac{\pi}{4} - \sin^{-1}\frac{1}{2\sqrt{2}}\right) - i\sin -\left(\frac{\pi}{4} - \sin^{-1}\frac{1}{2\sqrt{2}}\right)\right|.$$

Q.5 Plot the region represented by $\frac{\pi}{3} \le \arg\left(\frac{z+1}{z-1}\right) \le \frac{2\pi}{3}$ in the Argand plane.

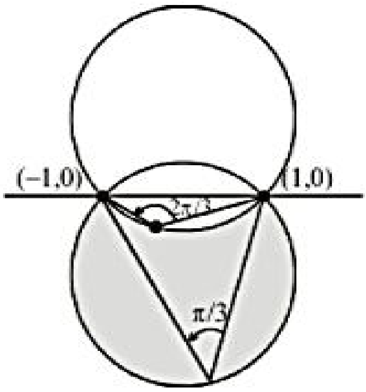
Sol. Let us take $\arg\left(\frac{z+1}{z-1}\right) = \frac{2\pi}{3}$,

Clearly z lies on the minor arc of the circle passing through (1, 0) and (– 1, 0).

Similarly, $\arg\left(\frac{z+1}{z-1}\right) = \frac{\pi}{3}$ means that 'z' is lying on the major arc of the circle passing through (1, 0) and (–1, 0).

Now if we take any point in the region included between the two arcs, say $P_1(z_1)$,

we get $\frac{\pi}{3} \le \arg\left(\frac{z+1}{z-1}\right) \le \frac{2\pi}{3}$.



Thus $\frac{\pi}{3} \le \arg\left(\frac{z+1}{z-1}\right) \le \frac{2\pi}{3}$ represents the shaded region (excluding the points (1, 0) and (– 1, 0).

Q.6 Consider the complex numbers $z_1 = 10 + 6i$ and $z_2 = 4 + 2i$. If $\arg\left(\frac{z-z_1}{z-z_2}\right) = \frac{\pi}{4}$, find the centre and radius of the circle traced by the complex number z.

Sol. Let $O(z_0)$ be the centre of the circle.

We hve $\angle AOB = \frac{\pi}{2}$. $AB = |z_1 - z_2| = |6+4i| = \sqrt{52}$.

Let $OA = OB = r \Rightarrow AB = r\sqrt{2} \Rightarrow r = \sqrt{26}$

Also $\frac{z_2 - z_0}{z_1 - z_0} = e^{-i\pi/2} = -i \Rightarrow z_2 - z_0 = -i(z_1 - z_0)$

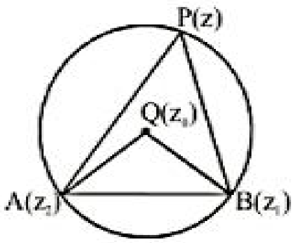
$\Rightarrow \quad z_0 = \frac{1}{2}(z_2 - iz_2 + iz_1 + z_1) = 5 + 7i.$

Alternative:

We have $\frac{\pi}{4} = \arg\frac{z-z_1}{z-z_2} = \arg(z-z_1) - \arg(z-z_2)$

$\Rightarrow \quad \tan^{-1} 1 = \tan^{-1}\frac{y-6}{x-10} - \tan^{-1}\frac{y-2}{x-4}$

$$\Rightarrow \quad \tan^{-1} 1 = \tan^{-1}\left[\frac{\frac{y-6}{x-10} - \frac{y-2}{x-4}}{1+\frac{(y-6)(y-2)}{(x-10)(x-4)}}\right]$$



$$\Rightarrow \quad 1 = \frac{6y-4x+4}{x^2+y^2-14x-8y+52}$$

$$\Rightarrow \quad x^2 + y^2 - 10x - 14y + 48 = 0$$

or $\quad (x-5)^2 + (y-7)^2 = 26 = \left(\sqrt{26}\right)^2$

or $\quad |x-5+i(y-7)| = \sqrt{26} \Rightarrow |z-5-7i| = \sqrt{26}$.

Q.7 Find all complex numbers z for which $\arg\left(\frac{3z-6-3i}{2z-8-6i}\right) = \frac{\pi}{4}$ and $|z-3+i| = 3$.

Sol. Now $\frac{3z-6-3i}{2z-8-6i} = \frac{3(x+iy)-6-3i}{2(x+iy)-8-6i}$

$$= \frac{3x-6+i(3y-3)}{(2x-8)+i(2y-6)} \frac{[(2x-8)-i(2y-6)]}{[(2x-8)-i(2y-6)]}$$

$$= \frac{6x^2+6y^2-36x-24y+66}{(2x-8)^2+(2y-6)^2} + \frac{(12x-12y-12)}{(2x-8)^2+(2y-6)^2} = a + ib \quad \text{(say)}$$

Since $\arg(a+ib) = \frac{\pi}{4}$ $\therefore$ $\tan\frac{\pi}{4} = \frac{b}{a}$ $\Rightarrow a = b$

$\Rightarrow \quad 6x^2 + 6y^2 - 36x - 24y + 66 = 12x - 12y - 12$

$\Rightarrow \quad x^2 + y^2 - 8x - 2y + 13 = 0$ ...(i)

Again $|z-3+i| = 3 \Rightarrow |x+iy-3+i| = 3$

$\Rightarrow \quad (x-3)^2 + (y+1)^2 = 9 \Rightarrow x^2 + y^2 - 6x + 2y + 1 = 0$ ...(ii)

(i) – (ii), $\Rightarrow -2x - 4y + 12 = 0$

$\Rightarrow x = -2y + 6$ ...(iii)

Putting the value of x in (ii), we get

$(-2y+6)^2 + y^2 - 6(-2y+6) + 2y + 1 = 0$

$\Rightarrow \quad 5y^2 - 10y + 1 = 0$

$\therefore \quad y = -\frac{10 \pm 4\sqrt{5}}{10} = 1 \pm \frac{2}{\sqrt{5}} \qquad \therefore \quad x = -2y + 6 = 4 \mp \frac{-4}{\sqrt{5}}$

$\therefore \quad z = x + iy = 4 \mp \frac{4}{\sqrt{5}} + i\left(1 \pm \frac{2}{\sqrt{5}}\right)$

Q.8 If $|a_n| < 2$ for $n = 1, 2, 3, \ldots\ldots$ and $1 + a_1z + a_2z^2 + \ldots\ldots + a_N z^N = 0$, show that z does not lie in the interior of the circle $|z| = \frac{1}{3}$

Sol. Given $|a_n| < 2$ for $n = 1, 2, 3, \ldots\ldots$

Again given $1 + a_1z + a_2z^2 + \ldots\ldots + a_Nz^N = 0$

$\Rightarrow \quad -1 = a_1z + a_2z^2 + \ldots\ldots + a_Nz^N$

$\Rightarrow \quad |-1| = |a_1z + a_2z^2 + \ldots\ldots + a_Nz^N| = |a_1|\ |z| + |a_2z^2| + \ldots\ldots |a_N|\ |z^N|$

$= |a_1|\ |z| + |a_2|\ |z|^2 + \ldots\ldots |a_N|\ |z|^N < 2\,|z| + 2|z|^2 + \ldots\ldots 2|z|^N$ [from (i)]

$< 2\,|z| + 2|z|^2 + \ldots\ldots + \text{to } \infty$ [From (iii), $|z| \neq 0$]

Case-I : When $|z| < 1$.

From (iii), $\frac{2|z|}{1-|z|} \quad \Rightarrow \quad 1 - |z| < 2|z| \quad \Rightarrow \quad 3|z| > 1 \quad \Rightarrow \quad |z| > \frac{1}{3}$

Case-II : When $|z| \geq 1$. I this case $|z| > \frac{1}{3}$ $[\because\ |z| \geq 1]$

Hence z does not lie in the interior of the circle $|z| = \frac{1}{3}$.

Note : Above question can also be asked as : if $|a_n| < 2$ for $1 \leq n \leq N$, then prove that there exists no z in the interior of the circle $|z| = \frac{1}{3}$ such that

$1 + a_1z + a_2z^2 + \ldots\ldots + a_Nz^N = 0$.

Q.9 Consider a triangle formed by the points $A\left(\frac{2}{\sqrt{3}} e^{i\frac{\pi}{2}}\right)$, $B\left(\frac{2}{\sqrt{3}} e^{-i\frac{\pi}{6}}\right)$, $C\left(\frac{2}{\sqrt{3}} e^{-i\frac{5\pi}{6}}\right)$. Let P (z) be any point on it's in-circle. Prove that $AP^2 + BP^2 + CP^2 = 5$.

Sol. Let $z_1 = \frac{2}{\sqrt{3}} e^{i\frac{\pi}{2}}, z_2 = \frac{2}{\sqrt{3}} e^{-i\frac{\pi}{6}}, z_3 = \frac{2}{\sqrt{3}} e^{-i\frac{5\pi}{6}}$.

Clearly the points lie on the circle $|z| = \frac{2}{\sqrt{3}}$

If l the length of side of the ΔABC, $AD = l \sin 60° = l\frac{\sqrt{3}}{2} \Rightarrow OD = \frac{1}{3}\ AD = \frac{l\sqrt{3}}{3}$

Now, $OA = \frac{2}{3}\ AD = \frac{2}{3} \cdot \frac{l\sqrt{3}}{2} = \frac{2}{\sqrt{3}}$

$\Rightarrow \quad l = 2 \text{ and } OD = \frac{2\sqrt{3}}{6} = \frac{1}{\sqrt{3}}$

Let P(z) be any point on the in circle, is $|z| = \frac{1}{\sqrt{3}}$

$\Rightarrow \quad z = \frac{1}{\sqrt{3}} e^{i\theta}$

Now, $AP^2 = |z - z_1|^2 = |z|^2 + |z_1|^2 - (z\bar{z}_1 + \bar{z}z_2)$

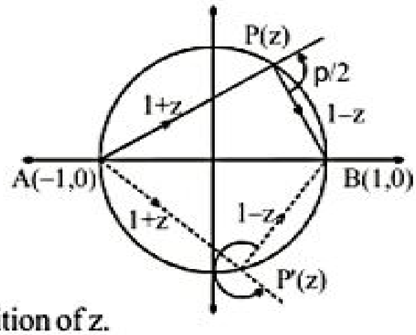
Similarly, $BP^2 = |z|^2 + |z_2|^2 - (z\bar{z}_2 + \bar{z}z_2)$ and $CP^2 = |z|^2 + |z_3|^2 - (z\bar{z}_3 + \bar{z}z_3)$

$AP^2 + BP^2 + CP^2 = 3|z|^2 + |z_1|^2 + |z_2|^2 + |z_3|^2 - z(\bar{z}_1 + \bar{z}_2 + \bar{z}_3) - \bar{z}(z_1 + z_2 + z_3)$

$= 1 + 3 \cdot \frac{4}{3} - z(0) - \bar{z}(0) = 5.$

Q.10 If $|z| = 1$, then prove that the points represented by $\sqrt{\frac{1+z}{1-z}}$ lie on one or other of two fixed perpendicular straight lines.

Sol. Since $|z| = 1$. z lies on a unit circle having centre at the origin



$$\arg\left(\frac{1+z}{1-z}\right) = +\frac{\pi}{2} \text{ or } +\frac{3\pi}{2}$$

$$\Rightarrow \quad \frac{1+z}{1-z} = ke^{i\pi/2} \text{ or } ke^{i3\pi/2}$$

where k is a real parameter and its value depends upon the position of z.

Let $\alpha = \sqrt{\frac{1+z}{1-z}} \quad \Rightarrow \quad \alpha = \sqrt{k}\, e^{i\pi/4}$ or $\sqrt{k}\, e^{i3\pi/4}$.

$\Rightarrow$ α lies on one or other of the two perpendicular lines.

Q.11 Dividing f(z) by $z - i$, we obtain the remainder i and dividing it by $z + i$, we get remainder $1 + i$. Find the remainder upon the division of f(z) by $z^2 + 1$.

Sol. $z - i = 0 \quad \Rightarrow \quad z = i$

Remainder when $f(z)$ is divided by $(z - i) = f(i)$

Similarly remainder when f(z) is divided by $(z + i) = f(-i)$ ...(i)

According to question $f(i) = i$

and $f(-i) = 1 + i$ ...(ii)

Since $z^2 + 1$ is a quadratic expression, therefore remainder when f(z) is divided by $z^2 + 1$ will be in general a linear expression.

Let g(z) be the quotient and $az + b$ the reaminder when g(z) is divided by $z^2 + 1$.

then $f(z) = g(z)(z^2 + 1) + az + b$ ...(iii)

$\therefore$ $f(i) = g(i)(i^2 + 1) + ai + b = ai + b$ ...(iv)

and $f(-i) = g(-i)(i^2 + 1) - ai + b = -ai + b$ ...(v)

From (i) and (iv), we have $b + ai = i$ ...(vi)

From (ii) and (v), we have $b - ai = 1 + i$ ...(vii)

Solving (vi) and (vii) we get , $b = \frac{1}{2} + i$ and $a = \frac{i}{2}$

Hence required remainder $= az + b = \frac{1}{2}iz + \frac{1}{2} + i$

Paragraph for question nos. 12 to 14

Let A, B, C be three sets of complex numbers as defined below.

$$A = \{z : |z+1| \le 2 + \text{Re}(z)\},\quad B = \{z : |z-1| \ge 1\} \text{ and } C = \left\{z : \left|\frac{z-1}{z+1}\right| \ge 1\right\}$$

Q.12 The number of point(s) having integral coordinates in the region $A \cap B \cap C$ is

(A) 4 (B) 5 (C) 6 (D) 10

Q.13 The area of region bounded by $A \cap B \cap C$ is

(A) $2\sqrt{3}$ (B) $\sqrt{3}$ (C) $4\sqrt{3}$ (D) 2

Q.14 The real part of the complex number in the region $A \cap B \cap C$ and having maximum amplitude is

(A) -1 (B) $\frac{-3}{2}$ (C) $\frac{1}{2}$ (D) -2

Sol. For A, $|z+1| \le 2 + \text{Re}(z)$

$\Rightarrow \quad (x+1)^2 + y^2 \le 4 + 4x + x^2$

$\Rightarrow \quad y^2 \le 3 + 2x$

$\Rightarrow \quad y^2 \le 2\left(x + \frac{3}{2}\right) \qquad .....(1)$

For B, $|z-1| \ge 1$

$\Rightarrow \quad (x-1)^2 + y^2 \ge 1 \qquad ......(2)$

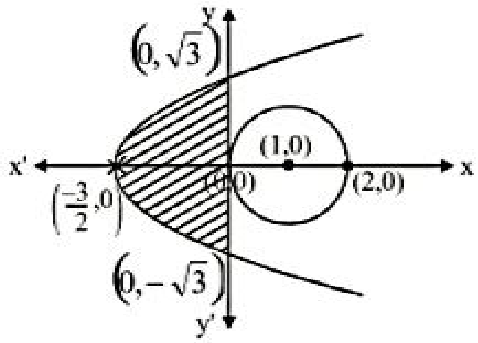
For C, $|z-1|^2 \ge |z+1|^2$

$\Rightarrow \quad (z-1)(\bar{z}-1) \ge (z+1)(\bar{z}+1)$

$\Rightarrow \quad (z\bar{z} - \bar{z} - z + 1) \ge (z\bar{z} + \bar{z} + z + 1)$

$\Rightarrow \quad z + \bar{z} \le 0$

i.e. $\quad x \le 0 \qquad .....(3)$



(i) $(-1,0), (-1,1), (-1,-1), (0,0), (0,1), (0,-1)$ but $z = -1$ is not in the domain in set C

$\therefore$ Total number of point(s) having integral coordinates in the region $A \cap B \cap C$ is 5.

(ii) Required area $= 2\int_{-3/2}^{0} \sqrt{2\left(x + \frac{3}{2}\right)}\, dx = 2\sqrt{3}$ (square units)

(iii) Clearly $z = \frac{-3}{2} + i0$ is the complex number in the region $A \cap B \cap C$ and having maximum amplitude.

$\therefore \quad \text{Re}(z) = \frac{-3}{2}$

Paragraph for question nos. 15 to 17

Let $A(z_1)$ be the point of intersection of curves $\arg(z-2+i)=\frac{3\pi}{4}$ and $\arg\left(z+i\sqrt{3}\right)=\frac{\pi}{3}$.

$B(z_2)$ be the point on the curve $\arg\left(z+i\sqrt{3}\right)=\frac{\pi}{3}$ such that $|z_2-5|$ is minimum and $C(z_3)$ be the centre of circle $|z-5|=3$.

[Note : $i^2=-1$**]**

Q.15 The area of triangle ABC is equal to

(A) $4\sqrt{3}$ (B) $\frac{3\sqrt{3}}{2}$ (C) $2\sqrt{3}$ (D) 4

Q.16 The equation of straight line passing through origin and perpendicular to line joining $A(z_1)$ and $B(z_2)$ on the complex plane is equal to

(A) $z=\lambda\left(2+i\sqrt{3}\right)$ (B) $z=\lambda\left(-\sqrt{3}+i\right)$ (C) $z=\lambda\left(1+i\sqrt{3}\right)$ (D) $z=\lambda\left(\sqrt{3}+i\right)$

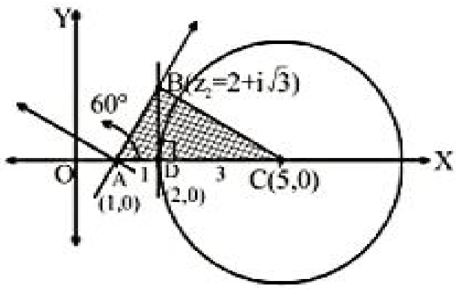
(where λ is real parameter.)

Q.17 If $|z-z_1|=1$ and $\omega=\text{Re}(z+2)$ then ω lie on

(A) real axis.
(B) line not passing through the origin.
(C) line segment joining (2, 0) and (4, 0).
(D) circle centred at (1, 0) and radius 2.

Sol.(i) Clearly, $A(z_1)$ is the point of intersection of $\arg(z-2+i)=\frac{3\pi}{4}$ and $\arg\left(z+i\sqrt{3}\right)=\frac{\pi}{3}$

$\Rightarrow \quad z_1=1$



Also, $B(z_2)$ is the point on $\arg\left(z+i\sqrt{3}\right)=\frac{\pi}{3}$

such that $|z_2-5|=\text{minimum}$, so $z_2=2+i\sqrt{3}$.

Ans $C(z_3)$ be the centre of the circle $|z-5|=3\Rightarrow z_3=5$.

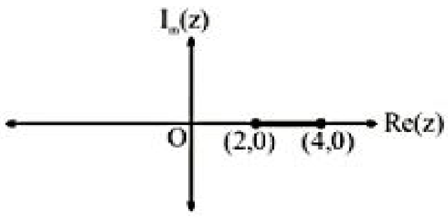
Hence, area $(\Delta ABC)=\frac{1}{2}(AC)(AD)=\frac{1}{2}(4)\left(\sqrt{3}\right)=2\left(\sqrt{3}\right)$ (square units.)

(ii) Clearly, required equation of straight line is

$z=\lambda i(z_2-z_1)=\lambda i\left(2+i\sqrt{3}-1\right)=\lambda i\left(1+i\sqrt{3}\right)=\lambda\left(-\sqrt{3}+i\right)$.

(iii) We have

$|z-1|=1\Rightarrow z=1+e^{i\theta},\ \theta\in[0,2\pi)$



$\Rightarrow \quad z=1+e^{i\theta}=1+\cos\theta+i\sin\theta$

$\therefore \quad \omega=\text{Re}(z+2)=3+\cos\theta=\text{purely real}$

Clearly, locus of ω is the line segment joining (2, 0) and (4, 0). **Ans.**

Q.18 z_1 & z_2 are two distinct points in an argand plane. If $a\,|z_1| = b\,|z_2|$, (where a, b $\in$ R) then prove that the point $\dfrac{a\,z_1}{b\,z_2} + \dfrac{b\,z_2}{a\,z_1}$ lies on the line segment [–2, 2] of the real axis.

Sol. Assuming $\arg z_1 = \theta$ and $\arg z_2 = \theta + \alpha$.

$$\frac{a\,z_1}{bz_2} + \frac{bz_2}{az_1} = \frac{a\,|z_1|\,e^{i\theta}}{b\,|z_2|\,e^{i(\theta+\alpha)}} + \frac{b\,|z_2|\,e^{i(\theta+\alpha)}}{a\,|z_1|\,e^{i\theta}} = e^{-i\alpha} + e^{i\alpha} = 2\cos\alpha$$

Alternatively: Let $\alpha = \dfrac{az_1}{bz_2}$; $\quad \dfrac{1}{\alpha} = \dfrac{bz_2}{az_1}$; Also $|\alpha| = \dfrac{|az_1|}{|bz_2|} = \dfrac{a\,|z_1|}{b\,|z_2|} = 1 \Rightarrow \alpha = \dfrac{1}{\overline{\alpha}}$

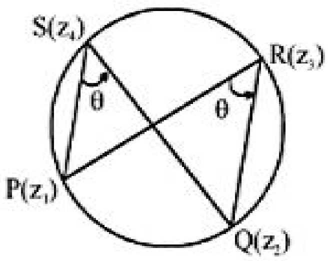
$$\Rightarrow \alpha + \frac{1}{\alpha} = \alpha + \overline{\alpha} = 2\,\text{Re.}(\alpha) = 2\cos\alpha$$

Q.19 If z_1, z_2, z_3 are complex numbers such that $\dfrac{2}{z_1} = \dfrac{1}{z_2} + \dfrac{1}{z_3}$, show that the points represented by z_1, z_2, z_3 lie on a circle passing through the origin.

Sol. Since $P(z_1)$, $Q(z_2)$, $P(z_3)$ and $S(z_4)$ are concyclic points,

$$\angle PSQ = \angle PRQ \qquad \Rightarrow \qquad \arg\frac{z_2 - z_4}{z_1 - z_4} = \arg\frac{z_2 - z_3}{z_1 - z_3}$$

$$\Rightarrow \qquad \arg\left[\left(\frac{z_2 - z_4}{z_1 - z_4}\right)\left(\frac{z_1 - z_3}{z_2 - z_3}\right)\right] = 0$$



$$\Rightarrow \qquad \frac{(z_2 - z_4)}{(z_1 - z_4)} \cdot \frac{(z_1 - z_3)}{(z_2 - z_3)} = \text{real}$$

If $\quad z_4 = 0 + i0$, then $\dfrac{z_2}{z_1} \cdot \dfrac{z_1 - z_3}{z_2 - z_3} = \text{real}$ ....(1)

We have $\dfrac{2}{z_1} = \dfrac{1}{z_2} + \dfrac{1}{z_3}$ from which $z_3 = \dfrac{z_1 z_2}{2z_2 - z_1}$ .....(2)

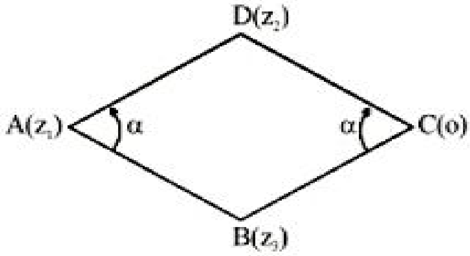
From (1) and (2), $\dfrac{z_2}{z_1} \times \dfrac{z_1 - \dfrac{z_1 z_2}{2z_2 - z_1}}{z_2 - \dfrac{z_1 z_2}{2z_2 - z_1}} = \text{real}$

$$\Rightarrow \qquad \frac{z_2 - z_1}{2(z_2 - z_1)} = \text{real} \Rightarrow \frac{1}{2} = \text{real, which is true.}$$

Therefore, z_1, z_2, z_3 and the origin are concylic.

Alternative

$$\frac{2}{z_1} = \frac{1}{z_2} + \frac{1}{z_3} \quad \Rightarrow \quad \frac{1}{z_1} - \frac{1}{z_2} = \frac{1}{z_3} - \frac{1}{z_1}$$



$$\Rightarrow \quad \frac{z_2 - z_1}{z_1 z_2} = \frac{z_1 - z_3}{z_1 z_3} \quad \Rightarrow \quad \frac{z_2 - z_1}{z_3 - z_1} = -\frac{z_2}{z_3}$$

$$\Rightarrow \quad \arg\left(\frac{z_2 - z_1}{z_3 - z_1}\right) = \arg\left(-\frac{z_2}{z_3}\right) = \pi + \arg\left(\frac{z_2}{z_3}\right)$$

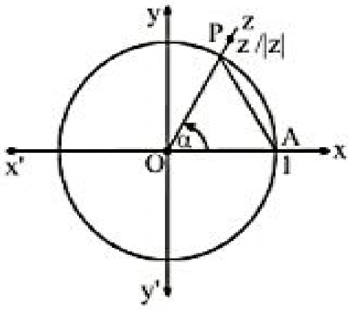
$\Rightarrow \quad \alpha = \pi - \beta \Rightarrow \alpha + \beta = \pi$

$\Rightarrow \quad$ points A, B, C, D are concyclic.

Q.20 Prove the following inequalities geometrically and analytically :

(a) $\left|\frac{z}{|z|} - 1\right| \le |\arg z|$ (b) $|z - 1| \le ||z| - 1| + |z||\arg z|$.

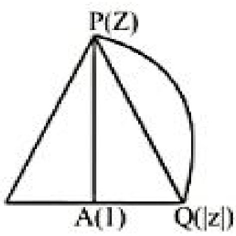
Sol.(a) The quantity $\frac{z}{|z|}$ lies on a unit circle centered at the origin.



It is clear that $\left|\frac{z}{|z|} - 1\right| = AP \le \text{arc }(AP) = \alpha = \arg(z)$

If α is negative, this gives $\left|\frac{z}{|z|} - 1\right| \le |\arg z|$.

To show this analytically, put $z = r(\cos\alpha + i\sin\alpha)$,



where $r = |z|$. Then $\left|\frac{z}{|z|} - 1\right| = |(\cos\alpha - 1) + i\sin\alpha|$

$$= \left[(\cos\alpha - 1)^2 + \sin^2\alpha\right]^{1/2} = [2(1 - \cos\alpha)]^{1/2}$$

$$= \left(4\sin^2\frac{\alpha}{2}\right)^{1/2} = 2\left|\sin\frac{\alpha}{2}\right| \le 2\left|\frac{\alpha}{2}\right| = |\alpha| = |\arg z| \cdot [\sin\theta \le \theta \text{ if } \theta \ge 0]$$

(b) Referring to the show figure and usign the result of part (a),

we have $|z-1| = AP \le AQ + QP = ||z|-1| + |z-|z||$

$$\Rightarrow \quad |z-1| \le ||z|-1| + |z|\left|\frac{z}{|z|}-1\right| \le ||z|-1| + |z||\arg z|.$$

Analytically, we can get this result as follows :

$$|z-1| = |(z-|z|)| + (|z|-1) \le |z-|z|| + ||z|-1|$$

$$\Rightarrow \quad |z-1| \le ||z|-1| + |z|\left|\frac{z}{|z|}-1\right| \le ||z|-1| + |z||\arg z|.$$

Q.21 If $z = 2 + t + i\sqrt{3-t^2}$, where t is real and $t^2 < 3$, show that the modulus of $\frac{(z+1)}{(z-1)}$ is independent of t. Also show that the locus of the points z for different values of t is a circle and fnd its centre and radius.

Sol. We have $\frac{z+1}{z-1} = \frac{3+i+i\sqrt{3-t^2}}{1+t+i\sqrt{3-t^2}}$

$$\therefore \quad \left|\frac{z+1}{z-1}\right|^2 = \frac{(3+t)^2+3-t^2}{(1+t^2)^2+3-t^2} = \frac{6(t+2)}{2(t+2)} = 3 \qquad \text{(independent of r)}$$

Let $z = x + iy$, then $x + iy = 2 + t + i\sqrt{3-t^2}$

Equating real and imaginary parts, we get

$x = 2 + t$ ...(i) and $y = \sqrt{3-t^2}$ ...(ii)

[In order to find the locus of z we will have a eliminate t]

Putting the value of t from (i) and (ii), we get

$y^2 = 3 - (x-2)^2$ or $(x-2)^2 + y^2 = 3$

Thus locus of z is a circle whose centre is (2, 0) and radius is $\sqrt{3}$.

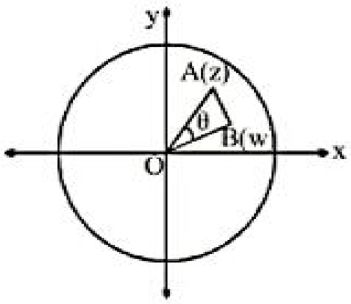
Q.22 $|z| \le 1$, $|w| \le 1$, show that $|z - w|^2 \le (|z| - |w|)^2 + (\arg z - \arg w)^2$.

Sol. Let us consider a unit circle will its centre as the origin.

Let $\angle AOX = \theta_1$ and $\angle BOX = \theta_2$

$\therefore \arg(z) = \theta_1$ and $\arg(\omega) = \theta_2$.

$z = \overline{OA}$, $w = \overline{OB}$



Now in ΔOAB,

$$\cos\theta = \frac{|\overline{OA}|^2 + |\overline{OB}|^2 - |\overline{BA}|^2}{2|\overline{OA}||\overline{OB}|^2}$$

$$\Rightarrow |\overline{BA}|^2 = |\overline{OA}|^2 + |\overline{OB}|^2 - 2|\overline{OA}||\overline{OB}|\cos\theta$$

$$\Rightarrow \quad |z - w|^2 = |z|^2 + |w|^2 - 2|z||w|\cos\theta = (|z| - |w|)^2 - 4|z||w|\sin^2\left(\frac{\theta}{2}\right)$$

We know $\sin\frac{\theta}{2} \le \frac{\theta}{2}$ $\quad \left[0 \le \frac{\theta}{2} < \frac{\pi}{2}\right]$

Hence $|z - w|^2 \le (|z| - |w|)^2 + (\arg z - \arg w)^2$

RELATION

This chapter deals with establishing binary relation between elements of one set and elements of another set according to some particular rule of relationship.

1. CARTESIAN PRODUCT:

The Cartesian product of two sets A, B is a non-void set of all ordered pairs (a, b), where $a \in A$ and $b \in B$. This is denoted by $A \times B$

$\therefore \quad A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\}$

e.g. $A = \{1, 2\}, \quad B = \{a, b\}$

$A \times B = \{(1, a), (1, b), (2, a), (2, b)\}$

Note : (i) $A \times B \neq B \times A$ (Non-commutative)

(ii) $n(A \times B) = n(A)\, n(B)$ and $n(P(A \times B)) = 2^{n(A)n(B)}$

(iii) $A = \phi$ and $B = \phi \Leftrightarrow A \times B = \phi$

(iv) If A and B are two non-empty sets having n elements in common then $(A \times B)$ and $(B \times A)$ have n^2 elements in common.

(v) $A \times (B \cup C) = (A \times B) \cup (A \times C)$

(vi) $A \times (B \cap C) = (A \times B) \cap (A \times C)$

(vii) $A \times (B - C) = (A \times B) - (A \times C)$

Illustration :

If $n(A) = 7$, $n(B) = 8$ and $n(A \cap B) = 4$, then match the following columns.

(i)	$n(A \cup B)$	*(a)*	56
(ii)	$n(A \times B)$	*(b)*	16
(iii)	$n((B \times A) \times A)$	*(c)*	392
(iv)	$n((A \times B) \cap (B \times A))$	*(d)*	96
(v)	$n((A \times B) \cup (B \times A))$	*(e)*	11

Sol. *(i)* $n(A \cup B) = n(A) + n(B) - n(A \cap B) = 7 + 8 - 4 = 11$ *Ans. (e)*

(ii) $n(A \times B) = n(A)\, n(B) = 7 \times 8 = 56 = n(B \times A)$ *Ans. (a)*

(iii) $n((B \times A) \times A) = n(B \times A).\, n(A) = 56 \times 7 = 392$ *Ans. (c)*

(iv) $n((A \times B) \cap (B \times A)) = (n((A \cap B))^2 = 4^2 = 16$ *Ans. (b)*

(v) $n((A \times B) \cup (B \times A)) = n(A \times B) + n(B \times A) - n(A \times B) \cap (B \times A)$

$= 56 + 56 - 16 = 96$ *Ans. (d)*

Illustration :

If A = {2, 4} and B = {3, 4, 5}, then $(A \cap B) \times (A \cup B)$ *is*

(1) {(2, 2), (3, 4), (4, 2), (5, 4)}

(2) {(2, 3), (4, 3), (4, 5)}

(3) {(2, 4), (3, 4), (4, 4), (4, 5)}

(4) {(4, 2), (4, 3), (4, 4), (4, 5)}

Sol. $A \cap B = \{4\}$ *and* $A \cup B = \{2, 3, 4, 5\}$

$\therefore$ $(A \cap B) \times (A \cup B) = \{(4, 2), (4, 3), (4, 4), (4, 5)\}$ *Ans. (4)*

2. RELATION:

Every non-zero subset of A × B defined a relation from set A to set B.

If R is a relation from $A \to B$

R : {(a, b) | (a, b) $\in$ A × B and a R b}

Highlights :

Let A and B be two non empty sets and R : $A \to B$ be a relation such that R : {(a, b) | (a, b) $\in$ R, a $\in$ A and b $\in$ B}.

(i) 'b' is called image of 'a' under R.

(ii) 'a' is called pre-image of 'b' under R.

(iii) **Domain of R :** Collection of all elements of A which has a image in B or Set of all first entries in A × B.

(iv) **Range of R :** Collection of all elements of B which has a pre-image in A or Set of all second entries in A × B.

Note :

(1) It is not necessary that each and every element of set A has a image in Set B and each and every element of set B has a preimage in Set A

(2) Elements of set A having image in B is not necessary unique.

(3)Basically relation is the number of subsets of A × B

<u>number of relations</u> = no. of ways of selecting a non zero subset of A × B

$$= {}^{mn}C_0 + {}^{mn}C_1 + \ldots\ldots\ldots + {}^{mn}C_{mn}$$
$$= 2^{mn}$$

Illustration :

Given A = {1, 2, 3, 4, 5} and B = {2, 4, 5}. A relation defined

aRb $\Rightarrow$ *a and b are relatively prime or co-prime (i.e. HCF is 1),*

find domain and range of R.

Sol. *R = {(1, 2), (1, 4), (1, 5), (2, 5), (3, 2), (3, 4), (3, 5), (4, 5), (5, 2), (5, 4)}]*

Domain of R {1,2,3,4,5}

Range of R {2,4,5}

Illustration :

A = {Jaipur, Patna, Kanpur, Lucknow} and B = {Rajasthan, Uttar Pradesh, Bihar}

$aRb \Rightarrow$ *a is capital of b,* $a \in A$ and $b \in B$

Sol. *R = {(Jaipur, Rajasthan), (Patna, Bihar), (Lucknow, Uttar Pradesh) }*

Illustration :

If $A = \{1, 3, 5, 7\}$, $B = \{2, 4, 6, 8\}$

Relation is $aRb \Rightarrow a > b$, $a \in A$, $b \in B$

Sol. $R = \{(3, 2), (5, 2), (5, 4), (7, 2), (7, 4), (7, 6)\}$

Domain $= \{3, 5, 7\}$

Range $= \{2, 4, 6\}$

Representation of a Relation :

1. **Roster form :** In this form we represent set of all ordered pairs (a,b) such that $(a,b) \in R$ where $a \in A$, $b \in B$
2. **Set builder notation :** Here we denote the relation by the rule which co relates the two set
3. **Arrow - diagram (Mapping):** This the pictorial notation of any relation .

Illustration :

Let $A = \{-2, -1, 4\}$ $B = \{1, 4, 9\}$

A relation from A to B i.e. a R b is defined as a is less than b.

This can be represented in the following ways.

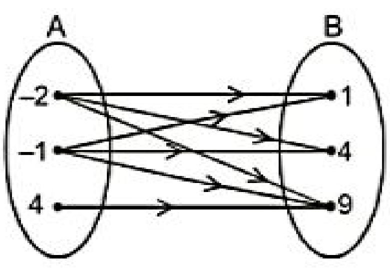
1. **Roster form :**

   $R = \{(-2, 1), (-2, 4), (-2, 9), (-1, 1), (-1, 4), (-1, 9), (4, 9)\}$

2. **Set builder notation :**

   $R = \{(a, b): a \in A \text{ and } b \in B, \text{ a is less than b}\}$

3. **Arrow - diagram :**



Empty relation: No elements of A is related to any elements of A.

Universal relation: Each elements of A is related to every element of A.

3. INVERSE RELATION:

If relation R is defined from A to B then the inverse relation would be defined from B to A, i.e

$R : A \rightarrow B \quad \Rightarrow \quad aRb$ where $a \in A, b \in B$

$R^{-1} : B \rightarrow A \quad \Rightarrow \quad bRa$ where $a \in A, b \in B$

Here Doman of R $=$ Range of R^{-1}

and Range of R $=$ Domain of R^{-1}

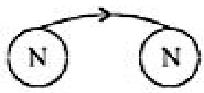
$\therefore \quad R^{-1} = \{(b, a) \mid (a, b) \in R\}$

A relation R is defined on the set of 1st ten natural numbers.

e.g. N is a set of first 10 natural nos. $\quad \therefore N = \{1, 2, 3, ..., 10\}$ & $a, b \in N$

$aRb \quad \Rightarrow \quad a + 2b = 10$

$R = \{(2, 4), (4, 3), (6, 2), (8, 1)\}$



$R^{-1} = \{(4, 2), (3, 4), (2, 6), (1, 8)\}$

4. IDENTITY RELATION:

A relation defined on a set A is said to be an Identity relation if each & every element of A is related to itself & only to itself.

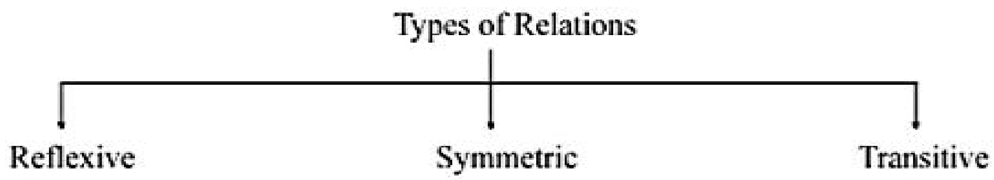
e.g. A relation defined on the set of natural nos. is

$aRb \quad \Rightarrow \quad a = b$ where a & $b \in N$

$R = \{(1, 1), (2, 2), (3, 3), ...\}$

R is an Identity relation

5. CLASSIFICATION OF RELATIONS:



(I) **<u>Reflexive</u>:** A relation defined on a set A is said to be an Identity relation if each & every element of A is related to itself.

i.e. if $(a, b) \in R$ then $(a, a) \in R$. However if there is a single ordered pair of $(a, b) \in R$ such $(a, a) \notin R$ then R is not reflexive.

e.g. A relation defined on (set of natural numbers)

aRb $\Rightarrow$ 'a' divides 'b' $a, b \in N$

R would always contain (a, a) because every natural number divides itself and hence it is a reflexive relation.

Note: Every Identity relation is a reflexive relation but every reflexive relation need not be an Identity.

(II) **Symmetric:** A relation defined on a set is said to be symmetric if aRb $\Rightarrow$ bRa.

If $(a, b) \in R$ then (b, a) must be necessarily there in the same relation.

EXAMPLES:

A relation defined on the set of lines.

(1) aRb $\Rightarrow$ $a \parallel b$

It is a symmetric relation because if line is || to 'b' then the line 'b' is || to 'a'.

where $(a, b) \in L$ {L is a set of || lines}

(2) L_1RL_2 $\Rightarrow$ $L_1 \perp L_2$ It is a symmetric relation

$L_1, L_2 \in L$ {L is a set of lines}

(3) aRb $\Rightarrow$ 'a' is borther of 'b' is not a symmetric relation as 'a' may be sister of 'b'.

(4) aRb $\Rightarrow$ 'a' is a cousin of 'b'. This is a symmetric relation.

If R is symmetric

(1) $R = R^{-1}$

(2) Rangle of R = Domain of R

(III) **Transitive:** A relation on set A is said to be transitive if aRb and bRc implies aRc

i.e. $(a, b) \in R$ and $(b, c) \in R$ then $(a, c) \in R$

Here a, b, c need not be distinct.

EXAMPLES:

(1) A relation R defined on a set of natural numbers N with rule aRb $\Rightarrow$ $a < b$

R: {(1, 2), (1, 1)}.

In this relation a, b, c are not distinct but it is transitive. It is neither reflexive nor symmetric as (2, 1) is missing. Minium number of ordered pair that must be added to make it reflexive, symmetric and transitive is 2 i.e. (2, 1) and (2, 2).

(2) Only Transitive $R = \{(x, y) \mid x < y, \quad x \in N, y \in N\}$

Only Symmetric $R = \{(x, y) \mid x + y = 10, \quad x \in N, y \in N\}$

6. EQUIVALENCE RELATION:

If a relation is Reflexive, Symmetric and Transitive then it is said to be an equivalence relation.

EXAMPLES:

(1) A relation defined on N

$xRy \Rightarrow x = y$

R is an equivalence relation.

(2) Relation defined on the set of integer (I)

Prove that: $xRy \Rightarrow (x - y)$ is even is an equivalence relation.

Asking: $A = \{1, 2, 3, 4\}$; $R = \{(1, 2), (2, 2), (1, 1), (4, 4), (1, 3), (3, 3), (3, 2)\}$ then

(A) $R \cap S \cap \overline{T}$ (B*) $R \cap T \cap \overline{S}$ (C) $S \cap T \cap \overline{R}$ (D) $R \cap S \cap T$

(3) $R = \{(1, 2), (2, 3)\}$ add minimum number of ordered pairs to make it an equivalence relation.

$\{(1, 1), (2, 2), (3, 3), (2, 1), (3, 2), (1, 3), (3, 1)\} = 7$

(4) $A = \{1, 2, 3, \ldots\ldots, 13, 14\}$

$R = \{(x, y) \mid 3x - y = 10\}$ $\overline{R} \cap \overline{S} \cap \overline{T}$

$R = \{(x, y) \mid x \text{ is factor of } y\}$ $R \cap \overline{S} \cap T$

$R = \{(x, y) \mid x \text{ is father of } y\}$ $\overline{R} \cap \overline{S} \cap \overline{T}$

MATHEMATICAL INDUCTION

INTRODUCTION :

In algebra, there are certain results that are formulated with n number of terms in them, where n is a natural number (i.e. a positive integer). Those results can be proved by a specific technique, known as the principle of mathematical induction. We use the symbol P(n) (read "P of n") to denote some proposition which depends on the positive integer n. For example, P(n) might denote the sum of the first n odd positive numbers, that is

$$1 + 3 + 5 + 7 + 9 + ... + (2n - 1) = n^2$$

where $n = 1, 2, 3, ..., n$.

• FIRST PRINCIPLE OF MATHEMATICAL INDUCTION :

The proof of the proposition P(n) by mathematical induction for all $n \in N$ consists of the following three steps :

Step-1. Verification step

Verify that the proposition P(n) is true for n = 1, i.e., the first natural number or the smallest positive integer. This is also called the basic step of the induction.

Step-2. Induction step

Assume that the proposition will also be true for some $n = k \geq 1$, i.e. we assume P(k) to be true. This is called the induction step.

Step-3. Generalization step

If P(k) is true, then prove that the proposition is also true for $n = (k + 1)$, which is the next positive integer (i.e. the next natural number), i.e. we have to prove that P(k + 1) must also be true. In this step we prove that the implication $P(k) \Rightarrow P(k + 1)$ is true.

Next we generalize the result by saying that, since the proposition is proved to be true for $n = k + 1$, then it must also be true for n = k, and hence the proposition will be true for all n belonging to the set of natural numbers.

• SECOND PRINCIPLE OF MATHEMATICAL INDUCTION :

Step-1. Verification step

Verify that the proposition P(n) is true for n = r, where r is some fixed integer.

Step-2. Induction step

Assume that the proposition P(n) is true for $n = r, r + 1, r + 2, ..., m$.

Step-3. Generalization step

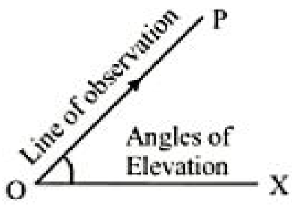
Prove that the proposition P(n) is true for $n = m + 1$. Thus, if true, we generalize the result by saying that since the proposition is true for $n = M + 1$, then it must also be true for $n = r, r + 1, r + 2, ..., m$ as assumed in Step 2. Thus, the proposition is true for all $n \geq r$ belonging to the set of natural numbers.

HEIGHT & DISTANCE

ANGLES OF ELEVATION AND DEPRESSION :

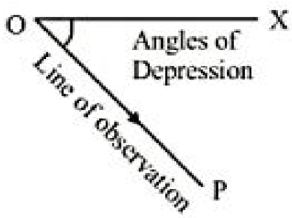
(i) Angle of elevation :

Let P be an object and OX be a horizontal line; an observer at O to perceive the object P has to elevate his eye from the direction OX to the direction OP. As such, we define ∠XOP as the **angle of elevation** of P at O.



(ii) Angle of depression :

Let P be an object and OX be a horizontal line; an observer at O to perceive the object P has to depress his eye from the direction OX to the direction OP. As such, we define ∠XOP as the **angle of depression** of P at O.



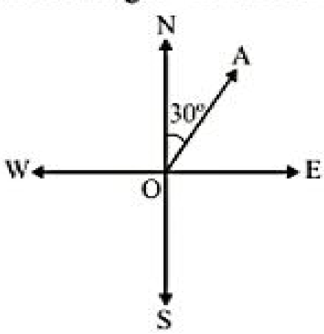
(iii) Angular elevation :

If PM is perpendicular to OX, then the angle XOP is also called the altitude or the **angular elevation** of MP at O.

(iv) Bearing of a point :

Let NS and EW stand for lines in the north-south and east-west directions respectively then the acute angle which the line OA makes with NS is called the **bearing** of the point A from O.

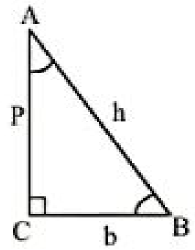
The bearing of A may be indicated precisely, by giving the size of the angle and specifying whether it is measured from ON (or OS) and whether to east (or west). For example, in figure, OA is in the direction 30° east of north. Thus, the bearing is written as N 30° E.



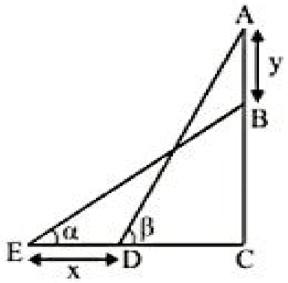
To express one side of a right angled triangle in terms of other sides :

Let $AB = h$ (hypotenuse) and let $\angle ABC = \theta$. The side opposite to θ is a perpendicular p and the remaining side is base b.

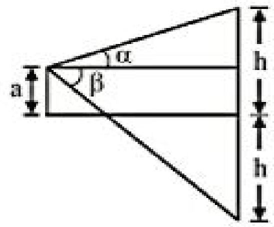
$b = p(\cot\theta)$
$p = b(\tan\theta)$
$p = h(\sin\theta)$
$b = h(\cos\theta)$



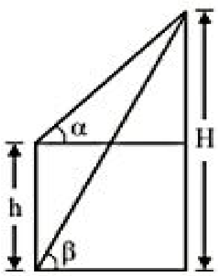
(a) If $AD = BE$, then $x = y\tan\left(\frac{\alpha+\beta}{2}\right)$



(b) $h = \frac{a\sin(\alpha+\beta)}{\sin(\beta-\alpha)}$



(c) $h = \frac{H\sin(\beta-\alpha)}{\cos\alpha\cos\beta}$ and $H = \frac{h\cot\alpha}{\cos\alpha-\cos\beta}$



STATISTICS

1. DEFINITION :

STATISTICS : A set of concepts, rules and procedures that help us to :

→ **Organize** numerical information in the form of tables, graphs and charts;

→ **Understand** statistical techniques underlying decisions that affect our lives and well-being; and

→ **Make** informed decisions.

2. DATA :

Facts, observations and information that come from investigations.

Generally three types of data are used

(i) Ungrouped data, Raw data or individual series :

(ii) Discrete frequency or ungrouped data :

Definition :

Data consist of n distinct values $x_1, x_2 \ldots\ldots, x_n$ occuring with frequency $f_1, f_2, \ldots\ldots, f_n$ respectively. This data in tabular form is called discrete freqency distribution.

(iii) Continuous frequency or grouped data :

Definition :

A continuous frequency Distribution is a series in which the data are classified into different class intervals without gaps along with their respective frequencies.

3. MEASURES OF CENTRAL VALUE :

Measure of central value gives rough idea about where data points are centred. Mean, mode, median are three measure of central tendency.

(A) MEAN :

The mean is the most common measure of central tendency and the one that can be mathematically manipulated. It is defined as the **average** of a distribution is equal to the $\Sigma X / N$. Simply, the mean is computed by summing all the scores in the distribution (ΣX) and dividing that sum by the total number of scores (N).

(I) Arithmetic mean of individual series (Ungrouped data) :

If the series in this case be $x_1, x_2, x_3, \ldots\ldots, x_n$; then the arithmetic mean $\bar{x}$ is given by

i.e., $$\bar{x} = \frac{\text{Sum of the series}}{\text{Number of terms}} = \frac{x_1 + x_2 + x_3 + \ldots\ldots + x_n}{N} = \frac{1}{N}\sum_{i=1}^{n} x_i .$$

(II) Arithmetic mean for discrete frequency distribution :

If the terms of the given series be $x_1, x_2, \ldots\ldots, x_n$ and the corresponding frequencies be $f_1, f_2, \ldots\ldots, f_n$, then the arithmetic mean $\bar{x}$ is given by,

$$\bar{x} = \frac{f_1x_1 + f_2x_2 + \ldots\ldots + f_nx_n}{N} = \frac{1}{N}\sum_{i=1}^{n} f_ix_i . \qquad \left(\sum_{i=1}^{n} f_i = N\right)$$

(III) Arithmetic mean for grouped or continuous frequency distribution :

$$\text{Arithmetic mean } (\bar{x}) = A + \frac{1}{N}\sum_{i=1}^{n} f_i(x_i - A),$$

where A = assumed mean, f = frequency and x − A = deviation of each item from the assumed mean.

(IV) Combined Arithmetic mean :

If $\bar{x}_i$ (i = 1, 2, ……, k) are the means of k-component series of sizes n_i, (i = 1, 2, …, k) respectively, then the mean $\bar{x}$ of the composite series obtained on combining the component series is given by the formula

$$\bar{x} = \frac{n_1\bar{x}_1 + n_2\bar{x}_2 + \ldots\ldots + n_k\bar{x}_k}{n_1 + n_2 + \ldots + n_k} = \frac{\sum_{i=1}^{n} n_i\bar{x}_i}{\sum_{i=1}^{n} n_i} .$$

(V) Weighted Arithmetic Mean :

Weighted arithmetic mean refers to the arithmetic mean calculated after assigning weights to different values of variable. It is suitable where the relative importance of difference items of variable is not same. Weighted Arithmetic Mean is give by

$$\bar{X}_W = \frac{\sum_{i=1}^{n} W_iX_i}{\sum_{i=1}^{n} W_i}$$

Properties of arithmetic mean :

If each of the values of a variable 'X' is increased of decreased by some constant k, then arithmetic mean also increased of decreased by k.

Similarly when the value of the variable 'X' are multiplied/divided by constant say k, arithmetic mean also multiplied /divided by the same quantity k.

Illustration :

The mean weight of 150 persons in a group is 60 kg. The mean weight of men in the group is 70 kg and that of the women is 55 kg. Find the number of men and women.

Sol. Number of person = 150; their mean weight = 60 kg;

mean weight of men $(\bar{x}_1) = 70$ kg and

mean weight of women $(\bar{x}_2) = 55$ kg

Let n_1 and n_2 be the number of men and number of women respectively.

We know that the total number of persons $(n_1 + n_2) = 150$ or $n_2 = 150 - n_1$.

We also know that the mean weight of all persons

$$(\bar{x}) = \frac{(n_1\bar{x}_1 + n_2\bar{x}_2)}{n_1 + n_2}$$

or $$60 = \frac{70n_1 + 55n_2}{150}$$

or $3n_1 = (1800 - 1650) = 150$

or $n_1 = 50$ and $n_2 = 100$

Illustration :

Find the mean of the following data :

Marks obtained	10 – 20	20 – 30	30 – 40	40 – 50	50 – 60	60 – 70	70 – 80
Number of students	2	3	8	14	8	3	2

Sol. <u>Method-1 :</u>

Marks obtained	Number of students	Mid - points	f_ix_i
10 – 20	2	15	30
20 – 30	3	25	75
30 – 40	8	35	280
40 – 50	14	45	630
50 – 60	8	55	440
60 – 70	3	65	195
70 – 80	2	75	150
	40		1800

$$N = \sum_{i=1}^{7} f_i = 40,\ \sum_{i=1}^{7} f_i x_i = 1800$$

$$\bar{x} = \frac{1}{N}\sum_{i=1}^{7} f_i x_i = \frac{1800}{40} = 45$$

Method-2 :

Asumed mean $a = \frac{10+80}{2} = 45, \quad h = 10$

Marks obtained	Number of students	Mid - points	$d_i = \frac{x_i - 45}{10}$	$f_i d_i$
10 – 20	2	15	–3	–6
20 – 30	3	25	–2	–6
30 – 40	8	35	–1	–8
40 – 50	14	45	0	0
50 – 60	8	55	1	8
60 – 70	3	65	2	6
70 – 80	2	75	3	6
	40			0

$$\bar{x} = a + \frac{\sum_{i=1}^{7} f_i d_i}{N} = 45 + \frac{0}{40} \times 10 = 45$$

(B) MEDIAN :

(a) **Definition :** The median is the score that divides the distribution into halves; half of the scores are above the median and half are below it when the data are arranged in numerical order. The median is also referred to as the score at the 50th percentile in the distribution.

Calculation of median :

(i) **Individual series :** If the data is raw, arrange in ascending or descending order. Let n be the number of observations.

If n is odd, Median = value of $\left(\frac{n+1}{2}\right)^{th}$ item.

If n is even, Median = $\frac{1}{2}\left[\text{value of } \left(\frac{n}{2}\right)^{th} \text{ item + value of } \left(\frac{n}{2}+1\right)^{th} \text{ item}\right]$.

(ii) **Discrete series :** In this case, we first find the cumulative frequencies of the variable arranged in ascending or descending order and the median is given by Median $= \left(\frac{n}{2}+1\right)^{th}$ observation, where n is the cumulative frequency.

(iii) **For grouped or continuous distributions :** In this case, following formula can be used.

$$\textbf{Median} = l + \frac{\left(\frac{N}{2} - C\right)}{f} \times i$$

where l = Lower limit of the median class

f = Frequency of the median class

N = The sum of all frequencies

i = The width of the median class

C = The cumulative frequency of the class preceding to median class.

(b) **Quartile :** As median, divides a distribution into two equal parts, similarly the quartiles, quantiles, deciles and percentiles divide the distribution respectively into 4, 5, 10 and 100 equal parts. The j^{th} quartile is given by $Q_j = l + \left(\frac{j\frac{N}{10} - C}{f}\right) i.$

Illustration :

The marks obtained by 10 students in an examination are 22, 26, 14, 30, 18, 11, 35, 41, 12, 32. What is the median mark?

Sol. Number of students (n) = 10 and marks obtained by them = 22, 26, 14, 30, 18, 11, 35, 41, 12, 32

Arranging the given marks in the ascending order, we get 11, 12, 14, 18, 22, 26, 30, 32, 35, 41.

Since the number of students is even, therefore median of their marks

= Arithmetic mean of $\left(\frac{10}{2}\right)$ and $\left(\frac{10+2}{2}\right)$ marks

= Arithmetic mean of 5^{th} and 6^{th} marks

$= \frac{22+26}{2} = 24$ **Ans.**

Illustration :

Calculate the median of the following data:

Wages per week (in Rs)	10 – 20	20 – 30	30 – 40	40 – 50	50 – 60	60 – 70	70 – 80
Number of worker	4	6	10	20	10	6	4

Sol. Calculation of Mean Deviation from Median

Wages per week (in Rs.)	Mid-Values (x_i)	Frequency (f_i)	Cumulative Frequency
10-20	15	4	4
20-30	25	6	10
30-40	35	10	20
40-50	45	20	40
50-60	55	10	50
60-70	65	6	56
70-80	75	4	60
		$N = \Sigma f_i = 60$	

Here, $N = 60$. So $\frac{N}{2} = 30$.

The cumulative frequency just greater than $\frac{N}{2} = 30$ is 40 and the corresponding class is 40-50.

So, 40-50 is the median class.

$\therefore \quad l = 40, f = 20, h = 10, F = 20$

Now, $\text{Median} = l + \frac{\frac{N}{2} - F}{f} \times h = 40 + \frac{30-20}{20} \times 10 = 55$ **Ans.**

(C) MODE :

Mode is the most frequent score in the distribution. A distribution where a single score is most frequent has one mode and is called unimodal. When there are ties for the most frequent score, the distribution is bimodal if two scores tie or multimodal if more than two scores tie.

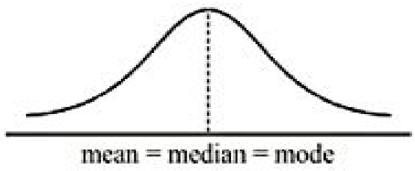
Mode for continuous series

$$\text{Mode} = l_1 + \left[\frac{f_1 - f_0}{2f_1 - f_0 - f_2}\right] \times i$$

Where, l_1 = The lower limit of the model class
f_1 = The frequency of the model class
f_0 = The frequency of the class preceding the model class
f_2 = The frequency of the class succeeding the model class
i = The size of the model class.

Symmetric distribution :

A distribution is a symmetric distribution if the values of mean, mode and median coincide. In a symmetric distribution frequencies are symmetrically distributed on both sides of the centre point of the frequency curve.

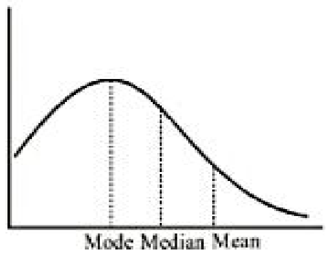


A distribution which is not symmetric is called a skewed distribution. In a moderately asymmetric distribution, the interval between the mean and the median is approximately one-third of the interval between the mean and the mode i.e., when have the following empirical relation between them,
Mean – Mode = 3 (Mean – Median) $\Rightarrow$ Mode = 3 Median – 2 Mean. it is known as Empirical relation.

Positively skewed :

A distribution is positively skewed when is has a tail extending out to the right (larger numbers) When a distribution is positively skewed, the mean is greater than the median reflecting the fact that the mean is sensitive to each score in the distribution and is subject to large shifts when the sample is small and contains extreme scores.

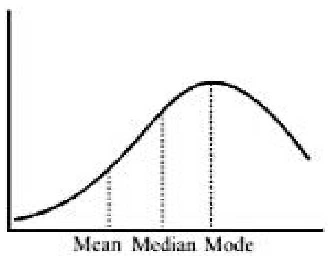
Mean > Median > Mode



Negatively skewed :

A negatively skewed distribution has an extended tail pointing to the left (smaller numbers) and reflects bunching of numbers in the upper part of the distribution with fewer scores at the lower end of the measurement scale.

Mean < Median < Mode.



In a moderately asymmetric distribution, the interval between the mean and the median is approximately one-third of the interval between the mean and the mode i.e., when have the following empirical relation between them,

Empirical formula : mode = 3 median – 2 mean

$$\text{Coefficient of skewness} = \frac{\text{Mean} - \text{Mode}}{\sigma}$$

Limitations of central values :

An average, such as the mean or the median only locates the centre of the data and does not tell us anything about the spread of the data.

4. MEASURES OF SPREAD OR DISPERSION :

Measures of variability provide information about the degree to which individual scores are clustered about or deviate from the average value in a distribution i.e.,

The degree to which numerical data tend to spread about an average value is called the dispersion of the data. The four measure of dispersion are

(i)	Range	(ii)	Mean deviation
(iii)	Variance	(iv)	Standard deviation

Important Note :

(a) A small value for a measure of dispersion indicate that the data are clustered closely (the mean is therefore representative of the data).

(b) A large value of dispersion indicates that the mean is not reliable (it is not representative of the data).

(i) Range :

The simplest measure of variability to compute and understand is the range. The range is the difference between the highest and lowest score in a distribution. Because it is based solely on the most extreme scores in the distribution and does not fully reflect the pattern of variation within a distribution, the range is a very limited measure of variability.

Coefficient of range : $\frac{L-S}{L+S}$

L = Largest value

S = Smallest value

(ii) Mean deviation :

The arithmetic average of the deviations (all taking positive) from the mean, median or mode is known as mean deviation.

(a) Mean deviation from ungrouped data (or individual series)

$$\text{Mean deviation} = \frac{1}{N}\sum_{i=1}^{n}|x_i - M|.$$

Where $\sum_{i=1}^{n}|x_i - M|$ is the sum of modulus of the deviation of the variate from the mean (mean, median or mode) and N is the number of terms.

(b) Mean deviation from continuous series :

Here first of all we find the mean from which deviation is to be taken. Then we find the deviation $|x_i - M|$ of each variate from the mean M and multiply these deviations by the corresponding frequency

$$\text{So, Mean deviation} = \frac{1}{N}\sum_{i=1}^{n} f_i|x_i - M|, \quad \text{where } N = \sum_{i=1}^{n} f_i.$$

Illustration :

The scores of a batsman in ten innings are : 38, 70, 48, 34, 42, 55, 63, 46, 54, 44. Find the mean deviation about the median.

Sol. Arranging the data in ascending order, we have

34, 38, 42, 44, 46, 48, 54, 55, 63, 70

Here n = 10. So, median is the A.M. of 5th and 6th observations.

$$\therefore \quad \text{Median, } M = \left(\frac{46+48}{2}\right) = 47$$

Calculation of Mean Deviation

x_i	$\lvert d_i\rvert = \lvert x_i - 47\rvert$
38	9
70	23
48	1
34	13
42	5
55	8
63	16
46	1
54	7
44	3
Total	$\Sigma \lvert d_i\rvert = 86$

$$\therefore \quad \text{M.D.} = \frac{1}{n}\Sigma|d_i| = \frac{86}{10} = 8.6 \quad \textbf{Ans.}$$

Illustration :

Calculate the mean deviation from the median of the following data:

Age	16 – 20	21 – 25	26 – 30	31 – 35	36 – 40	41 – 45	46 – 50	51 – 55
Number	5	6	12	14	26	12	16	9

Since given data is not continuous frequency distribution but we can make it continuous frequency distribution by subtracting lower limit by 0.5 and adding 0.5 to upper limit of every group.

Sol. Calculation of Mean Deviation from Median

Age	Mid-Values (x_i)	Frequency (f_i)	Cumulative Frequency	$\lvert d_i \rvert = \lvert x_i - 38 \rvert$	$f_i \lvert d_i \rvert$
15.5-20.5	18	5	5	20	100
20.5-25.5	23	6	11	15	90
25.5-30.5	28	12	23	10	120
30.5-35.5	33	14	37	5	70
35.5-40.5	38	26	63	0	0
40.5-45.5	43	12	75	5	60
45.5-50.5	48	16	91	10	160
50.5-55.5	53	9	100	15	135
		$N = \Sigma f_i = 100$			$\Sigma f_i \lvert d_i \rvert = 735$

Here, $N = 100$. So $\frac{N}{2} = 50$.

The cumulative frequency just greater than $\frac{N}{2} = 50$ is 63 and the corresponding class is 35.5-40.5.

So, 35.5-40.5 is the median class.

$\therefore \quad l = 35.5,\ f = 26,\ h = 5,\ C = 37$

Now, $\text{Median} = l + \frac{\frac{N}{2} - C}{f} \times h = 35.5 + \left(\frac{50 - 37}{26}\right) \times 5 = 38$ **Ans.**

Mean Deviation from median $= \frac{\Sigma f_i \lvert d_i \rvert}{N} = \frac{735}{100} = 7.35$ **Ans.**

(iii) Variance or Var(X) or σ^2 :

The variance is a measure based on the deviations of individual scores from the mean. As noted in the definition of the mean, however, simply summing the deviations will result in a value of 0. The get around this problem the variance is based on squared deviations of scores about the mean. When the deviations are squared, the rank order and relative distance of scores in the distribution is preserved while negative values are eliminated. Then to control for the number of subjects in the distribution, the sum of the squared deviations, $\sum(X-\overline{X})^2$, is divided by N(population). *The average of the sum of the squared deviations is called the variance.*

(a) Variance of individual observations :

If $x_1, x_2, \ldots\ldots, x_n$ are n values of a variable X, then

$$\text{Var}(X) = \frac{1}{n}\left|\sum_{i=1}^{n}(x_i-\overline{X})^2\right| = \frac{1}{n}\sum_{i=1}^{n}x_i^2 - \left(\frac{1}{n}\sum_{i=1}^{n}x_i\right)^2$$

= Mean of squares – Squares of Mean

(b) Variance of discrete frequency distribution :

If $x_1, x_2, \ldots\ldots, x_n$ are n values of a variable X and corresponding frequencies of them are $f_1, f_2, \ldots\ldots f_n$

$$\text{Var (X)} = \frac{1}{N}\left|\sum_{i=1}^{n}f_i(x_i-\overline{X})^2\right| = \frac{1}{N}\sum_{i=1}^{n}f_ix_i^2 - \left(\frac{1}{N}\sum_{i=1}^{n}f_ix_i\right)^2 \qquad \left(\sum_{i=1}^{n}f_i = N\right)$$

(c) Variance of a grouped or continuous frequency distribution :

$$\text{Var (X)} = h^2\left[\frac{1}{N}\sum f_iu_i^2 - \left(\frac{1}{N}\sum f_iu_i\right)^2\right] \qquad u_i = \frac{x_i-\overline{X}}{h}$$

where h = Class width

Properties :

(1) If $x_1, x_2, x_3 \ldots\ldots, x_n$ be n values of a variable X. If these values are changed to $x_1 + a, x_2 + a, \ldots. x_n + a$, where $a \in R$, then the variance remains unchanged.

(2) If $x_1, x_2, \ldots\ldots, x_n$ values of a variable X and let 'a' be a non-zero real number. Then, the variance of the observation $ax_1, ax_2, \ldots\ldots, ax_n$ is $\mathbf{a^2\,Var(X)}$.

(iv) Standard Deviation :

The standard deviation (s or σ) is defined as the positive square root of the variance. The variance is a measure in squared units and has little meaning with respect to the data. Thus, the standard deviation is a measure of variability expressed in the same units as the data. The standard deviation is very much like a mean or an "average" of these deviations.

Combined Standard Deviation :

If there are two sets of observations containing n_1 & n_2 items with respective mean $\bar{x}_1$ & $\bar{x}_2$ and standards deviations σ_1 & σ_2, then the mean $\bar{x}$ and the standard deviations of $n_1 + n_2$ observations, taken together, are

$$\bar{x} = \frac{n_1\bar{x}_1 + n_2\bar{x}_2}{n_1 + n_2}$$

$$\sigma^2 = \frac{1}{n_1 + n_2}\left[n_1\left(\sigma_1^2 + d_1^2\right) + n_2\left(\sigma_2^2 + d_2^2\right)\right]$$

where $d_1 = \bar{x} - \bar{x}_1$, $d_2 = \bar{x} - \bar{x}_2$

Illustration :

Calculate the mean and standard deviation of first n natural numbers.

Sol. Here $x_i = i = i = 1, 2, ..., n$. Let $\overline{X}$ be the mean and σ be the S.D. Then,

$$\overline{X} = \frac{1}{n}\sum_{i=1}^{n} x_i = \frac{1}{n}\sum_{i=1}^{n} i = \frac{1}{n}(1 + 2 + 3 + ... + n)$$

$$\Rightarrow \quad \overline{X} = \frac{n(n+1)}{2n} = \frac{n+1}{2}$$

and $\sigma^2 = \frac{1}{n}\left(\sum_{i=1}^{n} {x_i}^2\right) - \left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)^2 \quad \Rightarrow \quad \sigma^2 = \frac{1}{n}(1^2 + 2^2 + ... + n^2) - \left(\frac{n+1}{2}\right)^2$

$$\Rightarrow \quad \sigma^2 = \frac{n(n+1)(2n+1)}{6n} - \left(\frac{n+1}{2}\right)^2 \quad \Rightarrow \quad \sigma^2 = \frac{(n+1)(2n+1)}{6} - \frac{(n+1)^2}{4} = \frac{n^2-1}{12}$$ **Ans.**

Illustration :

The mean and variance of 7 observations are 8 and 16 respectively. If 5 of the observations are 2, 4, 10, 12, 14, find the remaining two observations.

Sol. Let x and y be the remaining two observation. Then,

Mean = 8

$$\Rightarrow \quad \frac{2+4+10+12+14+x+y}{7} = 8 \quad \Rightarrow \quad 42 + x + y = 56$$

$$\Rightarrow \quad x + y = 14 \qquad .....(i)$$

Variance = 16

$$\Rightarrow \quad \frac{1}{7}(2^2 + 4^2 + 10^2 + 12^2 + 14^2 + x^2 + y^2) - (\text{Mean})^2 = 16$$

$$\Rightarrow \quad \frac{1}{7}(4 + 16 + 100 + 144 + 196 + x^2 + y^2) - 64 = 16 \Rightarrow 460 + x^2 + y^2 = 7 \times 80$$

$$\Rightarrow \quad x^2 + y^2 = 100 \qquad .....(ii)$$

Now, $(x + y)^2 + (x - y)^2 = 2(x^2 + y^2)$

$\Rightarrow \quad 196 + (x - y)^2 = 2 \times 100 \Rightarrow (x - y)^2 = 4 \Rightarrow x - y = \pm 2$

If $x - y = 2$, then $x + y = 14$ and $x - y = 2 \Rightarrow x = 8, y = 6$

If $x - y = -2$, then $x + y = 14$ and $x - y = -2 \Rightarrow x = 6, y = 8$

Hence, the remaining two observations are 6 and 8.

Illustration :

Find the variance and standard deviation for the following distribution:

Classes	30 – 40	40 – 50	50 – 60	60 – 70	70 – 80	80 – 90	90 – 100
Frequency	3	7	12	15	8	3	2

Sol. Calculation of Variance and Standard Deviation

Class	Frequency (f_i)	Mid-point (x_i)	$y_i = \frac{x_i - 65}{10}$	y_i^2	$f_i y_i$	$f_i y_i^2$
30-40	3	35	–3	9	–9	27
40-50	7	45	–2	4	–14	28
50-60	12	55	–1	1	–12	12
60-70	15	65	0	0	0	0
70-80	8	75	1	1	8	8
80-90	3	85	2	4	6	12
90-100	2	95	3	9	6	18
	N = 50				–15	105

Therefore $\bar{x} = A + \frac{\sum f_i y_i}{50} \times h = 65 - \frac{15}{50} \times 10 = 62$

variance $\sigma^2 = \frac{h^2}{N^2}\left[N\sum f_i y_i^2 - \left(\sum f_i y_i\right)^2\right] = \frac{(10)^2}{(50)^2}\left[50 \times 105 - (-15)^2\right] = \frac{1}{25}[5250 - 225] = 201$

and standard deviation $(\sigma) = \sqrt{201} = 14.18$ **Ans.**

Illustration :

The mean and standard deviation of 20 observations are found to be 10 and 2 respectively. On rechecking, it was found that an observation 8 was incorrect. Calculate the correct mean and standard deviation in each of the following cases:

(i) If the wrong item is omitted.

Sol. We have , $n = 20$, $\overline{X} = 10$ and $\sigma = 2$

$$\therefore \quad \overline{X} = \frac{1}{n}\Sigma x_i \quad \Rightarrow \quad \Sigma x_i = n\overline{X} = 20 \times 10 = 200 \quad \Rightarrow \quad \text{Incorrect } \Sigma x_i = 200$$

$$\text{and,} \quad \sigma = 2 \quad \Rightarrow \quad \sigma^2 = 4 \quad \Rightarrow \quad \frac{1}{n}\Sigma x_i^2 - (\text{Mean})^2 = 4$$

$$\Rightarrow \quad \frac{1}{20}\Sigma x_i^2 - 100 = 4 \quad \Rightarrow \quad \Sigma x_i^2 = 104 \times 20 \quad \Rightarrow \quad \text{Incorrect } \Sigma x_i^2 = 2080$$

(i) When 8 is omitted from the data.

If 8 is omitted from the data, then 19 observations are left.

Now Incorrect $\Sigma x_i = 200$ $\Rightarrow$ Correct $\Sigma x_i + 8 = 200$ $\Rightarrow$ Correct $\Sigma x_i = 192$

and Incorrect $\Sigma x_i^2 = 2080$ $\Rightarrow$ Correct $\Sigma x_i^2 + 8^2 = 2080$ $\Rightarrow$ Correct $\Sigma x_i^2 = 2016$

$$\therefore \quad \text{Correct mean} = \frac{192}{19} = 10.10$$

$$\Rightarrow \quad \text{Correct variance} = \frac{1}{19}(\text{Correct } \Sigma x_i^2) - (\text{Correct mean})^2$$

$$\Rightarrow \quad \text{Correct variance} = \frac{2016}{19} - \left(\frac{192}{19}\right)^2$$

$$\text{Correct variance} = \frac{38304 - 36864}{361} = \frac{1440}{361}$$

$$\therefore \quad \text{Correct standard deviation} = \sqrt{\frac{1440}{361}} = \frac{12\sqrt{10}}{19} = 1.997$$

Analysis of Frequency Distributions :

Measures of dispersion are unable to compare two or more series which are measured in different units even if they have the same mean. Thus, we require those measures which are independent of the units. The measure of variability which is independent of units is called coefficient of variation (C.V.). The coefficient of variation is defined as

$$\text{C.V.} = \frac{\sigma}{\overline{X}} \times 100$$

where σ and $\overline{X}$ are the standard deviation and mean of the data.

For comparing the variability of two series, we calculate the coefficient of variation for each series. The series having greater C.V. is said to be more variable or conversely less consistent, less uniform less stable or less homogeneous than the other and the series having lesser C.V. is said to be more consistent (or homogeneous) than the other.

Illustration :

The following values are calculated in respect of heights and weights of the students of a section of Class XI :

	Height	Weight
Mean	162.6 cm	52.36
Variance	127.69 cm^2	23.1361 kg^2

Can we say that the weights show greater variation than the heights ?

Sol. To compare the variability, we have to calculate their coefficients of variation

Given Variance of height = 127.69 cm^2

Therefore Standard deviation of height $\sqrt{127.69}$ cm = 11.3 cm

Also Variance of weight = 23.1361 kg^2

Therefore Standard deviation of weight = $\sqrt{23.1361}$ kg = 4.81 kg

Now, the coefficient of variations (C.V.) are given by

$$\text{(C.V.) in heights} = \frac{\text{Standard Deviation}}{\text{Mean}} \times 100$$

$$= \frac{11.3}{162.6} \times 100 = 6.95$$

and $$\text{(C.V.) in weight} = \frac{4.81}{52.36} \times 1000 = 9.18$$

Clearly C.V. in weights is greater than the C.V. in heights

Therefore, we can say that weights show more variability than heights.

IMPORTANT DEFINITIONS :

1. Raw Data :

Data collected in original form.

2. Frequency :

The number of times a certain value or class of values occurs.

3. Frequency Distribution :

The organization of raw data in table form with classes and frequencies.

4. Categorical Frequency Distribution :

A frequency distribution in which the data is only nominal or ordinal.

5. Ungrouped Frequency Distribution :

A frequency distribution of numerical data. The raw data is not grouped.

6. Grouped Frequency Distribution :

A frequency distribution where several numbers are grouped into one class.

7. **Class Limits :**

Separate one class in a grouped frequency distribution from another. The limits could actually appear in the data and have gaps between the upper limit of one class and the lower limit of the next.

8. **Class Boundaries :**

Separate one class in a grouped frequency distribution from another. The boundaries have one more decimal place than the raw data and therefore do not appear in the data. There is no gap between the upper boundary of one class and the lower boundary of the next class. The lower class boundary is found by subtracting 0.5 units from the lower class limit and the upper class boundary is found by adding 0.5 units to the upper class limit.

9. **Class Width :**

The difference between the upper and lower boundaries of any class. The class width is also the difference between the lower limits of two consecutive classes or the upper limits of two consecutive classes. It is not the difference between the upper and lower limits of the same class.

10. **Class Mark (Midpoint) :**

The number in the middle of the class. It is found by adding the upper and lower limits and dividing by two. It can also be found by adding the upper and lower boundaries and dividing by two.

11. **Cumulative Frequency :**

The number of values less than the upper class boundary for the current class. This is a running total of the frequencies.

12. **Relative Frequency :**

The frequency divided by the total frequency. This gives the percent of values falling in that class.

13. **Cumulative Relative Frequency (Relative Cumulative Frequency) :**

The running total of the relative frequencies or the cumulative frequency divided by the total frequency. Gives the percent of the values which are less than the upper class boundary.

14. **Histogram :**

A graph which displays the data by using vertical bars of various heights to represent frequencies. The horizontal axis can be either the class boundaries, the class marks, or the class limits.

15. **Frequency Polygon :**

A line graph. The frequency is placed along the vertical axis and the class midpoints are placed along the horizontal axis. These points are connected with lines.

16. **Ogive :**

A frequency polygon of the cumulative frequency or the relative cumulative frequency. The vertical axis the cumulative frequency or relative cumulative frequency. The horizontal axis is the class boundaries. The graph always starts at zero at the lowest class boundary and will end up at the total frequency (for a cumulative frequency) or 1.00 (for a relative cumulative frequency).

MATHEMATICAL REASONING

STATEMENT:

A sentence is called a mathematically acceptable statement if it is either true or false but not both. A statement is neither imperative, nor interrogative nor exclamatory. A sentence which is a request, or a command is not a statement .

Ex : The following are the statements

(a) 6 is less than 8

(b) 2 is an odd number

(c) Every square is a rectangle

(d) New Delhi is in India

Note : A sentence which is both true and false simultaneously is not a statement. Such a sentence is called a **paradox**.

OPEN STATEMENT :

A declarative sentence containing variable (s) is an open statement if it becomes a statement when the variable(s) is (are) replaced by some definite value (s).

e.g. $s = x$ is an integer

s is true if x is integer and false if x is not integer.

COMPOUND STATEMENTS :

A compound statement is a statement which is made up of two or more statements. In this case, each statement is called a component statement.

e.g. All rational numbers are real and all real numbers are complex.

The component statement are

p : all rational numbers are real

q : all real numbers are complex numbers

CONJUNCTION :

Compound statement are combined by the word "and "($\wedge$) the resulting statement is called a conjunction denoted as $p \wedge q$.

e.g. A point occupies a position and its location can be determined.

The component statement are

p : A point occupies a position

q : Its location can be determined

Both statements are true.

Imp. : Do not think that a statement with "And" is always a compound statement.

e.g. A mixture of alcohol and water can be separated by chemical methods.

(Here "And" refers to two things).

Note :

(i) The compound statement with 'And' is true if all its component statements are true.

(ii) The compound statement with 'And' is false if any of its component statements is false (this includes the case that some of its component statements are false or all of its component statements are false).

The following truth table shows the truth values of $p \wedge q$ (p and q) and $q \wedge p$ (q and p).

Truth Table ($p \wedge q$, $q \wedge p$)			
p	q	$p \wedge q$	$q \wedge p$
T	T	T	T
T	F	F	F
F	T	F	F
F	F	F	F
Rule : $p \wedge q$ is true only when p and q are true.			

Remark : The above truth table shows that $p \wedge q = q \wedge p$.

DISJUNCTION OR ALTERNATION :

Compound statements p and q are combined by the connective 'OR' ($\vee$), then the compound statement denoted as $p \vee q$ (p or q) so formed is called a disjunction.
e.g. Two lines in a plane either intersect at one point or they are parallel.

Sometimes we use the connective 'either ... or ... ' to obtain $p \vee q$ and read $p \vee q$ as 'either p or q'.

Note :

(i) A compound statement with an 'Or' is true when one component statement is true or both the component statement are true.

(ii) A compound statement with an 'Or' is false when both the component statements are false.

Truth Table ($p \vee q$, $q \vee p$)			
p	q	$p \vee q$	$q \vee p$
T	T	T	T
T	F	T	T
F	T	T	T
F	F	F	F
Rule : $p \vee q$ is false only when both p and q are false.			

Imp. : e.g. A student who has taken biology or chemistry can apply for M.Sc. microbiology program. This means that student who have taken both biology and chemistry or only biology or only chemistry can apply for the microbiology program. This is example of **inclusive "Or"**. In this case truth table is same as $p \vee q$.

Imp. : e.g. Student can take French or Sanskrit as their third language.
This means that student have to choose only one subject from French and Sanskrit. It exclude the case when one student can choose both subject. This is case of **exclusive "Or"**. This is represented as $p \veebar q$ or $p \otimes q$. Truth table for exclusive or is as follows.

Truth Table ($p \veebar q$)		
p	q	$p \veebar q$ or $p \otimes q$
T	T	F
T	F	T
F	T	T
F	F	F
Rule : $p \veebar q$ is true only when one of p and q is true and the other is false.		

NEGATION (OR DENIAL) :

The denial of a statement is called the negation of the statement denoted as ~.
e.g. p : Everyone in Germany speaks German.
~ p : it is false that everyone in Germany speaks German.

While forming the negation of a statement, phrases like, "It is not the case" or "It is false that" are also used.
e.g. p : All integers are rational numbers".
~ p : Atleast one integer is not a rational number.

If p is true then ~ p must be false and if p is false then ~ p must be true

Truth Table (~ p)		
p	~ p	~ (~ p)
T	F	T
F	T	F
Rule : ~ is true only when p is false.		

It may be noticed that ~ (~p) = p. Also p and ~ p are contrary.
e.g. the statements 'x is an even number' and ' x is an odd number' are contrary if x is a whole number because both the statements cannot have the same truth value.

Imp. : If may be observed that negation is not a binary operation, it is a unary operation i.e. a modifier.
1. ~ p is true iff p is false.
2. ~ p is false iff p is true.

Quantifiers :

Quantifiers are phrases like "There exists" and "for all".

Negation of Quantifiers

(i) P = There exist a number which is equal to its square .

~ P = There does not exist a number which is not equal to its square.

(ii) P = For every real number x, x is less than x + 1.

~ P = There exist a number for which x is not less than x + 1.

IMPLICATION :

There are three types of implications :

(i) "If...... then"

(ii) "Only if"

(iii) "If and only if"

(1) "If...... then" type of compound statement is called **conditional statement**.

The statement 'if p then q' is denoted by $p \rightarrow q$ (to be read as 'p implies q') or by $p \Rightarrow q$. Note that $p \rightarrow q$ also means

(i) p is sufficient for q (ii) q is necessary for p (iii) p only if q

(iv) p lead to q (v) q if p (vi) q when p

(vii) if p, then q

e.g. p : a number is a multiple of 9

q : a number is a multiple of 3.

Then $p \rightarrow q$ or $p \Rightarrow q$

$p \rightarrow q$ is false only when p is true and q is false. Truth table for $p \rightarrow q$ is as follows.

Truth Table ($p \rightarrow q$, $q \rightarrow p$)			
p	q	$p \rightarrow q$	$q \rightarrow p$
T	T	T	T
T	F	F	T
F	T	T	F
F	F	T	T
Rule : $p \rightarrow q$ is false only when p is true and q is false.			

(2) "If and only if " type of compound statement is called **Biconditional or equivalence** or **'double implication'**. Symbolically 'p iff q' is represented by $p \leftrightarrow q$ or by $p \Leftrightarrow q$.

(i) p is a necessary and sufficient condition for q.

(ii) q is necessary and sufficient condition for p.

(iii) If p then q and if q then p

(iv) q if and only if p.

e.g. p : If the sum of digits of a number is divisible by 3, then the number is divisible by 3.

q : If a number is divisible by 3, then the sum of its digits is divisible by 3.

A number is divisible by 3 if and only if the sum of its digits is divisible by 3.

The following are other illustrations which actually do not appear to be so but they infact are biconditional.

(i) If you work hard only then you can succeed.

(ii) You can go on leave only if your boss permits. The truth table for biconditional is as follows:

Truth Table ($p \leftrightarrow q$, $q \leftrightarrow p$)			
p	**q**	**$p \leftrightarrow q$**	**$q \rightarrow p$**
T	T	T	T
T	F	F	F
F	T	F	F
F	F	T	T
Rule : $p \leftrightarrow q$ is true only when both p and q have the same truth value.			

CONTRAPOSITIVE AND CONVERSE :

Contrapositive and converse are certain other statements which can be formed from a given statement with "if..........then".

Contrapositive of $p \rightarrow q$ is $\sim q \rightarrow \sim p$

e.g. If a number is multiple of 6 then it is multiple of 2.

Contrapositive If a number is not multiple of 2 then it is not multiple of 6.

Converse of $p \rightarrow q$ is $q \rightarrow p$

e.g. If the angles of a triangle are equal then it is equilateral triangle.

Converse is if triangle is equilateral then angles of triangle are equal.

Truth Table ($p \rightarrow q$)				
p	q	$p \rightarrow q$	Contrapositive ($\sim q \rightarrow \sim p$)	Converse ($q \leftrightarrow p$)
T	T	T	T	T
T	F	F	F	T
F	T	T	T	F
F	F	T	T	T

Note : Truth table for $p \rightarrow q$ is same as its contrapositive.

NEGATION OF IMPLICATION :

If p and q are two statements, then

$\sim(p \Rightarrow q) = p \wedge \sim q$ $\qquad [\because\ p \Rightarrow q \equiv \sim p \wedge q]$

Proof :

p	q	$p \Rightarrow q$	$\sim(p \Rightarrow q)$
T	T	T	F
T	F	F	T
F	T	T	F
F	F	T	F

p	q	$\sim q$	$p \wedge \sim q$
T	T	F	F
T	F	T	T
F	T	F	F
F	F	T	F

Negation of Biconditional statement or equivalence theorem :

If p and q are two statement, then

$$\sim(p \Leftrightarrow q) = (p \wedge \sim q) \vee (q \wedge \sim p)$$

Proof :

$$p \Leftrightarrow q = (p \Rightarrow q) \wedge (q \Rightarrow p)$$

$$\therefore \quad \sim(p \Leftrightarrow q) = \{(p \Rightarrow q) \wedge (q \Rightarrow p)\}$$

$$= [\sim(p \Rightarrow q)] \vee [\sim(q \Rightarrow p)]$$

$$= (p \wedge \sim q) \vee (q \wedge \sim p)$$

TAUTOLOGIES AND FALLACIES :

The compound statements (or propositions) which are true for any truth value of their components are called '**Tautologies**'.

e.g. '$p \vee \sim p$' is a tautology, p being logical statement. This is illustrated by the truth table given below which shows only Ts in the last column.

Truth Table ($p \vee \sim p$)		
p	$\sim p$	$p \wedge \sim p$
T	F	T
F	T	T

The negation of tautology is called a **fallacy** or a **contradiction** i.e. a proposition which is false for any truth value of their components is called a fallacy. For example '$p \wedge \sim p$' is a fallacy, p being any logical statement. This is illustrated by the truth table given below which shows only Fs in the last column.

Truth Table ($p \wedge \sim p$)		
p	$\sim p$	$p \wedge \sim p$
T	F	F
F	T	F

Note :

(i) $p \vee q$ is true iff at least one of p and q is true.

(ii) $p \veebar q$ is true iff exactly one of p and q is true and the other is false.

(iii) $p \wedge q$ is true iff both p and q are true.

(iv) A tautology is always true.

(v) A fallacy is always false.

ALGEBRA OF STATEMENTS :

Statements satisfy many laws some of which are given below -

(1) **Idempotent Laws :** If p is any statement then

(i) $p \vee p = p$ (ii) $p \wedge p = p$

(2) **Associative Laws :** If p, q, r are any three statements, then

(i) $p \vee (q \vee r) = (p \vee q) \vee r$ (ii) $p \wedge (q \wedge r) = (p \wedge q) \wedge r$

(3) **Commutative Laws :** If p, q are any two statements, then

(i) $p \vee q = q \vee p$ (ii) $p \wedge q = q \wedge p$

(4) **Distributive Laws :** If p, q, r are any three statements, then

(i) $p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r)$ (ii) $p \vee (q \wedge r) = (p \vee q) \wedge (p \vee r)$

(5) **Identity Laws :** If p is any statement, t is tautology and c is a contradiction, then

(i) $p \vee t = t$ (ii) $p \wedge t = p$ (iii) $p \vee c = p$ (iv) $p \wedge c = c$

(6) **Complement Laws :** If t is a tautology, c is a contradiction and p is any statement, then

(i) $p \vee (\sim p) = t$ (ii) $p \wedge (\sim p) = c$ (iii) $\sim t = c$ (iv) $\sim c = t$

(7) **Involution Law :** If p is any statement, then $\sim(\sim p) = p$.

(8) **De-morgan's Law :** If p and q are two statements, then

(i) $\sim (p \vee q) \equiv (\sim p) \wedge (\sim q)$ (ii) $\sim (p \wedge q) \equiv (\sim p) \vee (\sim q)$

(i) **Proof :**

$\sim (p \vee q) \equiv (\sim p) \wedge (\sim q)$

p	q	$p \vee q$	$\sim(p \vee q)$	$\sim p$	$\sim q$	$(\sim p) \wedge (\sim q)$
T	T	T	F	F	F	F
T	F	T	F	F	T	F
F	T	T	F	T	F	F
F	F	F	T	T	T	T

(ii) **Proof :**

$\sim (p \wedge q) \equiv (\sim p) \vee (\sim q)$

p	q	$p \wedge q$	$\sim(p \wedge q)$	$\sim p$	$\sim q$	$(\sim p) \vee (\sim q)$
T	T	T	F	F	F	F
T	F	F	T	F	T	T
F	T	F	T	T	F	T
F	F	F	T	T	T	T

SOLVED EXAMPLES

Q.1 By mathematical induction, $\frac{1}{1\cdot2\cdot3}+\frac{1}{2\cdot3\cdot4}+...+\frac{1}{n(n+1)(n+2)}$ is equal to

(A) $\frac{n(n+1)}{4(n+2)(n+3)}$ (B) $\frac{n(n+3)}{4(n+1)(n+2)}$ (C) $\frac{n(n+2)}{4(n+1)(n+3)}$ (D) None of these

Sol. Let $P(n):\frac{1}{1\cdot2\cdot3}+\frac{1}{2\cdot3\cdot4}+...+\frac{1}{n(n+1)(n+2)}=\frac{n(n+3)}{4(n+1)(n+2)}$

(i) For n = 1

$$\text{L.H.S.}=\frac{1}{1\cdot2\cdot3}=\frac{1}{6}\text{ and R.H.S.}=\frac{1(1+3)}{4(1+1)(1+2)}=\frac{1}{6}$$

$\therefore$ P(1) is true.

(ii) Let P(k) be true, then

$$P(k):\frac{1}{1\cdot2\cdot3}+\frac{1}{2\cdot3\cdot4}+...+\frac{1}{k(k+1)(k+2)}$$

$$=\frac{k(k+3)}{4(k+1)(k+2)} \qquad ....(1)$$

(iii) For n = k + 1,

$$P(k+1):\frac{1}{1\cdot2\cdot3}+\frac{1}{2\cdot3\cdot4}+...+\frac{1}{k(k+1)(k+2)}+\frac{1}{(k+1)(k+2)(k+3)}=\frac{(k+1)(k+4)}{4(k+2)(k+3)}$$

$$\text{L.H.S.}=\frac{1}{1\cdot2\cdot3}+\frac{1}{2\cdot3\cdot4}+...+\frac{1}{k(k+1)(k+2)}+\frac{1}{(k+1)(k+2)(k+3)}$$

$$=\frac{k(k+3)}{4(k+1)(k+2)}+\frac{1}{(k+1)(k+2)(k+3)} \qquad \text{[From Eq. (1)]}$$

$$=\frac{k(k+3)^2+4}{4(k+1)(k+2)(k+3)}=\frac{k^3+6k^2+9k+4}{4(k+1)(k+2)(k+3)}=\frac{(k+1)^2(k+4)}{4(k+1)(k+2)(k+3)}$$

$$=\frac{(k+1)(k+4)}{4(k+2)(k+3)}=\text{R.H.S.}$$

Hence, P(k + 1) is true.

Hence, by the principle of mathematical induction for all $n\in N$, P(n) is true. **Ans. (B)**

Q.2 For all $n \in N$, $(3)(5^{2n+1}) + 2^{3n+1}$ is divisible by

(A) 17 (B) 19 (C) 21 (D) 23

Sol. $P(n) : 3(5^{2n+1}) + 2^{3n+1}$

$P(1) : 3(5^3) + 2^4 = 3(125) + 16$

$= 375 + 16 = 391 = 17(23)$

So, P(1) is divisible by 17.

Let $P(k) : 3(5^{2k+1}) + 2^{3k+1} = 17m$

$\therefore \quad P(k+1) : 3(5^{2k+3}) + 2^{3k+4} = 17\lambda$

L.H.S. of $P(k+1) = 3(5^{2k+3}) + 2^{3k+4}$

$= 3(5^{2k+1})(5^2) + (2^{3k+1})(2^3)$

$= (17m - 2^{3k+1})(25) + 8(2^{3k+1})$ [Assuming P(k) to be true]

$= 17(25m) - 25(2^{3k+1}) + 8(2^{3k+1})$

$= 17(25m) - 17(2^{3k+1})$

$= 17(25m - 2^{3k+1}) = 17\lambda$

Thus, $P(k+1)$ is divisible by 17 whenever $P(k)$ is divisible by 17. Hence $P(n)$ is divisible by 17 for all $n \in N$. **Ans. (A)**

Q.3 If $x^{2n-1} + y^{2n-1}$ is divisible by $x + y$, then n is

(A) a positive integer (B) an even positive integer

(C) an odd positive integer (D) none of these

Sol. $P(n) : x^{2n-1} + y^{2n-1}$

$P(1) : x^1 + y^1 = x + y$, which is divisible by $x + y$

Let $P(k) : x^{2k-1} + y^{2k-1} = (x + y)\,m$

$\therefore \quad P(k+1) : x^{2k+1} + y^{2k+1} = (x + y)\,\lambda$

L.H.S. of $P(k+1) = x^{2k+1} + y^{2k+1}$

$= x^2(x^{2k-1}) + y^2(y^{2k-1})$

$= x^2[m(x+y) - y^{2k-1}] + y^2(y^{2k-1})$ [Assuming P(k) to be true]

$= (x+y)(mx^2) - y^{2k-1}(x^2 - y^2)$

$= (x+y)[mx^2 - y^{2k-1}(x-y)] = (x+y)\lambda$

Thus, $P(k+1)$ is divisible by $(x+y)$ whenever $P(k)$ is divisible by $(x+y)$. Hence $P(n)$ is divisible by $x + y$ for all $n \in N$, i.e. for all positive integers. **Ans. (A)**

Q.4 When $P(n) = 9^n - 8^n$ is divided by 8, then the remainder is

(A) 2 (B) 3 (C) 1 (D) 7

Sol. $P(n) = 9^n - 8^n$

$\therefore$ $P(1) = 9 - 8 = 1$

$\therefore$ $P(1) - 1 = 0$ which is divisible by 8

$\therefore$ 1 is the remainder when P(n) is divided by 8

Now, $P(2) = 9^2 - 8^2 = 17 = 16 + 1.$

Remainder is 1, when divided by 8. **Ans.(C)**

Q.5 Let $P(n) = 2^{3n} - 7n - 1$, then P(n) is divisible by

(A) 63 (B) 36 (C) 49 (D) 25

Sol. $P(n) : 2^{3n} - 7n - 1$

$P(1) : 2^3 - 7 - 1 = 8 - 7 - 1 = 0$

$P(2) : 2^6 - 7(2) - 1 = 64 - 14 - 1 = 49.$

which is divisible by 49

Let $P(k) : 2^{3k} - 7k - 1 = 49m$

$\therefore$ $P(k + 1) = 2^{3k+3} - 7(k + 1) - 1 = 49\lambda$

L.H.S. of $P(k + 1) = 2^{3k+3} - 7(k + 1) - 1$

$= (2^{3k})(2^3) - 7k - 7 - 1$

$= (49m + 7k + 1)(8) - 7k - 8$ [Assuming P(k) to be true]

$= 49(8m) + 56k + 8 - 7k - 8$

$= 49(8m + k) = 49\lambda.$

Thus, P(k + 1) is divisible by 49 whenever P(k) is divisible by 49. Hence P(n) is divisible by 49 for all $n \in N$. **Ans.(C)**